An inverse resistivity problem: 1. Fréchet differentiability of the cost functional and Lipschitz continuity of the gradient

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Abstract. Mathematical model of vertical electrical sounding (VES) over a medium with continuously changing conductivity $\sigma(z)$ is studied by using a resistivity method. The considered model leads to an inverse problem of identification of the unknown leading coefficient $\sigma(z)$ of the elliptic equation $\frac{\partial}{\partial z}(\sigma(z)\frac{\partial u}{\partial z}) + \frac{\sigma(z)}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = 0$ in the layer $\Omega = \{(r,z) \in R^2 : \ 0 \le r < \infty, \ 0 < z < H\}$. The measured data $\psi(r) := (\partial u/\partial r)_{z=0}$ is assumed to be given on the upper boundary of the layer, in the form of the tangential derivative. The proposed approach is based on transformation of the inverse problem, by introducing the reflection function $p(z) = (\ln \sigma(z))'$ and then using the Bessel-Fourier transformation with respect to the variable $r \ge 0$. As a result the inverse problem is formulated in terms of the transformed potential $V(\lambda, z)$ and the reflection function $p(z)$. It is proved that the transformed cost functional is Fréchet differentiable with respect to the reflection function $p(z)$. Moreover, an explicit formula for the Fréchet gradient of the cost functional is derived. Then Lipschitz continuity of this gradient is proved in class of reflection functions $p(z)$ with Hölder class of derivative $p'(z)$.

Key words: vertical electrical sounding, coefficient inverse problem, Fréchet gradient, Lipschitz continuity of the gradient

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1. Introduction

We study the mathematical model of vertical electrical sounding by using a resistivity method. In geophysical sciences this method is defined to be the problem of interpretation of surface or near-surface measured data (see, [1,8-15] and references therein). Mathematical modeling of the vertical electrical sounding leads to the following inverse problem of determination of the unknown leading coefficient in a linear elliptic equation.

Find the function $\sigma(z)$ via the solution of the boundary value problem

$$\frac{\partial}{\partial z}\left(\sigma(z)\frac{\partial u}{\partial z}\right) + \frac{\sigma(z)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0, \quad (r,z) \in \Omega \subset R^2, \tag{1}$$

$$\begin{cases} \sigma(0)\frac{\partial u}{\partial z}|_{z=0} = \delta(r), \quad u(r,z)|_{z=H} = 0, \\ \lim\limits_{r\to\infty} u(r,z) = 0 \end{cases} \tag{2}$$

from the measured data $\psi(r)$ defined as follows:

$$\left.\frac{\partial u}{\partial r}\right|_{z=0} = \psi(r). \tag{3}$$

Here $\Omega = \{(r,z) \in R^2 : \ 0 \le r < \infty, \ 0 < z < H\}$, and the function $\sigma(z)$ satisfies the following conditions:

$$\sigma(z) \in \mathcal{S} := \{\sigma(z) \in C^2[0,H], \ \sigma(0) = \sigma_0, \ \sigma'(0) = 0, \quad 0 < \sigma_1 \le \sigma(z) \le \sigma_2 < \infty\}. \tag{4}$$

Note that physically conditions (4) mean that the conductivity $\sigma(z)$ is assumed to be known at the Earth surface, and its near-surface behaviour needs to be constant (the condition $\sigma'(0) = 0$).

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The problem (1)-(3) will be defined as a coefficient inverse problem or *VES-problem.* In this context, for a given $\sigma(z) \in \mathcal{S}$ from the set of admissible coefficients $\mathcal{S}$ the boundary value problem (1)-(2) will be refered as a *direct (or forward) problem.* The functions $\psi(r)$ and $\sigma(z)$ are defined to be the *measured output data* and the *input data*, accordingly [5-7].

In view of quasisolution (or least square) approach, this inverse problem can be formulated as a minimization problem for the corresponding cost functional $J(\sigma)$ ([3,17]). In most cases for the numerical solution of this minimization problem gradient methods are used ([2]). For this aim, in many applications various gradient formulas are either derived empirically, or computed numerically ([1],[9]). Note that the first attempt for the numerical solution of the inverse problem (1)-(3) by using the gradient method was given in [1]. Although an empirical gradient formula has been employed with regularization algorithm, there was no mathematical framework for this formula. At the same time, it is well known that any gradient method requires an estimation of the iteration parameter $\alpha_n > 0$ in the iteration process $\sigma^{(n+1)} = \sigma^{(n)} - \alpha_n J'(\sigma^{(n)})$, $n = 0, 1, 2, ...,$ where $\sigma^{(0)}$ is a given initial iteration. Choice of the parameter α_k defines various gradient methods, although in many situation estimations of this parameter is a difficult problem. However, in the case of Lipschitz continuity of the gradient $J'(\sigma)$ the parameter α_n can be estimated via the Lipschitz constant, which subsequently improves convergence properties of the iteration process (see, [7],[18] and references therein). In this paper we propose a new approach for the solution of the inverse problem (1)-(3). This approach is based on transformation of the inverse problem, by introducing the reflection function $p(z) = (\ln \sigma(z))'$ and then using the Bessel-Fourier transformation with respect to the variable $r \geq 0$. As a result the inverse problem is formulated in terms of the transformed potential $V(\lambda, z)$ and the reflection function $p(z)$. This transformation allows to prove the Fréchet differentiability of the transformed cost functional $J(p)$ with respect to the reflection function $p(z)$, and to obtain an explicit formula for the Fréchet gradient. Then Lipschitz continuity of this gradient is proved in class of reflection functions $p(z)$ with Hölder class of derivative $p'(z)$.

Let $\sigma(z) \in \mathcal{S}$ be a given coefficient. Denote by $u = u(r, z; \sigma)$ the unique solution of the direct problem (1)-(2), corresponding to the coefficient $\sigma(z)$. Further, we introduce the trace operator

$$\Lambda[\sigma] := \frac{\partial u(r, z; \sigma)}{\partial r}|_{z=0}. \tag{5}$$

Then the above inverse problem can be formulated in the following operator form [13, 14]:

$$(\Lambda\sigma)(r) = \psi(r), \quad r \in [0, \infty). \tag{6}$$

This operator form of the inverse problem (1)-(3) clearly shows that the problem of interpretation of surface or near-surface measured data consists of recovering the conductivity $\sigma(z)$ of the layer $0 < z < H$ from the knowledge of the measured output data $\psi(r)$.

