# Remarks on the Γ–regularization of Non–convex and Non–semi–continuous Functionals on Topological Vector Spaces

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**Abstract**

We show that the minimization problem of any non–convex and non–lower semi–continuous functional on a compact convex subset of a locally convex real topological vector space can be studied via an associated convex and lower semi–continuous functional $\Gamma(h)$. This observation uses the notion of Γ–regularization as a key ingredient. As an application we obtain, on any locally convex real space, a generalization of the Lanford III–Robinson theorem which has only been proven for separable real Banach spaces. The latter is a characterization of subdifferentials of convex continuous functionals.

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## 1. Introduction and Main Results

Minimization problems $\inf h(K)$ on compact convex subsets $K$ of a locally convex real (topological vector) space[1] $\mathcal{X}$ are extensively studied for convex and lower semi–continuous real functionals $h$.

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[1] For the precise definition of topological vector space used here, see Definition 4.1.

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Such variational problems are, however, not systematically studied for *non*–convex and *non*–lower semi–continuous real functionals $h$, except for a few specific functionals. The aim of this paper is to show that – independently of convexity or lower semi–continuity of functionals $h$ – the minimization problem $\inf h(K)$ on compact convex subsets $K$ of a locally convex real space $\mathcal{X}$ can be analyzed via another minimization problem $\inf \Gamma(h)(K)$ associated with a convex and lower semi–continuous functional $\Gamma(h)$, for which various methods of analysis are available.

We are particularly interested in characterizing the following set of generalized minimizers of any real functional $h$ on a compact convex set $K$:

**Definition 1.1 (Set of generalized minimizers).**
*Let $K$ be a (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and $h : K \to (-\infty, \infty]$ be any real functional. Then the set $\overline{\varOmega(h,K)} \subset K$ of generalized minimizers of $h$ is the closure of the set*

$$\varOmega(h,K) := \left\{ x \in K : \exists \{x_i\}_{i\in I} \subset K \text{ with } x_i \to x \text{ and } \lim_I h(x_i) = \inf h(K) \right\}$$

*of all limit points of approximating minimizers of $h$.*

Here, $\{x_i\}_{i\in I} \subset K$ is per definition a net of *approximating minimizers* when

$$\lim_I h(x_i) = \inf h(K).$$

Note that, for any compact set $K$, $\varOmega(h,K)$ is non–empty because any net $\{x_i\}_{i\in I} \subset K$ converges along a subnet.

In order to motivate the issue here, observe that $\inf h(K)$ can always be studied via a minimization problem associated with a (possibly not convex, but) lower semi–continuous functional $h_0$, known as the *lower semi–continuous hull* of $h$:

**Lemma 1.2 (Minimization of real functionals – I).**
*Let $K$ be any (non–empty) compact, convex, and metrizable subset of a locally convex real space $\mathcal{X}$ and $h : K \to [\mathrm{k}, \infty]$ be any real functional with $\mathrm{k} \in \mathbb{R}$. Then there is a lower semi–continuous functionals $h_0 : K \to [\mathrm{k}, \infty]$ such that*

$$\inf h(K) = \inf h_0(K) \quad \textit{and} \quad \varOmega(h_0,K) = \varOmega(h,K).$$

By lower semi–continuity, note that $\Omega(h_0, K)$ corresponds to the set of usual minimizers of $h_0$. Note further that Lemma 1.2 implies – in the case $K$ is metrizable – that $\Omega(h, K)$ is closed, again by lower semi–continuity of $h_0$. The proof of this lemma is straightforward and is given in Section 2.1 for completeness.

This result has two drawbacks: The compact convex set $K$ must be *metrizable* in the elementary proof we give here and, more important, the lower semi–continuous hull $h_0$ of $h$ is *generally not convex*. We give below a more elaborate result and show that both problems mentioned above can be overcome by using the so–called Γ–regularization of real functionals. The last is defined from the space $\mathrm{A}(\mathcal{X})$ of all affine continuous real valued functionals on a locally convex real space $\mathcal{X}$ as follows (cf. [1, Eq. (1.3) in Chapter I]):

**Definition 1.3 (Γ–regularization of real functionals).**
*For any real functional $h : K \to [\mathrm{k}, \infty]$ defined on a (non–empty) compact convex subset $K \subset \mathcal{X}$, its Γ–regularization $\Gamma(h)$ on $K$ is the functional defined as the supremum over all affine and continuous minorants $m : \mathcal{X} \to \mathbb{R}$ of $h$, i.e., for all $x \in K$,*

$$\Gamma(h)(x) := \sup \{m(x) : m \in \mathrm{A}(\mathcal{X}) \ \text{and}\ m|_K \leq h\}.$$

Since the Γ–regularization $\Gamma(h)$ of a real functional $h$ is a supremum over continuous functionals, $\Gamma(h)$ is a *convex* and *lower semi–continuous* functional on $K$. For convenience, note that we identify real functionals $g$ only defined on a convex compact subset $K \subset \mathcal{X}$ of the locally convex real space $\mathcal{X}$ with its (trivial) extension $g_{\mathrm{ext}}$ to the whole space $\mathcal{X}$ defined by

$$g_{\mathrm{ext}}(x) := \begin{cases} g(x) & \text{for } x \in K. \\ \infty & \text{else.} \end{cases}$$

With this prescription $g$ is lower semi–continuous (resp. convex) on $K$ iff $g$ is lower semi–continuous (resp. convex) on $\mathcal{X}$.

We prove in Section 2.2 the main result of this paper:

**Theorem 1.4 (Minimization of real functionals – II).**
*Let $K$ be any (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and $h : K \to [\mathrm{k}, \infty]$ be any real functional with $\mathrm{k} \in \mathbb{R}$. Then we have that:*
(i)

$$\inf\, h(K) = \inf\, \Gamma(h)(K).$$

(ii) *The set* $M$ *of minimizers of* $\Gamma(h)$ *over* $K$ *equals the closed convex hull of the set* $\Omega(h,K)$ *of generalized minimizers of* $h$ *over* $K$*, i.e.,*

$$M = \overline{\mathrm{co}\,(\Omega\,(h,K))}.$$

This general fact related to the minimization of non–convex and non–lower semi–continuous real functionals on compact convex sets has not been observed[2] before, at least to our knowledge. It turns out to be extremely useful. It is, for instance, an essential argument in the proof given in [2] of the validity of the so–called Bogoliubov approximation on the level of states for a class of models for fermions on the lattice. This problem, well–known in mathematical physics, was first addressed by Ginibre [3, p. 28] in 1968 and is still open for many physically important models.

