HIGHER ORDER ENERGY CONSERVATION, GAGLIARDO-NIRENBERG-SOBOLEV INEQUALITIES, AND GLOBAL WELL-POSEDNESS FOR GROSS-PITAEVSKII HIERARCHIES

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Abstract. We consider the cubic and quintic Gross-Pitaevskii (GP) hierarchy in d dimensions, for focusing and defocusing interactions. We introduce new higher order conserved energy functionals that allow us to prove global existence and uniqueness of solutions for defocusing GP hierarchies, with arbitrary initial data in the energy space. Moreover, we prove generalizations of the Sobolev and Gagliardo-Nirenberg inequalities for density matrices, which we apply to establish global existence and uniqueness of solutions for focusing and defocusing GP hierarchies on the L^2-subcritical level.

1. Introduction

In recent years, there has been impressive progress related to the derivation of nonlinear dispersive PDEs, such as the nonlinear Schrödinger (NLS) or nonlinear Hartree (NLH) equations, as effective theories describing the mean field dynamics of weakly interacting Bose gases, see [11, 12, 13, 21, 20, 25] and the references therein, and also [1, 3, 10, 14, 15, 16, 18, 17, 19, 27]. For advances in the mathematical theory of Bose-Einstein condensation in systems of interacting Bosons, we refer to the highly influencial works [2, 22, 23, 24] and the references therein.

In the landmark works [11, 12, 13], Erdös, Schlein, and Yau developed the following method to derive the NLS as a dynamical mean field limit of an interacting Bose gas. Starting with the solution of the Schrödinger equation describing N interacting bosons, one determines the BBGKY hierarchy of the associated marginal density matrices. The scaling of the system is chosen in such a way that the particle interaction potential tends to a delta distribution as $N \to \infty$, and, moreover, that the kinetic and potential energy in the system are both of the same order of magnitude, which is $O(N)$; see also [21, 26] for surveys. Subsequently, in the limit $N \to \infty$, one derives the Gross-Pitaevskii (GP), which is an infinite hierarchy of partial differential equations determining the dynamics of marginal k-particle density matrices, $k \in \mathbb{N}$. For *factorized* initial data, the solutions of the GP hierarchy are easily seen to remain factorized. The individual factors are governed by an NLS, which is cubic for systems with 2-body interactions, [11, 12, 13, 21], and quintic NLS for systems with 3-body interactions, [6]. The proof of the uniqueness of solutions of the GP hierarchy is the most difficult part of this analysis, and is obtained in [11, 12, 13] by use of highly sophisticated Feynman graph expansion methods inspired by quantum field theory.

Subsequently, Klainerman and Machedon presented an alternative method in [20] to prove the uniqueness of solutions for the cubic GP hierarchy in $d = 3$, using spacetime bounds on marginal density matrices and a sophisticated combinatorial result based on a certain "boardgame argument"; their analysis requires the assumption of an a priori spacetime bound which is not proven in [20]. Kirkpatrick, Schlein, and Staffilani then proved in [21] that this a priori spacetime bound is satisfied for the cubic GP hierarchy in $d = 2$, locally in time, by exploiting the conservation of energy in the BBGKY hierarchy, in the limit as $N \to \infty$. However, no explicit conserved energy functional on the level of the GP hierarchy was identified in [21], or in [11, 12].

It is currently not known how to obtain a GP hierarchy from the $N \to \infty$ limit of a BBGKY hierarchy with attractive interactions. Nevertheless, we have begun in [7] to use the level of GP hierarchies as our starting point, and to consider systems with both focusing and defocusing interactions. Accordingly, the corresponding GP hierarchies are referred to as *cubic*, *quintic*, *focusing*, or *defocusing GP hierarchies*, depending on the type of the NLS governing the solutions obtained from factorized initial conditions. Our interest lies in investigating the Cauchy problem for GP hierarchies without any factorization condition.

In [7], we prove the a priori bound conjectured in [20], and sharpen it by demonstrating it to correspond to an inequality of Strichartz type. For the proof, we introduce a natural topology on the space of sequences of k-particle marginal density matrices, and invoke a Picard fixed point argument. Accordingly, we prove in [7] local well-posedness for the cubic and quintic GP hierarchies, in various dimensions; moreover, we establish lower bounds on the blowup rate for blowup solutions of focusing GP-hierarchies. In [8], this result is sharpened, and we present a significantly improved and shorter proof. In the joint work [9] with Tzirakis, we identify a conserved energy functional and prove that on the L^2-critical and supercritical level, blowup occurs for focusing GP hierarchies whenever the average energy per particle is negative.

In the present paper, we continue our investigation of the Cauchy problem for the cubic and quintic GP hierarchy, with focusing and defocusing interactions. It is crucial that our results do not assume any factorization of the initial data. Our interest in this system is based on the fact that the GP hierarchy is an effective theory describing an interacting Bose gas on the quantum field theory level, which is richer than the set of factorized states parametrized by solutions of the NLS, but at the same time also more accessible than the original system before taking $N \to \infty$. It possesses an interesting combination of mean field features and characteristics of quantum manybody systems.

While in [7], we addressed the local well-posedness of solutions, we will in this work establish global well-posedness of solutions of GP hierarchies in various situations described below. To be more precise, we describe here some of the basic settings. Similarly to [7], we introduce Banach spaces $\mathcal{H}_\xi^\alpha = \{ \Gamma \in \mathfrak{G} \,|\, \| \Gamma \|_{\mathcal{H}_\xi^\alpha} < \infty \}$ where

$$\mathfrak{G} \, = \, \{ \, \Gamma \, = \, (\, \gamma^{(k)}(x_1, \dots, x_k; x_1', \dots, x_k') \,)_{k \in \mathbb{N}} \, | \, \mathrm{Tr} \gamma^{(k)} \, < \, \infty \, \} \tag{1.1}$$

is the space of sequences of k-particle density matrices, and

$$\| \Gamma \|_{\mathcal{H}_\xi^\alpha} := \sum_{k\in\mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} , \tag{1.2}$$

where

$$\|\gamma^{(k)}\|_{H_k^\alpha} := (\mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|^2))^{\frac{1}{2}} , \tag{1.3}$$

with $S^{(k,\alpha)} := \prod_{j=1}^k \langle\nabla_{x_j}\rangle^\alpha \langle\nabla_{x_j'}\rangle^\alpha$. These are the L^2-type norms used in [20, 6, 7]. In this paper, we will also at various instances make use of the norms

$$\|\gamma^{(k)}\|_{H_k^\alpha}^\sharp := \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|) \tag{1.4}$$

employed in [11, 12], mostly in connection with energy conservation.

The parameter $\xi > 0$ is determined by the initial condition, and it sets the energy scale of a given Cauchy problem. If $\Gamma \in \mathcal{H}_\xi^\alpha$, then ξ^{-1} an upper bound on the typical H^α-energy per particle; this notion is made precise in [7]. We note that small energy results are characterized by large $\xi > 1$, while results valid without any upper bound on the size of the energy can be proven for arbitrarily small values of $\xi > 0$; in the latter case, one can assume $0 < \xi < 1$ without any loss of generality.

The parameter α determines the regularity of the solution, and our results on the L^2-critical and supercritical level, with $p \geq p_{L^2} = \frac{4}{d}$, hold for

$$\alpha \in \mathfrak{A}(d,p) := \begin{cases} (\frac{1}{2},\infty) & \text{if } d = 1 \\ (\frac{d}{2} - \frac{1}{2(p-1)},\infty) & \text{if } d \geq 2 \text{ and } (d,p) \neq (3,2) \\ \left[1,\infty\right) & \text{if } (d,p) = (3,2)\,, \end{cases} \tag{1.5}$$

where $p = 2$ for the cubic, and $p = 4$ for the quintic GP hierarchy.

The main results proven in this paper are:

(1) We prove the global well-posedness of solutions to defocusing p-GP hierarchies with $\Gamma_0 \in \mathcal{H}_\xi^1$ for arbitrary $\xi > 0$ (see Section 7 for details). This result can be understood as an improvement of the global existence and uniqueness of solutions in $\mathcal{H}_\xi^1$ which was obtained in [7] under the assumption that an a priori bound $\|\Gamma(t)\|_{\mathcal{H}_\xi^1} < c$ holds for $\xi > 0$ sufficiently small. In the work at hand, we actually prove a related a-priori bound by identifying (in Section 4) a family of *conserved, higher order energy functionals*, generalizing those found in [9]. Moreover, another important tool, used in the proof of global well-posedness for defocusing p-GP hierarchies with $\Gamma_0 \in \mathcal{H}_\xi^1$, is a generalization of the Sobolev inequalities that we prove for marginal density matrices in Section 5.

(2) We prove the global well-posedness of solutions for focusing and defocusing p-GP hierarchies on the L^2-subcritical level, $p < \frac{4}{d}$ (see Section 8). For the proof, we establish (in Section 6) a generalization of the Gagliardo-Nirenberg inequality, valid for marginal density matrices.

The precise definition of the model studied in this paper is given in Section 2, and the main Theorems proven in this work are summarized in Section 3.

2. Definition of the model

In this section, we introduce the mathematical model analyzed in this paper. We will mostly adopt the notations and definitions from [7], and we refer to [7] for motivations and more details.