On the other hand, solution of the inverse problem (1)-(3) means inverting of the input-output map $\Lambda : \mathcal{S} \mapsto \Psi$, where Ψ is the set of measured data. To analyze this input-output map we introduce now the function

$$p(z) = (\ln(\sigma(z)))', \tag{7}$$

which represents an analogue of the reflection coefficient in the medium with continuously changing conductivity $\sigma(z)$. Denote by Λ_2 the right hand side operator in (7): $\Lambda_2\sigma(z) = p(z)$, and define the operator Λ_1 as follows

$$(\Lambda_1 p)(r) = \psi(r), \quad r \in [0, \infty). \tag{8}$$

Then the input-output map Λ can be represented as follows: $\Lambda = \Lambda_1\Lambda_2$. Evidently, the operator Λ_2 is invertible, and

$$\sigma(z) = \Lambda_2^{-1} p = \sigma(0) \exp\left(\int_0^z p(z)dz\right). \tag{9}$$

Therefore inverting of the input-output map $\Lambda : \mathcal{S} \mapsto \Psi$ can be reduced to inverting of the operator Λ_1, or solving the operator equation (8). Due to measurement errors this problem may not have a solution in any suitable class of admissible coefficients. For this reason we introduce the following auxiliary (cost) functional [17]

$$J(p) = \int_0^\infty (\Lambda_1(p) - \psi(r))^2 r dr, \quad p \in P, \tag{10}$$

and consider the minimization problem

$$J(p^*) = \inf_{p \in P} J(p), \quad p \in P, \tag{11}$$

in the class of admissible reflection coefficients

$$P = \{p(z) \in C^1[0, H],\, p(0) = 0,\, \Lambda_2^{-1} \in S\}.$$

The function $\sigma^*(z) = \Lambda_2^{-1} p^*$ will be defined to be a quasisolution of the inverse problem (1)-(3).

Note that existing in literature numerical methods are usually based on to direct reconstruction of the conductivity coefficient $\sigma(z)$ ([1], [13]). Computational realization of these methods have well-known difficulties related to ill-conditionedness of the considered inverse problem. In particular, the numerical algorithm proposed in [1] requires a regularization at each step of iterations. In the second part of this study we will show that the constructed here numerical algorithm, based on the factorization $\Lambda = \Lambda_1 \Lambda_2$ of the input-output map Λ, does not require any regularization.

The paper is organized as follows. Some preliminary results and estimations related to the transformed by the Bessel-Fourier transformation potential $V(\lambda, z)$ are given in Section 2. Fréchet differentiability of the cost functional (10) is derived in Section 3. In the final Section 4 the Lipschitz continuity of gradient of the cost functional (10) is proved in class of reflection functions $p(z)$, when the derivative $p'(z)$ is of Hölder class.

2. Preliminary estimations

Let us use the following representtion of the direct problem solution $u(r, z)$:

$$u(r, z) = -\frac{1}{\sigma(0)\sqrt{r^2 + z^2}} + \overline{u}(r, z), \quad \overline{u}(r, z) = O(r^{-1}), \quad r \to \infty, \tag{12}$$

where $\overline{u}(r, z)$ is a bounded and regular function. Based on this representation and condition (3), we define the set of measured output data Ψ as follows:

$$\Psi = \{\psi(r) | \psi(r) = \frac{1}{\sigma(0) r^2} + \overline{\psi}(r), \quad \int_0^\infty |\overline{\psi}(r)| \sqrt{r} dr < \infty, \;\; 0 < r < \infty\}. \tag{13}$$

Denote by $V(\lambda, z)$ the Bessel-Fourier transformation of the function $u(r, z)$ with respect to the variable $r \geq 0$:

$$V(\lambda, z) = \int_0^\infty u(r, z) J_0(\lambda r) r dr = -\frac{e^{-\lambda z}}{\sigma(0)\lambda} + \int_0^\infty \overline{u}(r, z) J_0(\lambda r) r dr.$$

Then we have

$$V(\lambda, z) = -\frac{e^{-\lambda z}}{\sigma(0)\lambda} + \overline{V}(\lambda, z). \tag{14}$$

It was shown in [16], the above right hand side integral exists, converges uniformly, and the function $V(\lambda, z)$ satisfies the following parameter-dependent two-point boundary value problem

$$\begin{cases} \frac{d}{dz}\left[\sigma(z)\frac{dV}{dz}\right] - \lambda^2\sigma(z)V = 0, \\ \sigma(0)\frac{dV}{dz}\Big|_{z=0} = 1, \quad V|_{z=H} = 0 \end{cases} \tag{15}$$

for ordinary differential equation. Now we define the Bessel-Fourier transformation of the measured output data $\psi(r)$,

$$\varphi(\lambda) = \int_0^\infty \psi(r)J_1(\lambda r)r dr, \tag{16}$$

by taking into account (13). Then the transformed the measured output data $\varphi(r)$ has the form:

$$\varphi(\lambda) = -\frac{1}{\sigma(0)} + \int_0^\infty \overline{\psi}(r)J_1(\lambda r)r dr. \tag{17}$$

This, with the unitarity of the Bessel-Fourier transformation, imply that in view of the function $V(\lambda, z)$ the cost functional has the following form:

$$J(\sigma(p(z))) := \int_0^\infty [\psi(r) - \frac{\partial u}{\partial r}(r,0)]^2 r dr = \int_0^\infty [\varphi(\lambda) - \lambda V(\lambda, 0)]^2 \lambda d\lambda. \tag{18}$$

Due to representation (12) and definition (13) the function $\psi(r)$ has the singularity at $r = 0$. However the difference of any two functions from the of measured output data Ψ is square integrable, which means that the right hand side integral in (18) exists.

To analyze the transformed functional (18) with help of problem (15) let us use the following transformation:

$$x(z) = \int_0^z \frac{d\xi}{\sigma(\xi)}, \quad x \in [0, H_1], \quad H_1 = \int_0^H \frac{dz}{\sigma(z)}. \tag{19}$$

Then equation (15) becomes

$$\frac{d^2y}{dx^2} - \lambda^2 s^2(x)y(x) = 0, \quad x \in [0, H_1]. \tag{20}$$

Here $s(x) = \sigma(z(x))$, $0 \le \sigma_1 < s(x) \le \sigma_2$, and $\sigma_1 = \min_{[0,H]}\sigma(z)$, $\sigma_2 = \max_{[0,H]}\sigma(z)$.

It is well-known that ([4],[19]) the transformed equation (20) has two fundamental solutions with the following asymptotic representations

$$\begin{cases} y_{1,2}(x,\lambda) = s^{-1/2}\exp(\pm\lambda z(x))(1 + \varepsilon_{1,2}(x,\lambda)/\lambda), \\ y'_{1,2}(x,\lambda) = \pm\lambda s^{1/2}\exp(\pm\lambda z(x))(1 + \varepsilon_{3,4}(x,\lambda)/\lambda), \end{cases} \tag{21}$$

when $\lambda \to \infty$ uniformly with respect to $x \in [0, H_1]$. Here

$$|\varepsilon_j(x,\lambda)| \le C_\varepsilon, \quad x \in [0, H_1], \quad \lambda \ge \lambda_0 > 0, \quad j = \overline{1,4}.$$