Then, by using the theory of compact convex subsets of locally convex real spaces $\mathcal{X}$ (see, e.g., [1]), Theorem 1.4 yields a characterization of the set $\overline{\Omega\,(h,K)}$ of all generalized minimizers of $h$ over $K$. Indeed, one important observation concerning locally convex real spaces $\mathcal{X}$ is that any compact convex subset $K \subset \mathcal{X}$ is the closure of the convex hull of the (non–empty) set $\mathcal{E}(K)$ of its extreme points, i.e., of the points which cannot be expressed as (non–trivial) convex combinations of other elements in $K$. This is the Krein–Milman theorem, see, e.g., [4, Theorems 3.4 (b) and 3.23]. In fact, among all subsets $Z \subset K$ generating $K$, $\mathcal{E}(K)$ is – in a sense – the smallest one. This is the Milman theorem, see, e.g., [4, Theorem 3.25]. It follows from Theorem 1.4 together with [4, Theorems 3.4 (b), 3.23, 3.25] that extreme points of the compact convex set[3] $M$ of minimizers of $\Gamma(h)$ over $K$ are generalized minimizers of $h$:

**Theorem 1.5 (Minimization of real functionals – III).**
*Let* $K$ *be any (non–empty) compact convex subset of a locally convex real space* $\mathcal{X}$ *and* $h : K \rightarrow [\mathrm{k},\infty]$ *be any real functional with* $\mathrm{k} \in \mathbb{R}$*. Then extreme points of the compact convex set* $M$ *belong to the set of generalized minimizers of* $h$*, i.e.,* $\mathcal{E}(M) \subseteq \overline{\Omega\,(h,K)}$.

This last result makes possible a *full characterization* of the closure of the set $\Omega\,(h,K)$ in the following sense: Since $M$ is compact and convex, we

[2] Assertion (i) is, however, trivial.

[3] The compacticity and convexity of $M$ are direct consequences of the lower semi–continuity and convexity of $\Gamma(h)$ on the compact convex set $K$.

can study the minimization problem $\inf\, h\left(K_M\right)$ for any closed (and hence compact) convex subset $K_M \subset M$. Applying Theorem 1.4 we get

$$\inf\, h\left(K_M\right) = \inf\, \Gamma\left(h|_{K_M}\right)\left(K_M\right). \tag{1}$$

If

$$\inf\, h\left(K_M\right) = \inf\, h\left(K\right)$$

then, by Theorem 1.5,

$$\mathcal{E}\left(M_{K_M}\right) \subseteq \overline{\Omega\left(h|_{K_M}, K_M\right)} \subseteq \overline{\Omega\left(h, K\right)},$$

where $M_{K_M}$ is the compact convex set of minimizers of $\Gamma\left(h|_{K_M}\right)$ over $K_M \subset M$. In general, $\mathcal{E}\left(M_{K_M}\right) \backslash \mathcal{E}\left(M\right) \neq \emptyset$ because $M_{K_M}$ is not necessarily a face of $M$. Thus we discover in this manner new points of $\overline{\Omega\left(h, K\right)}$ not contained in $\mathcal{E}\left(M\right)$. Choosing a sufficiently large family $\{K_M\}$ of closed convex subsets of $M$ we can exhaust the set $\overline{\Omega\left(h, K\right)}$ through the union $\cup\, \{\mathcal{E}\left(M_{K_M}\right)\}$. Note that this construction can be performed in an inductive way: For each set $M_{K_M}$ of minimizers consider further closed convex subsets $K'_M \subset M_{K_M}$. The art consists in choosing the family $\{K_M\}$ appropriately, i.e., it should be as small as possible and the extreme points of $M_{K_M}$ should possess some reasonable characterization. The latter is of course heavily dependant on the functional $h$ and on particular properties of the compact convex set $K$ (e.g., density of $\mathcal{E}(K)$, metrizability, etc.).

To close this section we recall that the Γ–regularization $\Gamma\left(h\right)$ of a functional $h$ on $K$ equals its twofold *Legendre–Fenchel transform* – also called the *biconjugate* (functional) of $h$. Indeed, $\Gamma\left(h\right)$ is the largest lower semi–continuous and convex minorant of $h$ (cf. Corollary 3.2). However, in contrast to the Γ–regularization the notion of Legendre–Fenchel transform requires the use of dual pairs (cf. Definition 4.2). Since, for any locally convex real space $\mathcal{X}$ together with its dual space $\mathcal{X}^*$ equipped with the weak$^*$–topology, $(\mathcal{X}, \mathcal{X}^*)$ is a dual pair, the Legendre–Fenchel transform can be defined on any locally convex real space $\mathcal{X}$ as follows:

**Definition 1.6 (The Legendre–Fenchel transform).**
*Let $K$ be a (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$. For any real functional $h : K \rightarrow (-\infty, \infty]$, its Legendre–Fenchel transform $h^*$ is the convex weak$^*$–lower semi–continuous functional from $\mathcal{X}^*$ to $(-\infty, \infty]$ defined, for any $x^* \in \mathcal{X}^*$, by*

$$h^*\left(x^*\right) := \sup_{x \in K} \left\{x^*\left(x\right) - h\left(x\right)\right\}.$$

Note that, together with its weak*–topology, the dual space $\mathcal{X}^*$ of any locally convex space $\mathcal{X}$ is also a locally convex space, see [4, Theorems 3.4 (b) and 3.10]. Therefore, in case nothing is further specified, the space $\mathcal{X}^*$ is always equipped with its weak*–topology.

The Legendre–Fenchel transform is strongly related to the notion of subdifferentials:

**Definition 1.7 (Subdifferentials).**
*Let $h : \mathcal{X} \rightarrow (-\infty, \infty]$ be any real functional on a real topological vector space $\mathcal{X}$. A linear functional $\mathrm{d}h_x \in \mathcal{X}^*$ is said to be a subgradient (or tangent) of the functional $h$ at $x \in \mathcal{X}$ iff, for all $x' \in \mathcal{X}$, $h(x+x') \geq h(x) + \mathrm{d}h_x(x')$. The set $\partial h(x) \subset \mathcal{X}^*$ of subgradients of $h$ at $x$ is called subdifferential of $h$ at $x$.*

Therefore, Theorem 1.4 establishes a link between generalized minimizers and subdifferentials:

**Theorem 1.8 (Subdifferentials of continuous convex functions – I).**
*Let $K$ be any (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and $h : K \rightarrow [\mathrm{k}, \infty]$ be any real functional with $\mathrm{k} \in \mathbb{R}$. Then the subdifferential $\partial h^*(x^*) \subset \mathcal{X}$ of $h^*$ at the point $x^* \in \mathcal{X}^*$ is the (non–empty) compact convex set*

$$\partial h^*(x^*) = \overline{\mathrm{co}\left(\Omega\left(h - x^*, K\right)\right)}.$$

This last result – proven in Section 2.3 – generalizes the Lanford III–Robinson theorem [5, Theorem 1] which has only been proven for separable real Banach spaces $\mathcal{X}$ and continuous convex functionals $h : \mathcal{X} \rightarrow \mathbb{R}$, cf. Theorem 4.8.