2.1. **The spaces.**

We introduce the space

$$\mathfrak{G} := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$$

of sequences of density matrices

$$\Gamma := (\gamma^{(k)})_{k\in\mathbb{N}}$$

where $\gamma^{(k)} \geq 0$, $\mathrm{Tr}\gamma^{(k)} = 1$, and where every $\gamma^{(k)}(\underline{x}_k, \underline{x}'_k)$ is symmetric in all components of $\underline{x}_k$, and in all components of $\underline{x}'_k$, respectively, i.e.

$$\gamma^{(k)}(x_{\pi(1)}, ..., x_{\pi(k)}; x'_{\pi'(1)}, ..., x'_{\pi'(k)}) = \gamma^{(k)}(x_1, ..., x_k; x'_1, ..., x'_k) \tag{2.1}$$

holds for all $\pi, \pi' \in S_k$.

Throughout the paper we will denote the vector $(x_1, \cdots, x_k)$ by $\underline{x}_k$ and similarly the vector $(x'_1, \cdots, x'_k)$ by $\underline{x}'_k$.

The k-particle marginals are assumed to be hermitean,

$$\gamma^{(k)}(\underline{x}_k; \underline{x}'_k) = \overline{\gamma^{(k)}(\underline{x}'_k; \underline{x}_k)}. \tag{2.2}$$

We call $\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}}$ admissible if $\gamma^{(k)} = \mathrm{Tr}_{k+1,...,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}$, that is,

$$\begin{aligned}\gamma^{(k)}&(\underline{x}_k; \underline{x}'_k) \\ &= \int dx_{k+1} \cdots dx_{k+\frac{p}{2}}\, \gamma^{(k+\frac{p}{2})}(\underline{x}_k, x_{k+1}, \ldots, x_{k+\frac{p}{2}}; \underline{x}'_k, x_{k+1}, \ldots, x_{k+\frac{p}{2}})\end{aligned} \tag{2.3}$$

for all $k \in \mathbb{N}$.

Let $0 < \xi < 1$. We define

$$\mathcal{H}^\alpha_\xi := \left\{ \Gamma \in \mathfrak{G} \,\middle|\, \|\Gamma\|_{\mathcal{H}^\alpha_\xi} < \infty \right\} \tag{2.4}$$

where

$$\|\Gamma\|_{\mathcal{H}^\alpha_\xi} = \sum_{k=1}^{\infty} \xi^k \| \gamma^{(k)} \|_{H^\alpha_k(\mathbb{R}^{dk}\times\mathbb{R}^{dk})},$$

with

$$\|\gamma^{(k)}\|_{H^\alpha_k} := \left(\mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|^2) \right)^{\frac{1}{2}} \tag{2.5}$$

where $S^{(k,\alpha)} := \prod_{j=1}^k \langle\nabla_{x_j}\rangle^\alpha \langle\nabla_{x'_j}\rangle^\alpha$.

Remark 2.1. *We remark that similar spaces are used in the isospectral renormalization group analysis of spectral problems in quantum field theory,* [4].

Remark 2.2. *We note that in* [11, 12], *the norm*

$$\|\gamma^{(k)}\|^{\sharp}_{H^{\alpha}_{k}} := \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|) \tag{2.6}$$

is used, which will also be employed here in connection with a priori energy bounds. We note that

$$\|\gamma^{(k)}\|_{H^{\alpha}_{k}} \,\leq\, C\,\|\gamma^{(k)}\|^{\sharp}_{H^{\alpha}_{k}} \tag{2.7}$$

as is proven in [7].

2.2. **The GP hierarchy.** Next, we introduce cubic, quintic, focusing, and defocusing GP hierarchies, adopting notations and definitions from [7].

Let $p \in \{2,4\}$. The p-GP (Gross-Pitaevskii) hierarchy is given by

$$i\partial_t\gamma^{(k)} \,=\, \sum_{j=1}^{k}[-\Delta_{x_j},\gamma^{(k)}] \,+\, \mu B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} \tag{2.8}$$

in d dimensions, for $k\in\mathbb{N}$. Here,

$$B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} \,=\, B^{+}_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} - B^{-}_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\,, \tag{2.9}$$

where

$$B^{+}_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} = \sum_{j=1}^{k} B^{+}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})},$$

and

$$B^{-}_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} = \sum_{j=1}^{k} B^{-}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})},$$

with

$$\begin{aligned}&\left(B^{+}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\ &\quad=\int dx_{k+1}\cdots dx_{k+\frac{p}{2}}dx'_{k+1}\cdots dx'_{k+\frac{p}{2}}\\ &\qquad\prod_{\ell=k+1}^{k+\frac{p}{2}}\delta(x_j-x_\ell)\delta(x_j-x_\ell')\gamma^{(k+\frac{p}{2})}(t,x_1,\dots,x_{k+\frac{p}{2}};x_1',\dots,x'_{k+\frac{p}{2}}),\end{aligned}$$

and

$$\begin{aligned}&\left(B^{-}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\ &\quad=\int dx_{k+1}\cdots dx_{k+\frac{p}{2}}dx'_{k+1}\cdots dx'_{k+\frac{p}{2}}\\ &\qquad\prod_{\ell=k+1}^{k+\frac{p}{2}}\delta(x_j'-x_\ell)\delta(x_j'-x_\ell')\gamma^{(k+\frac{p}{2})}(t,x_1,\dots,x_{k+\frac{p}{2}};x_1',\dots,x'_{k+\frac{p}{2}}).\end{aligned}$$

Moreover, we let

$$B^{\pm}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} \,:=\, B^{+}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} \,-\, B^{-}_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\,. \tag{2.10}$$

The operator $B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}$ accounts for $\frac{p}{2}+1$-body interactions between the Bose particles. We note that for factorized solutions, the corresponding 1-particle wave function satisfies the p-NLS $i\partial_t\phi = -\Delta\phi + \mu|\phi|^p\phi$.

As in [7, 9], we refer to (2.8) as the *cubic GP hierarchy* if $p = 2$, and as the *quintic GP hierarchy* if $p = 4$. Also we denote the L^2-critical exponent by $p_{L^2} = \frac{4}{d}$ and refer to (2.8) as a *L^2-critical GP hierarchy* if $p = p_{L^2}$ and as a *L^2-subcritical GP hierarchy* if $p < p_{L^2}$. Moreover, for $\mu = 1$ or $\mu = -1$ we refer to the GP hierarchies as being defocusing or focusing, respectively.

The p-GP hierarchy can be rewritten in the following compact manner:

$$\begin{aligned} i\partial_t \Gamma \, + \, \widehat{\Delta}_{\pm}\Gamma &= \mu \widehat{B}\Gamma \\ \Gamma(0) &= \Gamma_0 \, , \end{aligned} \tag{2.11}$$

where

$$\widehat{\Delta}_{\pm}\Gamma \, := \, (\, \Delta_{\pm}^{(k)} \gamma^{(k)} \,)_{k\in\mathbb{N}} \quad \text{with} \quad \Delta_{\pm}^{(k)} \, = \, \Delta_{\underline{x}_k} - \Delta_{\underline{x}_k'} ,$$

and

$$\widehat{B}\Gamma \, := \, (\, B_{k+\frac{p}{2}} \gamma^{(k+\frac{p}{2})} \,)_{k\in\mathbb{N}} \, . \tag{2.12}$$

Moreover, we will use the notation

$$\begin{aligned} \widehat{B}^{+}\Gamma &:= (\, B^{+}_{k+\frac{p}{2}} \gamma^{(k+\frac{p}{2})} \,)_{k\in\mathbb{N}}, \\ \widehat{B}^{-}\Gamma &:= (\, B^{-}_{k+\frac{p}{2}} \gamma^{(k+\frac{p}{2})} \,)_{k\in\mathbb{N}} \, . \end{aligned}$$

We refer to [7] for more detailed explanations.

3. Statement of the main Theorems

The main results of this paper, stated in Theorems 3.2 and 3.3 below, establish two important situations when solutions of (2.11) are globally wellposed. We also recall a new local well-posedness theorem (originally obtained in [7] in a weaker form) whose new and significantly simpler proof is given in [8].

In our arguments, we will make use of the following local wellposedness result proven in [8], which is an improvement on our result in [7].

Theorem 3.1. *Let $\alpha \in \mathfrak{A}(d,p)$ and $\Gamma_0 \in \mathcal{H}^\alpha_\xi$. Let $I = [0,T]$ for $0 < T < T_0(d,p,\xi)$. Then, there exists a unique solution $\Gamma \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi$ of the p-GP hierarchy, with*

$$\| \, \widehat{B}\Gamma \, \|_{L^1_{t\in I}\mathcal{H}^\alpha_\xi} \, < \, C(T,\xi,d,p) \, \|\Gamma_0\|_{\mathcal{H}^\alpha_\xi} \, , \tag{3.1}$$

in the space

$$\mathcal{W}(I,\xi) \, = \, \{ \, \Gamma \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi \, | \, \widehat{B}^{+}\Gamma \, , \, \widehat{B}^{-}\Gamma \in L^2_{t\in I}\mathcal{H}^\alpha_\xi \, \} \tag{3.2}$$

for the initial condition $\Gamma(0) = \Gamma_0$.

The key improvement of this local well-posedness result over the one established in [7] consists of the fact that the initial condition and the solution are in the same space $\mathcal{H}^\alpha_\xi$. In [7], the initial data Γ_0 was required to belong to $\mathcal{H}^\alpha_{\xi_1}$ for some $\xi_1 > 0$, while the solution $\Gamma(t)$ was shown to belong to $\mathcal{H}^\alpha_{\xi_2}$, for some $0 < \xi_2 < \xi_1$. For the proof of Theorem 3.1, we refer to [8].