For convenience, in subsequent we will use another equivalent system of fundamental solutions, satisfying the following boundary conditions:

$$y_1(0,\lambda) = 1, \quad {y_1}'(0,\lambda) = 0; \quad y_2(0,\lambda) = 0, \quad {y_2}'(0,\lambda) = 1, \tag{22}$$

with the following asymptotics:

$$
\begin{cases}
y_1(x,\lambda) = {s_0}^{1/2} s^{-1/2} \cosh(\lambda z(x))(1+\varepsilon_1(x,\lambda)/\lambda), \\
y_1'(x,\lambda) = \lambda^{-1} {s_0}^{-1/2} s^{1/2} \sinh(\lambda z(x))(1+\varepsilon_2(x,\lambda)/\lambda), \\
\\
y_2(x,\lambda) = {s_0}^{1/2} s^{-1/2} \lambda^{-1} \sinh(\lambda z(x))(1+\varepsilon_3(x,\lambda)/\lambda), \\
y_2'(x,\lambda) = {s_0}^{-1/2} s^{1/2} \cosh(\lambda z(x))(1+\varepsilon_4(x,\lambda)/\lambda),
\end{cases} \tag{23}
$$

where $|\varepsilon_j(x,\lambda)| \le C_\varepsilon$, $x \in [0,H_1]$, $\lambda \ge \lambda_0 > 0$, $j = \overline{1,4}$. The constant C_ε here depends on the norm $\|\sigma\|_{C^2}$ or, equivalently, on the norm $\|p\|_{C^1}$ (see, [4],[19]). Hence for all functions $p \in P_D$ from the class $P_D = \{p \ : \ \|p\|_{C^1} \le D\}$ the constant C_ε as well as the parameter λ_0 can be assumed to be the same.

Let us define now the conductivity ratio $\kappa := \sigma_2/\sigma_1$, which characterizes the contrastness of the medium.

Applying the comparison theorem to the pairs of functions $\langle y_1(x,\lambda), y_0(x) \equiv 1\rangle$, $\langle y_1(x,\lambda), u(x,\lambda) = \cosh(\lambda\sigma_2 x)\rangle$ we conclude that

$$
1 \le y_1(x,\lambda) \le \cosh(\lambda\sigma_2 x) \le \cosh(\lambda\kappa z), \quad x \in [0,H_1]. \tag{24}
$$

Further, it easily follows from the equation and boundary conditions that the function $y_1(x,\lambda)$ is a monotone increasing one.

To establish the necessary properties of the cost functional we need the following estimates.

LEMMA 2.1 *Let $p(z), \delta p(z) \subset P_D$, and the functions $V(\lambda,z)$ and $V(\lambda,z)+\delta V(\lambda,z)$ be the corresponding solutions of the boundary value problem (15). Then for $z \in [0,H]$ ($x \in [0,H_1]$), $\lambda \ge 0$ the following estimates, and for $\lambda_0 > 0$ the asymptotic estimates hold:*

e1) $V(\lambda,z) \le 0, \quad V'(\lambda,z) \ge 0;$

e2) $|V(\lambda,z)| \le \frac{(H-z)}{\sigma_1};$

$\forall\delta \in (0,1), \quad \exists\lambda_0 > 0, \quad |V(\lambda,z)| \le \frac{1+\delta}{\sigma_1\lambda}\exp(-\lambda z), \quad \forall\lambda \ge \lambda_0;$

e3) $|\sigma(z)V'(\lambda,z)| \le 1, |V'(\lambda,z)| \le \frac{1}{\sigma_1},$

$|\frac{dV(\lambda,z)}{dz}| \le \frac{1+\delta}{\sigma_1}\exp(-\lambda z), \quad \forall\lambda \ge \lambda_0;$

e4) $|\delta V(\lambda,z)| \le \frac{C_1(\lambda_0,\kappa,H)}{\sigma_1}\|\delta p(z)\|_C, \quad 0 \le \lambda \le \lambda_0,$

$|\delta V(\lambda,z)| \le \frac{C_2\kappa^3}{\sigma_1\lambda^3}\|\delta p(z)\|_{C_1}, \quad \forall\lambda \ge \lambda_0;$

e5) $|\delta V(\lambda,z)| \le \frac{C_2\kappa^3\Gamma(\alpha)}{\sigma_1\lambda^{3+\alpha}}\|\delta p(z)\|_{C^{1+\alpha}}, \quad \forall\delta p \in C^{1+\alpha}[0,H], \quad \forall\lambda \ge \lambda_0;$

e6) $|\sigma(z)\delta V'(\lambda,z)| \le \kappa C_3\|\delta p(z)\|_{C^1}, \quad \forall\lambda \ge 0,$

$|\sigma(z)\delta V'(\lambda,z)| \le \frac{C_4\kappa^4}{\lambda}\|\delta p(z)\|_{C^1}, \quad \forall\lambda \ge \lambda_0,$

where constants C_1, C_2, C_3, C_4 depend only on H, C_ε, λ_0.

Proof e1). Multiplying the equation (15) by the function $V(\lambda,z)$, integrating on $[z,H]$ and using the boundary conditions we obtain the following energy identity:

$$
\int_z^H \sigma(z)({V'}^2(\lambda,z) + \lambda^2 V^2(\lambda,z))dz = -\sigma(z)V'(\lambda,z)V(\lambda,z). \tag{25}
$$

Substituting in (25) $z = 0$ and taking into account the boundary condition we conclude $V(\lambda, 0) < 0$. To prove $V(\lambda, 0) \le 0$, $\forall z \in [0, H]$,we assume that $\exists z_0 \in (0, H)$ such that $V(\lambda, z_0) > 0$. Due continuity, $\exists z_1 \in (0, z_0)$, $V(\lambda, z_1) = 0$. By the boundary condition $V(\lambda, H) = 0$ and Rolle's theorem the function $V(\lambda, z)$ has a positive maximum at $z_2 \in (z_1, H)$, and $V(\lambda, z_2) > V(\lambda, z_0) > 0$. Then by the conditions $V'(\lambda, z_2) = 0$, $V''(\lambda, z_2) \le 0$ equation (25) at $z = z_2$ yields

$$\lambda^2 V(\lambda, z_2) = \sigma(z_2)V''(\lambda, z_2) + \sigma'(z_2)V'(\lambda, z_2) = \sigma(z_2)V''(\lambda, z_2) \le 0.$$

This contradiction implies that $V(\lambda, 0) \le 0$, $\forall z \in [0, H]$.

The second assertion of *e1)* follows from the positivity of the right hand side of (25).

e2). Let us apply transformation (19) to problem (15). Then the function $y(x, \lambda) = V(\lambda, z(x))$ will be the solution of equation (20) and satisfies the boundary conditions

$$y'(0, \lambda) = 1, \quad y(H_1, \lambda) = 0. \tag{26}$$

Consider the difference $\Delta y(x, \lambda) = y(x, \lambda) - (x - H_1)$, $x \in [0, H_1]$. Then equation (20) and the estimate $V(\lambda, 0) \le 0$ imply

$$\frac{d^2 \Delta y}{dx^2} = \lambda^2 s^2 y \le 0.$$

Hence

$$\frac{d\Delta y}{dx}(x, \lambda) \le \frac{d\Delta y}{dx}(0, \lambda) = 0, \quad \forall x \in [0, H_1]. \tag{27}$$

This, with the boundary condition, $y(H_1, \lambda) = 0$ implies

$$\Delta y = -\int_x^{H_1} \frac{d\Delta y}{dx} dx \ge 0, \quad \forall x \in [0, H_1],$$

which is equivalent to the condition: $V(\lambda, z) \ge x(z) - H_1$. Taking into account the sign of the function $V(\lambda, z)$ we obtain the first part of the assertion *e2)*:

$$|V(\lambda, z)| \le H_1 - x(z) = \int_z^H \frac{dz}{\sigma(z)} \le \frac{H - z}{\sigma_1}.$$

