Asymptotic stability of ground states in 2D nonlinear Schrödinger equation including subcritical cases

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Abstract

We consider a class of nonlinear Schrödinger equation in two space dimensions with an attractive potential. The nonlinearity is local but rather general encompassing for the first time both subcritical and supercritical (in L^2) nonlinearities. We study the asymptotic stability of the nonlinear bound states, i.e. periodic in time localized in space solutions. Our result shows that all solutions with small initial data, converge to a nonlinear bound state. Therefore, the nonlinear bound states are asymptotically stable. The proof hinges on dispersive estimates that we obtain for the time dependent, Hamiltonian, linearized dynamics around a careful chosen one parameter family of bound states that "shadows" the nonlinear evolution of the system. Due to the generality of the methods we develop we expect them to extend to the case of perturbations of large bound states and to other nonlinear dispersive wave type equations.

1 Introduction

In this paper we study the long time behavior of solutions of the nonlinear Schrödinger equation (NLS) with potential in two space dimensions (2-d):

$$i\partial_t u(t,x) = [-\Delta_x + V(x)]u + g(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2 \tag{1.1}$$

$$u(0,x) = u_0(x) \tag{1.2}$$

where the local nonlinearity is constructed from the real valued, odd, C^2 function $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying

$$g(0) = g'(0) = 0 \quad \text{and} \quad |g''(s)| \le C(|s|^{\alpha_1} + |s|^{\alpha_2}), \quad s \in \mathbb{R}, \ \frac{1}{2} < \alpha_1 \le \alpha_2 < \infty \tag{1.3}$$

which is then extended to a complex function via the gauge symmetry:

$$g(e^{i\theta}s) = e^{i\theta}g(s), \qquad \theta \in \mathbb{R}. \tag{1.4}$$

The equation has important applications in statistical physics, optics and water waves. It describes certain limiting behavior of Bose-Einstein condensates [8, 15] and propagation of time harmonic waves in wave guides [14, 16, 18]. In the latter, t plays the role of the coordinate along the axis of symmetry of the wave guide.

It is well known that this nonlinear equation admits periodic in time, localized in space solutions (bound states or solitary waves). They can be obtained via both variational techniques [1, 28, 22] and bifurcation methods [20, 22, 13]. Moreover the set of periodic solutions can be organized as a C^2 manifold (center manifold), see [11, 12] or next section. Orbital stability of solitary waves, i.e. stability modulo the group of symmetries $u \mapsto e^{-i\theta}u$, was first proved in [22, 30], see also [9, 10, 24].

The main result of this paper is that solutions of (1.1)-(1.2) with small initial data asymptotically converge to a bound state, see Theorem 3.1. While asymptotic stability results for bound states in NLS have first appeared in the work of A. Soffer and M. I. Weinstein [25, 26], and continued in [20, 29, 2, 3, 4, 7, 11], our main contribution is to allow for subcritical and critical (L^2) nonlinearities, $\frac{1}{2} < \alpha_1 \le 1$ in (1.3). To

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accomplish this we carefully project the nonlinear dynamics onto the center manifold of bound states and use linearization around this time changing projection to study the motion in the radiative directions, i.e. directions that are not in the tangent space of the center manifold. Previously, linearization around a fixed bound state has been used, see the papers cited above. By continuously adapting the linear dynamics to the actual nonlinear evolution of the solution we can more precisely capture the effective potential induced by the nonlinearity g into a time dependent linear operator. Once we have a good understanding of this time dependent linear dynamics, i.e. we have good dispersive estimates for its semigroup of operators, see Section 4, we obtain information for the nonlinear dynamics via Duhamel formula and contraction principles for integral equations, see Section 3. Note that we have recently used a similar technique to show that in the critical (cubic) case, (1.1) with $g(s) = s^3,\ s \in \mathbb{R}$, the center manifold of bound states is an attractor for small initial data, see [13]. In this paper the technique is much refined, we use a better projection of the dynamics on the center manifold and sharper estimates for the linear dynamics. The refinements not only allow us to treat a much larger spectrum of nonlinearities including, for the first time, the subcritical ones but also allow us to obtain actual convergence of the solution to a bound state.

However, the main challenge for our approach is to obtain good dispersive estimates for the semigroup of operators generated by the time dependent linearization that we use. This is accomplished in Section 4 via a perturbative method similar to the one we developed in [13]. As described in that section, we could have obtained sharper estimates by using a generalized Fourier multiplier technique to remove the singularity of

$$\|e^{i(\Delta-V)t}\|_{L^1\mapsto L^\infty} \sim |t|^{-1},$$

see [12, Section 4]. We chose not to do it because it requires stronger hypotheses on V without allowing us to enlarge the spectrum of nonlinearities that we can treat.

Finally, we remark that our method is quite general, based solely on linearization around nonlinear bound states and estimates for integral operators with dispersive kernels. Therefore we expect it to generalize to the case of large nonlinear 2D ground states, see for example [7], the presence of multiple families of bound states, see for example [27], or to the case of time dependent nonlinearity, see [6]. In all three cases our method will not only allow to treat the less dispersive environment, 2D compared to 3D, but it should greatly reduce the restrictions on the nonlinearity. The first author and collaborators are currently working on adapting the method to other dimensions and other dispersive wave type equations. The work in 3-D is complete, see [12].

Notations: $H = -\Delta + V$;

$L^p = \{f : \mathbb{R}^2 \mapsto \mathbb{C} \mid f$ measurable and $\int_{\mathbb{R}^2} |f(x)|^p dx < \infty\},\ 1 \le p < \infty$, endowed with the standard norm $\|f\|_{L^p} = \left(\int_{\mathbb{R}^2} |f(x)|^p dx\right)^{1/p}$, while for $p = \infty$, $L^\infty = \{f : \mathbb{R}^2 \mapsto \mathbb{C} \mid f$ measurable and $\text{essup}|f(x)| < \infty\}$, and it is endowed with the norm: $\|f\|_{L^\infty} = \text{essup}|f(x)|$;

$< x >= (1 + |x|^2)^{1/2}$, and for $\sigma \in \mathbb{R},\ 1 \le p < \infty$, L^p_σ denotes the L^p space with weight $< x >^{p\sigma}$, i.e. the space of functions $f(x)$ such that $(< x >^\sigma f(x))^p$ are integrable endowed with the norm $\|f(x)\|_{L^p_\sigma} = \| < x >^\sigma f(x)\|_p$, while for $p = \infty$, L^∞_σ denotes the vector space of measurable functions $f(x)$ such that $\text{essup}| < x >^\sigma f(x)| < \infty$ endowed with the norm $\|f(x)\|_{L^\infty_\sigma} = \| < x >^\sigma f(x)\|_{L^\infty}$;

$\langle f, g\rangle = \int_{\mathbb{R}^2} \overline{f}(x)g(x)dx$ is the scalar product in L^2 where $\overline{z}$ = the complex conjugate of the complex number f;

P_c is the projection associated to the continuous spectrum of the self adjoint operator H on L^2, range$P_c = \mathcal{H}_0$;

H^n denote the Sobolev spaces of measurable functions having all distributional partial derivatives up to order n in L^2, $\|\cdot\|_{H^n}$ denotes the standard norm in this spaces.

2 Preliminaries. The center manifold.

The center manifold is formed by the collection of periodic solutions for (1.1):

$$u_E(t, x) = e^{-iEt}\psi_E(x) \tag{2.1}$$

where $E \in \mathbb{R}$ and $0 \not\equiv \psi_E \in H^2(\mathbb{R}^2)$ satisfy the time independent equation:

$$[-\Delta + V]\psi_E + g(\psi_E) = E\psi_E \tag{2.2}$$

Clearly the function constantly equal to zero is a solution of (2.2) but (iii) in the following hypotheses on the potential V allows for a bifurcation with a nontrivial, one (complex) parameter family of solutions:

(H1) Assume that

(i) There exists $C > 0$ and $\rho > 3$ such that:

$$|V(x)| \leq C < x >^{-\rho}, \quad \text{for all } x \in \mathbb{R}^2;$$

(ii) 0 is a regular point[1] of the spectrum of the linear operator $H = -\Delta + V$ acting on L^2;

(iii) H acting on L^2 has exactly one negative eigenvalue $E_0 < 0$ with corresponding normalized eigenvector ψ_0. It is well known that $\psi_0(x)$ can be chosen strictly positive and exponentially decaying as $|x| \to \infty$.

Conditions (i)-(ii) guarantee the applicability of dispersive estimates of Murata [17] and Schlag [23] to the Schrödinger group e^{-iHt}. These estimates are used for obtaining Theorems 4.1 and 4.2, see also [13, section 4]. In particular (i) implies the local well posedness in H^1 of the initial value problem (1.1-1.2), see section 3.

By the standard bifurcation argument in Banach spaces [19] for (2.2) at $E = E_0$, condition (iii) guarantees existence of nontrivial solutions. Moreover, these solutions can be organized as a C^2 manifold (center manifold) for $x \in \mathbb{R}^n$, see [12, section 2] or [11]. The proofs for the following results can be found in [12, section 2] or [11]:

Proposition 2.1 *There exist $\delta > 0$, the C^2 function*

$$h : \{a \in \mathbb{R} \times \mathbb{R} \ : \ |a| < \delta\} \mapsto L^2_\sigma \cap H^2, \ \sigma \in \mathbb{R}$$

and the C^1 function $E : (-\delta, \delta) \mapsto \mathbb{R}$ such that for $|E - E_0| < \delta$ and $|\langle \psi_0, \psi_E \rangle| < \delta$ the eigenvalue problem (2.2) has a unique solution up to multiplication with $e^{i\theta}$, $\theta \in [0, 2\pi)$, which can be represented as a center manifold:

$$\psi_E = a\psi_0 + h(a), \ E = E(|a|), \quad \langle \psi_0, h(a) \rangle = 0, \quad h(e^{i\theta} a) = e^{i\theta} h(a), \ |a| < \delta. \tag{2.3}$$

Moreover $E(|a|) = \mathcal{O}(|a|^{1+\alpha_1})$, $h(a) = \mathcal{O}(|a|^{2+\alpha_1})$, and for $a \in \mathbb{R}$, $|a| < \delta$, $h(a)$ is a real valued function with $\frac{d^2 h}{da^2}(a) = \mathcal{O}(|a|^{\alpha_1})$ and $\frac{dh}{da}(0) = 0$.

Since $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$ the proposition implies that $\psi_E \in L^2_\sigma$. A regularity argument, see [25], gives a stronger result:

Corollary 2.1 *For any $\sigma \in \mathbb{R}$, there exists a finite constant C_σ such that:*

$$\| < x >^\sigma \psi_E \|_{H^2} \leq C_\sigma \|\psi_E\|_{H^2}.$$

We are going to decompose the solution of (1.1)-(1.2) into a projection onto the center manifold and a correction. To insure that the correction disperses to infinity on long times we require that the correction is always in the invariant subspace of the linearized dynamics at the projection that complements the tangent space to the center manifold. A short description of the decomposition follows, for more details and the proofs see [12].