The main results proven in this work are:

(1) We establish global well-posedness for defocusing p-GP hierarchies with arbitrary H^1-data.

Theorem 3.2. *Assume that* $p \leq \frac{2d}{d-2}$ *and* $1 \in \mathfrak{A}(d,p)$*, and that* $\|\Gamma_0\|^\sharp_{\mathcal{H}^1_{\xi'}} < \infty$ *for* $0 < \xi' < 1$*. Let*

$$\xi \;\leq\; (1 + \frac{2}{p+2} C_{Sob}(d,p))^{-\frac{1}{k_p}} \, \xi' \,, \tag{3.3}$$

where C_{Sob} *is the constant in Theorem 5.1 (generalized Sobolev inequality). Moreover, let* $I_j := [jT, (j+1)T]$ *with* $T < T_0(d,p,\xi)$ *(see Theorem 3.1). Then, there exists a unique global solution* $\Gamma \in \cup_{j\in\mathbb{Z}} \mathcal{W}(I_j,\xi)$ *of the p-GP hierarchy with initial condition* $\Gamma(0) = \Gamma_0$*, satisfying*

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \;\leq\; \|\Gamma_0\|^\sharp_{\mathcal{H}^1_{\xi'}} \tag{3.4}$$

for all $t \in \mathbb{R}$*.*

For the proof, we identify higher order conserved energy functionals, generalizing those found in [9]. Moreover, we prove a generalization of the Sobolev inequalities, applied to marginal density matrices.

(2) We prove global well-posedness for focusing and defocusing p-GP hierarchies on the L^2-subcritical level, $p < \frac{4}{d}$.

Theorem 3.3. *Assume that* $p < p_{L^2} = \frac{4}{d}$*, and that* $\Gamma_0 \in \mathcal{H}^1_{\xi'}$ *for some* $0 < \xi' < 1$*. Moreover, let* $\frac{d}{2}(\frac{1}{2} - \frac{1}{2k_p}) = \frac{1-2\delta}{k_p} < \frac{1}{k_p} < 1$ *where* $k_p = 1 + \frac{p}{2}$ *and* $\delta > 0$*. Moreover, let* $I_j := [jT, (j+1)T]$ *with* $T < T_0(d,p,\xi)$ *(see Theorem 3.1). Then, for* $0 < \xi < 1$ *sufficiently small, there exists a unique global solution* $\Gamma \in \cup_{j\in\mathbb{Z}} \mathcal{W}(I_j,\xi)$ *of the p-GP hierarchy with initial condition* $\Gamma(0) = \Gamma_0$*, and there exists a positive constant* $C(d,p,\gamma_0,\xi,\delta,\Gamma_0) < \infty$ *such that the a priori bound*

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \;\leq\; C(d,p,\gamma_0,\xi,\delta,\Gamma_0) \tag{3.5}$$

holds, for all $t \in \mathbb{R}$*.*

An explicit upper bound on the constant $C(d,p,\gamma_0,\xi,\delta,\Gamma_0)$ is provided in Theorem 8.1 below. For the proof, we establish a generalization of the Gagliardo-Nirenberg inequality for marginal density matrices, to control the norm of $\widehat{B}\Gamma$.

The remainder of this paper is dedicated to the proofs of these results.

4. Higher order energy conservation

In this section, we introduce a higher order generalization of the energy functional introduced in [9]. We prove that it is a conserved quantity for solutions of the p-GP hierarchy. As a main application, it will be used to enhance local well-posedness of solutions to global well-posedness.

Let

$$k_p \;:=\; 1 \,+\, \frac{p}{2}\,. \tag{4.1}$$

We define the operators

$$K_\ell \;:=\; \frac{1}{2}\,(1\,-\,\Delta_{x_\ell})\,\mathrm{Tr}_{\ell+1,\dots,\ell+\frac{p}{2}} \;+\; \frac{\mu}{p+2}\,B^+_{\ell;\ell+1,\dots,\ell+\frac{p}{2}}$$

for $\ell \in \mathbb{N}$. This operator is related to the average energy per particle $E_1(\Gamma)$ (introduced in [9]) through

$$\begin{aligned}\frac{1}{2} &+ E_1(\Gamma)\\ &= \;\; \mathrm{Tr}_{1,\dots,\ell,\ell+k,\cdots,j} K_\ell \gamma^{(j)}\\ &= \;\; \frac{1}{2} \,+\, \frac{1}{2}\,\mathrm{Tr}_1(-\Delta_x\gamma^{(1)}) \,+\, \frac{\mu}{p+2}\int dx\,\gamma^{(k_p)}(x,\dots,x;x,\dots,x)\,,\end{aligned} \tag{4.2}$$

using the admissibility of $\Gamma = (\gamma^j)_{j\in\mathbb{N}}$, see also [9].

Moreover, we introduce the operator

$$\mathcal{K}^{(m)} \quad:=\quad K_1 K_{k_p+1}\cdots K_{(m-1)k_p+1} \tag{4.3}$$

where the m factors are mutually commuting, in the sense that

$$K_{jk_p+1}K_{j'k_p+1}\gamma^{(j)} \;=\; K_{j'k_p+1}K_{jk_p+1}\gamma^{(j)} \tag{4.4}$$

holds for $0 \le j \ne j' \le m-1$.

Theorem 4.1. *Assume that $\Gamma = (\gamma^{(j)})$ is admissible and solves the GP hierarchy. Let $m \in \mathbb{N}$. Then,*

$$\big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \;:=\; \mathrm{Tr}_{1,k_p+1,2k_p+1,\dots,(m-1)k_p+1}\big(\,\mathcal{K}^{(m)}\,\gamma^{(mk_p)}(t)\,\big) \tag{4.5}$$

is a conserved quantity,

$$\partial_t \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \;=\; 0\,. \tag{4.6}$$

We note that replacing $\gamma^{(mk_p)}$ by any $\gamma^{(j)}$ with $j \ge mk_p$ yields the same value of $\big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)}$.

In particular, for the defocusing p-GP hierarchy, with $\mu = +1$, the a priori bound

$$\mathrm{Tr}(\,S^{(m,1)}\gamma^{(m)}(t)\,) \;\le\; \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma_0} \tag{4.7}$$

holds for all $t \in \mathbb{R}$, and any solution $\Gamma(t)$ of the p-GP hierarchy with $\big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma_0} < \infty$.

Proof. To prove (4.6), we note that

$$i\partial_t\gamma^{(mk_p)} \quad=\quad \sum_{\ell=1}^{m}\Big(\,h^\pm_\ell\gamma^{(mk_p)} \,+\, \mu\, b^\pm_\ell\gamma^{((m+1)k_p)}\Big) \tag{4.8}$$

where

$$\begin{aligned}&h^\pm_\ell\gamma^{(mk_p)}(\underline{x}_{mk_p};\underline{x}'_{mk_p})\\ &:= \;\; -\sum_{j=(\ell-1)k_p+1}^{\ell k_p}(\Delta_{x_j} - \Delta_{x'_j})\,\gamma^{(mk_p)}(\underline{x}_{mk_p};\underline{x}'_{mk_p})\end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & b_\ell^\pm \gamma^{(mk_p)}(\underline{x}_{mk_p};\underline{x}'_{mk_p}) \\ & := \sum_{j=(\ell-1)k_p+1}^{\ell k_p} (\, B^\pm_{j;mk_p+1,\dots,(m+1)k_p}\,\gamma^{((m+1)k_p)})(\underline{x}_{mk_p};\underline{x}'_{mk_p})\,. \end{aligned} \tag{4.10}$$

Accordingly,

$$\partial_t \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \;=\; \sum_{\ell=1}^m \Big[\, A_h(\ell;m) \;+\; \mu\, A_b(\ell;m)\,\Big]\,,$$

where

$$A_h(\ell;m) \quad := \quad \mathrm{Tr}_{1,k_p+1,2k_p+1,\dots,(m-1)k_p+1}(\, \mathcal{K}^{(m)} h_\ell^\pm \gamma^{(mk_p)}\,)\,, \tag{4.11}$$

and

$$A_b(\ell;m) \quad := \quad \mathrm{Tr}_{1,k_p+1,2k_p+1,\dots,(m-1)k_p+1}(\, \mathcal{K}^{(m)} b_\ell^\pm \gamma^{((m+1)k_p)}\,)\,. \tag{4.12}$$

Next, we claim that

$$A_h(\ell;m) \;+\; \mu\, A_b(\ell;m) \;=\; 0 \tag{4.13}$$

for every $\ell \in \{1,\dots,m\}$.

To prove this, we first of all note that by symmetry of $\gamma^{(mk_p)}(\underline{x}_{mk_p};\underline{x}'_{mk_p})$ with respect to the components of $\underline{x}_{mk_p}$ and $\underline{x}'_{mk_p}$, it suffices to assume that $\ell = 1$. The other cases are similar.