To prove the second part of the assertion *e2)* we introduce the function

$$y(x(z), \lambda) = y_2(x(z), \lambda) - \frac{y_2(H_1, \lambda)}{y_1(H_1, \lambda)} y_1(x(z), \lambda),$$

i.e. the linear combination of the fundamental solutions. Evidently this function satisfies the boundary condition (26). Then the function

$$V(\lambda, z) = y_2(x(z), \lambda) - \frac{y_2(H_1, \lambda)}{y_1(H_1, \lambda)} y_1(x(z), \lambda), \tag{28}$$

obtained from the function $y(x(z), \lambda)$ by transformation (19) will be the solution of problem (15): $V(\lambda, z) = y(x(z), \lambda)$. Hence we may use the asymptotic formula (23) for the function $V(\lambda, z)$, $\forall \lambda > \lambda_0$, $\lambda_0 > 0$:

$$V(\lambda, z) = y(x(z), \lambda) = \frac{1}{\lambda\sqrt{\sigma(0)\sigma(x)}} \frac{\sinh \lambda(z - H)}{\cosh \lambda H} \left[1 + \frac{\varepsilon(x, \lambda)}{\lambda}\right], \quad |\varepsilon(x, \lambda)| \le C_\varepsilon.$$

By the asymptotic behaviour of the hyperbolic functions we obtain the required second estimate of *e2)*:

$$|V(\lambda, z)| \leq \frac{1+\beta}{\lambda\sqrt{\sigma(0)\sigma(x)}} \exp(-\lambda z) \leq \frac{1+\beta}{\lambda\sigma_1} \exp(-\lambda z), \quad \forall\beta \in (0,1), \quad \forall\lambda > \lambda_0.$$

e3). From inequality (27) we may conclude $(\Delta y(x,\lambda))' = y'(x,\lambda) - 1 \leq 0$ which implies

$$y'(x,\lambda) = \sigma(z)V'(\lambda, z) = |\sigma(z)V'(\lambda, z)| \leq 1.$$

In particular, $|V'(\lambda, z)| \leq 1/\sigma_1$, $\sigma_1 > 0$.

To prove the third assertion of *e3)* let us differentiate (28) and use the asymptotic representation (23). Then we have:

$$y'(x(z),\lambda) = \sigma(z)\frac{dV(\lambda,z)}{dz} = \frac{\sqrt{\sigma(z)}}{\sqrt{\sigma(0)}}\frac{\cosh\lambda(H-z)}{\cosh\lambda H}[1+\frac{\varepsilon(z,\lambda)}{\lambda}], \quad |\varepsilon(z,\lambda)| \leq C_\varepsilon \tag{29}$$

Due to the asymptotic of hyperbolic cosine we conclude that $\exists\lambda_0 > 0$ such that

$$\left|\frac{dV(\lambda,z)}{dz}\right| \leq \frac{1+\beta}{\sqrt{\sigma(z)}\sqrt{\sigma_0}}(\exp(-\lambda z) \leq \frac{1+\beta}{\sigma_1}\exp(-\lambda z), \quad \forall\lambda \geq \lambda_0, \quad \forall\beta \in (0,1). \tag{30}$$

Evidently choice of the parameter $\lambda_0 > 0$ depends on the constant $C_\varepsilon > 0$, since $\lambda_0 > C_\varepsilon/\beta$. For this reason in subsequence the dependency on the parameter $\lambda_0 > 0$ will be replaced by the dependency on the constant $C_\varepsilon > 0$.

e4). Let us rewrite equation (15) taking into account transformation (7):

$$\frac{d^2V}{dz^2} + p(z)\frac{dV}{dz} - \lambda^2 V = 0.$$

Evidently the function $\delta V(\lambda, z)$ satisfies the nonhomogeneous equation

$$\frac{d^2\delta V}{dz^2} + p(z)\frac{d\delta V}{dz} - \lambda^2\delta V = -\frac{d(V+\delta V)}{dz}\delta p(z). \tag{31}$$

Using here transformation (19) we conclude that the function $g(x,\lambda) := \delta V(\lambda, z(x))$ satisfies the following second order nonhomegeneous equation and the boundary conditions:

$$\begin{cases} \frac{d^2g}{dx^2} - \lambda^2 s^2(x)g(x,\lambda) = f(x,\lambda), \quad x \in [0, H_1], \\ g'(0,\lambda) = 0, \quad g(H_1,\lambda) = 0, \end{cases} \tag{32}$$

where the source term is defined as follows:

$$f(x,\lambda) = -s^2(x)z(x)\delta p(z(x))\frac{d(V+\delta V)}{dz}. \tag{33}$$

The solution

$$\delta V(\lambda, z(x)) := g(x,\lambda) = -\frac{y_2(H_1,\lambda)}{y_1(H_1,\lambda)}y_1(x,\lambda)\int_0^{H_1} y_1(t,\lambda)f(t,\lambda)dt +$$

$$y_1(x,\lambda)\int_x^{H_1} y_2(t,\lambda)f(t,\lambda)dt + y_2(x,\lambda)\int_0^x y_1(t,\lambda)f(t,\lambda)dt$$

of the two point problem (32) is obtained by the method of variation constans, which can be easily verified. Using here formula (28) we get

$$\delta V(\lambda, z(x)) = y(x,\lambda)\int\limits_0^x f(t,\lambda)y_1(t,\lambda)dt + y_1(x,\lambda)\int\limits_x^{H_1} f(t,\lambda)y(t,\lambda)dt. \tag{34}$$

Substituting in the right hand side formula (34) for the source term $f(x,\lambda)$ we obtain the following representation for the function $\delta V(\lambda, z(x))$:

$$\delta V(\lambda, z) = -V(\lambda, z)\int\limits_0^z \sigma(V' + \delta V')\delta p(z)y_1(x(z))dz - y_1(x(z))\int\limits_z^H \sigma(V' + \delta V')V(\lambda, z)\delta p(z)dz. \tag{35}$$

By the estimates obtained in *e1)* we get

$$|\delta V(\lambda, z)| \le |V(\lambda, z)|\int\limits_0^z \sigma(V' + \delta V')|\delta p(z)|y_1(x(z),\lambda)dz + y_1(x(z),\lambda)\int\limits_z^H \sigma(V' + \delta V')|V(\lambda, z)||\delta p(z)|dz. \tag{36}$$

