A NOTE ON EXISTENCE THEOREM OF PEANO II

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ABSTRACT. An ODE with non-Lipschitz right hand side has been considered. A family of solutions with Borel measurable dependence of the initial data has been obtained.

1. Introduction

Consider a system of ordinary differential equations of the following form

$$\dot{x} = v(t, x), \quad x \in \mathbb{R}^m. \tag{1.1}$$

The vector-function v is defined in the cross product of some interval $[-T, T]$ and a domain $D \subseteq \mathbb{R}^m$.

The simplest and often occurred situation is when the vector field v is continuous and fulfills the Lipschitz condition in the second variable:

$$\|v(t, x') - v(t, x'')\| \le c\|x' - x''\|. \tag{1.2}$$

In such a case problem (1.1) has a unique solution $x(t)$ that satisfies the initial condition $x(0) = x_0 \in D$. This result is known as Cauchy-Picard existence theorem. (All the classical facts we mention without reference are contained in [8].)

In general, the solution $x(t)$ is defined not in the whole interval $[-T, T]$ but in its smaller subinterval. In the described above conditions the solution $x(t)$ depends continuously on the initial data x_0.

The Cauchy-Picard existence theorem as well as its proof transmit literally from the case $x \in \mathbb{R}^m$ to the case when x belongs to an infinite dimensional Banach space.

If we refuse Lipschitz hypothesis (1.2) then our problem becomes widely complicated. Particularly, it is known that in an infinite dimensional Banach space problem (1.1) may have no solutions [19], [7]. In the finite dimensional case the existence is guaranteed by Peano's theorem.

2000 *Mathematics Subject Classification.* 34A12.

Key words and phrases. Peano existence theorem, Non-Lipschitz nonlinearity, non-uniqueness, IVP, ODE, Cauchy problem.

Partially supported by grants RFBR 08-01-00681, Science Sch.-8784.2010.1.

So, when the function v is only continuous in $[-T,T]\times D$ then for the same initial datum x_0 there may be several solutions. Nevertheless if by some reason for any initial condition x_0 the solution is unique then it depends continuously on the initial data.

There are a lot of works devoted to investigating of different types of the uniqueness conditions. As far as the author knows this activity has been started from Kamke [9] and Levy [14]. Their results have been generalized in different directions. See for example [15], [1] and references therein. Anther approach is contained in [13], [2].

The problem of existence of individual solutions to ODE with measurable in t and continuous in x right-hand side has been considered by Caratheodory in [3].

The case when the vector field belongs to Sobolev spaces (at least $H^{1,1}$) has been studied in [5] in connection with the Navier-Stokes equation. In this article the results on existence and dependence on the initial data have been obtained.

When problem (1.1) admits non-uniqueness then for some initial data x_0 there are many ways to pick up a solution $x(t)$ such that $x(0)=x_0$. Actually we even do not know how many ways to do this we have and how many such points x_0 are there. An attempt to clarify the last question has been done in [17]. The main result of that article is as follows: the initial data with non-unique solution form a Borel set of the class $F_{\sigma\delta}$.

Anyway for each x_0 we can choose one of the solutions $x(t)$ such that $x(0)=x_0$ and write

$$x(t)=x(t,x_0),\quad x(0,x_0)=x_0.$$

At this moment our argument is heavily rested on the Axiom of Choice.

From analysis we know that the Axiom of Choice is the best device to produce very irregular functions. It is sufficient to recall that all the examples of non-measurable functions are based on the Choice Axiom.

Thus a priori we should not expect anything good from the function $x(t,x_0)$.

The aim of this article is to show that under suitable choice of the correspondence $x(t,x_0)$ the function $x_0\mapsto x(t,x_0)$ possesses good properties.

2. Main Theorems

Equip the space $\mathbb{R}^m=\{x=(x^1,\dots,x^m)\}$ with a norm

$$\|x\|=\max_{k=1,\dots,m}|x^k|.$$

Let B_R stands for the closed ball of $\mathbb{R}^m$ with radius R and the center at the origin. By I_T denote an interval $I_T=[-T,T]$.

Introduce a vector-function $f(t,x)=(f^1,\dots,f^m)(t,x)\in C(\mathbb{R}_t\times\mathbb{R}^m_x,\mathbb{R}^m)$. Suppose that

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}\|f(t,x)\|=M<\infty.$$

This assumption is made only for simplicity, actually it is sufficient to have f defined in the closure of a bounded domain. The reader may consider f to be continuously extendable outside B_R.

We will look for solutions to the following IVP.

$$u_t(t,x) = f(t,u(t,x)), \quad u(0,x) = x. \tag{2.1}$$

In such a setup problem (2.1) is no longer a Cauchy problem for finite-dimensional ODE, it is an infinite dimensional Cauchy problem. Indeed, for any fixed t the function $u(t,x)$ is a function of variable x i.e. $t \mapsto u(t,x)$ is a curve of an infinite dimensional functional space.

In the Introduction it has already been noted that such an infinite dimensional Cauchy problems may have no solutions. All the existence results concerning this type of IVP use some compactness argument. For example in [18] it is imposed that f is weakly continuous mapping of a reflexive Banach space. Another approach see for example in [10].

Theorem 1. *For any positive constants T and R problem (2.1) has a solution $w(t,x)$ such that the functions $x \mapsto w(t,x)$, $\quad x \mapsto w_t(t,x)$ are Borel measurable mappings of B_R to the Banach space $C(I_T)$.*

2.1. **Proof of Theorem 1.** Consider a set

$$K = \{u(\cdot) \in C^1(I_T) \mid u_t(t) = f(t,u(t)), \quad u(0) \in B_R\}.$$

First, we intent to show that K is a compact set in $C(I_T)$.