Indeed, for any real map $h$ from a compact convex subset $K \subset \mathcal{X}$ of a locally convex real space $\mathcal{X}$ to $(-\infty, \infty]$, let

$$\mathcal{Y}^* := \{x^* \in \mathcal{X}^* : h^* \text{ has a unique subgradient } \mathrm{d}h^*_{x^*} \in \mathcal{X} \text{ at } x^*\}\,.$$

For all $x^* \in \mathcal{X}^*$ and any open neighborhood $\mathcal{V}$ of $\{0\} \subset \mathcal{X}^*$, we also define the set

$$\mathcal{T}_{x^*,\mathcal{V}} := \overline{\left\{\mathrm{d}h^*_{y^*} : y^* \in \mathcal{Y}^* \cap (x^* + \mathcal{V})\right\}} \subset \mathcal{X} \tag{2}$$

and denote by $\mathcal{T}_{x^*}$ the intersection

$$\mathcal{T}_{x^*} := \bigcap_{\mathcal{V} \ni 0 \text{ open}} \mathcal{T}_{x^*,\mathcal{V}}. \tag{3}$$

Then we observe first that Theorem 1.8 implies that the set $\partial h^*(x^*) \subset \mathcal{X}$ of subgradients of $h^*$ at the point $x^* \in \mathcal{X}^*$ is included in the closed convex hull of the set $\mathcal{T}_{x^*}$ provided $\mathcal{Y}^*$ is dense in $\mathcal{X}^*$ (cf. Section 2.4):

**Corollary 1.9 (Subdifferentials of continuous convex functions – II).**
*Let $K$ be any (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and $h : K \rightarrow [\mathrm{k}, \infty]$ be any real functional with $\mathrm{k} \in \mathbb{R}$. If $\mathcal{Y}^*$ is dense in $\mathcal{X}^*$ then, for any $x^* \in \mathcal{X}^*$,*

$$\partial h^*(x^*) \subseteq \overline{\mathrm{co}\,(\mathcal{T}_{x^*})}.$$

This last result applied on separable Banach spaces yields, in turn, the following assertion (cf. Section 2.5):

**Corollary 1.10 (The Lanford III–Robinson theorem).**
*Let $\mathcal{X}$ be a separable Banach space and $h : \mathcal{X} \rightarrow \mathbb{R}$ be any convex functional which is globally Lipschitz continuous. If the set*

$$\mathcal{Y} := \{x \in \mathcal{X} : h \text{ has a unique subgradient } \mathrm{d}h_x \in \mathcal{X}^* \text{ at } x\}$$

*is dense in $\mathcal{X}^*$ then the subdifferential $\partial h(x)$ of $h$, at any $x \in \mathcal{X}$, is the weak*–closed convex hull of the set $\mathcal{Z}_x$. Here, at fixed $x \in \mathcal{X}$, $\mathcal{Z}_x$ is the set of functionals $x^* \in \mathcal{X}^*$ such that there is a net $\{x_i\}_{i \in I}$ in $\mathcal{Y}$ converging to $x$ with the property that the unique subgradient $\mathrm{d}h_{x_i} \in \mathcal{X}^*$ of $h$ at $x_i$ converges towards $x^*$ in the weak*–topology.*

Recall that the Mazur theorem shows that the set $\mathcal{Y}$ on which a continuous convex functional $h$ is Fréchet differentiable, i.e., the set $\mathcal{Y}$ for which $h$ has exactly one subgradient $\mathrm{d}h_x \in \mathcal{X}^*$ at any $x \in \mathcal{Y}$, is dense in a separable Banach space $\mathcal{X}$, cf. Theorem 4.6 and Remark 4.7. Therefore, for globally Lipschitz continuous and convex functionals, the Lanford III–Robinson theorem [5, Theorem 1] (cf. Theorem 4.8) directly follows from Corollary 1.10. Observe that, in which concerns subdifferentials of convex continuous functionals on Banach spaces, the case of global Lipschitz continuous functionals is already the most general case: For any continuous convex functional $h$ on a Banach space $\mathcal{X}$ and any $x \in \mathcal{X}$, there are $\varepsilon > 0$ and a globally Lipschitz continuous convex functional $g$ such that $g\,(y) = h\,(y)$ whenever $\|x - y\| < \varepsilon$. In particular, $g$ and $h$ have the same subgradients at $x$. Remark, indeed, that continuous convex functionals $h$ on a Banach space $\mathcal{X}$ are locally Lipschitz continuous and an example of such a global Lipschitz continuous convex functional is given by

$$g\left(x\right) := \inf\left\{z \in \mathbb{R} : (x,z) \in \left[\mathrm{epi}\left(h\right) + \mathcal{C}_{\alpha}\right]\right\},$$

for sufficiently small $\alpha > 0$. Here,

$$\mathcal{C}_{\alpha} := \left\{(x,z) \in \mathcal{X} \times \mathbb{R} : z \geq 0, \|x\| \leq \alpha z\right\}$$

and $\mathrm{epi}\left(h\right)$ is the epigraph of $h$ defined by

$$\mathrm{epi}\left(h\right) := \left\{(x,z) \in \mathcal{X} \times \mathbb{R} : z \geq f\left(x\right)\right\}.$$

The rest of the paper is structured as follows. Section 2 gives the detailed proofs of Lemma 1.2, Theorems 1.4, 1.8, and Corollaries 1.9–1.10. Then, Section 3 discusses an additional observation which is relevant in the context of minimization of non–convex or non–semi–continuous functionals and which does not seem to have been observed before. Indeed, Lemma 3.4 gives an extension of the Bauer maximum principle (Lemma 3.3). Finally, Section 4 is a concise appendix about locally convex real spaces, dual pairs, barycenters in relation with the $\Gamma$–regularization, the Mazur theorem, and the Lanford III–Robinson theorem.

## 2. Proofs

This section gives the detailed proofs of Lemma 1.2, Theorems 1.4, 1.8, and Corollaries 1.9–1.10. Up to Corollary 1.10, we will always assume that $K$ is a (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and $h : K \rightarrow [\mathrm{k}, \infty]$ is any real functional with $\mathrm{k} \in \mathbb{R}$. In Lemma 1.2 the metrizability of the topology on $K$ is also assumed. In Corollary 1.10 $\mathcal{X}$ is a separable Banach space and $h : \mathcal{X} \rightarrow \mathbb{R}$ is any globally Lipschitz continuous convex functional.

### *2.1. Proof of Lemma 1.2*

Because the subset $K \subset \mathcal{X}$ is metrizable and compact, it is sequentially compact and we can restrict ourselves to sequences instead of more general nets. Using any metric $d(x,y)$ on $K$ generating the topology we define, at fixed $\delta > 0$, the real map $h_{\delta}$ from $K$ to $[\mathrm{k}, \infty]$ by

$$h_{\delta}\left(x\right) := \inf\ h(\mathcal{B}_{\delta}\left(x\right))$$

for any $x \in K$, where

$$\mathcal{B}_{\delta}(x) := \{y \in K : \ d(x,y) < \delta\} \qquad (4)$$

is the ball (in $K$) of radius $\delta > 0$ centered at $x \in K$. The family $\{h_{\delta}(x)\}_{\delta>0}$ of real functionals is clearly increasing as $\delta \searrow 0$ and is bounded from above by $h(x)$. Therefore, for any $x \in K$, the limit of $h_{\delta}(x) \geq \mathrm{k}$ as $\delta \searrow 0$ exists and defines a real map

$$x \mapsto h_0(x) := \lim_{\delta \searrow 0} h_{\delta}(x)$$

from $K$ to $[\mathrm{k}, \infty]$.