Assume that $0 < \lambda \le \lambda_0$ for some $\lambda_0 > 0$. Then using estimates *e1)*, *e2)* and (24) we get:

$$|\delta V(\lambda, z(x))| \le \frac{\sigma_2 H}{\sigma_1^2}\int\limits_0^x \cosh(\lambda\sigma_2 t)|\delta p|dt + \frac{\sigma_2 H}{\sigma_1^2}\cosh(\lambda\sigma_2 x)\int\limits_x^H |\delta p|dt \le \frac{\kappa H}{\sigma_1}\cosh(\lambda\sigma_2 x)\int\limits_0^H |\delta p|dt,$$

which implies the first estimate of*e4)*:

$$|\delta V(\lambda, z)| \le \frac{\kappa H^2}{\sigma_1}\cosh(\lambda_0\kappa H)\|\delta p\|_C \le \frac{\kappa C_1(C_\varepsilon, \kappa, H)}{\sigma_1}\|\delta p\|_C, \quad \forall\lambda \in (0, \lambda_0].$$

To obtain the second estimate of *e4)* we note that the function $\delta p(z)$ satisfies the condition $\delta p(0) = 0$ in the class of functions P_D. Hence

$$|\delta p(z)| \le z\|\delta p\|_{C^1}, \quad \forall z \in [0, H]. \tag{37}$$

This estimate, with the monotonicity and the sign of the function $V(\lambda, z(x))$, established in *e1)*, as well as the monotonicity of the fundamental solution $y_1(x,\lambda)$, permits one to improve the above obtained estimate (36):

$$\begin{aligned} |\delta V(\lambda, z)| &\le |V(\lambda, z)|y_1(x(z))\int\limits_0^z \sigma(\zeta)(V' + \delta V')|\delta p|d\zeta \\ &+|V(\lambda, z)|y_1(x(z))\int\limits_z^H \sigma(\zeta)(V' + \delta V')|\delta p|d\zeta \\ &\le |V(\lambda, z)|y_1(x(z))\sigma_2\|\delta p\|_{C^1}\int\limits_0^H \zeta(V' + \delta V')d\zeta. \end{aligned} \tag{38}$$

On the other hand, it follows from asymptotics (23) that $\forall\beta \in (0, 1)$, there exists $\lambda_0 > 0$ such that $\forall\lambda \ge \lambda_0$

$$y_1(x) \le \frac{\sqrt{s_0}}{\sqrt{s(x)}}\exp(\lambda z)(1 + \beta) \le \sqrt{\kappa}\,(1 + \beta)\exp(\lambda z).$$

We use this estimate with the second estimate of *e1)*, and estimate (30), applied to the function $V' + \delta V'$ on the rignt hand side of (38). Then we arrive to the second estimate of *e4)*:

$$|\delta V| \le \frac{(1+\beta)^3}{\lambda\sigma_1}\exp(-\lambda z)\sqrt{\kappa}\exp(\lambda z)\|\delta p(z)\|_{C_1}\frac{\sigma_2}{\sigma_1}\int_0^H z\exp(-\lambda z)dz \le \frac{C_2\kappa^{3/2}}{\lambda^3\sigma_1}\|\delta p(z)\|_{C^1}, \quad \forall\lambda \ge \lambda_0.$$

e5). Let now $\delta p \in C^{1+\alpha}[0,H]$. Then as in (37),

$$|\delta p(z)| \le z^{1+\alpha}\|\delta p\|_{C^{1+\alpha}}.$$

Substitute this in (36), use the second estimates of *e2)* and *e3)* for the functions $V(\lambda,z)$, $V'(\lambda,z) + (\delta V(\lambda,z))'$, correspondingly, and use also the above estimate for the fundamental solution $y_1(x,\lambda)$. Then we obtain the required assertion of *e5)*:

$$|\delta V(\lambda,z)| \le \frac{(1+\beta)^3}{\lambda\sigma_1}\exp(-\lambda z)\sqrt{\kappa}\|\delta p(z)\|_{C_1}\frac{\sigma_2}{\sigma_1}\int_0^H z^{1+\alpha}\exp(-\lambda z)dz \le \frac{C_2\kappa^{3/2}\Gamma(\alpha)}{\lambda^{3+\alpha}\sigma_1}\|\delta p(z)\|_{C^{1+\alpha}}.$$

e6). We rewrite the nonhomogeneous equation (31) for the function $\delta V(\lambda,z)$:

$$(\sigma\delta V')' - \lambda^2\sigma\delta V = -\sigma\delta p(V+\delta V)'.$$

Integrate this equation on $[0,H]$ and use the boundary conditions (32):

$$\sigma(z)\delta V'(z) = \int_0^z \lambda^2\sigma(\zeta)\delta V(\lambda,\zeta)d\zeta - \int_0^z \sigma(\zeta)\delta p(\zeta)(V'(\lambda,\zeta)+\delta V'(\lambda,\zeta))d\zeta. \tag{39}$$

Now we estimate the right hand side of (39) separately, for the cases $\lambda \le \lambda_0$ and $\lambda > \lambda_0$, using estimate (37). Taking into account estimates of *e2),e3)* and *e5)* we get:

$$|\sigma(z)\delta V'(z)| \le \frac{\sigma_2 C_1}{\sigma_1}\|\delta p\|_C H\lambda_0^2 + \sigma_2\|\delta p\|_C\frac{H}{\sigma_1} \le \kappa C_3(C_\varepsilon,\kappa,H)\|\delta p\|_{C^1}, \quad \lambda \le \lambda_0;$$

$$|\sigma(z)\delta V'(z)| \le \frac{\sigma_2 C_2\kappa^3}{\sigma_1\lambda}\|\delta p\|_{C^1} + \sigma_2\frac{(1+\delta)}{\sigma_1}\|\delta p\|_{C^1}\int_0^z \zeta\exp(-\lambda\zeta)d\zeta \le$$

$$\frac{C_2\kappa^4}{\lambda}\|\delta p\|_{C^1} + 2\kappa\|\delta p\|_{C^1}\frac{1}{\lambda^2} \le \frac{C_4\kappa^4}{\lambda}\|\delta p\|_{C^1}, \quad \lambda > \lambda_0,$$

which are the assertions of *e6).*

The lemma is proved. □

The right hand side of the functional (18) contains the difference $\lambda V(\lambda,0) - \varphi(\lambda)$. To estimate this difference we need the following

LEMMA 2.2 *Let $\sigma_0(z) \subset S$, $z \in [0,H]$, be the conductivity coefficient corresponding to the measured data $\varphi(\lambda)$, and $p_0(z) = \ln(\sigma_0'(z)) \subset P$. Assume that $\sigma(z) \in S$ is an arbitrary coefficient from the set of admissible coefficients S, $p(z) = \ln(\sigma'(z)) \in P$ and $V(\lambda,z)$ is the corresponding solutions of problem (15). Then the following estimates hold:*

$$\begin{cases} |\lambda V(\lambda,0) - \varphi(\lambda)| \le \frac{C_5}{\sigma_1}, & \forall\lambda \in [0,\lambda_0]; \\ |\lambda V(\lambda,0) - \varphi(\lambda)| \le \frac{C_6}{\sigma_1\lambda^2}, & \forall\lambda \in (\lambda_0,\infty), \end{cases} \tag{40}$$

where the constants C_5 and C_6 depend only on the difference $\|p - p_0\|_{C^1}$, and on the positive constants C_ε, H, κ.