Consider the linearization of (1.1) at a function on the center manifold $\psi_E = a\psi_0 + h(a)$, $a = a_1 + ia_2 \in \mathbb{C}$, $|a| < \delta$:

$$\frac{\partial w}{\partial t} = -iL_{\psi_E}[w] - iEw \tag{2.4}$$

where

$$L_{\psi_E}[w] = (-\Delta + V - E)w + Dg_{\psi_E}[w] = (-\Delta + V - E)w + \lim_{\varepsilon \in \mathbb{R}, \ \varepsilon \to 0} \frac{g(\psi_E + \varepsilon w) - g(\psi_E)}{\varepsilon} \tag{2.5}$$

[1] see [23, Definition 7] or $M_\mu = \{0\}$ in relation (3.1) in [17]

Remark 2.1 *Note that for $a \in \mathbb{R}$ we have $\psi_E = a\psi_0 + h(a)$ is real valued and*

$$Dg_{\psi_E}[w] = g'(\psi_E)\Re w + i\frac{g(\psi_E)}{\psi_E}\Im w = \frac{1}{2}\left(g'(\psi_E) + \frac{g(\psi_E)}{\psi_E}\right) w + \frac{1}{2}\left(g'(\psi_E) - \frac{g(\psi_E)}{\psi_E}\right)\overline{w}$$

hence

$$|Dg_{\psi_E}[w]| \leq |w| \max\left\{|g'(\psi_E)|, \left|\frac{g(\psi_E)}{\psi_E}\right|\right\} \leq C(|\psi_E|^{1+\alpha_1} + |\psi_E|^{1+\alpha_2})|w| \tag{2.6}$$

where we used (1.3). For $a = |a|e^{i\theta} \in \mathbb{C}$ we have, using the equivariant symmetry (1.4), $\psi_E = a\psi_0 + h(a) = e^{i\theta}(|a|\psi_0 + h(|a|) = e^{i\theta}\psi_E^{\text{real}}$, where ψ_E^{real} is real valued, and $Dg_{\psi_E}[w] = e^{i\theta}Dg_{\psi_E^{\text{real}}}[e^{-i\theta}w]$, hence (2.6) is valid for any ψ_E on the manifold of ground states.

Properties of the linearized operator:

1. L_{ψ_E} is real linear and symmetric with respect to the real scalar product $\Re\langle\cdot,\cdot\rangle$, on $L^2(\mathbb{R}^2)$, with domain $H^2(\mathbb{R}^2)$.
2. Zero is an e-value for $-iL_{\psi_E}$ and its generalized eigenspace includes $\left\{\frac{\partial\psi_E}{\partial a_1}, \frac{\partial\psi_E}{\partial a_2}\right\}$
3. $\text{span}_{\mathbb{R}}\left\{\frac{\partial\psi_E}{\partial a_1}, \frac{\partial\psi_E}{\partial a_2}\right\}$ and $\mathcal{H}_a = \left\{-i\frac{\partial\psi_E}{\partial a_2}, i\frac{\partial\psi_E}{\partial a_1}\right\}^{\perp}$, where orthogonality is with respect to the real scalar product in $L^2(\mathbb{R}^2)$, are invariant subspaces for $-iL_{\psi_E}$ and, by possible choosing $\delta > 0$ smaller than the one in Proposition 2.1, we have:

   $$L^2(\mathbb{R}^2) = \text{span}_{\mathbb{R}}\left\{\frac{\partial\psi_E}{\partial a_1}, \frac{\partial\psi_E}{\partial a_2}\right\} \oplus \mathcal{H}_a, \qquad \text{for all } a \in \mathbb{C},\ |a| < \delta.$$

   Note that $\mathcal{H}_0$ coincides with the subspace of L^2 associated to the continuous spectrum of the self-adjoint operator $H = -\Delta + V$.
4. the above decomposition can be extended to $H^{-1}(\mathbb{R}^2)$:

   $$H^{-1}(\mathbb{R}^2) = \text{span}_{\mathbb{R}}\left\{\frac{\partial\psi_E}{\partial a_1}, \frac{\partial\psi_E}{\partial a_2}\right\} \oplus \mathcal{H}_a, \qquad \text{for all } a \in \mathbb{C},\ |a| < \delta, \tag{2.7}$$

   where

   $$\mathcal{H}_a = \left\{\phi \in H^{-1} \mid \Re\langle -i\frac{\partial\psi_E}{\partial a_2},\ \phi\rangle = 0,\ \text{and } \Re\langle i\frac{\partial\psi_E}{\partial a_1},\ \phi\rangle = 0\right\}$$

Our goal is to decompose the solution of (1.1) at each time into:

$$u = \psi_E + \eta = a\psi_0 + h(a) + \eta, \qquad \eta \in \mathcal{H}_a$$

which insures that η is not in the non-decaying directions of the linearized equation (2.4) at ψ_E. The fact that this can be done in an unique manner is a consequence of the following lemma:

Lemma 2.1 *There exists $\delta/2 > \delta_1 > 0$ such that any $\phi \in H^{-1}(\mathbb{R}^2)$ satisfying $\|\phi\|_{H^{-1}} \leq \delta_1$ can be uniquely decomposed:*

$$\phi = \psi_E + \eta = a\psi_0 + h(a) + \eta$$

where $a = a_1 + ia_2 \in \mathbb{C}$, $|a| < \delta$, $\eta \in \mathcal{H}_a$. Moreover the maps $\phi \mapsto a$ and $\phi \mapsto \eta$ are C^1 and there exists the constant C independent on ϕ such that

$$|a| \leq 2\|\phi\|_{H^{-1}}, \qquad \|\eta\|_{H^{-1}} \leq C\|\phi\|_{H^{-1}},$$

while for $\phi \in L^2(\mathbb{R}^2)$ we have $\eta \in L^2(\mathbb{R}^2)$ and:

$$|a| \leq 2\|\phi\|_{L^2}, \qquad \|\eta\|_{L^2} \leq C\|\phi\|_{L^2}.$$

Remark 2.2 *The above lemma uses the implicit function theorem applied to*

$$F:\mathbb{R}^2\times H^{-1}(\mathbb{R}^2)\mapsto\mathbb{R}^2 \qquad F(a_1,a_2,\phi)=\begin{bmatrix}\Re\langle\Psi_1,\ \psi_E-\phi\rangle\\ \Re\langle\Psi_2,\ \psi_E-\phi\rangle\end{bmatrix}$$

where $\psi_E=(a_1+ia_2)\psi_0+h(a_1+ia_2)$ *and*

$$\begin{aligned}\Psi_1(a_1,a_2) &= -i\frac{\partial\psi_E}{\partial a_2}\left(\Re\langle -i\frac{\partial\psi_E}{\partial a_2},\ \frac{\partial\psi_E}{\partial a_1}\rangle\right)^{-1}\\ \Psi_2(a_1,a_2) &= i\frac{\partial\psi_E}{\partial a_1}\left(\Re\langle i\frac{\partial\psi_E}{\partial a_1},\ \frac{\partial\psi_E}{\partial a_2}\rangle\right)^{-1}\end{aligned}$$

form the dual basis of $\left\{\frac{\partial\psi_E}{\partial a_1},\frac{\partial\psi_E}{\partial a_2}\right\}$ *with respect to the decomposition* (2.7). *Note that*

$$\frac{\partial F}{\partial(a_1,a_2)}(a_1,a_2,\phi)=\mathbb{I}_{\mathbb{R}^2}-M(a_1,a_2,\phi)$$

where the entries of the two by two matrix M are

$$M_{ij}=\Re\langle\frac{\partial\Psi_i}{\partial a_j},\ \phi-\psi_E\rangle$$

and, consequently, $M(0,0,0)$ is the zero matrix. Thus the implicit function theorem applies to $F=0$, in a neighborhood of $(a_1,a_2,\phi)=(0,0,0)$ and the number δ_1 in the above lemma is chosen such that:

$$\left|\Re\langle i\frac{\partial\psi_E}{\partial a_1},\ \frac{\partial\psi_E}{\partial a_2}\rangle\right|\geq\frac{1}{2},\qquad \text{whenever } |(a_1,a_2)|\leq 2\delta_1,$$

and the norm of the matrix M as a linear, bounded operator on $\mathbb{R}^2$ satisfies:

$$\|M_\phi\|=\|M(a_1(\phi),a_2(\phi),\phi)\|\leq\frac{1}{2},\qquad \text{whenever } \|\phi\|_{H^{-1}}\leq\delta_1, \tag{2.8}$$

see [12, section 2] for details.

We need one more technical result relating the spaces $\mathcal{H}_a$ and the space corresponding to the continuous spectrum of $-\Delta+V$:

Lemma 2.2 *With δ_1 given by the previous lemma we have that for any $a\in\mathbb{C},\ |a|\leq 2\delta_1$, the linear map $P_c|_{\mathcal{H}_a}:\mathcal{H}_a\mapsto\mathcal{H}_0$ is invertible, and its inverse $R_a:\mathcal{H}_0\mapsto\mathcal{H}_a$ satisfies:*

$$\begin{aligned}\|R_a\zeta\|_{L^2_{-\sigma}} &\leq C_{-\sigma}\|\zeta\|_{L^2_{-\sigma}}, &&\sigma\in\mathbb{R}\text{ and for all }\zeta\in\mathcal{H}_0\cap L^2_{-\sigma}\\ \|R_a\zeta\|_{L^p} &\leq C_p\|\zeta\|_{L^p}, &&1\leq p\leq\infty\text{ and for all }\zeta\in\mathcal{H}_0\cap L^p\\ \overline{R_a\zeta} &= R_a\overline{\zeta}\end{aligned}$$

where the constants $C_{-\sigma},\ C_p>0$ are independent of $a\in\mathbb{C},\ |a|\leq 2\delta_1$.

We are now ready to prove our main result.

3 The Main Result

Theorem 3.1 *If hypothesis* (1.3), (1.4), $(H1)$ *hold and*

$$\frac{1}{2}<\alpha_1$$

then there exists $q_0'<\frac{4+2\alpha_2}{3+2\alpha_2}$ and $\varepsilon_0>0$ such that for all initial conditions $u_0(x)$ satisfying

$$\max\{\|u_0\|_{L^{q_0'}},\|u_0\|_{H^1}\}\leq\varepsilon_0$$

the initial value problem (1.1)-(1.2) is globally well-posed in H^1, and the solution decomposes into a radiative part and a part that asymptotically converges to a ground state.

More precisely, there exist a C^1 function $a : \mathbb{R} \mapsto \mathbb{C}$ such that, for all $t \in \mathbb{R}$ we have:

$$u(t,x) = \underbrace{a(t)\psi_0(x) + h(a(t))}_{\psi_E(t)} + \eta(t,x)$$

where $\psi_E(t)$ is on the central manifold (i.e it is a ground state) and $\eta(t,x) \in \mathcal{H}_{a(t)}$, see Proposition 2.1 and Lemma 2.1. Moreover, there exists the ground states $\psi_{E_{\pm\infty}}$ and the C^1 function $\tilde{\theta} : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{|t|\to\infty} \theta(t) = 0$ *and:*

$$\lim_{t\to\pm\infty} \|\psi_E(t) - e^{-it(E_\pm - \theta(t))}\psi_{E_{\pm\infty}}\|_{H^2 \bigcap L^2_\sigma} = 0, \ \sigma \in \mathbb{R} \tag{3.1}$$

while η satisfies the following decay estimates. Fix $p_0 > \max\{\frac{2}{\alpha_1 - 1/2}, \ (4+2\alpha_2)\frac{q_0-2}{q_0-(4+2\alpha_2)}\}$, where $q_0 = \frac{q_0'}{q_0'-1} > 4+2\alpha_2$. Then for $2 \le p \le \frac{p_0 q_0}{p_0+q_0-2}$ we have:

$$\|\eta(t)\|_{L^p} \le \begin{cases} C\varepsilon_0 \frac{\log^{\frac{1-2/p}{1-2/p_0}}(2+|t|)}{(1+|t|)^{1-2/p}} & \text{if } \alpha_1 \ge 1 \text{ or } \alpha_1 < 1 \text{ and } \ p \le \frac{2}{1-\alpha_1+2/p_0}, \\ C\varepsilon_0 \frac{\log^{\frac{\alpha_1-2/p_0}{1-2/p_0}}(2+|t|)}{(1+|t|)^{\alpha_1-2/p_0}} & \text{if } \alpha_1 < 1 \text{ and } p > \frac{2}{1-\alpha_1+2/p_0}, \end{cases} \tag{3.2}$$

for some constant $C = C(p_0)$.

