ON THE EVOLUTION OF A REFLECTION COEFFICIENT UNDER THE KDV FLOW

ALEXEI RYBKIN

ABSTRACT. We are concerend with the KdV equation on the full-line with real non-decaying initial profile. We find the time evolution of a (relative) reflection coefficient. An inverse spectral formalism is also considered for a certain mixed problems on the full-line.

1. INTRODUCTION

Consider the initial value problem (Cauchy problem) for the *Korteweg-de Vries (KdV) equation* on the domain $-\infty < x < \infty, t \geq 0$:

$$\begin{cases} \partial_t q - 6q\partial_x q + \partial_x^3 q = 0 \\ q(x,0) = q_0(x) \end{cases} \tag{1.1}$$

where q_0, called an *initial profile*, is a real-valued Schwarz function. It was discovered by Gardner-Greene-Kruskal-Miura in 1965 that equation (1.1) can be linearized using a procedure commonly referred to as the *Inverse Scattering Transform (IST)* or inverse scattering formalism. To agree upon our notation we briefly outline some of the principal ideas of IST (see, e.g. the 1991 book [1] by Ablowitz-Clarkson or the 2005 concise survey [2] by Aktosun).

The inverse scattering formalism goes as follows. Consider the one-dimensional *Schrodinger equation*

$$-\partial_x^2 u + q_0(x) u = k^2 u, \quad k \in \mathbb{R}, \tag{1.2}$$

where $q_0(x)$, called a *potential*, is the same Schwarz function as the initial profile in (1.1). The operator $H = -\partial_x^2 + q_0(x)$ defined on $L^2(\mathbb{R})$ has a simple finite negative spectrum $\{-\kappa_n^2\}_{n=1}^N$ and a twofold purely absolutely continuous (a.c.) spectrum filling $\mathbb{R}_+ := [0,\infty)$. Equation (1.2) has the so-called *Jost* (scattering) *solution*[1] $\psi_\pm(x,k)$ subject to

$$\begin{cases} \psi_\pm(x,k) = T(k) e^{\pm ikx} + o(1), x \to \pm\infty, \\ \psi_\pm(x,k) = e^{\pm ikx} + R_\mp(k) e^{\mp ikx} + o(1), x \to \mp\infty. \end{cases} \tag{1.3}$$

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[1]Throughout the paper, we have the following agreement on $\pm$ statements: $P_\pm \Rightarrow Q_\pm$ means two separete statements $P_+ \Rightarrow Q_+, P_- \Rightarrow Q_-$. Unless otherwise stated, we use $P_\pm$ as a single noun.

The coefficients $R_{\pm}(k)$ and $T(k)$ are called right (left) *reflection* and *transition* respectively. Upon solving problem $(1.2)-(1.3)$ (called the direct scattering problem) one finds the *scattering data*:

$$\mathcal{S}_{\pm} := \{R_{\pm}(k), k \in \mathbb{R}; \{-\kappa_n^2, c_n^{\pm}\}_{n=1}^{N}\}, \tag{1.4}$$

where $c_n^{\pm}$ are the so-called morning constants:

$$c_n^{\pm} := \left(\int_{\mathbb{R}} \left(T\left(i\kappa_n\right)^{-1} \psi_{\mp}\left(x, i\kappa_n\right) \right)^2 dx \right)^{-1}.$$

Consider now the one-parametric set

$$\mathcal{S}_{\pm}\left(t\right) := \{R_{\pm}(k,t), k \in \mathbb{R}; \{-\kappa_n^2, c_n^{\pm}\left(t\right)\}_{n=1}^{N}\}, \mathcal{S}_{\pm}\left(0\right) = \mathcal{S}_{\pm}, \tag{1.5}$$

where

$$R_{\pm}(k;t) := R_{\pm}(k)e^{\pm 8ik^3 t}, \kappa_n(t) = \kappa_n, c_n^{\pm}\left(t\right) := c_n^{\pm} e^{\pm 8ik^3 t}. \tag{1.6}$$