In fact, this construction is well–known and the functional $h_0$ is called the *lower semi–continuous hull* of $h$ as it is a lower semi–continuous real map from $K$ to $[\mathrm{k}, \infty]$. Indeed, for all $\delta > 0$ and any sequence $\{x_n\}_{n=1}^{\infty} \subset K$ converging to $x \in K$, there is $N_{\delta} > 0$ such that, for all $n > N_{\delta}$, $x_n \in \mathcal{B}_{\delta/2}(x)$ which implies that $\mathcal{B}_{\delta/2}(x_n) \subset \mathcal{B}_{\delta}(x)$. In particular, $h_{\delta}(x) \leq h_{\delta/2}(x_n)$ for all $\delta > 0$ and $n > N_{\delta}$. Since the family $\{h_{\delta}(x)\}_{\delta>0}$ defines an increasing sequence as $\delta \searrow 0$, it follows that

$$h_{\delta}(x) \leq \liminf_{n\to\infty} h_0(x_n)$$

for any $\delta > 0$ and $x \in K$. In the limit $\delta \searrow 0$ the latter yields the lower semi–continuity of the real functional $h_0$ on $K$. Moreover,

$$h_0(x) \geq h_{\delta}(x) \geq \inf\, h(K) \geq \mathrm{k} > -\infty \qquad (5)$$

for any $x \in K$ and $\delta > 0$.

We observe now that $h$ and $h_0$ have the same infimum on $K$:

$$\inf h_0(K) = \inf h(x). \qquad (6)$$

This can be seen by observing first that there is $y \in K$ such that

$$\inf h_0(K) = h_0(y) \qquad (7)$$

because of the lower semi–continuity of $h_0$. Since $h_{\delta} \leq h$ on $K$ for any $\delta > 0$, we have $h_0 \leq h$ on $K$, which combined with (5) and (7) yields Equality (6).

Additionally, for all $\delta > 0$ and any minimizer $y \in K$ of $h_0$ over $K$, there is a sequence $\{x_{\delta,n}\}_{n=1}^{\infty} \subset \mathcal{B}_{\delta}(y)$ of approximating minimizers of $h$ over $\mathcal{B}_{\delta}(y)$, that is,

$$h_{\delta}(y) := \inf\, h(\mathcal{B}_{\delta}(y)) = \lim_{n\to\infty} h(x_{\delta,n}) \leq h(y).$$

We can assume without loss of generality that

$$d(x_{\delta,n}, y) \leq \delta \quad \text{and} \quad |h(x_{\delta,n}) - h_\delta\,(y)\,| \leq 2^{-n}$$

for all $n \in \mathbb{N}$ and all $\delta > 0$. Note that $h_\delta\,(y) \to h_0\,(y)$ as $\delta \searrow 0$. Thus, by taking any function $p(\delta) \in \mathbb{N}$ satisfying $p(\delta) > \delta^{-1}$ we obtain that $x_{\delta,p(\delta)}$ converges to $y \in K$ as $\delta \searrow 0$ with the property that $h(x_{\delta,p(\delta)})$ converges to $h_0\,(y)$. Using Equalities (6) and (7) we obtain that all minimizers of (7) are generalized minimizers of $h$, i.e.,

$$\mathit{\Omega}\,(h_0, K) \subseteq \mathit{\Omega}\,(h, K)\,.$$

The converse inclusion

$$\mathit{\Omega}\,(h, K) \subseteq \mathit{\Omega}\,(h_0, K)$$

is straightforward because one has the inequality $h_0 \leq h$ on $K$ as well as Equality (6).

*2.2. Proof of Theorem 1.4*

The assertion (i) of Theorem 1.4 is a standard result. Indeed, by Definition 1.3, $\Gamma\,(h) \leq h$ on $K$ and thus

$$\inf\,\Gamma\,(h)\,(K) \leq \inf\,h\,(K)\,.$$

The converse inequality is derived by restricting the supremum in Definition 1.3 to constant maps $m$ from $\mathcal{X}$ to $\mathbb{R}$ with $\mathrm{k} \leq m \leq h$ on $K$.

By Definition 1.3, we also observe that $\Gamma\,(h)$ is a lower semi–continuous functional. This implies that the variational problem $\inf\,\Gamma\,(h)\,(K)$ has minimizers and the set $M = \mathit{\Omega}\,(\Gamma\,(h)\,, K)$ of all minimizers of $\Gamma\,(h)$ is compact. Moreover, again by Definition 1.3, the functional $\Gamma\,(h)$ is convex which obviously yields the convexity of the set $M$.

For any $y \in \mathit{\Omega}\,(h, K)$, there is a net $\{x_i\}_{i \in I} \subset K$ of approximating minimizers of $h$ on $K$ converging to $y$. In particular, since the functional $\Gamma\,(h)$ is lower semi–continuous and $\Gamma\,(h) \leq h$ on $K$, we have that

$$\Gamma\,(h)\,(y) \leq \liminf_I \Gamma\,(h)\,(x_i) \leq \lim_I h(x_i) = \inf\,h(K) = \inf\,\Gamma\,(h)\,(K),$$

i.e., $y \in M$. As $M$ is convex and compact, we obtain that

$$M \supset \overline{\mathrm{co}\,(\mathit{\Omega}\,(h, K))}. \tag{8}$$

So, we prove now the converse inclusion. We can assume without loss of generality that $\overline{\mathrm{co}\,(\Omega\,(h,K))} \neq K$ since otherwise there is nothing to prove. We show next that, for any $x \in K \backslash \overline{\mathrm{co}\,(\Omega\,(h,K))}$, we have $x \notin M$.

As $\overline{\mathrm{co}\,(\Omega\,(h,K))}$ is a closed set of a locally convex real space $\mathcal{X}$, for any $x \in K \backslash \overline{\mathrm{co}\,(\Omega\,(h,K))}$, there is an open and convex neighborhood $\mathcal{V}_x \subset \mathcal{X}$ of $\{0\} \subset \mathcal{X}$ which is symmetric, i.e., $\mathcal{V}_x = -\mathcal{V}_x$, and which satisfies

$$\mathcal{G}_x \cap [\{x\} + \mathcal{V}_x] = \emptyset$$

with

$$\mathcal{G}_x := K \cap \left[\overline{\mathrm{co}\,(\Omega\,(h,K))} + \mathcal{V}_x\right].$$

This follows from [4, Theorem 1.10] together with the fact that each neighborhood of $\{0\} \subset \mathcal{X}$ contains some open and convex neighborhood of $\{0\} \subset \mathcal{X}$ because $\mathcal{X}$ is locally convex. Observe also that any one–point set $\{x\} \subset \mathcal{X}$ is compact.