Remark 3.1 *The estimates on η show that the component of the solution that does not converge to a ground states disperses like the solution of the free Schrödinger equation except for a logarithmic correction in L^p spaces for critical and supercritical regimes, $\alpha_1 \ge 1$. In subcritical regimes, $\alpha_1 < 1$, the decay rate remains comparable to the free Schrödinger one in L^p spaces for $2 \le p < 2/(1-\alpha_1)$, while it saturates to $|t|^{\alpha_1-1-0}$ in $L^p, \ p \ge 2/(1-\alpha_1)$.*

Proof of Theorem 3.1. It is well known that under hypothesis $(H1)(i)$ the initial value problem (1.1)-(1.2) is locally well posed in the energy space H^1 and its L^2 norm is conserved, see for example [5, Corollary 4.3.3. at p. 92]. Global well posedness follows via energy estimates from $\|u_0\|_{H^1}$ small, see [5, Corollary 6.1.5 at p. 165].

We choose $\varepsilon_0 \le \delta_1$ given by Lemma 2.1. Then, for all times, $\|u(t)\|_{H^{-1}} \le \|u(t)\|_{L^2} \le \varepsilon_0 \le \delta_1$ and, via Lemma 2.1, we can decompose the solution into a solitary wave and a dispersive component:

$$u(t) = a(t)\psi_0 + h(a(t)) + \eta(t) = \psi_E(t) + \eta(t), \qquad \text{where } |a(t)| = |a_1(t) + ia_2(t)| \le 2\varepsilon_0 \le 2\delta_1 \ \forall t \in \mathbb{R}. \tag{3.3}$$

Note that since $a \mapsto h(a)$ is C^2, see Proposition 2.1, and a is uniformly bounded in time we deduce that there exists the constant $C_H > 0$ such that:

$$\max\left\{\|\psi_E(t)\|_{H^2}, \|\frac{\partial\psi_E}{\partial a_1}(t)\|_{H^2}, \|\frac{\partial\psi_E}{\partial a_2}(t)\|_{H^2}\right\} \le C_H\varepsilon_0, \qquad \text{for all } t \in \mathbb{R},$$

which combined with Corollary 2.1 implies that for any $\sigma \in \mathbb{R}$ there exists a constant $C_{H,\sigma} > 0$ such that:

$$\max\left\{\| <x>^\sigma \psi_E(t)\|_{H^2}, \| <x>^\sigma \frac{\partial\psi_E}{\partial a_1}(t)\|_{H^2}, \| <x>^\sigma \frac{\partial\psi_E}{\partial a_2}(t)\|_{H^2}\right\} \le C_{H,\sigma}\varepsilon_0, \qquad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

Consequently, using the continuous imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2), \ 2 \le p \le \infty$ and $L^2_\sigma(\mathbb{R}^2) \hookrightarrow L^1(\mathbb{R}^2), \ \sigma > 1$ we have that for all $1 \le p \le \infty$ and all $\sigma \in \mathbb{R}$, there exists the constants $C_{p,\sigma}$ such that

$$\sup_{t\in\mathbb{R}} \max\left\{\|\psi_E(t)\|_{L^p_\sigma}, \|\frac{\partial\psi_E}{\partial a_1}(t)\|_{L^p_\sigma}, \|\frac{\partial\psi_E}{\partial a_2}(t)\|_{L^p_\sigma}, \|\Psi_1(a(t))\|_{L^p_\sigma}, \|\Psi_1(a(t))\|_{L^p_\sigma}\right\} \le C_{p,\sigma}\varepsilon_0, \tag{3.5}$$

see Remark 2.2 for the definitions of $\Psi_j(a), \ j = 1, 2$. In addition, since

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)),$$

and $u \mapsto a$ respectively $u \mapsto \eta$ are C^1, see Lemma 2.1, we get that $a(t)$ is C^1 and $\eta \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$.

The solution is now described by the C^1 function $a : \mathbb{R} \mapsto \mathbb{C}$ and $\eta(t) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. To obtain estimates for them it is useful to first remove their dominant phase. Consider the C^2 function:

$$\theta(t) = \int_0^t E(|a(s)|)ds \tag{3.6}$$

and

$$\tilde{u}(t) = e^{i\theta(t)}u(t), \tag{3.7}$$

then $\tilde{u}(t)$ satisfies the differential equation:

$$i\partial\tilde{u}(t) = -E(|a(t)|)\tilde{u}(t) + (-\Delta + V)\tilde{u}(t) + g(\tilde{u}(t)), \tag{3.8}$$

see (1.1) and (1.4). Moreover, like $u(t)$, $\tilde{u}(t)$ can be decomposed:

$$\tilde{u}(t) = \underbrace{\tilde{a}(t)\psi_0 + h(\tilde{a}(t))}_{\tilde{\psi}_E(t)} + \tilde{\eta}(t) \tag{3.9}$$

where

$$\tilde{a}(t) = e^{i\theta(t)}a(t), \qquad \tilde{\eta}(t) = e^{i\theta(t)}\eta(t) \in \mathcal{H}_{\tilde{a}(t)} \tag{3.10}$$

By plugging in (3.9) into (3.8) we get

$$\begin{aligned} i\frac{\partial\tilde{\eta}}{\partial t} + iD\tilde{\psi}_E|_{\tilde{a}}\frac{d\tilde{a}}{dt} &= (-\Delta + V - E(|a|)(\tilde{\psi}_E + \tilde{\eta}) + g(\tilde{\psi}_E) + g(\tilde{\psi}_E + \tilde{\eta}) - g(\tilde{\psi}_E) \\ &= L_{\tilde{\psi}_E}\tilde{\eta} + g_2(\tilde{\psi}_E, \tilde{\eta}) \end{aligned}$$

or, equivalently,

$$\frac{\partial\tilde{\eta}}{\partial t} + \underbrace{\frac{\partial\tilde{\psi}_E}{\partial a_1}\frac{d\tilde{a}_1}{dt} + \frac{\partial\tilde{\psi}_E}{\partial a_2}\frac{d\tilde{a}_2}{dt}}_{\in \mathrm{span}_{\mathbb{R}}\{\frac{\partial\tilde{\psi}_E}{\partial a_1}, \frac{\partial\tilde{\psi}_E}{\partial a_2}\}} = \underbrace{-iL_{\tilde{\psi}_E}\tilde{\eta}}_{\in\mathcal{H}_{\tilde{a}}} - ig_2(\tilde{\psi}_E, \tilde{\eta}) \tag{3.11}$$

where $L_{\tilde{\psi}_E}$ is defined by (2.5):

$$L_{\tilde{\psi}_E}\tilde{\eta} = (-\Delta + V - E(|\tilde{a}|))\tilde{\eta} + \frac{d}{d\varepsilon}g(\tilde{\psi}_E + \varepsilon\tilde{\eta})|_{\varepsilon=0}$$

and we used $|a| = |\tilde{a}|$, while g_2 is defined by:

$$g_2(\tilde{\psi}_E, \tilde{\eta}) = g(\tilde{\psi}_E + \tilde{\eta}) - g(\tilde{\psi}_E) - \frac{d}{d\varepsilon}g(\tilde{\psi}_E + \varepsilon\tilde{\eta})|_{\varepsilon=0} \tag{3.12}$$

and we also used the fact that $\tilde{\psi}_E$ is a solution of the eigenvalue problem (2.2). Note that g_2 is at least quadratic in the second variable, more precisely:

Lemma 3.1 *There exists a constant $C > 0$ such that for all $a, z \in \mathbb{C}$ we have:*

$$|g_2(a, z)| \leq C(|a|^{\alpha_1} + |a|^{\alpha_2} + |z|^{\alpha_1} + |z|^{\alpha_2})|z|^2$$

Proof: From the definition (3.12) of g_2 we have:

$$g_2(a, z) = g(a + z) - g(a) - Dg_a[z] = \int_0^1 (Dg_{a+\tau z} - Dg_a)[z]d\tau = \int_0^1 \int_0^1 D^2 g_{a+s\tau z}[\tau z][z]d\tau ds.$$

Now (1.3) and (1.4) imply that there exists a constant $C_1 > 0$ such that the bilinear form Dg on $\mathbb{C} \times \mathbb{C}$ satisfies:

$$\|D^2 g_b\| \leq C_1(|b|^{\alpha_1} + |b|^{\alpha_2}), \qquad \forall b \in \mathbb{C}. \tag{3.13}$$

Hence

$$|g_2(a,z)| \le C_1\left((2\max(|a|,|z|))^{\alpha_1} + (2\max(|a|,|z|))^{\alpha_2}\right)\frac{1}{2}|z|^2,$$

which proves the lemma. □

We now project (3.11) onto the invariant subspaces of $-iL_{\tilde{\psi}_E}$, namely $\mathrm{span}_{\mathbb{R}}\{\frac{\partial\tilde{\psi}_E}{\partial a_1}, \frac{\partial\tilde{\psi}_E}{\partial a_2}\}$, and $\mathcal{H}_{\tilde{a}}$. More precisely, we evaluate both the left and right hand side of (3.11) which are functionals in $H^{-1}(\mathbb{R}^2)$ at $\Psi_j = \Psi_j(\tilde{a}(t)),\ j=1,2$, see Remark 2.2, and take the real parts. We obtain:

$$\begin{bmatrix} \Re\langle\Psi_1, \frac{\partial\tilde{\eta}}{\partial t}\rangle \\ \Re\langle\Psi_2, \frac{\partial\tilde{\eta}}{\partial t}\rangle \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = \begin{bmatrix} g_{21}(\tilde{\psi}_E,\tilde{\eta}) \\ g_{22}(\tilde{\psi}_E,\tilde{\eta}) \end{bmatrix}$$

where

$$g_{2j}(\tilde{\psi}_E,\tilde{\eta}) = \Re\langle\Psi_j, -ig_2(\tilde{\psi}_E,\tilde{\eta})\rangle, \qquad j=1,2. \tag{3.14}$$