Accordingly, letting $\ell = 1$, we introduce the notations

$$\widetilde{\gamma}^{(k_p)}(\underline{x}_{k_p};\underline{x}'_{k_p}) \;:=\; \mathrm{Tr}_{k_p,\dots,mk_p}(\, K_{k_p}\cdots K_{(m-1)k_p}\gamma^{(mk_p)}\,) \tag{4.14}$$

and

$$\begin{aligned} & \widetilde{\gamma}^{(2k_p)}(\underline{x}_{k_p},\underline{y}_{k_p};\underline{x}'_{k_p},\underline{y}'_{k_p}) \\ & := \quad \mathrm{Tr}_{k_p,\dots,mk_p}(\, K_{k_p}\cdots K_{(m-1)k_p}\gamma^{((m+1)k_p)}\,)(\underline{x}_{k_p},\underline{y}_{k_p};\underline{x}'_{k_p},\underline{y}'_{k_p}) \end{aligned} \tag{4.15}$$

where $y_i = x_{mk_p+i}$ and $y'_i = x'_{mk_p+i}$, for $i \in \{1,\dots,k_p\}$.

We recall that

$$K_1 \;=\; K_1^{(1)} \;+\; K_1^{(2)} \tag{4.16}$$

where

$$K_1^{(1)} \;:=\; \frac{1}{2}\,(1-\Delta_{x_1})\,\mathrm{Tr}_{2,\dots,k_p} \tag{4.17}$$

and

$$K_1^{(2)} \;:=\; \frac{\mu}{p+2}\, B^+_{1;2,\dots,k_p}\,. \tag{4.18}$$

Accordingly, we consider

$$
\begin{aligned}
A_h^{(1)} &:= \mathrm{Tr}_1(\, K_1^{(1)} h_1^{\pm} \widetilde{\gamma}^{(k_p)}\,) \\
&= -\frac{1}{2} \mathrm{Tr}_{1,2,\dots,k_p}(\,(1-\Delta_{x_1}) \sum_{j=1}^{k_p} (\Delta_{x_j} - \Delta_{x_j'})\, \widetilde{\gamma}^{(k_p)}\,) \\
&= \frac{1}{2} \int du_1 \dots du_{k_p} du_1' \dots du_{k_p}' \int dx_1 \dots dx_{k_p} dx_1' \dots dx_{k_p}' \\
&\qquad \delta(x_1 - x_1') \cdots \delta(x_{k_p} - x_{k_p}')\, (1 + u_1^2) \sum_{j=1}^{k_p} \big(u_j^2 - (u_j')^2\big) \\
&\qquad\qquad (\prod_{l=1}^{k_p} e^{i(u_l x_l - u_l' x_l')}\,) \widehat{\widetilde{\gamma}^{(k_p)}}(\underline{u}_{k_p}; \underline{u}_{k_p}') \\
&= \frac{1}{2} \int du_1 \dots du_{k_p} du_1' \dots du_{k_p}' \, (\prod_{l=1}^{k_p} \delta(u_l - u_l')\,) \\
&\qquad\qquad (1 + u_1^2) \sum_{j=1}^{k_p} \big(u_j^2 - (u_j')^2\big)\, \widehat{\widetilde{\gamma}^{(k_p)}}(\underline{u}_{k_p}; \underline{u}_{k_p}') \\
&= 0
\end{aligned}
$$

and

$$
A_b^{(1)} \quad := \quad \mathrm{Tr}_1(\, K_1^{(1)} b_1^{\pm} \widetilde{\gamma}^{(2k_p)}\,)\,, \tag{4.19}
$$

and

$$
A_h^{(2)} \quad := \quad \mathrm{Tr}_1(\, K_1^{(2)} h_1^{\pm} \widetilde{\gamma}^{(k_p)}\,)\,, \tag{4.20}
$$

as well as

$$
\begin{aligned}
A_b^{(2)} &:= \mathrm{Tr}_1(\, K_1^{(2)} b_1^{\pm} \widetilde{\gamma}^{(2k_p)}\,) \\
&= \frac{\mu}{p+2} \mathrm{Tr}_1(\, B^{+}_{1;2,\dots,k_p}\, b_1^{\pm} \widetilde{\gamma}^{(2k_p)}\,) \\
&= \frac{\mu}{p+2} \sum_{j=1}^{k_p} \mathrm{Tr}_1(\, B^{+}_{1;2,\dots,k_p}\, B^{\pm}_{j;k_p+1,\dots,2k_p} \widetilde{\gamma}^{(2k_p)}\,) \\
&= \frac{\mu}{p+2} \sum_{j=1}^{k_p} \int dx_1 \dots dx_{k_p} dx_1' \dots dx_{k_p}'\, \delta(x_1 = \dots = x_{k_p} = x_1' = \dots = x_{k_p}') \\
&\qquad \Big(B^{\pm}_{j;k_p+1,\dots,2k_p} \widetilde{\gamma}^{(2k_p)} \Big)\, (\underline{x}_{k_p}, \underline{x}_{k_p}')\,,
\end{aligned}
$$

where we used the notation

$$
\begin{aligned}
&\delta(x_1 = \dots = x_{k_p} = x_1' = \dots = x_{k_p}') \\
&\quad := \delta(x_1 - x_1') \prod_{\ell=2}^{k_p} (\, \delta(x_1 - x_\ell)\, \delta(x_1 - x_\ell')\,)
\end{aligned} \tag{4.21}
$$

also employed in [9].

Using the definition of $B^{\pm}_{j;k_p+1,\dots,2k_p}$, we find

$$\begin{aligned} A_b^{(2)} &= \frac{\mu}{p+2}\sum_{j=1}^{k_p}\int dx_1\dots dx_{k_p}dx'_1\dots dx'_{k_p}\,\delta(x_1=\cdots=x_{k_p}=x'_1=\cdots=x'_{k_p}) \\ &\qquad [\,\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_j,\dots,x_j}_{k_p};x'_1,\dots,x'_{k_p},\underbrace{x_j,\dots,x_j}_{k_p}) \\ &\qquad\quad -\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x'_j,\dots,x'_j}_{k_p};x'_1,\dots,x'_{k_p},\underbrace{x'_j,\dots,x'_j}_{k_p})\,] \\ &= 0 \end{aligned}$$

(see also [9]). Next, we claim that

$$A_h(1;m)\;+\;\mu\,A_b(1;m)\;=\;A_h^{(2)}\;+\;\mu\,A_b^{(1)}\;=\;0 \tag{4.22}$$

holds.

To this end, we note that

$$\begin{aligned} A_b^{(1)} \;&=\; \frac{1}{2}\,\mathrm{Tr}_1(\,(1-\Delta_{x_1})\,\mathrm{Tr}_{2,\dots,k_p}\sum_{j=1}^{k_p}B^{\pm}_{j;k_p+1,\dots,2k_p}\widetilde{\gamma}^{(2k_p)}\,) \\ &=\;\frac{1}{2}\,\sum_{j=2}^{k_p}\mathrm{Tr}_{1,2,\dots,k_p}(\,(1-\Delta_{x_1})\,B^{\pm}_{j;k_p+1,\dots,2k_p}\widetilde{\gamma}^{(2k_p)}\,) \\ &\qquad +\;\frac{1}{2}\,\mathrm{Tr}_{1,2,\dots,k_p}(\,(1-\Delta_{x_1})\,B^{\pm}_{1;k_p+1,\dots,2k_p}\widetilde{\gamma}^{(2k_p)}\,) \end{aligned} \tag{4.23}$$

$$\begin{aligned} &=\;\frac{1}{2}\,\mathrm{Tr}_{1,2,\dots,k_p}(\,(1-\Delta_{x_1})\,B^{\pm}_{1;k_p+1,\dots,2k_p}\widetilde{\gamma}^{(2k_p)}\,) \\ &=\;\frac{1}{2}\int dx_1\dots dx_{k_p}dx'_1\dots dx'_{k_p}\,\delta(x_1-x'_1)\cdots\delta(x_{k_p}-x'_{k_p}) \end{aligned} \tag{4.24}$$

$$\Big((1-\Delta_{x_1})B^{\pm}_{1;k_p+1,\dots,2k_p}\widetilde{\gamma}^{(2k_p)}\Big)\,(\underline{x}_{k_p},\underline{x}'_{k_p})\,. \tag{4.25}$$

Using the definition of $B^{\pm}_{1;k_p+1,\dots,2k_p}$, this equals

$$\begin{aligned}
A_b^{(1)} = &\ \frac{1}{2}\int dx_1\dots dx_{k_p}dx_1'\dots dx_{k_p}'\,\delta(x_1-x_1')\cdots\delta(x_{k_p}-x_{k_p}')\\
&\big(1-\nabla_{x_1}\nabla_{x_1'}\big)\\
&\quad[\,\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_1,\dots,x_1}_{k_p};x_1',\dots,x_{k_p}',\underbrace{x_1,\dots,x_1}_{k_p})\\
&\quad\quad-\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_1',\dots,x_1'}_{k_p};x_1',\dots,x_{k_p}',\underbrace{x_1',\dots,x_1'}_{k_p})\,]\\
= &\ \frac{1}{2}\int dx_1\dots dx_{k_p}dx_1'\dots dx_{k_p}'\,\delta(x_1-x_1')\cdots\delta(x_{k_p}-x_{k_p}')\\
&\quad[\,\Delta_{x_1'}\,\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_1,\dots,x_1}_{k_p};x_1',\dots,x_{k_p}',\underbrace{x_1,\dots,x_1}_{k_p})\\
&\quad\quad-\Delta_{x_1}\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_1',\dots,x_1'}_{k_p};x_1',\dots,x_{k_p}',\underbrace{x_1',\dots,x_1'}_{k_p})\,]. \qquad (4.26)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A_h^{(2)} = &-\frac{\mu}{p+2}\mathrm{Tr}_1(\,B^{+}_{1;2,\dots,k_p}\sum_{j=1}^{k_p}(\Delta_{x_j}-\Delta_{x_j'})\widetilde{\gamma}^{(k_p)}\,)\\
= &-\frac{\mu}{p+2}\int dx_1\dots dx_{k_p}dx_1'\dots dx_{k_p}'\,\delta(x_1=\dots=x_{k_p}=x_1'=\dots=x_{k_p}')\\
&\sum_{j=1}^{k_p}(\Delta_{x_j}-\Delta_{x_j'})\,\widetilde{\gamma}^{(k_p)}\,(\underline{x}_{k_p};\underline{x}_{k_p}') \qquad (4.27)
\end{aligned}$$