Proof Consider the difference $\Delta V(\lambda, z) = V_0(\lambda, z) - V(\lambda, z)$ satisfying the following equation

$$\frac{d^2 \Delta V}{dz^2} + p_0(z)\frac{d\Delta V}{dz} - \lambda^2 \Delta V = -\frac{dV}{dz}(p(z) - p_0(z)), \tag{41}$$

similar to equation (31). Applied the transformation (19) to the function $g(x, \lambda) = V_0(\lambda, z) - V(\lambda, z)$ we conclude that this function satisfies the boundary value problem (32) with the source function

$$f(x, \lambda) = -s_0^2(x)[p(z(x) - p_0(z(x))]\frac{dV}{dz}. \tag{42}$$

The solution of the bounfary value problem (32) and (42) can also be represented in the form of (34), assuming here the source function (42). For $z = 0$ this formula implies

$$\Delta V(\lambda, 0) = -\int_0^H \sigma_0(z)V(\lambda, z){V_0}'(\lambda, z)(p(z) - p_0(z))dz. \tag{43}$$

Then from the estimate *e4)* for $0 \le \lambda \le \lambda_0$ we get

$$|\lambda \Delta V(\lambda, 0)| \le \frac{C_1}{\sigma_1}\|p(z) - p_0(z)\|_C \equiv \frac{C_5}{\sigma_1},$$

which is the first assertion of the lemma. To derive the second assertion of the lemma for $\lambda > \lambda_0$, we use the second assertions of *e2)* and *e3)* for the functions $V_0(\lambda, z)$ and $V(\lambda, z)$. Then $\forall \beta \in (0, 1)$, $\exists \lambda_0 > 0$ such that $\forall \lambda \ge \lambda_0$

$$|V_0(\lambda, z)| \le \frac{(1+\beta)}{\lambda \sigma_1}\exp(-\lambda z), \tag{44}$$

$$\sigma_0(z)\frac{dV}{dz} \le \frac{(1+\beta)\sigma_0(z)}{\sqrt{\sigma(z)\sigma(0)}}\exp(-\lambda z) \le \kappa(1+\beta)\exp(-\lambda z). \tag{45}$$

Now we use estimate (37) for the function $p - p_0$. Since $p(0) = p_0(0) = 0$ from the definition of the set P we get:

$$|p - p_0| \le z\|p - p_0\|_{C^1}.$$

Applying estimates (44) and (45) in (43) we obtain:

$$|\lambda \Delta V(\lambda, 0)| \le \frac{(1+\beta)^2 \kappa}{\sigma_1}\|p - p_0\|_{C^1}\int_0^H \exp(-2\lambda z)z dz$$
$$\le \frac{(1+\beta)^2\kappa}{4\lambda^2\sigma_1}\|p - p_0\|_{C^1} \equiv \frac{C_6(\kappa, C_\varepsilon, \|p - p_0\|_{C^1})}{\sigma_1 \lambda^2}.$$

Since $\lambda \Delta V(\lambda, 0) = \lambda V(\lambda, 0) - \varphi(\lambda)$,the last estimate with (44) implies the second assertion of the lemma. □

3. The Fréchet differentiability of the cost functional

The following result shows that the transformed cost functional $J(p)$ is of Fréchet differentiable. Moreover, it permits to derive the Fréchet derivative explicitly, in the integral form.

THEOREM 3.1 *Let $\mathcal{S}$ be the set of admissible coefficients defined by (4), and $P = \{p(z) \in C^1[0,H],\ p(0) = 0,\ \Lambda_2^{-1} \in S\}$ be the set of reflection functions. Then the cost functional $J(p)$, $p(z) \in P$ is a Fréchet differentiable one, and its Fréchet derivative is:*

$$\nabla J(p) = -2\sigma_0 \exp\left(\int_0^z p(\zeta)d\zeta\right)\int_0^\infty (\lambda V(\lambda,0) - \varphi(\lambda))V(\lambda,z)V'(\lambda,z)\lambda^2 d\lambda. \tag{46}$$

Proof Let $\delta p(z) \in P$ be an admissible increment. We need to show that the increment of the cost functional $J(p)$, given by (18), has the form

$$\Delta J(p) := J(p+\delta p) - J(p) =< \nabla J, \delta p > +o(\|\delta p\|) = \int_0^H q(z)\delta p(z)dz + o(\|\delta p\|_{C([0,H])}), \tag{47}$$

for some function $q(z) \in L_2[0,H]$. Let us first rewrite the increment of the cost functional $J(p)$ in the following form:

$$\Delta J(p) = J(p+\delta p) - J(p) = 2\int_0^\infty [\lambda V(\lambda,0) - \varphi(\lambda)]\delta V(\lambda,0)\lambda^2 d\lambda + \int_0^\infty \lambda^2 \cdot \delta V^2(\lambda,0)\lambda d\lambda. \tag{48}$$

We derive now the increment $\delta V(\lambda,0)$ via the increment $\delta p(z)$. For this aim we use formula (35) at $z=0$ by taking into account the boundary condition (15)

$$\delta V(\lambda,0) = -\int_0^H \sigma(z)V(\lambda,z)(V' + \delta V')\delta p(z)dz \equiv -\int_0^H \sigma(z)V(\lambda,z)V'(\lambda,z)\delta p(z)dz + r_3(\lambda), \tag{49}$$

where the residual term $r_3(\lambda)$ is

$$r_3(\lambda) = -\int_0^H \sigma(z)V(\lambda,z)\delta V'(\lambda,z)\delta p(z).$$

Formulas (48) and (49) imply that the increment of the cost functional $J[p]$ has the following form:

$$\Delta J = -2\int_0^\infty (\lambda V(\lambda,0) - \varphi(\lambda))\lambda^2 \int_0^H \sigma(z)V(\lambda,z)V'(\lambda,z)\delta p(z)dzd\lambda + r_0(\lambda), \tag{50}$$

with the residual function

$$r_0(\lambda) = 2\int_0^\infty (\lambda V(\lambda,0) - \varphi(\lambda))\lambda^2 r_3(\lambda)d\lambda + \int_0^\infty \beta V^2(\lambda,0)\lambda^3 d\lambda. \tag{51}$$

According to estimates *e2)*, *e3)* and (37), for enough large $\lambda > \lambda_0$ the interior integral in (50) can be estimated as follows:

$$\left|\int_0^H \sigma(z)V(\lambda,z)V'(\lambda,z)\beta p(z)dz\right| \le \frac{\kappa(1+\beta)^2}{\lambda\sigma_1}\|\delta p\|_{C^1}\int_0^H z\exp(-2\lambda z)dz = O(\lambda^{-3})\|\delta p\|_{C^1}.$$

Hence the improper integral in (50) converges, due to the first estimate (40). Let us change the order of integration in (50). By estimates *e2)*, *e3)*, (37) and (40), this integral, depending on the variable z, uniformly converges when $z \in [0, H]$, since for enough large $\lambda > \lambda_0$, the majorant integral converges and does not depend on $z \in [0, H]$:

$$\left| \int_{\lambda_0}^{H} \sigma(z) V(\lambda, z) V'(\lambda, z) \delta p(z) dz (\lambda V(\lambda, 0) - \varphi(\lambda)) \lambda^2 d\lambda \right|$$

$$\le \frac{\kappa(1+\delta)^2 C_6}{\sigma_1^2} \|\delta p\|_{C^1} \int_{\lambda_0}^{\infty} \frac{z \exp(-2\lambda z)}{\lambda} d\lambda \le \frac{4\kappa C_6}{\sigma_1^2} \|\delta p\|_{C^1} \int_{\lambda_0}^{\infty} \frac{\exp(-1) d\lambda}{2\lambda^2}.$$