For any neighborhood $\mathcal{V}_x$ of $\{0\} \subset \mathcal{X}$ in a locally convex real space, there is another convex, symmetric, and open neighborhood $\mathcal{V}_x'$ of $\{0\} \subset \mathcal{X}$ such that $[\mathcal{V}_x' + \mathcal{V}_x'] \subset \mathcal{V}_x$, see proof of [4, Theorem 1.10]. Let

$$\mathcal{G}_x' := K \cap \left[\overline{\mathrm{co}\,(\Omega\,(h,K))} + \mathcal{V}_x'\right].$$

Then the following inclusions hold:

$$\overline{\mathrm{co}\,(\Omega\,(h,K))} \subset \mathcal{G}_x' \subset \overline{\mathcal{G}_x'} \subset \mathcal{G}_x \subset \overline{\mathcal{G}_x} \subset K \backslash \{x\}. \tag{9}$$

Since $K$, $\mathcal{V}_x$, $\mathcal{V}_x'$, and $\overline{\mathrm{co}\,(\Omega\,(h,K))}$ are all convex sets, $\mathcal{G}_x$ and $\mathcal{G}_x'$ are also convex. Seen as subsets of $K$ they are open neighborhoods of $\overline{\mathrm{co}\,(\Omega\,(h,K))}$.

By Definition 4.1 and [4, Theorem 1.12], the set $\mathcal{X}$ is a Hausdorff space and thus any compact subset $K$ of $\mathcal{X}$ is a normal space. By Urysohn lemma, there is a continuous function

$$f_x : K \to [\inf h(K), \inf h(K \backslash \mathcal{G}_x')]$$

satisfying $f_x \leq h$ and

$$f_x\,(y) = \begin{cases} \inf h(K) & \text{for } y \in \overline{\mathcal{G}_x'}. \\ \inf h(K \backslash \mathcal{G}_x') & \text{for } y \in K \backslash \mathcal{G}_x. \end{cases}$$

By compacticity of $K \backslash \mathcal{G}_x'$ and the inclusion $\Omega\,(h,K) \subset \mathcal{G}_x'$, observe that

$$\inf h(K \backslash \mathcal{G}_x') > \inf h(K).$$

Then we have per construction that

$$f_x(\overline{\mathrm{co}\,(\Omega\,(h,K))}) = \{\inf h(K)\} \tag{10}$$

and

$$f_x^{-1}(\inf h(K)) = \Omega\,(f_x, K) \subset \mathcal{G}_x \tag{11}$$

for any $x \in K \backslash \overline{\mathrm{co}\,(\Omega\,(h,K))}$.

We use now the $\Gamma$–regularization $\Gamma\,(f_x)$ of $f_x$ on the set $K$ and denote by $M_x = \Omega\,(\Gamma(f_x), K)$ its non–empty set of minimizers over $K$. Applying Theorem 4.5, for any $y \in M_x$, we have a probability measure $\mu_y \in M_1^+(K)$ on $K$ with barycenter $y$ such that

$$\Gamma\,(f_x)\,(y) = \int_K \mathrm{d}\mu_y(z)\; f_x\,(z)\,. \tag{12}$$

As $y \in M_x$, i.e.,

$$\Gamma\,(f_x)\,(y) = \inf\,\Gamma\,(f_x)\,(K) = \inf f_x(K), \tag{13}$$

we deduce from (12) that

$$\mu_y(\Omega\,(f_x, K)) = 1$$

and it follows that $y \in \overline{\mathrm{co}\,(\Omega\,(f_x, K))}$, by Theorem 4.4. Using (11) together with the convexity of the open neighborhood $\mathcal{G}_x$ of $\overline{\mathrm{co}\,(\Omega\,(h,K))}$ we thus obtain

$$M_x \subset \overline{\mathrm{co}\,(\Omega\,(f_x, K))} \subset \overline{\mathcal{G}_x} \tag{14}$$

for any $x \in K \backslash \overline{\mathrm{co}\,(\Omega\,(h,K))}$.

We remark now that the inequality $f_x \leq h$ on $K$ yields $\Gamma\,(f_x) \leq \Gamma\,(h)$ on $K$ because of Corollary 3.2. As a consequence, it results from (i) and (10) that the set $M$ of minimizers of $\Gamma\,(h)$ over $K$ is included in $M_x$, i.e., $M \subset M_x$. Hence, by (9) and (14), we have the inclusions

$$M \subset \overline{\mathcal{G}_x} \subset K \backslash \{x\}. \tag{15}$$

Therefore, we combine (8) with (15) for all $x \in K \backslash \overline{\mathrm{co}\,(\Omega\,(h,K))}$ to obtain the desired equality in the assertion (ii) of Theorem 1.4.

*2.3. Proof of Theorem 1.8*

The proof of Theorem 1.8 is a simple consequence of Theorem 1.4 together with the following lemma:

**Lemma 2.1 (Subgradients as minimizers).**
*Let $(\mathcal{X}, \mathcal{X}^*)$ be a dual pair and $h$ be any real functional from a (non–empty) convex subset $K \subseteq \mathcal{X}$ to $(-\infty, \infty]$. Then the subdifferential $\partial h^*(x^*) \subset \mathcal{X}$ of $h^*$ at the point $x^* \in \mathcal{X}^*$ is the (non–empty) set $M_{x^*}$ of minimizers over $K$ of the map*

$$y \mapsto \Gamma(h)(y) - x^*(y)$$

*from $K \subseteq \mathcal{X}$ to $(-\infty, \infty]$.*

*Proof.* The proof is standard and simple, see, e.g., [6, Theorem I.6.6]. Indeed, any subgradient $x \in \mathcal{X}$ of the Legendre–Fenchel transform $h^*$ at the point $x^* \in \mathcal{X}$ satisfies the inequality:

$$x^*(x) + h^*(y^*) - y^*(x) \geq h^*(x^*) \tag{16}$$

for any $y^* \in \mathcal{X}^*$, see Definition 1.7. Since $h^* = h^{***}$ and $\Gamma(h) = h^{**}$ (cf. Corollary 3.2 and [7, Proposition 51.6]), we have (16) iff

$$x^*(x) + \inf_{y^* \in \mathcal{X}^*} \{h^*(y^*) - y^*(x)\} = x^*(x) - \Gamma(h)(x) \geq \sup_{y \in K} \{x^*(y) - \Gamma(h)(y)\},$$

see Definition 1.6. ∎

We combine now Theorem 1.4 with Lemma 2.1 to characterize the subdifferential $\partial h^*(x^*) \subset \mathcal{X}$ of $h^*$ at the point $x^* \in \mathcal{X}^*$ as the closed convex hull of the set $\Omega(h - x^*, K)$ of generalized minimizers of $h$ over a compact convex subset $K$, see Definition 1.1. Indeed, for any $x^* \in \mathcal{X}^*$,

$$\Gamma(h - x^*) = \Gamma(h) - x^*,$$

see Definition 1.3.