The functions from K satisfy the integral equation

$$u(t) = u(0) + \int_0^t f(s,u(s))\,ds. \tag{2.2}$$

Thus the set K is uniformly bounded: for every $u(t) \in K$ it follows that $\|u(t)\| \le R + MT$ and uniformly continuous: for every $t', t'' \in I_T$ one has

$$\|u(t') - u(t'')\| \le M|t' - t''|.$$

Thus by Ascoli theorem [16] the set K is relatively compact in $C(I_T)$.

It remains to note that K is closed in $C(I_T)$. Indeed, if a sequence $\{u_n(t)\} \subseteq K$ and this sequence is uniformly convergent to the function $u(t)$ then from standard theorems of analysis we know that $u \in C(I_T)$ and u satisfies equation (2.2). Thus $u \in K$.

The following proposition is a consequence from the Measurable Selection Theorem [11].

Proposition 1. *Let K be a compact metric space and let Y be a separable Hausdorff topological space. Then for any continuous mapping $g : K \to Y$ there exists a Borel set $B \subseteq K$ such that $g(B) = g(K)$ and $g\mid_B$ is an injection and $g^{-1} : g(K) \to B$ is Borel measurable.*

On a role Y we take B_R and let $g(u(\cdot)) = u(0)$. By Proposition 1 we obtain the Borel function $x \mapsto w(t,x)$ that solves problem (2.1). The mapping $x \mapsto w_t(t,x)$ is Borel measurable as a composition of measurable functions.

Theorem 1 is proved.

2.2. **Corollaries.** Let $G : B_R \to K$ stands for the papping $x \mapsto w(t,x)$.

Corollary 1. *There is a set $U \subset B_R$ of the first Baire category such that G is a continuous function of $B_R \backslash U$ to K.*

This directly follows from the properties of the Borel functions [12].

Let μ stands for the standard Lebesgue measure in B_R.

As a consequence of Lusin's theorem [16] we have the following assertion.

Corollary 2. *For any $\varepsilon > 0$ there is a closed set $P_\varepsilon \subset B_R$ such that $\mu(B_R \backslash P_\varepsilon) < \varepsilon$ and the mapping $G\mid_{P_\varepsilon}$ is continuous.*

Theorem 2. *For any $p > 0$ the mapping $t \mapsto w(t,x)$ belongs to the space $C^1(I_T, L^p(B_R))$.*

2.2.1. *Proof of Teorem 2.* According to Corollary 2 there are closed sets $P_n \subseteq B_R, \quad n \in \mathbb{N}$ such that $\mu(B_R \backslash P_n) < 1/n$ and $w(t,x) \in C(P_n, C(I))$. Introduce a sequence of functions $w_n(t,x)$ by the rule $w_n(t,x) = w(t,x)$ provided $x \in P_n$ and $w_n(t,x) = 0$ otherwise. Then we have

$$w_n(t,x) \in C(P_n, C(I)) \subset C(I \times P_n),$$

and for each $t \in I_T$ the sequence $\{w_n(t,\cdot)\}$ converges to $w(t,\cdot)$ in measure.

Let us check that the functions $\{w_n\}$ are uniformly continuous in t in the following sence:

$$\sup_{n \in \mathbb{N}} \|w_n(t',\cdot) - w_n(t'',\cdot)\|_{L^p(B_R)} \le M(\mu(B_R))^{1/p}|t' - t''|. \tag{2.3}$$

Indeed,

$$\begin{aligned} &\int_{B_R} \|w_n(t',x) - w_n(t'',x)\|^p dx \\ &\quad = \int_{P_n} \|w_n(t',x) - w_n(t'',x)\|^p dx + \int_{B_R \backslash P_n} \|w_n(t',x) - w_n(t'',x)\|^p dx \\ &\quad = \int_{P_n} \|w(t',x) - w(t'',x)\|^p dx \end{aligned}$$

From formula (2.2) we know that $\|w(t',x) - w(t'',x)\| \le M|t' - t''|$. This implies formula (2.3).

The following proposition is a corollary from the Vitali convergence theorem [6].

Proposition 2. *Let $(X, \mathfrak{S}, \mu)$ be a measure space, $\mu(X) < \infty$. And a sequence of measurable functions $\{f_n\}$ is such that for all $n \in \mathbb{N}$ and for almost all $x \in X$ we have $|f_n(x)| \le \mathrm{const}$. Assume that $\{f_n\}$ is a Cauchy sequence in measure. Then it converges in measure to a measurable function f and $\int_X (f_n - f) d\mu \to 0$.*

From Proposition 2 it follows that for each $t \in I_T$ the function $w(t,\cdot)$ is measurable and

$$\|w_n(t,\cdot) - w(t,\cdot)\|_{L^p(B_R)} \to 0.$$

The functions $w_n(t,\cdot)$ are uniformly continuous as mappings of I_T to $L^p(B_R)$ thus the pointwise convergence implies the uniform one [16]:

$$\sup_{t\in I_T}\|w_n(t,\cdot)-w(t,\cdot)\|_{L^p(B_R)}\to 0.$$

This implies that $w(t,x)\in C(I_T,L^p(B_R))$. To finish the proof it remains to observe that $w_t(t,x)=f(t,w(t,x))\in C(I_T,L^p(B_R))$.

Theorem 2 is proved.

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