By symmetry of $\widetilde{\gamma}^{(k_p)}$ with respect to the components of $\underline{x}_{k_p}$ and $\underline{x}_{k_p}'$, this yields

$$\begin{aligned}
A_h^{(2)} = &-\frac{\mu\,k_p}{p+2}\int dx_1\dots dx_{k_p}dx_1'\dots dx_{k_p}'\,\delta(x_1=\dots=x_{k_p}=x_1'=\dots=x_{k_p}')\\
&(\Delta_{x_1}-\Delta_{x_1'})\,\widetilde{\gamma}^{(k_p)}\,(\underline{x}_{k_p};\underline{x}_{k_p}') \qquad (4.28)\\
= &-\frac{\mu}{2}\int dx_1\dots dx_{k_p}dx_1'\dots dx_{k_p}'\,\delta(x_1-x_1')\cdots\delta(x_{k_p}-x_{k_p}')\\
&[\,\Delta_{x_1}\,\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_1',\dots,x_1'}_{k_p};x_1',\dots,x_{k_p}',\underbrace{x_1',\dots,x_1'}_{k_p})\\
&\quad-\Delta_{x_1'}\widetilde{\gamma}^{(2k_p)}(x_1,\dots,x_{k_p},\underbrace{x_1,\dots,x_1}_{k_p};x_1',\dots,x_{k_p}',\underbrace{x_1,\dots,x_1}_{k_p})\,], \qquad (4.29)
\end{aligned}$$

where in order to obtain (4.29) we used the admissibility of $\Gamma = (\,\gamma^{(j)}\,)_{j\in\mathbb{N}}$.

The proof that (4.22) now follows from (4.26) and (4.29).

This proves (4.6).

It is clear that if $\mu = +1$, then

$$\begin{aligned} \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma_0} &= \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \\ &\geq \mathrm{Tr}_{1,k_p,\dots,(m-1)k_p}\big(K_1^{(1)} K_{k_p}^{(1)} \cdots K_{(m-1)k_p}^{(1)} \gamma^{(mk_p)} \big) \\ &= 2^{-m}\, \mathrm{Tr}_{1,\dots,m}\big(S^{(m,1)} \gamma^{(m)} \big) . \end{aligned} \tag{4.30}$$

This proves (4.7). □

5. Generalized Sobolev inequality

As a preparation for our discussion in Section 7 where we use higher order energy conservation to enhance local to global well-posedness for defocusing GP-hierarchies, we present a generalization of Sobolev inequalities for density matrices.

Theorem 5.1. *(Sobolev inequality) Assume that $f \in \dot{H}^\alpha(\mathbb{R}^{qd})$. Then, there exists $C_{Sob} = C_{Sob}(d,q)$ such that*

$$\begin{aligned} \Big(\int dx |f(\underbrace{x,\dots,x}_{q})|^2 \Big)^{\frac{1}{2}} &\leq C_{Sob} \Big(\int dx_1 \cdots dx_q \, \big| |\nabla_{x_1}| \cdots |\nabla_{x_q}| \, f(x_1,\dots,x_q) \big|^2 \Big)^{\frac{1}{2}} \\ &= \| f \|_{\dot{H}^\alpha_{x_1,\dots,x_q}} \end{aligned} \tag{5.1}$$

for $\alpha = \frac{(q-1)d}{2q}$ and $x_i \in \mathbb{R}^d$. The statement is also valid for H^α in place of $\dot{H}^\alpha$.

Proof. We perform a Littlewood-Paley decomposition $1 = \sum_j P_j$ where P_j acts in frequency space as multiplication with the characteristic function on the dyadic annulus $A_i := \{\xi \in \mathbb{R}^d | 2^j \leq |\xi| < 2^{j+1}\}$, and P_0 is the characteristic function on the unit ball.

Let $j_1, \dots, j_q \in \mathbb{N}_0$, and

$$f_{j_1 \dots j_q}(x_1,\dots,x_q) := (P_{j_1}^{(1)} \cdots P_{j_q}^{(q)} f)(x_1,\dots,x_q) \tag{5.2}$$

where the superscript in $P_i^{(m)}$ signifies that it acts on the m-th variable.

Then, clearly, the Fourier transform satisfies

$$\widehat{f_{j_1 \dots j_q}}(\xi_1,\dots,\xi_q) = h_{j_1}(\xi_1) \cdots h_{j_q}(\xi_q) \widehat{f_{j_1 \dots j_q}}(\xi_1,\dots,\xi_q) \tag{5.3}$$

where h_i are Schwartz class functions with $P_i h_i = P_i$.

We note that

$$\begin{aligned} h_j^\vee(x) &= \int d\xi \, h_j(\xi) \, e^{2\pi i \xi x} \\ &= 2^{jd} \int d\xi \, h_1(\xi) \, e^{2\pi i \xi (2^j x)} \\ &= 2^{jd} \, h_1^\vee(2^j x) . \end{aligned} \tag{5.4}$$

Therefore, $h_j^\vee$ is a smooth delta function with amplitude 2^{jd}, and supported on a ball of radius 2^{-j}. In particular,

$$\| h_j^\vee \|_{L^\infty_x} \leq c \, 2^{jd} \ , \ \| h_j^\vee \|_{L^1_x} = \| h_1^\vee \|_{L^1_x} \leq c' , \tag{5.5}$$

for constants c, c' independent of j. Because h_j is an even function for all j, it follows that $h_j^\vee \in \mathbb{R}$.

Accordingly, performing the inverse Fourier transform,

$$f_{j_1\dots j_q}(x_1,\dots,x_q) = \int dy_1 \cdots dy_q \, f_{j_1\dots j_q}(y_1,\dots,y_q) \, h_{j_1}^\vee(y_1+x_1)\cdots h_{j_q}^\vee(y_q+x_q)\,. \tag{5.6}$$

In particular,

$$\begin{aligned}&\int dx\,|f_{j_1\dots j_q}(x,\dots,x)|^2 \\ &= \int dy_1\cdots dy_q\,dy_1'\cdots dy_q'\,f_{j_1\dots j_q}(y_1,\dots,y_q)\overline{f_{j_1\dots j_q}(y_1',\dots,y_q')} \\ &\quad\int dx\,h_{j_1}^\vee(y_1+x)\cdots h_{j_q}^\vee(y_q+x) \\ &\quad\quad h_{j_1}^\vee(y_1'+x)\cdots h_{j_q}^\vee(y_q'+x)\,. \end{aligned} \tag{5.7}$$

Using Cauchy-Schwarz only on $f_{j_1\dots j_q}\overline{f_{j_1\dots j_q}}$, this is bounded by

$$\begin{aligned}\int dx\,|f_{j_1\dots j_q}(x,\dots,x)|^2 &\le \int dy_1\cdots dy_q\,|\,f_{j_1\dots j_q}(y_1,\dots,y_q)\,|^2 \\ &\quad\int dx\int dy_1'\cdots dy_q'\,|\,h_{j_1}^\vee(y_1+x)\cdots h_{j_q}^\vee(y_q+x) \\ &\quad\quad h_{j_1}^\vee(y_1'+x)\cdots h_{j_q}^\vee(y_q'+x)\,|\,. \end{aligned} \tag{5.8}$$

Thus, integrating out $y_1',\dots,y_q'$ and using $\|h_j^\vee\|_{L^1_x} < c'$,

$$\begin{aligned}\int dx\,|f_{j_1\dots j_q}(x,\dots,x)|^2 &\le C\int dy_1\cdots dy_q\,|\,f_{j_1\dots j_q}(y_1,\dots,y_q)\,|^2 \\ &\quad\int dx\,|\,h_{j_1}^\vee(y_1+x)\cdots h_{j_q}^\vee(y_q+x)\,|\,. \end{aligned} \tag{5.9}$$

Now, we assume without any loss of generality that $j_i \le j_q$ for all $i < q$. Then,

$$\begin{aligned}\int dx\,|\,h_{j_1}^\vee(y_1+x)\cdots h_{j_q}^\vee(y_q+x)\,| &\le \|h_{j_1}^\vee\|_{L^\infty_x}\cdots\|h_{j_{q-1}}^\vee\|_{L^\infty_x}\|h_{j_q}^\vee\|_{L^1_x} \\ &\le 2^{(j_1+\cdots+j_{q-1})d}\,c'\,. \end{aligned} \tag{5.10}$$