*2.4. Proof of Corollary 1.9*

For $x^* \in \mathcal{X}^*$ and any open neighborhood $\mathcal{V}$ of $\{0\} \subset \mathcal{X}^*$, we define the map $g_{\mathcal{V},x^*}$ from $\mathcal{X}$ to $[\mathrm{k}, \infty]$ with $\mathrm{k} \in \mathbb{R}$ by

$$g_{\mathcal{V},x^*}(x) := \begin{cases} \Gamma(h)(x) & \text{for } x = \mathrm{d}h^*_{y^*} \text{ with } y^* \in \mathcal{Y}^* \cap (x^* + \mathcal{V}). \\ \infty & \text{else.} \end{cases}$$

For any $y^* \in \mathcal{Y}^* \cap (x^* + \mathcal{V})$, one has the equality $g^*_{\mathcal{V},x^*}(y^*) = h^*(y^*)$. This easily follows from the fact that

$$\begin{aligned} h^*(y^*) &= \sup_{z\in K}\{y^*(z) - \Gamma(h)(z)\} = y^*(x) - \Gamma(h)(x) \\ &= \sup_{z\in K}\{y^*(z) - g_{\mathcal{V},x^*}(z)\} = g^*_{\mathcal{V},x^*}(y^*) \end{aligned}$$

with $x := \mathrm{d}h^*_{y^*}$, see proof of Lemma 2.1. Let $\mathcal{W}$ be any open neighborhood of $\{0\} \subset \mathcal{X}^*$. Then, for any $z \in K$, the set

$$\{\delta^*(z) : \delta^* \in \mathcal{W}\} \subset \mathbb{R}$$

is bounded, by continuity of the linear map $\delta^* \mapsto \delta^*(z)$. From the the principle of uniform boundedness for compact convex sets, i.e., the variant of the Banach–Steinhaus theorem stated, for instance, in [4, Theorem 2.9], the set

$$\{\delta^*(z) : \delta^* \in \mathcal{W},\, z \in K\} \subset \mathbb{R}$$

is also bounded. Thus, for any $z^* \in \mathcal{X}^*$,

$$\begin{aligned} \lim_{s\searrow 0}\sup\{|h^*(z^*) - h^*(z^* + \delta^*)| \,:\, \delta^* \in s\mathcal{W}\} &= 0, \\ \lim_{s\searrow 0}\sup\left\{|g^*_{\mathcal{V},x^*}(z^*) - g^*_{\mathcal{V},x^*}(z^* + \delta^*)| \,:\, \delta^* \in s\mathcal{W}\right\} &= 0. \end{aligned}$$

This implies the continuity of $h^*$ and $g^*_{\mathcal{V},x^*}$. Hence, from the density of $\mathcal{Y}^*$, $h^* = g^*_{\mathcal{V},x^*}$ on the open neighborhood $(x^* + \mathcal{V})$ of $\{x^*\} \subset \mathcal{X}^*$. In particular, $h^*$ and $g^*_{\mathcal{V},x^*}$ have the same subgradients at the point $x^*$. From Theorems 1.5 and 1.8, for each open neighborhood $\mathcal{V}$ of $\{0\} \subset \mathcal{X}^*$, the extreme subgradients of $h^*$ at $x^*$ are all contained in the set $\mathcal{T}_{x^*,\mathcal{V}}$ defined by (2). Corollary 1.9 thus follows.

*2.5. Proof of Corollary 1.10*

Note that $h^{**} = h$ because the functional $h$ is continuous and convex. By the global Lipschitz continuity of $h$,

$$h(x) = \sup_{x^*\in\mathcal{X}^*}\{x^*(x) - h^*(x^*)\} = \sup_{x^*\in K}\{x^*(x) - h^*(x^*)\}$$

with $K := \mathcal{B}_R(0) \subset \mathcal{X}^*$ being some ball of sufficiently large radius $R > 0$ centered at 0. The set $K$ is weak*–compact, by the Banach–Alaoglu theorem.

Now, for any fixed $x \in \mathcal{X}$ and all $x^* \in \mathcal{Z}_x \subset \mathcal{X}^*$, by definition of the set $\mathcal{Z}_x$, there is a net $\{x_i\}_{i \in I}$ in $\mathcal{Y}$ converging to $x$ with the property that the unique subgradient $x_i^* := \mathrm{d}h_{x_i} \in \mathcal{X}^*$ of $h$ at $x_i$ converges towards $x^*$ in the weak$^*$–topology. Therefore, by continuity of $h$, for any fixed $x \in \mathcal{X}$ and all $x^* \in \mathcal{Z}_x$,

$$h(x) = \sup_{y^* \in \mathcal{X}^*} \{y^*(x) - h^*(y^*)\} = \lim_I h(x_i) = \lim_I \{x_i^*(x) - h^*(x_i^*)\},$$

with $\{x_i^*\}_{i \in I}$ converging to $x^*$. In other words,

$$\mathcal{Z}_x \subset \Omega(h^* - x, K),$$

see Definition 1.1. Thus, by Theorem 1.8 and Corollary 1.9, it suffices to prove that $\mathcal{T}_x \subset \mathcal{Z}_x$.

By density of $\mathcal{Y}$ in $\mathcal{X}$, observe that the set

$$\mathcal{T}_{x,\mathcal{V}} := \overline{\{\mathrm{d}h_y : y \in \mathcal{Y} \cap (x + \mathcal{V})\}} \subset \mathcal{X}^*$$

is non–empty for any open neighborhood $\mathcal{V}$ of $\{0\} \subset \mathcal{X}$. Meanwhile, the weak$^*$–compact set $K$ is metrizable with respect to (w.r.t.) the weak$^*$–topology, by separability of $\mathcal{X}$, see [4, Theorem 3.16]. In particular, $K$ is sequentially compact and we can restrict ourselves to sequences instead of more general nets. In particular, by (2)–(3), one has

$$\mathcal{T}_x = \bigcap_{n \in \mathbb{N}} \mathcal{T}_{x,\mathcal{B}_{1/n}(0)} \tag{17}$$

with $\mathcal{B}_\delta(x)$ being the ball (in $K$) of radius $\delta > 0$ centered at $x \in K$. Here, $\mathcal{B}_\delta(x)$ is defined by (4) for any metric $d$ on $K$ generating its weak$^*$–topology. For any $x^* \in \mathcal{T}_x \subset K$ and any $n \in \mathbb{N}$, there are per definition a sequence $\{x_{n,m}^*\}_{m=1}^\infty$ converging to $x^*$ in $K$ as $m \to \infty$ and an integer $N_n > 0$ such that, for all $m \geq N_n$, $d(x^*, x_{n,m}^*) \leq 2^{-n}$ and $x_{n,m}^* = \mathrm{d}h_{x_{n,m}}$ for some $x_{n,m} \in \mathcal{Y} \cap [x + \mathcal{B}_{1/n}(0)]$. Taking any function $p(n) \in \mathbb{N}$ satisfying $p(n) > N_n$ and converging to $\infty$ as $n \to \infty$ we obtain a sequence $\{x_{n,p(n)}^*\}_{n=1}^\infty$ converging to $x^* \in \mathcal{Z}_x$ as $n \to \infty$. This yields the inclusion $\mathcal{T}_x \subset \mathcal{Z}_x$.

## 3. Further Remarks

We give here an additional observation which is not necessarily directly related to the main results of the paper. It concerns an extension of the

Bauer maximum principle [1, Theorem I.5.3.]. See [2] for an application to statistical mechanics.