Note that from Lemma 3.1 and Hölder inequality we have for all $t\in\mathbb{R}$:

$$\begin{aligned} |g_{2j}(\tilde{\psi}_E(t),\tilde{\eta}(t))| &\le C\int_{\mathbb{R}^2}|\Psi_j(t,x)|\left(|\tilde{\psi}_E(t,x)|^{\alpha_1}+|\tilde{\psi}_E(t,x)|^{\alpha_2}+|\tilde{\eta}(t,x)|^{\alpha_1}+|\tilde{\eta}(t,x)|^{\alpha_2}\right)|\tilde{\eta}(t,x)|^2dx \qquad (3.15)\\ &\le C\left[\|\Psi_j(t)\|_{L^{r_0}}\left(\|\tilde{\psi}_E(t)\|_{L^\infty}^{\alpha_1}+\|\tilde{\psi}_E(t)\|_{L^\infty}^{\alpha_2}\right)\|\tilde{\eta}(t)\|_{L^{p_2}}^2+\|\Psi_j(t)\|_{L^{r_1}}\|\tilde{\eta}(t)\|_{L^{p_2}}^{2+\alpha_1}+\|\Psi_j(t)\|_{L^{r_2}}\|\tilde{\eta}(t)\|_{L^{p_2}}^{2+\alpha_2}\right],\end{aligned}$$

where $r_0^{-1}+(p_2/2)^{-1}=1,\ r_j^{-1}+(p_2/(2+\alpha_j))^{-1}=1,\ j=1,2.$

To calculate $\Re\langle\Psi_j,\frac{\partial\tilde{\eta}}{\partial t}\rangle,\ j=1,2$ we use the fact that $\tilde{\eta}(t)\in\mathcal{H}_{\tilde{a}}$, for all $t\in\mathbb{R}$, i.e.

$$\Re\langle\Psi_j(\tilde{a}(t)),\tilde{\eta}(t)\rangle \equiv 0.$$

Differentiating the latter with respect to t we get:

$$\Re\langle\Psi_j,\frac{\partial\tilde{\eta}}{\partial t}\rangle = -\Re\langle\frac{\partial\Psi_j}{\partial a_1}\frac{d\tilde{a}_1}{dt}+\frac{\partial\Psi_j}{\partial a_2}\frac{d\tilde{a}_2}{dt},\tilde{\eta}\rangle \qquad j=1,2$$

which replaced into above leads to:

$$\frac{d}{dt}\begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = (\mathbb{I}_{\mathbb{R}^2}-M_{\tilde{a}})^{-1}\begin{bmatrix} g_{21}(\tilde{\psi}_E,\tilde{\eta}) \\ g_{22}(\tilde{\psi}_E,\tilde{\eta}) \end{bmatrix}, \tag{3.16}$$

where the two by two matrix $M_{\tilde{a}}$ is defined in Remark 2.2. In particular

$$\begin{bmatrix} \Re\langle\Psi_1, \frac{\partial\tilde{\eta}}{\partial t}\rangle \\ \Re\langle\Psi_2, \frac{\partial\tilde{\eta}}{\partial t}\rangle \end{bmatrix} = -M_{\tilde{a}}(\mathbb{I}_{\mathbb{R}^2}-M_{\tilde{a}})^{-1}\begin{bmatrix} g_{21}(\tilde{\psi}_E,\tilde{\eta}) \\ g_{22}(\tilde{\psi}_E,\tilde{\eta}) \end{bmatrix},$$

which we use to obtain the component in $\mathcal{H}_{\tilde{a}}=\{\Psi_1(\tilde{a}),\Psi_2(\tilde{a})\}^\perp$ of (3.11):

$$\frac{\partial\tilde{\eta}}{\partial t} = -iL_{\tilde{\psi}_E}\tilde{\eta} - ig_2(\tilde{\psi}_E,\tilde{\eta}) - (\mathbb{I}-M_{\tilde{a}})^{-1}g_3(\tilde{\psi}_E,\tilde{\eta}),$$

where g_3 is the projection of $-ig_2$ onto $\mathrm{span}_{\mathbb{R}}\{\frac{\partial\tilde{\psi}_E}{\partial a_1},\frac{\partial\tilde{\psi}_E}{\partial a_2}\}$ relative to the decomposition (2.7):

$$g_3(\tilde{\psi}_E,\tilde{\eta}) = g_{21}(\tilde{\psi}_E,\tilde{\eta})\frac{\partial\tilde{\psi}_E}{\partial a_1} + g_{22}(\tilde{\psi}_E,\tilde{\eta})\frac{\partial\tilde{\psi}_E}{\partial a_2}, \tag{3.17}$$

see (3.14) for the definitions of $g_{2j},\ j=1,2$, and $\mathbb{I}-M_{\tilde{a}}$ is the linear operator on the two dimensional real vector space $\mathrm{span}_{\mathbb{R}}\{\frac{\partial\tilde{\psi}_E}{\partial a_1},\frac{\partial\tilde{\psi}_E}{\partial a_2}\}$ whose matrix representation relative to the basis $\{\frac{\partial\tilde{\psi}_E}{\partial a_1},\frac{\partial\tilde{\psi}_E}{\partial a_2}\}$ is $\mathbb{I}_{\mathbb{R}^2}-M_{\tilde{a}}$. It is easier to switch back to the variable $\eta(t)=e^{-i\theta(t)}\tilde{\eta}(t)\in\mathcal{H}_a$:

$$\frac{\partial\eta}{\partial t} = -i(-\Delta+V)\eta - iDg_{\psi_E}\eta - ig_2(\psi_E,\eta) - (\mathbb{I}-M_u)^{-1}g_3(\psi_E,\eta), \tag{3.18}$$

where we used the equivariant symmetry (1.4) and its obvious consequences for the symmetries of $Dg,\ g_2,\ g_3$ and M. Since by Lemma 2.2 it is sufficient to get estimates for $\zeta(t) = P_c\eta(t)$, we now project (3.18) onto the continuous spectrum of $-\Delta + V$:

$$\frac{\partial \zeta}{\partial t} = -i(-\Delta + V)\zeta - iP_c Dg_{\psi_E} R_a \zeta - iP_c g_2(\psi_E, R_a\zeta) - P_c(\mathbb{I} - M_u)^{-1} g_3(\psi_E, R_a\zeta), \tag{3.19}$$

where $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ is the inverse of P_c restricted to $\mathcal{H}_a$, see Lemma 2.2.

Consider the initial value problem for the linear part of (3.19):

$$\begin{aligned} \frac{\partial z}{\partial t} &= -i(-\Delta + V)z - iP_c Dg_{\psi_E(t)} R_{a(t)} z(t) \\ z(s) &= v \in \mathcal{H}_0 \end{aligned} \tag{3.20}$$

and write its solution in terms of a family of operators:

$$\Omega(t,s) : \mathcal{H}_0 \mapsto \mathcal{H}_0, \qquad \Omega(t,s)v = z(t), \quad t,\ s \in \mathbb{R}. \tag{3.21}$$

In Section 4 we show that such a family of operators exists, is uniformly bounded in $t,\ s$ with respect to the L^2 norm and it has very similar properties with the unitary group of operators $e^{-i(-\Delta+V)(t-s)}P_c$ generated by the Schrödinger operator $-i(-\Delta + V)P_c$. In particular $\Omega(t,s)$ satisfies certain dispersive decay estimates in weighted L^2 spaces and $L^p,\ p > 2$ spaces, see Theorem 4.1 and Theorem 4.2. For all these results to hold we only need to choose ε_0 small enough such that $\varepsilon_0 C_{H,4\sigma/3} \le \varepsilon_1$, where $\sigma > 1$ and $\varepsilon_1 > 0$ are fixed in Section 4 and the constant $C_{H,4\sigma/3}$ is the one from (3.4).

Using Duhamel formula, the solution $\zeta \in C(\mathbb{R}, H^1 \cap \mathcal{H}_0) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2) \cap \mathcal{H}_0)$ of (3.19) also satisfies:

$$\begin{aligned} \zeta(t) &= \Omega(t,0)\zeta(0) - i\int_0^t \Omega(t,s)P_c g_2(\psi_E(s), R_{a(s)}\zeta(s))ds \\ &\quad - \int_0^t \Omega(t,s)P_c(\mathbb{I} - M_{u(s)})^{-1} g_3(\psi_E(s), R_{a(s)}\zeta(s))ds. \end{aligned} \tag{3.22}$$

Note that the right hand side of (3.22) contains only terms that are quadratic and higher order in ζ, see Lemma 3.1 and (3.15). As in [13, 12] this is essential in controlling low power nonlinearities and it is the main difference between our approach and the existing literature on asymptotic stability of coherent structures for dispersive nonlinear equations, see [13, p. 449] for a more detailed discussion.

To obtain estimates for ζ we apply a contraction mapping argument to the fixed point problem (3.22) in the following Banach space. Fix $p_0 > 2$ such that

$$p_0 > \max\left\{\frac{2}{\alpha_1 - 1/2},\ (4 + 2\alpha_2)\frac{q_0 - 2}{q_0 - (4 + 2\alpha_2)}\right\}, \tag{3.23}$$

and let

$$p_2 = \frac{p_0 q_0}{p_0 + q_0 - 2}, \tag{3.24}$$

and

$$p_1 = \frac{2}{1 - \alpha_1 + 2/p_0}, \qquad \text{if } \alpha_1 < 1, \tag{3.25}$$

then

Case I if $\alpha_1 \ge 1$, or $1/2 < \alpha_1 < 1$ and $p_1 \ge p_2$, let:

$$Y = \left\{ v \in C(\mathbb{R}, L^2 \cap L^{p_2})\ :\ \sup_{t\in\mathbb{R}} \|v(t)\|_{L^2} < \infty,\ \sup_{t\in\mathbb{R}} \frac{(1+|t|)^{1-\frac{2}{p_2}}}{[\log(2+|t|)]^{\frac{1-\frac{2}{p_2}}{1-\frac{2}{p_0}}}} \|v(t)\|_{L^{p_2}} < \infty \right\};$$

Case II if $1/2 < \alpha_1 < 1$ and $p_1 < p_2$, let:

$$Y = \Big\{ v \in C\left(\mathbb{R}, L^2 \cap L^{p_1} \cap L^{p_2}\right) \ : \ \sup_{t\in\mathbb{R}} \|v(t)\|_{L^2} < \infty,$$
$$\sup_{t\in\mathbb{R}} \frac{(1+|t|)^{1-\frac{2}{p_1}}}{[\log(2+|t|)]^{\frac{1-\frac{2}{p_1}}{1-\frac{2}{p_0}}}} \|v(t)\|_{L^{p_1}} < \infty, \ \sup_{t\in\mathbb{R}} \frac{(1+|t|)^{\alpha_1-\frac{2}{p_0}}}{[\log(2+|t|)]^{\frac{\alpha_1-\frac{2}{p_0}}{1-\frac{2}{p_0}}}} \|v(t)\|_{L^{p_2}} < \infty \Big\};$$

endowed with the norm

$$\|v\|_Y = \max\left\{ \sup_{t\in\mathbb{R}} \|v(t)\|_{L^2}, \ \sup_{t\in\mathbb{R}} \frac{(1+|t|)^{1-\frac{2}{p_2}}}{[\log(2+|t|)]^{\frac{1-\frac{2}{p_2}}{1-\frac{2}{p_0}}}} \|v(t)\|_{L^{p_2}} \right\}$$

in Case I, while in Case II

$$\|v\|_Y = \max\left\{ \sup_{t\in\mathbb{R}} \|v(t)\|_{L^2}, \ \sup_{t\in\mathbb{R}} \frac{(1+|t|)^{1-\frac{2}{p_1}}}{[\log(2+|t|)]^{\frac{1-\frac{2}{p_1}}{1-\frac{2}{p_0}}}} \|v(t)\|_{L^{p_1}}, \ \sup_{t\in\mathbb{R}} \frac{(1+|t|)^{\alpha_1-\frac{2}{p_0}}}{[\log(2+|t|)]^{\frac{\alpha_1-\frac{2}{p_0}}{1-\frac{2}{p_0}}}} \|v(t)\|_{L^{p_2}} \right\}.$$