Now, since by assumption, $j_i \le j_q$ for all $i < q$,

$$j_1+\cdots+j_{q-1} \le \frac{q-1}{q}(j_1+\cdots+j_q)\,. \tag{5.11}$$

Therefore,

$$\begin{aligned}&\int dx\,|f_{j_1\dots j_q}(x,\dots,x)|^2 \\ &\le C\,2^{2(j_1+\cdots+j_q)\alpha}\int dy_1\cdots dy_q\,|\,f_{j_1\dots j_q}(y_1,\dots,y_q)\,|^2\,, \end{aligned} \tag{5.12}$$

where

$$\alpha = \frac{(q-1)d}{2q}\,. \tag{5.13}$$

We thus find that

$$
\begin{aligned}
&\| f(x,\dots,x) \|_{L^2_x(\mathbb{R}^d)} \\
&= \| \sum_{j_1,\dots,j_q} f_{j_1\dots j_q}(x,\dots,x) \|_{L^2_x(\mathbb{R}^d)} \\
&\leq \sum_{j_1,\dots,j_q} \Big(\int dx \, |f_{j_1,\dots,j_q}(x,\dots,x)|^2 \Big)^{\frac{1}{2}} \\
&\leq C \sum_{j_1,\dots,j_q} 2^{(j_1+\cdots+j_q)\alpha} \Big(\int dy_1 \cdots dy_q \, | f_{j_1,\dots,j_q}(y_1,\dots,y_q) |^2 \Big)^{\frac{1}{2}} \\
&= C \sum_{j_1,\dots,j_q} 2^{(j_1+\cdots+j_q)\alpha} \| f_{j_1,\dots,j_q} \|_{L^2_{x_1,\dots,x_q}} && (5.14) \\
&= C \| f \|_{\dot{H}^\alpha_{x_1,\dots,x_q}} . && (5.15)
\end{aligned}
$$

This is the asserted result. □

6. Generalized Gagliardo-Nirenberg inequality

We can easily generalize our proof of the generalized Sobolev inequalities to a generalization of the Gagliardo-Nirenberg inequality, which will be useful to us in Section 8 where we prove global well-posedness for L^2 subcritical GP hierarchies.

Theorem 6.1. *(Gagliardo-Nirenberg inequality) Assume that* $f \in \dot{H}^1(\mathbb{R}^{qd})$ *and* $\alpha = \frac{(q-1)d}{2q} < 1$. *Then,*

$$
\Big(\int dx \, | f(\underbrace{x,\dots,x}_{q}) |^2 \Big)^{\frac{1}{2}} \leq C \| f \|^\alpha_{\dot{H}^1_{x_1,\dots,x_q}} \| f \|^{1-\alpha}_{L^2_{x_1,\dots,x_q}} \tag{6.1}
$$

where $x_i \in \mathbb{R}^d$. *The statement is also valid for* H^1 *in place of* $\dot{H}^1$.

Proof. From (5.14), the Hölder estimate yields

$$
\begin{aligned}
&\| f(x,\dots,x) \|_{L^2_x(\mathbb{R}^d)} \\
&\leq \sum_{j_1,\dots,j_q} \| f_{j_1\dots j_q}(x,\dots,x) \|_{L^2_x(\mathbb{R}^d)} \\
&\leq C \sum_{j_1,\dots,j_q} 2^{(j_1+\cdots+j_q)\alpha} \| f_{j_1,\dots,j_q} \|_{L^2} \\
&= C \sum_{j_1,\dots,j_q} 2^{(j_1+\cdots+j_q)\alpha} \| f_{j_1,\dots,j_q} \|^\alpha_{L^2} \| f_{j_1,\dots,j_q} \|^{1-\alpha}_{L^2} \\
&\leq C \Big[\sum_{j_1,\dots,j_q} \Big(2^{(j_1+\cdots+j_q)\alpha} \| f_{j_1,\dots,j_q} \|^\alpha_{L^2} \Big)^{\frac{1}{\alpha}} \Big]^\alpha \\
&\qquad \Big[\sum_{j_1,\dots,j_q} \Big(\| f_{j_1,\dots,j_q} \|^{1-\alpha}_{L^2} \Big)^{\frac{1}{1-\alpha}} \Big]^{1-\alpha} \\
&= C \| f \|^\alpha_{\dot{H}^1_{x_1,\dots,x_q}} \| f \|^{1-\alpha}_{L^2_{x_1,\dots,x_q}} , && (6.2)
\end{aligned}
$$

for $\alpha = \frac{(q-1)d}{2q} < 1$ (noting that $\frac{1}{\alpha}, \frac{1}{1-\alpha} > 1$ are Hölder conjugate exponents). □

7. Global well-posedness of solutions for defocusing GP hierarchies and arbitrary H^1 data

For energy subcritical, defocusing GP hierarchies, we can now deduce a priori energy bounds based on which we establish global well-posedness of solutions independent of the size of the initial data.

Theorem 7.1. *Assume that $\mu = +1$ (defocusing p-GP hierarchy), $p \leq \frac{2d}{d-2}$, $1 \in \mathfrak{A}(d,p)$, and that $\|\Gamma_0\|^{\sharp}_{\mathcal{H}^1_{\xi'}} < \infty$ for $0 < \xi' < 1$. Let*

$$\xi \;\leq\; (1 + \frac{2}{p+2} C_{Sob}(d,p))^{-\frac{1}{k_p}} \, \xi' \,. \tag{7.1}$$

Moreover, let $I_j := [jT, (j+1)T]$ with $T < T_0(d,p,\xi)$ (see Theorem 3.1). Then, there exists a unique global solution $\Gamma \in \cup_{j\in\mathbb{Z}} \mathcal{W}(I_j, \xi)$ of the p-GP hierarchy with initial condition $\Gamma(0) = \Gamma_0$, satisfying

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \;\leq\; \|\Gamma_0\|^{\sharp}_{\mathcal{H}^1_{\xi'}} \tag{7.2}$$

for all $t \in \mathbb{R}$.

To prove this result, we first note that the following Lemma is an immediate corollary of Theorem 4.1.

Lemma 7.2. *Assume that $\mu = +1$ (defocusing p-GP hierarchy), and that*

$$\sum_{m\in\mathbb{N}} (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma_0} \;<\; \infty \,, \tag{7.3}$$

for $\xi > 0$. Assume that $\Gamma(t) \in \mathcal{H}^1_\xi$ solves the p-GP hierarchy. Then,

$$\begin{aligned} \|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi} &\leq \sum_{m\in\mathbb{N}} (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \\ &= \sum_{m\in\mathbb{N}} (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma_0} \\ &< \infty \end{aligned} \tag{7.4}$$

for all $t \in \mathbb{R}$.

Moreover, we use the following proposition which expresses that the interaction energy in Γ_0 is bounded by the kinetic energy, in the energy critical and subcritical case, $p \leq \frac{2d}{d-2}$.

Proposition 7.3. *Assume that $\mu = +1$ or $\mu = -1$ (defocusing or focusing p-GP hierarchy), $p \leq \frac{2d}{d-2}$, and that $\|\Gamma\|^{\sharp}_{\mathcal{H}^1_{\xi'}} < \infty$ for $0 < \xi' < 1$. Let*

$$\xi \;\leq\; (1 + \frac{2}{p+2} C_{Sob}(d,p))^{-\frac{1}{k_p}} \, \xi' \,. \tag{7.5}$$

Then,

$$\sum_{m\in\mathbb{N}} (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma} \;\leq\; \|\Gamma\|^{\sharp}_{\mathcal{H}^1_{\xi'}} \tag{7.6}$$

holds.

Proof. The Sobolev inequalities for the GP hierarchy proven in Theorem 5.1 imply that for $p \leq \frac{2d}{d-2}$, the interaction energy is bounded by the kinetic energy,

$$\mathrm{Tr}_1(\, B_{1;2,\dots,k_p} \widetilde{\gamma}^{(k_p)}\,) \;\leq\; C_{Sob}(d,p)\, \mathrm{Tr}_{1,\dots,k_p}(\, S^{(k_p,1)} \widetilde{\gamma}^{(k_p)}\,)\,. \tag{7.7}$$

This follows from the fact that writing $\widetilde{\gamma}^{(k_p)}$ with respect to an orthonormal basis $(\phi_j)_j$,

$$\widetilde{\gamma}^{(k_p)}(\underline{x}_{k_p}, \underline{x}'_{k_p}) \;=\; \sum_j \lambda_j \big|\phi_j(\underline{x}_{k_p})\big\rangle\big\langle\phi_j(\underline{x}'_{k_p})\big|\,, \tag{7.8}$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$, we have

$$\mathrm{Tr}_1(\, B^+_{1;2,\dots,k_p} \gamma^{(k_p)}\,) \;=\; \sum_j \lambda_j \int dx |\phi_j(x,\dots,x)|^2\,. \tag{7.9}$$

Accordingly, Theorem 5.1 implies that

$$\begin{aligned} &\mathrm{Tr}_1(\, B^+_{1;2,\dots,k_p} \widetilde{\gamma}^{(k_p)}\,) \\ &\leq \; C_{Sob} \sum_j \lambda_j \|\phi_j\|^2_{H^1_{\underline{x}_{k_p}}} \\ &= \; C_{Sob}\, \mathrm{Tr}_{1,\dots,k_p}(\, S^{(k_p,1)} \widetilde{\gamma}^{(k_p)}\,). \end{aligned} \tag{7.10}$$