First, recall that the $\Gamma$–regularization $\Gamma(h)$ of a real functional $h$ is a convex and lower semi–continuous functional on a compact convex subset $K$. Moreover, every convex and lower semi–continuous functional on $K$ equals its own $\Gamma$–regularization on $K$ (see, e.g., [1, Proposition I.1.2.]):

**Proposition 3.1 ($\Gamma$–regularization of lower semi–cont. conv. maps).** *Let $h$ be any functional from a (non–empty) compact convex subset $K \subset \mathcal{X}$ of a locally convex real space $\mathcal{X}$ to $(-\infty, \infty]$. Then the following statements are equivalent:*
(i) $\Gamma(h) = h$ *on $K$.*
(ii) *$h$ is a lower semi–continuous convex functional on $K$.*

This proposition is a standard result. The compacticity of $K$ is in fact not necessary but $K$ should be a closed convex set. This result can directly be proven without using the fact that the $\Gamma$–regularization $\Gamma(h)$ of a functional $h$ on $K$ equals its twofold *Legendre–Fenchel transform* – also called the *biconjugate* (functional) of $h$. Indeed, $\Gamma(h)$ is the largest lower semi–continuous and convex minorant of $h$:

**Corollary 3.2 (Largest lower semi–cont. convex minorant of $h$).** *Let $h$ be any functional from a (non–empty) compact convex subset $K \subset \mathcal{X}$ of a locally convex real space $\mathcal{X}$ to $(-\infty, \infty]$. Then its $\Gamma$–regularization $\Gamma(h)$ is its largest lower semi–continuous convex minorant on $K$.*

*Proof.* For any lower semi–continuous convex real functional $f$ defined on $K$ satisfying $f \leq h$, we have, by Proposition 3.1, that

$$f(x) = \sup\{m(x) : m \in \mathrm{A}(\mathcal{X}) \text{ and } m|_K \leq f \leq h\} \leq \Gamma(h)(x)$$

for any $x \in K$. ∎

In particular, if $(\mathcal{X}, \mathcal{X}^*)$ is a dual pair and $h$ is any functional from $K$ to $(-\infty, \infty]$ then $\Gamma(h) = h^{**}$, see [7, Proposition 51.6].

Proposition 3.1 has another interesting consequence: An extension of the Bauer maximum principle [1, Theorem I.5.3.] which, in the case of convex functionals, is:

**Lemma 3.3 (Bauer maximum principle).**
*Let $\mathcal{X}$ be a locally convex real space. An upper semi–continuous convex real functional $h$ over a compact convex subset $K \subset \mathcal{X}$ attains its maximum at an extreme point of $K$, i.e.,*

$$\sup\ h\,(K) = \max\ h\,(\mathcal{E}(K))\,.$$

*Here, $\mathcal{E}(K)$ is the (non–empty) set of extreme points of $K$.*

Indeed, by combining Proposition 3.1 with Lemma 3.3 it is straightforward to check the following statement which does not seem to have been observed before:

**Lemma 3.4 (Extension of the Bauer maximum principle).**
*Let $h_{\pm}$ be two convex real functionals from a locally convex real space $\mathcal{X}$ to $(-\infty, \infty]$ such that $h_-$ and $h_+$ are respectively lower and upper semi–continuous. Then the supremum of the sum $h := h_- + h_+$ over a compact convex subset $K \subset \mathcal{X}$ can be reduced to the (non–empty) set $\mathcal{E}(K)$ of extreme points of $K$, i.e.,*

$$\sup\ h\,(K) = \sup\ h\,(\mathcal{E}(K))\,.$$

*Proof.* We first use Proposition 3.1 in order to write $h_- = \Gamma\,(h_-)$ as a supremum over affine and continuous functionals. Then we commute this supremum with the one over $K$ and apply the Bauer maximum principle to obtain that

$$\sup\ h\,(K) = \sup\left\{\sup\ [m + h_+]\,(\mathcal{E}(K)) : m \in \mathrm{A}\,(\mathcal{X})\ \text{ and } m|_K \leq h_-|_K\right\}.$$

The lemma follows by commuting again both suprema and by using $h_- = \Gamma\,(h_-)$. ∎

Observe, however, that under the conditions of the lemma above, the supremum of $h = h_- + h_+$ is generally not attained on $\mathcal{E}(K)$.

## 4. Appendix

For the reader's convenience we give here a short review on the following subjects:

- Dual pairs and locally convex real spaces, see, e.g., [4];

- Barycenters and Γ–regularization of real functionals, see, e.g., [1];
- The Mazur and Lanford III–Robinson theorems, see [5, 8].

These subjects are rather standard. Therefore, we keep the exposition as short as possible and only concentrate on results used in this paper.

*4.1. Dual Pairs and Locally Convex Real Spaces*

The definition of topological vector space used here corresponds to Rudin's definition [4, Section 1.6]:

**Definition 4.1 (Topological vector spaces).**
*A topological vector space $\mathcal{X}$ is a vector space equipped with a topology $\tau$ for which the vector space operations of $\mathcal{X}$ are continuous and every point of $\mathcal{X}$ defines a closed set.*

The fact that each point of $\mathcal{X}$ is a closed set is not part of the definition of topological vector spaces in many textbooks. It is used here because it is satisfied in most applications and in this case, the space $\mathcal{X}$ is automatically a Hausdorff space, by [4, Theorem 1.12].

The notion of *dual pairs* is defined as follow:

**Definition 4.2 (Dual pairs).**
*For any locally convex space $(\mathcal{X}, \tau)$, let $\mathcal{X}^*$ be its dual space, i.e., the set of all continuous linear functionals on $\mathcal{X}$. Let $\tau^*$ be any locally convex topology on $\mathcal{X}^*$. $(\mathcal{X}, \mathcal{X}^*)$ is called a dual pair iff, for all $x \in \mathcal{X}$, the functional $x^* \mapsto x^*(x)$ on $\mathcal{X}^*$ is continuous w.r.t. $\tau^*$, and all linear functionals which are continuous w.r.t. $\tau^*$ have this form.*

By [4, Theorems 3.4 (b) and 3.10], a typical example of a dual pair $(\mathcal{X}, \mathcal{X}^*)$ is given by any locally convex real space $\mathcal{X}$ equipped with a topology $\tau$ and $\mathcal{X}^*$ equipped with the $\sigma(X^*, X)$–topology $\tau^*$, i.e., the weak$^*$–topology. We also observe that if $(\mathcal{X}, \mathcal{X}^*)$ is a dual pair w.r.t. $\tau$ and $\tau^*$ then $(\mathcal{X}^*, \mathcal{X})$ is a dual pair w.r.t. $\tau^*$ and $\tau$.

*4.2. Barycenters and Γ–regularization*

The theory of compact convex subsets of a locally convex real (topological vector) space $\mathcal{X}$ is standard. For more details, see, e.g., [1]. An important observation is the Krein–Milman theorem (see, e.g., [4, Theorems 3.4 (b) and 3.23]) which states that any compact convex subset $K \subset \mathcal{X}$ is the closure of the convex hull of the (non–empty) set $\mathcal{E}(K)$ of its extreme points. Restricted to finite dimensions this theorem corresponds to a classical result of Minkowski which, for any $x \in K$ in a (non–empty) compact convex subset $K \subset \mathcal{X}$, states the existence of a finite number of extreme points $\hat{x}_1, \ldots, \hat{x}_k \in \mathcal{E}(K)$ and positive numbers $\mu_1, \ldots, \mu_k \geq 0$ with $\Sigma_{j=1}^{k} \mu_j = 1$ such that

$$x = \sum_{j=1}^{k} \mu_j \hat{x}_j. \tag{1}$$

To this simple decomposition we can associate a probability measure, i.e., a *normalized positive Borel regular measure*, $\mu$ on $K$.