Consider now the nonlinear operator in (3.22):

$$N(v)(t) = -i \int_0^t \Omega(t,s) P_c g_2(\psi_E(s), R_{a(s)} v(s)) ds - \int_0^t \Omega(t,s) P_c (\mathbb{I} - M_{u(s)})^{-1} g_3(\psi_E(s), R_{a(s)} v(s)) ds. \quad (3.26)$$

We have:

Lemma 3.2 *$N : Y \to Y$ is well defined, and locally Lipschitz, i.e. there exists $\tilde{C} > 0$, such that*

$$\|Nv_1 - Nv_2\|_Y \le \tilde{C}(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1} + \|v_1\|_Y^{1+\alpha_2} + \|v_2\|_Y^{1+\alpha_2})\|v_1 - v_2\|_Y.$$

Assuming that the lemma has been proven then we can apply the contraction principle for (3.22) in a closed ball in the Banach space Y in the following way. Let

$$v = \Omega(t,0)\zeta(0)$$

then by Theorem 4.2

$$\|v\|_Y \le \max\{C_2, C_{p_0,p_1}, C_{p_0,p_2}\} \|\zeta(0)\|_{L^2 \cap L^{q_0'}}$$

where we used the interpolation $\|\zeta(0)\|_{L^r} \le \|\zeta(0)\|_{L^2 \cap L^{q_0'}}, \ q_0' \le r \le 2$ with $r = q'$ and $r = p'$ defined in theorem 4.2 for $p = p_j, \ j = 1, 2$. Recall that

$$\zeta(0) = P_c \eta(0) = P_c u_0 - h(a(0)) = u_0 - \langle \psi_0, u_0 \rangle \psi_0 - h(a(0))$$

where $u_0 = u(0)$ is the initial data, see also (3.3). Hence

$$\|\zeta(0)\|_{L^2 \cap L^{q_0'}} \le \|u_0\|_{L^2 \cap L^{q_0'}} + \|u_0\|_{L^2} \|\psi_0\|_{L^{q_0'}} + D_1 \|u_0\|_{L^2} \le D\varepsilon_0$$

where $D_1, \ D > 0$ are constants independent on u_0 and the estimate on $h(a(0))$ follows from Proposition 2.1 and $|a(0)| \le 2\|u_0\|_{L^2}$ see Lemma 2.1.

Therefore we can choose ε_0 small enough such that $R = 2\|v\|_Y$ satisfies

$$Lip \stackrel{def}{=} 2\tilde{C}(R + R^{1+\alpha_1} + R^{1+\alpha_2}) < 1.$$

In this case the integral operator given by the right hand side of (3.22):

$$K(\zeta) = v + N(\zeta)$$

leaves $B(0,R)=\zeta\in Y:\|\zeta\|_Y\le R$ invariant and it is a contraction on it with Lipschitz constant Lip defined above. Consequently the equation (3.22) has a unique solution in $B(0,R)$ and because $\zeta(t)\in C(\mathbb{R},H^1)\hookrightarrow C(\mathbb{R},L^2,L^{p_1},L^{p_2})$ already verified the equation we deduce that $\zeta(t)$ is in $B(0,R)$, in particular it satisfies the estimates (3.2).

Then $\eta(t)=R_a(t)\zeta(t)$ satisfies (3.2) because of Lemma 2.2. Moreover, the system of ODE's (3.16) has integrable in time right hand side because the matrix has norm bounded by 2, see (2.8), while g_{2j} satisfy (3.15) where $\tilde{\eta}(t)$ differs from $\eta(t)$ by only a phase and the L^p, $1\le p\le\infty$ norms of $\Psi_j(t)$, $\psi_E(t)$ are uniformly bounded in time, see (3.5). Consequently $\tilde{a}_1(t)$ and $\tilde{a}_2(t)$ converge as $t\to\pm\infty$, and there exist the constants $C,\epsilon>0$ such that:

$$\lim_{t\to\pm\infty}\tilde{a}(t)=\lim_{t\to\pm\infty}\tilde{a}_1(t)+i\tilde{a}_2(t)\overset{def}{=}a_{\pm\infty},\qquad |\tilde{a}(\pm t)-a_{\pm\infty}|\le C(1+t)^{-\epsilon},\ \text{for all } t\ge 0.$$

We can now define

$$\psi_{E_{\pm\infty}}=a_{\pm\infty}\psi_0+h(a_{\pm\infty}),\tag{3.27}$$

and we have

$$\lim_{t\to\pm\infty}\|\tilde{\psi}_E(t)-\psi_{E_{\pm\infty}}\|_{H^2\cap L^2_\sigma}=0,\ \text{for } \sigma\in\mathbb{R}\tag{3.28}$$

where we used (3.9) and the continuity of $h(a)$, see Proposition 2.1. In addition, since $E:[-2\delta_1,\delta_1]\mapsto(-\delta,\delta)$ is a C^1 function, see Proposition 2.1, the following limits exist together with the constant $C_1>0$ such that:

$$\lim_{t\to\pm\infty}E(|\tilde{a}(t)|)=E_{\pm\infty},\qquad |E(|\tilde{a}(\pm t)|)-E_{\pm\infty}|\le C_1(1+t)^{-\epsilon}\ \text{for all } t\ge 0.$$

If we define

$$\tilde{\theta}(t)=\begin{cases}\frac{1}{t}\int_0^t E(|\tilde{a}(s)|)-E_{+\infty}ds & \text{if } t>0\\ 0 & \text{if } t=0\\ \frac{1}{t}\int_0^t E(|\tilde{a}(s)|)-E_{-\infty}ds & \text{if } t<0\end{cases}\tag{3.29}$$

then

$$\lim_{|t|\to\infty}\tilde{\theta}(t)=0$$

and

$$\theta(t)=\int_0^t E(|a(s)|)ds=\begin{cases}t(E_{+\infty}+\tilde{\theta}(t)) & \text{if } t\ge 0\\ t(E_{+\infty}+\tilde{\theta}(t)) & \text{if } t<0\end{cases}\tag{3.30}$$

where we used $|a(t)|=|\tilde{a}(t)|$, see (3.10).

In conclusion, since $\psi_E(t)=e^{i\theta(t)}\tilde{\psi}_E(t)$, see (3.3), (3.9) and (3.10), we get from (3.28) and (3.30) the convergence (3.1).

It remains to prove Lemma 3.2:

Proof of Lemma 3.2: It suffices to prove the estimate:

$$\|Nv_1-Nv_2\|_Y\le\tilde{C}(\|v_1\|_Y+\|v_2\|_Y+\|v_1\|_Y^{1+\alpha_1}+\|v_2\|_Y^{1+\alpha_1}+\|v_1\|_Y^{1+\alpha_2}+\|v_2\|_Y^{1+\alpha_2})\|v_1-v_2\|_Y,\tag{3.31}$$

because plugging in $v_2\equiv 0$ and using $N(0)\equiv 0$, see (3.26), will then imply $N(v_1)\in Y$ whenever $v_1\in Y$.

Note that via interpolation in L^p spaces we have for all $v\in Y$ and any $2\le p\le p_2$:

$$\|v(t)\|_{L^p}\le\begin{cases}\|v\|_Y\frac{\log^{\frac{1-2/p}{1-2/p_0}}(2+|t|)}{(1+|t|)^{1-2/p}} & \text{if } \alpha_1\ge 1 \text{ or } \alpha_1<1 \text{ and } p\le\frac{2}{1-\alpha_1+2/p_0},\\ \\ \|v\|_Y\frac{\log^{\frac{\alpha_1-2/p_0}{1-2/p_0}}(2+|t|)}{(1+|t|)^{\alpha_1-2/p_0}} & \text{if } \alpha_1<1 \text{ and } p>\frac{2}{1-\alpha_1+2/p_0}.\end{cases}\tag{3.32}$$

Now, from (3.12), we have for any $v_1,\ v_2\in Y$:

$$\begin{aligned}g_2(\psi_E,R_av_1)-g_2(\psi_E,R_av_2)&=g(\psi_E+R_av_1)-g(\psi_E+R_av_2)-Dg_{\psi_E}[R_a(v_1-v_2)]\\&=\int_0^1\left(Dg_{\psi_E+R_a(\tau v_1+(1-\tau)v_2)}-Dg_{\psi_E}\right)[R_a(v_1-v_2)]d\tau\\&=\int_0^1\int_0^1 D^2g_{\psi_E+sR_a(\tau v_1+(1-\tau)v_2)}[R_a(\tau v_1+(1-\tau)v_2)][R_a(v_1-v_2)]d\tau ds\\&=A_1(\psi_E,v_1,v_2)+A_2(\psi_E,v_1,v_2)+A_3(\psi_E,v_1,v_2),\end{aligned}\tag{3.33}$$

where we consider $\chi_j(t,x),\ j=1,2$ to be the characteristic function of the set $S_1=\{(t,x)\in\mathbb{R}\times\mathbb{R}^2\ :\ |\psi_E(t,x)|\geq \max(|R_{a(t)}v_1(t,x)|,|R_{a(t)}v_2(t,x)|)\}$, respectively
$S_2=\{(t,x)\in\mathbb{R}\times\mathbb{R}^2\ :\ \max(|R_{a(t)}v_1(t,x)|,|R_{a(t)}v_2(t,x)|)\leq 1\}$ and

$$\begin{aligned}
A_1(\psi_E,v_1,v_2) &= \int_0^1\int_0^1 \chi_1 D^2 g_{\psi_E+sR_a(\tau v_1+(1-\tau)v_2)}[R_a(\tau v_1+(1-\tau)v_2)][R_a(v_1-v_2)]d\tau ds,\\
A_2(\psi_E,v_1,v_2) &= \int_0^1\int_0^1 (1-\chi_1)\chi_2 D^2 g_{\psi_E+sR_a(\tau v_1+(1-\tau)v_2)}[R_a\tau v_1+(1-\tau)v_2)][R_a(v_1-v_2)]d\tau ds,\\
A_3(\psi_E,u_1,u_2) &= \int_0^1\int_0^1 (1-\chi_1)(1-\chi_2) D^2 g_{\psi_E+sR_a(\tau v_1+(1-\tau)v_2)}[R_a(\tau v_1+(1-\tau)v_2)][R_a(v_1-v_2)]d\tau ds.
\end{aligned}$$