Now we observe that (7.7) implies

$$\mathrm{Tr}_1(\, K_1 \widetilde{\gamma}^{(k_p)}\,) \;\leq\; (\,\frac{1}{2} + \frac{1}{p+2} C_{Sob}(d,p)\,)\, \mathrm{Tr}_{1,\dots,k_p}(\, S^{(k_p,1)} \widetilde{\gamma}^{(k_p)}\,)\,. \tag{7.11}$$

By iteration, this is easily seen to imply that

$$\big\langle \mathcal{K}^{(m)} \big\rangle_\Gamma \quad\leq\quad (\,\frac{1}{2} + \frac{1}{p+2} C_{Sob}(d,p)\,)^m\, \mathrm{Tr}_{1,\dots,mk_p}(\, S^{(mk_p,1)} \gamma^{(mk_p)}\,)\,. \tag{7.12}$$

Therefore,

$$\begin{aligned} \sum_\ell (2\xi)^\ell \big\langle \mathcal{K}^{(\ell)} \big\rangle_\Gamma \quad &\leq \quad \sum_\ell \Big(\,(\,1 + \frac{2}{p+2} C_{Sob}(d,p)\,)^{\frac{1}{k_p}}\, \xi\Big)^\ell \,\|\, \gamma^{(\ell)} \,\|^\sharp_{H^1_\ell} \\ &\leq \quad \|\, \Gamma \,\|_{\mathcal{H}^1_{\xi'}} \end{aligned} \tag{7.13}$$

with $\xi \leq (\,1 + \frac{2}{p+2} C_{Sob}(d,p)\,)^{-\frac{1}{k_p}}\, \xi'$. Hence, the claim follows. □

We may now prove Theorem 7.1, by enhancing local well-posedness to global well-posedness using the a priori H^1 bound provided by Proposition 7.3.

Proof. To begin with, the bound $\|\Gamma\|_{\mathcal{H}^\alpha_\xi} \leq \|\Gamma\|^\sharp_{\mathcal{H}^\alpha_\xi}$ is proven in [7].

Let ξ satisfy

$$0 \;<\; \xi \;\leq\; (1 + \frac{2}{p+2} C_{Sob}(d,p))^{-\frac{1}{k_p}}\, \xi'\,. \tag{7.14}$$

Given Γ_0 with $\|\Gamma_0\|^\sharp_{\mathcal{H}^1_{\xi'}} < \infty$, we have that

$$\|\Gamma_0\|_{\mathcal{H}^1_\xi} \leq \|\Gamma_0\|^\sharp_{\mathcal{H}^1_\xi} \leq \|\Gamma_0\|^\sharp_{\mathcal{H}^1_{\xi'}} < \infty . \tag{7.15}$$

We now let $T > 0$ be small enough that Theorem 3.1 holds, and $I := [0, T]$. We recall that Theorem 3.1 requires that $1 \in \mathfrak{A}(d, p)$. Then, $\Gamma(\,\cdot\,) \in \mathcal{W}(I, \xi)$ is the unique solution of the p-GP hierarchy with initial condition Γ_0; see (3.2) for the definition of $\mathcal{W}(I, \xi)$.

Next, Lemma 7.2 and Proposition 7.3 imply that

$$\begin{aligned} \|\Gamma(t)\|_{\mathcal{H}^1_\xi} &\leq \|\Gamma(t)\|^\sharp_{\mathcal{H}^1_\xi} \\ &\leq \sum_\ell (2\xi)^\ell \big\langle \mathcal{K}^{(\ell)} \big\rangle_{\Gamma(t)} \\ &\leq \|\Gamma(t)\|^\sharp_{\mathcal{H}^1_{\xi'_1}} \end{aligned} \tag{7.16}$$

for all $t \in I$.

However, Theorem 4.1 implies that $\big\langle \mathcal{K}^{(\ell)} \big\rangle_{\Gamma(s)} = \big\langle \mathcal{K}^{(\ell)} \big\rangle_{\Gamma(s')}$ for all $s, s' \in I$, and thus, we also have that

$$\|\Gamma(t)\|^\sharp_{\mathcal{H}^1_\xi} \leq \|\, \Gamma_0 \,\|^\sharp_{\mathcal{H}^1_{\xi'}} . \tag{7.17}$$

Accordingly, writing $I_0 := I$,

$$\|\Gamma(t)\|^\sharp_{L^\infty_{t\in I_0} \mathcal{H}^1_\xi} , \, \|\Gamma(T)\|^\sharp_{\mathcal{H}^1_\xi} \leq \|\, \Gamma_0 \,\|^\sharp_{\mathcal{H}^1_{\xi'}} . \tag{7.18}$$

We may thus use $\Gamma(T)$ as the initial condition for the p-GP hierarchy with $t \in I_1 := [T, 2T]$, to find that

$$\|\Gamma\|^\sharp_{L^\infty_{t\in I_1} \mathcal{H}^1_\xi} , \, \|\Gamma(2T)\|^\sharp_{\mathcal{H}^1_\xi} \leq \|\, \Gamma_0 \,\|^\sharp_{\mathcal{H}^1_{\xi'}} , \tag{7.19}$$

so that we can use $\Gamma(2T)$ as the initial condition for the interval $I_2 := [2T, 3T]$, and so on. Extending this argument also to $t < 0$, we conclude that

$$\|\Gamma\|_{L^\infty_{t\in\mathbb{R}} \mathcal{H}^1_\xi} \leq \|\Gamma\|^\sharp_{L^\infty_{t\in\mathbb{R}} \mathcal{H}^1_\xi} \leq \|\, \Gamma_0 \,\|^\sharp_{\mathcal{H}^1_{\xi'}} . \tag{7.20}$$

Using Theorem 3.1, we conclude that the solution $\Gamma(\,\cdot\,) \in \mathcal{W}(I_j, \xi)$ is unique for every $j \in \mathbb{Z}$, and hence globally in time. This implies the claim. □

8. Global well-posedness for L^2 subcritical GP hierarchies

In this section, we prove that the Cauchy problem for any p-GP hierarchy on the L^2-subcritical level, $p < p_{L^2} = \frac{4}{d}$, is globally well-posed. For the NLS, this is a well-known result. Due to the generalized Gagliardo-Nirenberg inequality, we can now obtain analogous result in the context of the L^2-subcritical p-GP hierarchy.

Theorem 8.1. *Assume that $p < p_{L^2} = \frac{4}{d}$, and that $\Gamma_0 \in \mathcal{H}^1_{\xi'}$ for some $0 < \xi' < 1$. Moreover, let $\alpha = \frac{(k_p-1)d}{2k_p} = \frac{1-2\delta}{k_p} < \frac{1}{k_p} < 1$ where $k_p = 1 + \frac{p}{2}$. Moreover, let $I_j := [jT, (j+1)T]$ with $T < T_0(d, p, \xi)$ (see Theorem 3.1). Then, for $0 < \xi < 1$*

sufficiently small, there exists a unique global solution $\Gamma \in \cup_{j\in\mathbb{Z}}\mathcal{W}(I_j,\xi)$ *of the p-GP hierarchy with initial condition* $\Gamma(0)=\Gamma_0$*, and the a priori bound*

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \le \max\{ \left(\frac{2}{1-\xi^\delta}\right)^{\frac{1}{1-\alpha}} , 2\sum_m (2\xi)^m \langle \mathcal{K}^{(m)}\rangle_{\Gamma_0} \} \tag{8.1}$$

holds, for all $t\in\mathbb{R}$*.*

Proof. The bound

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \le \|\Gamma(t)\|^\sharp_{\mathcal{H}^1_\xi} = \sum_{k\ge1}\xi^k \mathrm{Tr}(\, S^{k,1}\gamma^{(k)}\,) \tag{8.2}$$

is proven in [7]; see also (2.7).

To prove that $\|\Gamma(t)\|^\sharp_{\mathcal{H}^1_\xi}$ is bounded by the right hand side in (8.1), we note the following.

Given $\alpha = \frac{(k_p-1)d}{2k_p} < 1$, we obtain from the generalized Gagliardo-Nirenberg inequality given in Theorem 6.1 that

$$\mathrm{Tr}_1(\, B^+_{1;2,\dots,k_p}\gamma^{(k_p)}\,) \le C\,(\mathrm{Tr}(\, S^{(k_p,1)}\gamma^{(k_p)}\,)\,)^\alpha (\mathrm{Tr}(\,\gamma^{(k_p)}\,)\,)^{1-\alpha}. \tag{8.3}$$

More precisely, this follows from the fact that writing $\gamma^{(k_p)}$ with respect to an orthonormal basis $(\phi_j)_j$,

$$\gamma^{(k_p)}(\underline{x}_{k_p},\underline{x}'_{k_p}) = \sum_j \lambda_j \big|\phi_j(\underline{x}_{k_p})\big\rangle\big\langle\phi_j(\underline{x}'_{k_p})\big|, \tag{8.4}$$

where $\lambda_j\ge0$ and $\sum_j\lambda_j=1$, it is clear that

$$\mathrm{Tr}_1(\, B^+_{1;2,\dots,k_p}\gamma^{(k_p)}\,) = \sum_j \lambda_j\int dx|\phi_j(x,\dots,x)|^2. \tag{8.5}$$

Therefore, Theorem 6.1 implies that

$$\begin{aligned}
&\mathrm{Tr}_1(\, B^+_{1;2,\dots,k_p}\gamma^{(k_p)}\,)\\
&\le C\sum_j(\lambda_j\|\phi_j\|^2_{L^2_{\underline{x}_{k_p}}})^{1-\alpha}(\lambda_j\|\phi_j\|^2_{H^1_{\underline{x}_{k_p}}})^\alpha\\
&\le C\,(\sum_j\lambda_j\|\phi_j\|^2_{L^2_{\underline{x}_{k_p}}})^{1-\alpha}(\sum_j\lambda_j\|\phi_j\|^2_{H^1_{\underline{x}_{k_p}}})^\alpha,
\end{aligned} \tag{8.6}$$

where in order to arrive at the last line, we used the Hölder inequality with dual exponents $\frac{1}{1-\alpha}$ and $\frac{1}{\alpha}$. This implies (8.3).