Therefore we may change the order of integration in (50), and rewrite the linear (with respect to δp) part of the increment $\delta J(p)$ of the cost functional in the following form:

$$\delta J = -2 \int_0^H \sigma(z) \big(\int_0^{\infty} (\lambda V(\lambda, 0) - \varphi(\lambda)) \lambda^2 V(\lambda, z) V'(\lambda, z) d\lambda \big) \delta p(z) dz. \qquad (52)$$

By the defintion of the Fréchet differential we need to prove that the residual function $r_0(\lambda)$ is of order $o(\|\delta p\|_{C^1}^q)$, $q \geq 2$. For this aim first we estimate the residual function $r_3(\lambda)$, using the estimates *e2)*, *e6)* and (37). We have

$$\begin{aligned} |r_3(\lambda)| &\le \left| \int_0^H \sigma(z) V(\lambda, z) \delta V'(\lambda, z) \delta p(z) dz \right| \\ &\le \tfrac{C_4 \kappa^4}{\lambda} \|\delta p\|_{C^1}^2 \tfrac{(1+\beta)}{\lambda \sigma_1} \int_0^H z \exp(-2\lambda z) dz \le O(\lambda^{-4}) \|\delta p\|_{C^1}^2. \end{aligned} \qquad (53)$$

The second term in (51) can be estimated by using estimates *e4)*:

$$|\delta V^2(\lambda, 0) \lambda^3| = O(\lambda^{-3}) \|\delta p\|_{C^1}^2, \quad \forall \lambda \geq 0. \qquad (54)$$

Hence $r_0(\lambda) = o(\|\delta p\|_{C^1}^q)$, and the lemma is proved. □

4. Lipschitz continuity of gradient of the cost functional

To apply any gradient method for the numerical solution of the minimization problem (10)-(11), one needs an estimation of the iteration parameter $\alpha_n > 0$ in the iteration process $p^{(n+1)} = p^{(n)} - \alpha_n \nabla J(p^{(n)})$, $n = 0, 1, 2, ...$, where $p^{(0)}$ is a given initial iteration. In the case of Lipschitz continuity of the gradient $\nabla J(p)$ the parameter α_n can be estimated via the Lipschitz constant, which subsequently improves convergence properties of the iteration process ([18]). The following theorem shows that in the subset $\mathcal{P} = \{p(z) \in P : \ p(z) \in C^{1+\alpha}[0, H], \|p(z)\|_{C^{1+\alpha}[0,H]} \le D, \ \ D < \infty\}$ of reflection functions the gradient $\nabla J(p)$ of the transformed cost functional is a Lipschitz continuous one.

THEOREM 4.1 *Let $\mathcal{P} = \{p(z) \in P : \ p(z) \in C^{1+\alpha}[0, H], \|p(z)\|_{C^{1+\alpha}[0,H]} \le D, \ \ D < \infty\}$ be the subset of reflection functions in the set of admissible reflection coefficients P. Then gradient $\nabla J(p)$ of the transformed cost functional, deefined by (18) is Lipschitz continuous in $\mathcal{P} \subset P$, i.e.*

$$\|\nabla J[p_1] - \nabla J[p]\|_{C[0,H]} \le L \|p_1 - p\|_{C^{1+\alpha}[0,H]}, \quad \forall p, p_1 \in \mathcal{P},$$

where the Lipschitz constant L is increasing function of the conductivity ratio $\kappa = \sigma_2/\sigma_1$, inversely proportional to σ_1^2, and depends only on the positive constants H, C_ε α.

Proof We rewrite the gradient formula (46) in the following convenient form:

$$\nabla J[p] = 2 \int_0^{\infty} (\lambda V(\lambda, 0) - \varphi(\lambda)) I(\lambda, z) \lambda^2 d\lambda,$$

where

$$I(\lambda, z) = -\sigma_0 V(\lambda, z) V'(\lambda, z) \exp\left(\int_0^z p(t)dt\right) = -\sigma(z) V(\lambda, z) V'(\lambda, z),$$

by transformation (9). Assume that the pairs $V(\lambda, z)$, $I(\lambda, z)$ and $V_1(\lambda, z)$, $I_1(\lambda, z)$ correspond to the above given functions $p, p_1 \in \mathcal{P}$. Denote by $\delta V(\lambda, z) = V_1(\lambda, z) - V(\lambda, z)$, $\delta I(\lambda, z) = I_1(\lambda, z) - I(\lambda, z)$. Transforming the difference $\nabla J[p_1] - \nabla J[p]$ as

$$\nabla J[p_1] - \nabla J[p] = 2\int_0^\infty \lambda^3 \delta V(\lambda, 0) I_1(\lambda, z) d\lambda + 2\int_0^\infty (\lambda V(\lambda, 0) - \varphi(\lambda)) \delta I(\lambda, z) \lambda^2 d\lambda, \quad \forall p, p_1 \in \mathcal{P}$$

we can estimate the right hand side as follows:

$$\nabla J[p_1] - \nabla J[p] \le 2\int_0^\infty |\delta V(\lambda, 0) I_1(\lambda, z)| \lambda^3 d\lambda + 2\int_0^\infty |(\lambda V(\lambda, 0) - \varphi(\lambda)) \delta I(\lambda, z)| \lambda^2 d\lambda. \tag{55}$$

To estimate the first right hand side integral we use Lemma 2.1:

$$|I_1(\lambda, z)| \le \frac{H}{\sigma_1}, \quad \lambda \le \lambda_0,$$

$$|I_1(\lambda, z)| \le |(\sigma + \delta\sigma) {V_1}' V_1| \le 1 \cdot \frac{(1+\beta)}{\lambda \sigma_1} \exp(-\lambda z), \quad \lambda \ge \lambda_0.$$

Then we get

$$2\int_0^\infty |\delta V(\lambda, 0) I_1(\lambda, z)| \lambda^3 d\lambda \le 2\int_0^{\lambda_0} |\delta V(\lambda, 0) I_1(\lambda, z)| \lambda^3 d\lambda + 2\int_{\lambda_0}^\infty |\delta V(\lambda, 0) I_1(\lambda, z)| \lambda^3 d\lambda \le$$

$$\frac{2C_1 H}{\sigma_1^2} \|\delta p(z)\|_{C^1} \int_0^{\lambda_0} \lambda^3 d\lambda + \frac{2C_2 \kappa^3 \Gamma(\alpha)}{\sigma_1} \frac{(1+\beta)}{\sigma_1} \|\delta p\|_{C^{1+\alpha}} \int_{\lambda_0}^\infty \frac{\exp(-\lambda z)}{\lambda^{1+\alpha}} d\lambda.$$