Note that there exists a constant $C>0$ such that for any $\psi_E,\ v_1,\ v_2\in Y$, any $t\in\mathbb{R}$ and almost all $x\in\mathbb{R}^2$ we have the pointwise estimates:

$$\begin{aligned}
|A_1(\psi_E(t,x),v_1(t,x),v_2(t,x))| &\leq C\left(2^{\alpha_1}|\psi_E(t,x)|^{\alpha_1}+2^{\alpha_2}|\psi_E(t,x)|^{\alpha_2}\right)\left(|R_{a(t)}v_1(t,x)|+|R_{a(t)}v_2(t,x)|\right)\\
&\qquad\times|R_{a(t)}(v_1(t,x)-v_2(t,x))|\\
|A_2(\psi_E(t,x),v_1(t,x),v_2(t,x))| &\leq 2^{\alpha_1}C\left(|R_{a(t)}v_1(t,x)|^{1+\alpha_1}+|R_{a(t)}v_2(t,x)|^{1+\alpha_1}\right)|R_{a(t)}(v_1(t,x)-v_2(t,x))|\\
|A_3(\psi_E(t,x),v_1(t,x),v_2(t,x))| &\leq 2^{\alpha_2}C\left(|R_{a(t)}v_1(t,x)|^{1+\alpha_2}+|R_{a(t)}v_2(t,x)|^{1+\alpha_2}\right)|R_{a(t)}(v_1(t,x)-v_2(t,x))|
\end{aligned}$$

where we used (3.13). Consequently, for any $\sigma\in\mathbb{R}$ there exists a constant $C_\sigma>0$ such that for any $t\in\mathbb{R}:$

$$\begin{aligned}
\|A_1(\psi_E(t),v_1(t),v_2(t))\|_{L^2_\sigma} &\leq C\|2^{\alpha_1}|\psi_E(t)|^{\alpha_1}+2^{\alpha_2}|\psi_E(t)|^{\alpha_2}\|_{L^s_\sigma}(\|R_{a(t)}v_1(t)\|_{L^{p_2}}+\|R_{a(t)}v_2(t)\|_{L^{p_2}})\\
&\qquad\times\|R_{a(t)}(v_1(t)-v_2(t))\|_{L^{p_2}}\\
&\leq \frac{C_\sigma\log^{a_1}(2+|t|)}{(1+|t|)^{b_1}}(\|v_1\|_Y+\|v_2\|_Y)\|v_1-v_2\|_Y \qquad (3.34)
\end{aligned}$$

$$(3.35)$$

where $\frac{1}{s}+\frac{2}{p_2}=\frac{1}{2}$, and, for $\Psi_j,\ j=1,2$ defined in Remark 2.2:

$$\begin{aligned}
|\Re\langle\Psi_j(a(t)),-iA_1(\psi_E(t),v_1(t),v_2(t))\rangle| &\leq \|\Psi_j(a(t))\|_{L^2_{-\sigma}}\|A_1(\psi_E(t),v_1(t),v_2(t))\|_{L^2_\sigma}\\
&\leq C_{2,-\sigma}\frac{C_\sigma\log^{a_1}(2+|t|)}{(1+|t|)^{b_1}}(\|v_1\|_Y+\|v_2\|_Y)\|v_1-v_2\|_Y \qquad (3.36)
\end{aligned}$$

where

$$b_1=\begin{cases}2-\frac{4}{p_2} & \text{in Case I,}\\ 2\alpha_1-\frac{4}{p_0} & \text{in Case II,}\end{cases}\qquad a_1=\begin{cases}2\frac{1-2/p_2}{1-2/p_0} & \text{in Case I,}\\ 2\frac{\alpha_1-2/p_0}{1-2/p_0} & \text{in Case II,}\end{cases} \qquad (3.37)$$

see the definition of the Banach space Y, and we used Hölder inequality together with (3.5) and Lemma 2.2.

Similarly, for any $1\leq r'\leq 2$ we have $(2+\alpha_1)r'\leq(2+\alpha_2)r'\leq p_2$, hence the above pointwise estimates and (3.32) imply that there exists a constant $C_{r'}>0$ such that for any $t\in\mathbb{R}:$

$$\begin{aligned}
\|A_2(\psi_E(t),v_1(t),v_2(t))\|_{L^{r'}} &\leq 2^{\alpha_1}C\|\,|R_{a(t)}v_1(t)|^{1+\alpha_1}+|R_{a(t)}v_2(t)|^{1+\alpha_1}\|_{L^{\frac{(2+\alpha_1)r'}{1+\alpha_1}}}\|R_{a(t)}(v_1(t)-v_2(t))\|_{L^{(2+\alpha_1)r'}}\\
&\leq \frac{C_{r'}\log^{a_2(r')}(2+|t|)}{(1+|t|)^{b_2(r')}}(\|v_1\|_Y^{1+\alpha_1}+\|v_2\|_Y^{1+\alpha_1})\|v_1-v_2\|_Y, \qquad (3.38)
\end{aligned}$$

where

$$\begin{aligned}
&b_2(r')=\alpha_1+\tfrac{2}{r}, && a_2(r')=\tfrac{\alpha_1+2/r}{1-2/p_0}, && \text{if } \alpha_1\geq 1 \text{ or } \alpha_1<1 \text{ and } (2+\alpha_1)r'\leq p_1,\\
&b_2(r')=(2+\alpha_1)(\alpha_1-\tfrac{2}{p_0}), && a_2(r')=(2+\alpha_1)\tfrac{\alpha_1-2/p_0}{1-2/p_0}, && \text{if } \alpha_1<1 \text{ and } (2+\alpha_1)r'>p_1,
\end{aligned} \qquad (3.39)$$

with $1/r+1/r'=1$, and

$$\begin{aligned}
\|A_3(\psi_E(t),v_1(t),v_2(t))\|_{L^{r'}} &\leq 2^{\alpha_2}C\|\,|R_{a(t)}v_1(t)|^{1+\alpha_2}+|R_{a(t)}v_2(t)|^{1+\alpha_2}\|_{L^{\frac{(2+\alpha_2)r'}{1+\alpha_2}}}\|R_{a(t)}(v_1(t)-v_2(t))\|_{L^{(2+\alpha_2)r'}}\\
&\leq \frac{C_{r'}\log^{a_3(r')}(2+|t|)}{(1+|t|)^{b_3(r')}}(\|v_1\|_Y^{1+\alpha_2}+\|v_2\|_Y^{1+\alpha_2})\|v_1-v_2\|_Y, \qquad (3.40)
\end{aligned}$$

where

$$b_3(r') = \alpha_2 + \tfrac{2}{r}, \qquad a_3(r') = \tfrac{\alpha_2+2/r}{1-2/p_0}, \qquad \text{if } \alpha_1 \geq 1 \text{ or } \alpha_1 < 1 \text{ and } (2+\alpha_2)r' \leq p_1,$$
$$b_3(r') = (2+\alpha_2)(\alpha_1 - \tfrac{2}{p_0}), \quad a_3(r') = (2+\alpha_2)\tfrac{\alpha_1-2/p_0}{1-2/p_0}, \quad \text{if } \alpha_1 < 1 \text{ and } (2+\alpha_2)r' > p_1. \tag{3.41}$$

Moreover, using Cauchy-Schwartz inequality and (3.5) we have:

$$\begin{aligned} |\Re\langle\Psi_j(a(t)), -iA_2(\psi_E(t), v_1(t), v_2(t))\rangle| &\leq \|\Psi_j(a(t))\|_{L^2}\|A_2(\psi_E(t), v_1(t), v_2(t))\|_{L^2} \\ &\leq C_{2,0}\frac{C_2\log^{a_2(2)}(2+|t|)}{(1+|t|)^{b_2(2)}}(\|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1})\|v_1 - v_2\|_Y \end{aligned} \tag{3.42}$$

and

$$|\Re\langle\Psi_j(a(t)), -iA_3(\psi_E(t), v_1(t), v_2(t))\rangle| \leq C_{2,0}\frac{C_2\log^{a_3(2)}(2+|t|)}{(1+|t|)^{b_3(2)}}(\|v_1\|_Y^{1+\alpha_2} + \|v_2\|_Y^{1+\alpha_2})\|v_1 - v_2\|_Y. \tag{3.43}$$

Now, from (3.17) and (3.14) we have

$$\begin{aligned} &g_3(\psi_E, R_a v_1) - g_3(\psi_E, R_a v_2) \\ &= \Re\langle\Psi_1(a), -i(g_2(\psi_E, R_a v_1) - g_2(\psi_E, R_a v_2))\rangle\frac{\partial\psi_E}{\partial a_1} + \Re\langle\Psi_2(a), -i(g_2(\psi_E, R_a v_1) - g_2(\psi_E, R_a v_2))\rangle\frac{\partial\psi_E}{\partial a_2} \\ &= \Re\langle\Psi_1(a), -i(A_1 + A_2 + A_3)(\psi_E, v_1, v_2)\rangle\frac{\partial\psi_E}{\partial a_1} + \Re\langle\Psi_2(a), -i(A_1 + A_2 + A_3)(\psi_E, v_1, v_2)\rangle\frac{\partial\psi_E}{\partial a_2}. \end{aligned}$$

Consequently, for

$$A_4(\psi_E, v_1, v_2) \stackrel{def}{=} (\mathbb{I} - M_u)^{-1}(g_3(\psi_E, R_a v_1) - g_3(\psi_E, R_a v_2)) \tag{3.44}$$

we have that for any $\sigma \in \mathbb{R}$ there exists a constant $C_\sigma > 0$ such that:

$$\begin{aligned} \|A_4(\psi_E(t), v_1(t), v_2(t))\|_{L^2_\sigma} &\leq \max\left\{\|\frac{\partial\psi_E}{\partial a_1}(t)\|_{L^2_\sigma}, \|\frac{\partial\psi_E}{\partial a_2}(t)\|_{L^2_\sigma}\right\}\sqrt{2}\|(\mathbb{I} - M_{u(t)})^{-1}\|_{\mathbb{R}^2\mapsto\mathbb{R}^2} \\ &\quad\times\sqrt{|\Re\langle\Psi_1(a(t)), -i(A_1+A_2+A_3)(t)\rangle|^2 + |\Re\langle\Psi_2(a(t)), -i(A_1+A_2+A_3)(t)\rangle|^2} \\ &\leq \frac{C_\sigma\log^{a_4}(2+|t|)}{(1+|t|)^{b_4}}(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1} + \|v_1\|_Y^{1+\alpha_2} + \|v_2\|_Y^{1+\alpha_2})\|v_1 - v_2\|_Y \end{aligned} \tag{3.45}$$

where

$$b_4 = \min\{b_1, b_2(2), b_3(2)\}, \qquad a_4 = \max\{a_1, a_2(2), a_3(3)\}, \tag{3.46}$$

and we used (3.5), (2.8), (3.36), (3.42), and (3.43).

We are now ready to prove the Lipschitz estimate for the nonlinear operator N, (3.31). From its definition (3.26) and (3.33), (3.44) we have for any $v_1, v_2 \in Y$, any $2 \leq p \leq p_2$, and a fixed $\sigma > 1$:

$$\begin{aligned} \|N(v_1)(t) - N(v_2)(t)\|_{L^p} &= \left\|\int_0^t \Omega(t,s)P_c(-iA_1 - iA_2 - iA_3 - A_4)(\psi_E(s), v_1(s), v_2(s))ds\right\|_{L^p} \\ &\leq \int_0^t \|\Omega(t,s)\|_{L^2_\sigma\mapsto L^p}\left(\|A_1(\psi_E(s), v_1(s), v_2(s))\|_{L^2_\sigma} + \|A_4(\psi_E(s), v_1(s), v_2(s))\|_{L^2_\sigma}\right)ds \\ &+ \int_0^{|t|} \|\Omega(t,s)\|_{L^{q'}\cap L^{p'}\mapsto L^p}\left(\|A_2(\psi_E(s), v_1(s), v_2(s))\|_{L^{q'}\cap L^{p'}} + \|A_3(\psi_E(s), v_1(s), v_2(s))\|_{L^{q'}\cap L^{p'}}\right)ds. \end{aligned}$$

where

$$1/p' + 1/p = 1,\ q' = p'(p_0 - 2)/(p_0 - p'),\ 1/q + 1/q' = 1. \tag{3.47}$$

From Theorem 4.1 and estimates (3.34), (3.45) we get:

$$\begin{aligned} &\int_0^{|t|} \|\Omega(t,s)\|_{L^2_\sigma\mapsto L^p}\left(\|A_1(\psi_E(s), v_1(s), v_2(s))\|_{L^2_\sigma} + \|A_4(\psi_E(s), v_1(s), v_2(s))\|_{L^2_\sigma}\right)ds \\ &\leq (\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1} + \|v_1\|_Y^{1+\alpha_2} + \|v_2\|_Y^{1+\alpha_2})\|v_1 - v_2\|_Y \\ &\quad\times\int_0^t \frac{C_p}{|t-s|^{1-2/p}}\left[\frac{C_\sigma\log^{a_1}(2+|s|)}{(1+|s|)^{b_1}} + \frac{C_\sigma\log^{a_4}(2+|s|)}{(1+|s|)^{b_4}}\right]ds \end{aligned}$$

while from Theorem 4.2 and estimates (3.38), (3.40) we get:

$$\int_0^{|t|} \|\Omega(t,s)\|_{L^{q'}\cap L^{p'}\mapsto L^p} \|A_2(\psi_E(s), v_1(s), v_2(s))\|_{L^{q'}\cap L^{p'}} ds \le (\|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1})\|v_1 - v_2\|_Y$$

$$\times \int_0^t \frac{C_{p_0,p} \log^{\frac{1-2/p}{1-2/p_0}}(2+|t-s|)}{|t-s|^{1-2/p}} \max\left\{ \frac{C_{q'} \log^{a_2(q')}(2+|s|)}{(1+|s|)^{b_2(q')}}, \frac{C_{p'} \log^{a_2(p')}(2+|s|)}{(1+|s|)^{b_2(p')}} \right\} ds$$

and

$$\int_0^{|t|} \|\Omega(t,s)\|_{L^{q'}\cap L^{p'}\mapsto L^p} \|A_3(\psi_E(s), v_1(s), v_2(s))\|_{L^{q'}\cap L^{p'}} ds \le (\|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1})\|v_1 - v_2\|_Y$$

$$\times \int_0^t \frac{C_{p_0,p} \log^{\frac{1-2/p}{1-2/p_0}}(2+|t-s|)}{|t-s|^{1-2/p}} \max\left\{ \frac{C_{q'} \log^{a_3(q')}(2+|s|)}{(1+|s|)^{b_3(q')}}, \frac{C_{p'} \log^{a_3(p')}(2+|s|)}{(1+|s|)^{b_3(p')}} \right\} ds.$$

In Case I, i.e. $\alpha_1 \ge 1$, or $1/2 < \alpha_1 < 1$ and $p_1 \ge p_2$, since $\alpha_2 \ge \alpha_1$ and $p_2 \ge 4 + 2\alpha_2 > 4$, we have from (3.37), (3.39), (3.41) and (3.46) for $r' \in \{q', p', 2\}$ and $1/r + 1/r' = 1$:

$$b_1 = 2 - \frac{4}{p_2} > 1,\ b_2(r') = \alpha_1 + \frac{2}{r} > 1,\ b_3(r') = \alpha_2 + \frac{2}{q} > 1,\ b_4 = \min\{b_1, b_2(2), b_3(2)\} > 1.$$

We now use the following known convolution estimate:

$$\int_0^{|t|} \frac{\log^a(2+|t-s|)}{|t-s|^b} \frac{\log^c(2+|s|)}{(1+|s|)^d} ds \le C(a,b,c,d) \frac{\log^a(2+|t|)}{(1+|t|)^b}, \qquad \text{for } d > 1,\ b < 1, \tag{3.48}$$

to bound the integral terms above and obtain for all $2 \le p \le p_2$:

$$\|N(v_1)(t) - N(v_2)(t)\|_{L^p} \le C_p \frac{\log^{\frac{1-2/p}{1-2/p_0}}(2+|t|)}{(1+|t|)^{1-2/p}}$$

$$\times(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1} + \|v_1\|_Y^{1+\alpha_2} + \|v_2\|_Y^{1+\alpha_2})\|v_1 - v_2\|_Y \tag{3.49}$$

(3.50)

which, upon moving the time dependent terms to the left hand side and taking supremum over $t \in \mathbb{R}$ when $p \in \{2, p_2\}$, leads to (3.31) for $\tilde{C} = \max\{C_2, C_{p_2}\}$.

In Case II, i.e. $1/2 < \alpha_1 < 1$ and $p_1 < p_2$, we have from (3.37) $b_1 = 2(\alpha_1 - \frac{2}{p_0}) > 1$ because $p_0 > 2/(\alpha_1 - 1/2)$, see (3.23). From (3.39), under the restriction $2 \le p \le p_1$, with $p',\ q',\ q$ defined by (3.47), we have either:

$$b_2(p') > b_2(q') = \alpha_1 + 2/q > 1,$$

or

$$b_2(p') = b_2(q') = (2+\alpha_1)(\alpha_1 - 2/p_0) > (2+\alpha_1)/2 > 1.$$

Since $\alpha_2 \ge \alpha_1$ implies $b_3(\cdot) \ge b_2(\cdot)$ we deduce that, under the restriction $2 \le p \le p_1$, we also have

$$b_3(p') \ge b_3(q') \ge b_2(q') > 1,$$

and

$$b_4 = \min\{b_1, b_2(2), b_3(2)\} > 1.$$

We can again apply (3.48) to the above integral terms and get for $2 \le p \le p_1$ the estimate (3.49). For $p > p_1$ one can show that $(2+\alpha_1)q' < p_1$ hence $b_2(q') = \alpha_1 + 2/q$, and, in the particular case of $p = p_2$, we get

$$b_2(q_2') = \alpha_1 + 2/q_2 < 1,$$

where $q_2',\ q_2$ are given by (3.47). We now have from convolution estimates:

$$\int_0^{|t|} \frac{\log^{\frac{1-2/p_2}{1-2/p_0}}(2+|t-s|)}{|t-s|^{1-2/p_2}} \frac{\log^{a_2(q_2')}(2+|s|)}{(1+|s|)^{b_2(q_2')}} ds \le C(p_2) \frac{\log^{\frac{1-2/p_2}{1-2/p_0}+a_2(q_2')}(2+|t|)}{(1+|t|)^{\alpha_1+2/q_2-2/p_2}} \le \tilde{C}(p_2) \frac{\log^{\frac{\alpha_1-2/p_0}{1-2/p_0}}(2+|t|)}{(1+|t|)^{\alpha_1-2/p_0}},$$

where we used (3.47) and $p_2 \le p_0$ to obtain:

$$\frac{2}{p_2} - \frac{2}{q_2} = \frac{2}{p_0}\left(\frac{1-2/p_2}{1-2/p_0}\right) \le \frac{2}{p_0}.$$

Since $b_2(p_2') > b_2(q_2')$ and $b_3(p_2') \ge b_3(q_2') \ge b_2(q_2')$ we deduce

$$\begin{aligned}\|N(v_1)(t) - N(v_2)(t)\|_{L^{p_2}} &\le \tilde{C}_{p_2} \frac{\log^{\frac{\alpha_1 - 2/p_0}{1-2/p_0}}(2+|t|)}{(1+|t|)^{\alpha_1 - 2/p_0}} \\ &\times(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{1+\alpha_1} + \|v_2\|_Y^{1+\alpha_1} + \|v_1\|_Y^{1+\alpha_2} + \|v_2\|_Y^{1+\alpha_2})\|v_1 - v_2\|_Y\end{aligned} \tag{3.51}$$

which, combined with (3.49) for $p \in \{2, p_1\}$, after moving the time dependent terms on the left hand side and taking supremum over $t \in \mathbb{R}$, gives (3.31) in the Case II with $\tilde{C} = \max\{C_2, C_{p_1}, \tilde{C}_{p_2}\}$.

This finishes the proof of Lemma 3.2 and of Theorem 3.1. □

4 Linear Estimates

Consider the linear Schrödinger equation with a potential in two space dimensions:

$$\begin{cases} i\frac{\partial u}{\partial t} = (-\Delta + V(x))u \\ u(0) = u_0. \end{cases}$$

It is known that if V satisfies hypothesis (H1)(i) and (ii) then the radiative part of the solution, i.e. its projection onto the continuous spectrum of $H = -\Delta + V$, satisfies the estimates:

$$\|e^{-iHt}P_c u_0\|_{L^2_{-\sigma}} \le C_M \frac{1}{(1+|t|)\log^2(2+|t|)}\|u_0\|_{L^2_\sigma}, \qquad t \in \mathbb{R}, \tag{4.1}$$

for any $\sigma > 1$ and some constant $C_M > 0$ depending only on σ see [17, Theorem 7.6 and Example 7.8], and

$$\|e^{-iHt}P_c u_0\|_{L^p} \le \frac{C_p}{|t|^{1-2/p}}\|u_0\|_{L^{p'}} \tag{4.2}$$

for some constant $C_p > 0$ depending only on $p \ge 2$ and p' given by $p'^{-1} + p^{-1} = 1$. The case $p = \infty$ in (4.2) is proven in [23]. The conservation of the L^2 norm, see [5, Corollary 4.3.3], gives the $p = 2$ case:

$$\|e^{-iHt}P_c u_0\|_{L^2} = \|u_0\|_{L^2}.$$

The general result (4.2) follows from Riesz-Thorin interpolation.

We would like to extend these estimates to the linearized dynamics around the center manifold. In other words we consider the linear equation (3.20), with initial data at time s :

$$\begin{aligned}\frac{\partial z}{\partial t} &= -i(-\Delta + V)z - iP_c Dg_{\psi_E(t)} R_{a(t)} z(t) \\ z(s) &= v \in \mathcal{H}_0\end{aligned}$$

Note that this is a nonautonomous problem as the bound state ψ_E around which we linearize may change with time.

By Duhamel's principle we have:

$$z(t) = e^{-iH(t-s)}P_c v - i\int_s^t e^{-iH(t-\tau)}P_c Dg_{\psi_E(\tau)} R_{a(\tau)} z(\tau)d\tau \tag{4.3}$$

As in (3.21) we denote

$$\Omega(t,s)v \stackrel{def}{=} z(t). \tag{4.4}$$

Relying on the fact that $\psi_E(t)$ is small and localized uniformly in $t \in \mathbb{R}$, we have shown in [13, Section 4] for the particular case of cubic nonlinearity, $g(s) = s^3,\ s \in \mathbb{R}$, that estimates of type (4.1)-(4.2) can be extended to the operator $\Omega(t,s)$. Due to (2.6) which implies for $\sigma \geq 0$ and $1 \leq p' \leq 2$:

$$\begin{aligned}
\|Dg_{\psi_E} R_a z\|_{L^2_\sigma} &\leq C\left(\|\psi_E\|^{1+\alpha_1}_{L^\infty_{2\sigma/(1+\alpha_1)}} + \|\psi_E\|^{1+\alpha_2}_{L^\infty_{2\sigma/(1+\alpha_2)}}\right) C_{-\sigma}\|z\|_{L^2_{-\sigma}} && (4.5)\\
\|Dg_{\psi_E} R_a z\|_{L^{p'}} &\leq C\left(\|\psi_E\|^{1+\alpha_1}_{L^{(1+\alpha_1)q}_{\sigma/(1+\alpha_1)}} + \|\psi_E\|^{1+\alpha_2}_{L^{(1+\alpha_2)q}_{\sigma/(1+\alpha_2)}}\right) C_{-\sigma}\|z\|_{L^2_{-\sigma}},\quad \frac{1}{p'} = \frac{1}{q} + \frac{1}{2} && (4.6)\\
\|Dg_{\psi_E} R_a z\|_{L^{p'}} &\leq C\left(\|\psi_E\|^{1+\alpha_1}_{L^{(1+\alpha_1)q}} + \|\psi_E\|^{1+\alpha_2}_{L^{(1+\alpha_2)q}}\right) C_r\|z\|_{L^r},\quad \frac{1}{p'} = \frac{1}{q} + \frac{1}{r} && (4.7)
\end{aligned}$$

see also Lemma 2.2, we can use, with obvious modifications, the arguments in [13, Section 4] to show that:

Theorem 4.1 *Fix $\sigma > 1$. There exists $\varepsilon_1 > 0$ such that if $\| < x >^{4\sigma/3} \psi_E(t)\|_{H^2} < \varepsilon_1$ for all $t \in \mathbb{R}$, then there exist constants $C,\ C_p > 0$ with the property that for any $t,\ s \in \mathbb{R}$ the following hold:*

$$\begin{aligned}
\|\Omega(t,s)\|_{L^2_\sigma \mapsto L^2_{-\sigma}} &\leq \frac{C}{(1+|t-s|)\log^2(2+|t-s|)},\\
\|\Omega(t,s)\|_{L^{p'} \mapsto L^2_{-\sigma}} &\leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}},\ \text{for any } 2 \leq p < \infty \text{ where } p'^{-1} + p^{-1} = 1,\\
\|\Omega(t,s)\|_{L^2_\sigma \mapsto L^p} &\leq \frac{C_p}{|t-s|^{1-\frac{2}{p}}},\ \text{for any } p \geq 2 && (4.8)
\end{aligned}$$

and, for:

$$T(t,s) = \Omega(t,s) - e^{-iH(t-s)}P_c, \tag{4.9}$$

Lemma 4.1 *Assume that $\| < x >^{4\sigma/3} \psi_E(t)\|_{H^2} < \varepsilon_1,\ t \in \mathbb{R}$, where ε_1 is the one used in Theorem 4.1. Then for each $1 < q' \leq 2$ and $2 < p < \infty$ there exist the constants $C_{q'},\ C_{p,q'} > 0$ such that for all $t,\ s \in \mathbb{R}$ we have:*

$$\begin{aligned}
\|T(t,s)\|_{L^1 \cap L^{q'} \mapsto L^2_{-\sigma}} &\leq \frac{C_{q'}}{1+|t-s|},\\
\|T(t,s)\|_{L^1 \cap L^{q'} \mapsto L^p} &\leq \frac{C_{p,q'}\log(2+|t-s|)}{(1+|t-s|)^{1-\frac{2}{p}}}.
\end{aligned}$$

Note that according to the proofs in [13, Section 4] $C_{q'} \to \infty$ as $q' \to 1$ and $C_{p,q'} \to \infty$ as $q' \to 1$ or $p \to \infty$. These could be prevented and an estimate of the type

$$\|T(t,s)\|_{L^1 \mapsto L^\infty} \leq \frac{C\log(2+|t-s|)}{1+|t-s|} \tag{4.10}$$

can be obtained by avoiding the singularity of $\|e^{-iHt}\|_{L^1 \mapsto L^\infty} \sim t^{-1}$ at $t = 0$ via a generalized Fourier multiplier technique developed in [12, Appendix and Section 4]. We choose not to use it here because it requires stronger restrictions on the potential $V(x)$ like its Fourier transform should be in L^1 while its gradient should be in L^p, for some $p \geq 2$, and convergent to zero as $|x| \to \infty$.

We now present an improved L^2 estimate for the family of operators $T(t,s)$:

Lemma 4.2 *Assume that $\| < x >^{4\sigma/3} \psi_E(t)\|_{H^2} < \varepsilon_1,\ t \in \mathbb{R}$, where ε_1 is the one used in Theorem 4.1. Then there exists the constants $C_2 > 0$ such that for all $t,\ s \in \mathbb{R}$ we have:*

$$\|T(t,s)\|_{L^2 \mapsto L^2} \leq C_2$$

Proof: We are going to use a Kato type smoothing estimate:

$$\| < x >^{-\sigma} e^{-iHt}P_c f(x)\|_{L^2_t(\mathbb{R},L^2_x)} \leq C_K\|f\|_{L^2}, \tag{4.11}$$

see for example [21]. We claim that the previous estimate still holds if we replace $e^{-iH(t-s)}P_c$ by $\Omega(t,s)$, namely, there exists a constant $\tilde{C}_K > 0$ such that for any $s \in \mathbb{R}$:

$$\| < x >^{-\sigma} \Omega(\cdot,s) f\|_{L^2_t(\mathbb{R},L^2_x)} \le \tilde{C}_K \|f\|_{L^2}. \tag{4.12}$$

Indeed, from (4.4) and (4.3), we have

$$< x >^{-\sigma} \Omega(t,s)v =< x >^{-\sigma} e^{-H(t-s)} P_c v + \int_s^t < x >^{-\sigma} e^{-iH(t-\tau)} P_c D g_{\psi_E(\tau)}[R_a(\tau)\Omega(\tau,s)v] d\tau$$

and using (4.5):

$$\begin{aligned}
\|\Omega(t,s)v\|_{L^2_{-\sigma}} &\le \|e^{-H(t-s)}P_c v\|_{L^2_{-\sigma}} + \int_s^t \|e^{-iH(t-\tau)}P_c\|_{L^2_\sigma \mapsto L^2_{-\sigma}} \|Dg_{\psi_E(\tau)}\Omega(\tau,s)v(s)\|_{L^2_\sigma} d\tau \\
&\le \|e^{-iH(t-s)}v\|_{L^2_{-\sigma}} + C\sup_{\tau\in\mathbb{R}} \left(\|\psi_E(\tau)\|^{1+\alpha_1}_{L^\infty_{2\sigma/(1+\alpha_1)}} + \|\psi_E(\tau)\|^{1+\alpha_2}_{L^\infty_{2\sigma/(1+\alpha_2)}} \right) \\
&\times \int_{\mathbb{R}} \frac{\|\Omega(\tau,s)v(s)\|_{L^2_{-\sigma}}}{(1+|t-\tau|)\log^2(2+|t-\tau|)} d\tau.
\end{aligned}$$

By Young inequality: $\|f * g\|_{L^2(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}$ and (4.11) we get

$$\|\Omega(\cdot,s)v\|_{L^2(\mathbb{R},L^2_{-\sigma})} \le C_K\|v\|_{L^2_x} + C\varepsilon_1\|\Omega(\cdot,s)v\|_{L^2(\mathbb{R},L^2_{-\sigma})}$$

which implies (4.12).

Finally we turn to the estimate in L^2_x for $T(t,s)$:

$$\begin{aligned}
&\|T(t,s)v\|^2_{L^2_x} = \\
&= \langle \int_s^t e^{-iH(t-\tau)}P_c D g_{\psi_E}[R_a\Omega(\tau,s)v]d\tau, \int_s^t e^{-iH(t-\tau')}P_c D g_{\psi_E}[R_a\Omega(\tau',s)v]d\tau' \rangle \\
&= \int_s^t \int_s^t d\tau d\tau' \langle Dg_{\psi_E}[R_a\Omega(\tau,s)v], e^{-iH(\tau-\tau')}P_c D g_{\psi_E}[R_a\Omega(\tau',s)v]\rangle \\
&\le C\sup_{\tau\in\mathbb{R}} \left(\|\psi_E(\tau)\|^{1+\alpha_1}_{L^\infty_{2\sigma/(1+\alpha_1)}} + \|\psi_E(\tau)\|^{1+\alpha_2}_{L^\infty_{2\sigma/(1+\alpha_2)}} \right)^2 \\
&\times \int_s^t \int_s^t d\tau d\tau' \underbrace{\|\Omega(\tau,s)v\|_{L^2_{-\sigma}}}_{\in L^2(\mathbb{R})} \underbrace{\|e^{-iH(\tau-\tau')}P_c\|_{L^2_{-\sigma}\mapsto L^2_{-\sigma}}}_{\in L^1(\mathbb{R})} \underbrace{\|\Omega(\tau',s)v\|_{L^2_{-\sigma}}}_{\in L^2(\mathbb{R})}.
\end{aligned}$$

Using (4.1) combined with Young then Hölder inequalities the integral above is bounded by

$$C_M\|\Omega(\cdot,s)v\|^2_{L^2(\mathbb{R},L^2_{-\sigma})} \le C_M\tilde{C}^2_K\|v\|_{L^2_x}.$$

where, for the last inequality we employed (4.12). Consequently, there exist a constant C_2 such that for any $t,\ s\in\mathbb{R}$:

$$\|T(t,s)v\|_{L^2_x} \le C_2\|v\|_{L^2_x}.$$

This finishes the proof of the Lemma. □

Fix now $2 < p_0 < \infty$ and let $p_0' = p_0/(p_0-1)$. By applying Riesz-Thorin interpolations to the operators $T(t,s)$ satisfying for all $t,\ s\in\mathbb{R}$:

$$\begin{aligned}
\|T(t,s)\|_{L^2\mapsto L^2} &\le C_2 \\
\|T(t,s)\|_{L^1\cap L^{p_0'}\mapsto L^{p_0}} &\le \frac{C_{p_0}\log(2+|t-s|)}{(1+|t-s|)^{1-\frac{2}{p_0}}}
\end{aligned}$$

we obtain that for any $2\le p\le p_0$ there exists a constant $C_{p_0,p}$ between C_2 and C_{p_0} such that:

$$\|T(t,s)\|_{L^{q'}\cap L^{p'}\mapsto L^p} \le \frac{C_{p_0,p}\log^{\frac{1-2/p}{1-2/p_0}}(2+|t-s|)}{(1+|t-s|)^{1-\frac{2}{p}}},\ \text{where } p' = \frac{p}{p-1},\ q' = p'\frac{p_0-2}{p_0-p'}.$$

Finally, using (4.9) and the estimates for the Schrödinger group (4.2) we get:

Theorem 4.2 *Fix $2 < p_0 < \infty$ and assume that $\| < x >^{4\sigma/3} \psi_E(t)\|_{H^2} < \varepsilon_1,\ t \in \mathbb{R}$ where ε_1 is the constant obtained in Theorem 4.1. Then there exists the constants $C_2,\ C_{p_0,p} > 0$ such that for all $2 \leq p \leq p_0$ and $t,\ s \in \mathbb{R}$ the following estimates hold:*

$$\begin{aligned} \|\Omega(t,s)\|_{L^2 \mapsto L^2} &\leq C_2; \\ \|\Omega(t,s)\|_{L^{q'} \cap L^{p'} \mapsto L^p} &\leq \frac{C_{p_0,p} \log(2 + |t-s|)^{\frac{1-2/p}{1-2/p_0}}}{|t-s|^{1-\frac{2}{p}}}, \text{ where } p' = \frac{p}{p-1},\ q' = p' \frac{p_0 - 2}{p_0 - p'}. \end{aligned}$$

Note that the estimates for the family of operators $\Omega(t,s)$ given by the above theorem are similar to the standard $L^{p'} \mapsto L^p$ estimates for Schrödinger operators (4.2) except for the logarithmic correction and a smaller domain of definition $L^{q'} \cap L^{p'} \subset L^{p'}$ where $q' < p'$ when $p' < 2$. If we would have proven (4.10) then we could use $p_0 = \infty$, hence $q' = p'$ in the above theorem and obtain:

$$\|\Omega(t,s)\|_{L^{p'} \mapsto L^p} \leq \frac{C_p \log(2 + |t-s|)^{1-2/p}}{|t-s|^{1-\frac{2}{p}}} \qquad \text{where } p' = \frac{p}{p-1}.$$

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