We recall the definition of the operators

$$K_\ell = K_\ell^{(1)} + K_\ell^{(2)} \tag{8.7}$$

where

$$K_\ell^{(1)} := \frac{1}{2}(1-\Delta_{x_\ell})\,\mathrm{Tr}_{\ell+1,\dots,\ell+\frac{p}{2}} \tag{8.8}$$

and

$$K_\ell^{(2)} := \frac{\mu}{p+2}\,B^+_{\ell;\ell+1,\dots,\ell+\frac{p}{2}},$$

for $\ell \in \mathbb{N}$. Moreover, we have previously introduced

$$\mathcal{K}^{(m)} \;\; := \;\; K_1 K_{k_p+1} \cdots K_{(m-1)k_p+1} \,, \tag{8.9}$$

and we proved that

$$\big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} := \mathrm{Tr}_{1,k_p+1,2k_p+1,\dots,(m-1)k_p+1}(\, \mathcal{K}^{(m)} \gamma^{(mk_p)}(t)\,) \tag{8.10}$$

is a conserved quantity, for every $m \in \mathbb{N}$, provided that $\Gamma(t)$ solves the p-GP hierarchy.

In order to simplify the presentation below, we introduce the notation

$$\mathrm{Tr}^{1,m} := \mathrm{Tr}_{1,k_p+1,2k_p+1,\dots,(m-1)k_p+1} \,. \tag{8.11}$$

In the L^2 subcritical case, where $p < p_{L^2} = \frac{4}{d}$, we will now use the sequence of conserved quantities $(\langle \mathcal{K}^{(m)} \rangle_{\Gamma(t)})_{m \in \mathbb{N}}$ to obtain an a priori bound on $\|\Gamma(t)\|_{\mathcal{H}^1_\xi}$, for $\xi > 0$ sufficiently small.

To this end, we observe that, clearly,

$$\begin{aligned} &\big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} - \mathrm{Tr}^{1,m}(\, K^{(1)}_1 K^{(1)}_{k_p+1} \cdots K^{(1)}_{(m-1)k_p+1} \gamma^{(mk_p)}\,) \\ &\quad = \sum_{\ell=0}^{m-1} \binom{m}{\ell} \mathrm{Tr}^{1,m}(\, K^{(1)}_1 \cdots K^{(1)}_{(\ell-1)k_p+1} K^{(2)}_{\ell k_p+1} \cdots K^{(2)}_{(m-1)k_p+1} \gamma^{(mk_p)}\,) \,. \end{aligned} \tag{8.12}$$

Using (8.3) repeatedly, we find that

$$\begin{aligned} &\mathrm{Tr}^{1,m}(\, K^{(1)}_1 \cdots K^{(1)}_{(\ell-1)k_p+1} K^{(2)}_{\ell k_p+1} \cdots K^{(2)}_{(m-1)k_p+1} \gamma^{(mk_p)}\,) \\ &\leq C_1 \Big[\mathrm{Tr}^{1,m}(\, K^{(1)}_1 \cdots K^{(1)}_{(\ell-1)k_p+1} K^{(2)}_{\ell k_p+1} \cdots K^{(2)}_{(m-2)k_p+1} S^{(k_p,1)}_{(m-1)k_p+1} \gamma^{(mk_p)}\,) \Big]^{\alpha} \\ &\leq \cdots\cdots \\ &\leq C_1^{m-\ell} \Big[\mathrm{Tr}^{1,m}(\, K^{(1)}_1 \cdots K^{(1)}_{(\ell-1)k_p+1} S^{((k_p,1))}_{\ell k_p+1} \cdots S^{(k_p,1)}_{(m-1)k_p+1} \gamma^{(mk_p)}\,) \Big]^{\alpha^{m-\ell}} \end{aligned} \tag{8.13}$$

Due to the admissibility of $\gamma^{(mk_p)}$, the last line equals

$$\begin{aligned} (8.13) \;\; &= \;\; C_1^{m-\ell} \Big[\mathrm{Tr}(\, S^{(\ell+(m-\ell)k_p,1)} \gamma^{(\ell+(m-\ell)k_p)}\,) \Big]^{\alpha^{m-\ell}} \\ &\leq \;\; C_1^{m-\ell} \Big[\mathrm{Tr}(\, S^{(\ell+(m-\ell)k_p,1)} \gamma^{(\ell+(m-\ell)k_p)}\,) \Big]^{\alpha} \end{aligned} \tag{8.14}$$

where we used the fact that

$$\mathrm{Tr}(\, S^{(\ell+(m-\ell)k_p,1)} \gamma^{(\ell+(m-\ell)k_p)}\,) \; \geq \; 1 \,,$$

and $0 < \alpha < \frac{1}{k_p} < 1$, and moreover, $m - \ell \geq 1$.

Clearly,

$$(8.14) \;\; \leq \;\; \xi^{-(\ell+(m-\ell)k_p)\alpha} \, C_1^{m-\ell} \, (\, \|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha} \,. \tag{8.15}$$

Therefore, by the admissibility of $\Gamma(t)$,

$$\begin{aligned} |(8.12)| &= \big|\, 2^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \,-\, \mathrm{Tr}(\, S^{(m,1)}\gamma^{(m)}\,)\,\big| \\ &\leq \sum_{\ell=0}^{m-1} \binom{m}{\ell} \xi^{-(\ell+(m-\ell)k_p)\alpha}\, C_1^{m-\ell}\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha} \\ &= \sum_{\ell=0}^{m-1} \binom{m}{\ell} (\xi^{-\alpha})^{\ell} (C_1\,\xi^{-k_p\alpha})^{m-\ell}\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha} \\ &\leq (\xi^{-\alpha} \,+\, C_1\,\xi^{-k_p\alpha})^m\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha}\,. \end{aligned} \tag{8.16}$$

Since $\alpha < \frac{1}{k_p} < 1$, we find that for sufficiently small $\xi > 0$,

$$\xi\,(\xi^{-\alpha} \,+\, C_1\,\xi^{-k_p\alpha}) \,<\, \xi^{\delta} \,<\, 1\,, \tag{8.17}$$

where $\delta > 0$ is defined by $\alpha = \frac{1-2\delta}{k_p} < \frac{1}{k_p} < 1$.

Accordingly,

$$\begin{aligned} &\Big|\sum_m (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \,-\, \sum_m \xi^m \mathrm{Tr}(\, S^{(m,1)}\gamma^{(m)}\,)\,\Big| \\ &= \Big|\sum_m (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \,-\, \|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,\Big| \\ &\leq \Big(\,\sum_m \xi^{\delta m}\,\Big)\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha} \\ &= \frac{1}{1-\xi^{\delta}}\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha}\,. \end{aligned} \tag{8.18}$$

Because $\alpha < 1$, this immediately yields an a priori upper bound on $\|\Gamma(t)\|_{\mathcal{H}^1_\xi}$, as we show next.

It follows from (8.18) that

$$\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi} \Big(1 \,-\, \frac{1}{1-\xi^{\delta}}\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha-1}\Big) \,\leq\, \sum_m (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)}\,. \tag{8.19}$$

Herefrom, we deduce the following:

- If $\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi} < \Big(\frac{2}{1-\xi^{\delta}}\Big)^{\frac{1}{1-\alpha}}$, then $\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}$ is trivially bounded.
- On the other hand, if $\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi} \geq \Big(\frac{2}{1-\xi^{\delta}}\Big)^{\frac{1}{1-\alpha}}$, it follows from $\alpha < 1$ that

  $$\Big(1 \,-\, \frac{1}{1-\xi^{\delta}}\,(\,\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi}\,)^{\alpha-1}\Big) \,\geq\, \frac{1}{2}\,, \tag{8.20}$$

  so that

  $$\begin{aligned} \|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi} &\leq 2\sum_m (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma(t)} \\ &= 2\sum_m (2\xi)^m \big\langle \mathcal{K}^{(m)} \big\rangle_{\Gamma_0} \end{aligned} \tag{8.21}$$

  where the right hand side is a conserved quantity, as was established in Theorem 4.1. In particular, the right hand side converges for $0 < \xi <$

1 sufficiently small, if $\Gamma_0 \in \mathcal{H}^1_{\xi'}$ for some $0 < \xi' < 1$, as follows from Proposition 7.3.

This implies that for $0 < \xi < 1$ sufficiently small,

$$\|\Gamma(t)\|^{\sharp}_{\mathcal{H}^1_\xi} \leq \max\{ \left(\frac{2}{1-\xi^\delta}\right)^{\frac{1}{1-\alpha}} , 2\sum_m (2\xi)^m \left\langle \mathcal{K}^{(m)} \right\rangle_{\Gamma_0} \} \tag{8.22}$$

is a priori bounded, for all $t \in \mathbb{R}$. This proves the claim of the theorem. □

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