Borel sets of any set $K$ are elements of the $\sigma$–algebra $\mathfrak{B}$ generated by closed – or open – subsets of $K$. Positive Borel regular measures are the positive countably additive set functions $\mu$ over $\mathfrak{B}$ satisfying

$$\mu(B) = \sup\{\mu(C) : C \subset B,\ C \text{ closed}\} = \inf\{\mu(O) : B \subset O,\ O \text{ open}\}$$

for any Borel subset $B \in \mathfrak{B}$ of $K$. If $K$ is compact then any positive Borel regular measure $\mu$ (one–to–one) corresponds to an element of the set $M^+(K)$ of Radon measures with $\mu(K) = \|\mu\|$, and we write

$$\mu(h) = \int_K \mathrm{d}\mu(\hat{x})\ h(\hat{x}) \tag{2}$$

for any continuous functional $h$ on $K$. A probability measure $\mu \in M_1^+(K)$ is per definition a positive Borel regular measure $\mu \in M^+(K)$ which is *normalized*: $\|\mu\| = 1$.

Therefore, using the probability measure $\mu_x \in M_1^+(K)$ on $K$ defined by

$$\mu_x = \sum_{j=1}^{k} \mu_j \delta_{\hat{x}_j}$$

with $\delta_y$ being the Dirac – or point – mass[4] at $y$, Equation (1) can be seen as an integral defined by (2) for the probability measure $\mu_x \in M_1^+(K)$:

$$x = \int_K \mathrm{d}\mu_x(\hat{x})\ \hat{x}\ . \tag{3}$$

The point $x$ is in fact the *barycenter* of the probability measure $\mu_x$. This notion is defined in the general case as follows (cf. [1, Eq. (2.7) in Chapter I]):

**Definition 4.3 (Barycenters of probability measures in convex sets).** *Let $K \subset \mathcal{X}$ be any (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and let $\mu \in M_1^+(K)$ be a probability measure on $K$. We say that $x \in K$ is the barycenter[5] of $\mu$ if, for all $z^* \in \mathcal{X}^*$,*

$$z^*(x) = \int_K \mathrm{d}\mu(\hat{x})\ z^*(\hat{x})\,.$$

Barycenters are well–defined for *all* probability measures in convex compact subsets of locally convex real spaces (cf. [4, Theorems 3.4 (b) and 3.28]):

**Theorem 4.4 (Well-definiteness and uniqueness of barycenters).** *Let $K \subset \mathcal{X}$ be any (non–empty) compact subset of a locally convex real space $\mathcal{X}$ such that $\overline{\mathrm{co}(K)}$ is also compact. Then, for any probability measure $\mu \in M_1^+(K)$ on $K$, there is a unique barycenter $x_\mu \in \overline{\mathrm{co}(K)}$.*

Note that Barycenters can also be defined in the same way via affine continuous functionals instead of continuous linear functionals, see, e.g., [1, Proposition I.2.2.] together with [4, Theorem 1.12].

It is natural to ask whether, for any $x \in K$ in the compact convex set $K$, there is a (possibly not unique) probability measure $\mu_x$ on $K$ (pseudo–) supported on $\mathcal{E}(K)$ with barycenter $x$. Equation (3) already gives a first positive answer to that problem in the finite dimensional case. The general case, which is a remarkable refinement of the Krein–Milman theorem, has been proven by Choquet–Bishop–de Leeuw (see, e.g., [1, Theorem I.4.8.]).

[4] $\delta_y$ is the Borel measure such that, for any Borel subset $B \in \mathfrak{B}$ of $K$, $\delta_y(B) = 1$ if $y \in B$ and $\delta_y(B) = 0$ if $y \notin B$.

[5] Other terminologies existing in the literature: "$x$ is represented by $\mu$", "$x$ is the resultant of $\mu$".

We conclude now by a crucial property concerning the Γ–regularization of real functionals in relation with the concept of barycenters (cf. [1, Corollary I.3.6.]):

**Theorem 4.5 (Barycenters and Γ–regularization).**
*Let $K \subset \mathcal{X}$ be any (non–empty) compact convex subset of a locally convex real space $\mathcal{X}$ and $h : K \to (-\infty, \infty]$ be a continuous real functional. Then, for any $x \in K$, there is a probability measure $\mu_x \in M_1^+(K)$ on $K$ with barycenter $x$ such that*

$$\Gamma(h)(x) = \int_K \mathrm{d}\mu_x(\hat{x})\, h(\hat{x}) .$$

This theorem is a very important statement used to prove Theorem 1.4.

*4.3. The Mazur and Lanford III–Robinson Theorems*

If $\mathcal{X}$ is a separable real Banach space and $h$ is a continuous convex real functional on $\mathcal{X}$ then it is well–known that $h$ has, on each point $x \in \mathcal{X}$, at least one subgradient $\mathrm{d}h \in \mathcal{X}^*$. The Mazur theorem describes the set $\mathcal{Y}$ on which a continuous convex functional $h$ is Fréchet differentiable, i.e., the set $\mathcal{Y}$ for which $h$ has exactly one subgradient $\mathrm{d}h_x \in \mathcal{X}^*$ at any $x \in \mathcal{Y}$:

**Theorem 4.6 (Mazur).**
*Let $\mathcal{X}$ be a separable real Banach space and let $h : \mathcal{X} \to \mathbb{R}$ be a continuous convex functional. The set $\mathcal{Y} \subset \mathcal{X}$ of elements where $h$ is Fréchet differentiable is residual, i.e., a countable intersection of dense open sets.*

**Remark 4.7.** *By Baire category theorem, the set $\mathcal{Y}$ is dense in $\mathcal{X}$.*

The Lanford III–Robinson theorem [5, Theorem 1] completes the Mazur theorem by characterizing the subdifferential $\partial h(x) \subset \mathcal{X}^*$ at any $x \in \mathcal{X}$:

**Theorem 4.8 (Lanford III – Robinson).**
*Let $\mathcal{X}$ be a separable real Banach space and let $h : \mathcal{X} \to \mathbb{R}$ be a continuous convex functional. Then the subdifferential $\partial h(x) \subset \mathcal{X}^*$ of $h$, at any $x \in \mathcal{X}$, is the weak*–closed convex hull of the set $\mathcal{Z}_x$. Here, at fixed $x \in \mathcal{X}$, $\mathcal{Z}_x$ is the set of functionals $x^* \in \mathcal{X}^*$ such that there is a net $\{x_i\}_{i \in I}$ in $\mathcal{Y}$ converging to $x$ with the property that the unique subgradient $\mathrm{d}h_{x_i} \in \mathcal{X}^*$ of $h$ at $x_i$ converges towards $x^*$ in the weak*–topology.*

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