Hence

$$2\int_0^\infty |\delta V(\lambda, 0) I_1(\lambda, z)| \lambda^3 d\lambda \le \frac{C_7(C_\varepsilon, \kappa, H, \alpha)\kappa^3}{\sigma_1^2} \|\delta p(z)\|_{C^{1+\alpha}} \tag{56}$$

To estimate the second right hand side integral of (55) we rewrite the increment $\delta I(\lambda, z)$ in the following form:

$$\delta I(\lambda, z) = -\delta\sigma(z) V' V - \delta(V' V)\sigma(z) \equiv \delta A + \delta B, \tag{57}$$

and estimate the terms δA, δB separately. For this aim let first express the increment $\delta\sigma(z)$ via the increment $\delta p(z)$, by using (9). We have:

$$\delta\sigma(z) = \sigma_0 \exp\left(\int_0^z p(t)dt\right)\left[\exp\left(\int_0^z \delta p(t)dt\right) - 1\right] = \sigma(z)\left[\exp\left(\int_0^z \delta p(t)dt\right) - 1\right]. \tag{58}$$

Introducing the auxiliary function

$$f(\theta) = \exp\left(\theta \int_0^z \delta p(t)dt\right), \quad \theta \in R,$$

for a given $z \in [0, H]$, and using the Taylor's formula

$$f(\theta) = 1 + \theta \int_0^z \delta p(t)dt \cdot \exp\left(\xi \int_0^z \delta p(t)dt\right), \quad \xi \in [0, \theta],$$

we obtain that the right hand side of (58) is $\sigma(z)[f(1) - 1]$. Hence:

$$\delta\sigma(z) = \sigma(z) \exp\left(\xi \int_0^z \delta p(t)dt\right) \left(\int_0^z \delta p(t)dt\right), \quad \xi \in [0, \theta].$$

Introducing here the new function

$$\sigma_\xi(z) = \sigma(z) \exp\left(\xi \int_0^z \delta p(t)dt\right), z \in [0, H], \tag{59}$$

we conclude that the increment $\delta\sigma(z)$ has the form:

$$\delta\sigma_\xi(z) = \sigma_\xi(z) \left(\int_0^z \delta p(t)dt\right), \quad \xi \in [0, 1]. \tag{60}$$

Evidently, for $p(z), p(z) + \delta p(z) \in P$, the function $\sigma_\xi(z)$, given by (59), has the same properties, as the function $\sigma(z) \in S$. Specifically, if

$$\int_0^z \delta p(t)dt < 0, \quad \forall z \in [0, H],$$

then formulas (59)-(60) imply that $\sigma_1 \leq \sigma(z) + \delta\sigma(z) \leq \sigma_\xi(z) \leq \sigma(z) \leq \sigma_2$, and hence $\sigma_\xi(z) \in S$. The same conclusion is obtained in the converse case, if the above integral is positive.

Let us estimate now the terms δA, δB in (57). We use estimations *e2)*, *e3)*, and auxiliary formulas (59-(60) to estimate the term δA for the cases $\lambda \leq \lambda_0$ and $\lambda \geq \lambda_0$, separately:

$$|\delta A| \leq |\delta\sigma(z)V'V| \leq \frac{\sigma_2 H}{\sigma_1^2} \int_0^z |\delta p(t)|dt \leq \frac{\kappa H^2}{\sigma_1} \|\delta p\|_C, \quad \lambda \leq \lambda_0; \tag{61}$$

$$\begin{aligned} |\delta A| \leq |\delta\sigma(z)V'V| &\leq \sigma_2 \int_0^z |\delta p(t)|dt \cdot \frac{(1+\beta)^2}{\lambda\sigma_1^2} \exp(-2\lambda z) \\ \leq \frac{\kappa(1+\beta)^2}{\lambda\sigma_1} z \exp(-2\lambda z)\|\delta p\|_C &\leq \frac{\kappa(1+\beta)^2}{2\lambda^2\sigma_1} \|\delta p\|_C, \quad \lambda \geq \lambda_0. \end{aligned} \tag{62}$$

To estimate the term δB we rewrite it in the form $\delta B = -(\delta V'V + V_1'\delta V)\sigma(z)$. Then we have: $\delta B \leq |\sigma(z)\delta V'|\,|V_1| + |\sigma(z)V_1'|\,|\delta V|$. Applying the estimates *e2)*, *e3)*, *e5)*, *e6)* we conclude

$$|\delta B| \leq \kappa C_3 \|\delta p\|_C \frac{H}{\sigma_1} + \frac{\sigma_2}{\sigma_1} \frac{C_1}{\sigma_1} \|\delta p\|_C \leq \frac{\kappa C_8}{\sigma_1} \|\delta p\|_C, \quad \lambda \leq \lambda_0, \tag{63}$$

$$|\delta B| \leq \frac{\kappa^4 C_4}{\lambda} \|\delta p\|_{C^1} \frac{(1+\beta)}{\lambda\sigma_1} + \frac{\sigma_2}{\sigma_1} \frac{C_2\kappa^3}{\sigma_1\lambda^3} \|\delta p\|_{C^1} \leq \frac{\kappa^4 C_9}{\lambda^2\sigma_1} \|\delta p\|_{C^1}, \quad \lambda \geq \lambda_0. \tag{64}$$

Having estimates (61)-(64), and using Lemma 2.2, we may now estimate the second right hand side integral in (55):

$$2\int_0^\infty |\lambda V(\lambda,0)-\varphi(\lambda)||\delta I|\lambda^2 d\lambda \le 2\int_0^{\lambda_0} \frac{C_5}{\sigma_1}\left[\frac{\kappa H^2}{\sigma_1}\|\delta p(t)\|_C + \frac{\kappa C_8}{\sigma_1}\|\delta p\|_C\right]\lambda^2 d\lambda$$

$$+2\int_{\lambda_0}^\infty \frac{C_6}{\sigma_1}\left[\frac{\kappa(1+\beta)^2}{2\sigma_1\lambda^2}\|\delta p(t)\|_C + \frac{C_9\kappa^4}{\sigma_1\lambda^2}\|\delta p\|_{C^1}\right]d\lambda \le \frac{\kappa^4 C_{10}(C_\varepsilon,\kappa,H,\alpha)}{\sigma_1^2}\|\delta p\|_{C^{1+\alpha}}.$$

This result, with (56), allows to estimate the difference $\nabla J[p_1]-\nabla J[p]$, given by (55):

$$\|\nabla J[p_1]-\nabla J[p]\|_{C[0,H]} \le \frac{C_7\kappa^3}{\sigma_1^2}\|\delta p(z)\|_{C^{1+\alpha}} + \frac{C_{10}\kappa^4}{\sigma_1^2}\|\delta p\|_{C^{1+\alpha}},$$

where $C_{10}=C_{10}(C_\varepsilon,\kappa,H,\alpha)>0$, $C_7=C_7(C_\varepsilon,\kappa,H,\alpha)>0$. Here we define the Lipschitz constant to be $L=C_7\kappa^3/\sigma_1^2+C_{10}\kappa^4/\sigma_1^2>0$. Then we have the proof. $\square$

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