It is the crucial fact of the inverse scattering formalism that the set $\mathcal{S}_{\pm}\left(t\right)$ is the scattering data for

$$-\partial_x^2 u + q\left(x,t\right) u = k^2 u \tag{1.7}$$

where $q\left(x,t\right)$ is the solution to (1.1). Now one solves the inverse problem with the scattering data $\mathcal{S}\left(t\right)$ to recover the potential $q(x,t)$ by means of the Marchenko procedure[2]. More specifically, consider the function

$$F_{\pm}\left(x,t\right) := \sum_{n=1}^{N} c_n^{\pm}\left(t\right) e^{\mp \kappa_n x} + \frac{1}{2\pi} \int_{\mathbb{R}} e^{\pm ikx} R_{\pm}(k;t) dk,$$

which is made of scattering data $\mathcal{S}_{\pm}\left(t\right)$ and form the *Marchenko* (essentially linear of Volterra type) equation,

$$\begin{aligned} K_{\pm}\left(x,y,t\right) + F_{\pm}\left(x+y,t\right) \pm \int_x^{\pm\infty} F_{\pm}\left(s+y,t\right) K_{\pm}\left(x,s,t\right) ds &= 0, \\ y &\in \left(x, \pm\infty\right). \end{aligned} \tag{1.8}$$

Solve (1.8) for $K_{\pm}\left(x,y,t\right)$. The function

$$q(x,t) = \mp 2 \frac{d}{dx} K_{\pm}\left(x,x,t\right) \tag{1.9}$$

solves (1.1); $q(x,t)$ being independent of the choice of $\pm$.

The IST formalism can be extended to any short-range initial profile q_0:

$$\int_{\mathbb{R}} \left(1 + |x|\right) \left|q_0\left(x\right)\right| dx < \infty. \tag{1.10}$$

Similar approach has been discovered for many other physically important evolution equations (see, e.g. [1] for a fairly complete account of integrable equations). The historically second equation was nonlinear Schrodinger which inverse scattering approach is based upon a Dirac system. The IST method can also be described in terms of the Riemann-Hilbert problem (see, e.g. [1]).

Relaxation of condition (1.10) leads to serious complications. In-depth discussions of this interesting topic are beyond the scope of our short note and we restrict ourselves to mentioning just a few well-established trends. We mention first the periodic initial value problem (see, e.g. the 1999 survey [9] by Krichever-Noviko and

[2] This procedure is also referred to as the Gelfand-Levitan-Marchenko.

the 2003 book [4] by Gesztesy-Holden). The IST methods in the periodic context are quite different from the standard IST assuming (1.10) and based upon the Bloch-Floquet spectral theory of the Hill operator, Riemann surfaces, and theta-function. Another important direction is the IST for step-like initial profiles (see, e.g. the 1994 detailed survey [7] by Khruslov-Kotlyarov). The original Gardner-Greene-Kruskal-Miura inverse scattering formalism admits a fairly direct modification in this case but shows a striking new phenomenon related to a possible finite interval of simple a.c. spectrum of the underlying Schrodinger operator This simple a.c. spectrum causes an infinite train of non-interacting solitons (even in the absence of eigenvalues). And last but not least we mention a spectral class of slowly decaying solutions called positons[3] (we refer, e.g. to the 2002 paper [11] by Matveev and the detailed 2005 paper [8] by Kovalyov where a comprehensive review of positon solutions and further extensive literature is given). Positon solutions are related to so-called Wigner-von Neumann potentials q_0 in (1.1), i.e. $q_0(x) = O(1/|x|), x \to \pm\infty$, and oscillatory The main feature of the Schrodinger operator with Wigner-von Neumann potentials is embedded eigenvalues[4]. The problem with the IST for such solutions is that there always appear stable strong local singularities and the very definition of the associated Schrodinger operator becomes problematic (see also the interesting 1999 paper [10] by Kurasov-Packalen).

The present paper is concerned with the KdV equation with quite general initial profiles q_0. We do not challenge here the intriguing problem of understanding how far beyond (1.10) one can go. Instead we show that there exists a natural generalization of the reflection coefficient, a central ingredient of the IST, which is defined for an extremely broad class of initial profiles q_0 but still has a relatively simple time evolution. Our approach is based upon the study of the so-called Titchmarsh-Weyl m-function, a central object of spectral theory of one-dimensional Schrodinger operator. In the short-range case the Titchmarsh-Weyl m-function[5] $m_\pm$ can be introduced by

$$m_\pm\left(k^2\right) := \frac{\partial_x \psi_\pm(0,k)}{\psi_\pm(0,k)}, \quad \operatorname{Im} k^2 > 0, \tag{1.11}$$

where $\psi_\pm$ is the Jost solution satisfying (1.3). Note that the Titchmarsh-Weyl m-function does have much of physical sense and in the context of short-range scattering the Titchmarsh-Weyl m-function is not useful as it is easier to work directly with the scattering quantities $R_\pm$ and T. However, definition (1.11) of $m_\pm$ can be easily extended to virtually any real potential for which scattering theory (and hence $R_\pm$ and T) does not make much sense. To this end, one replaces the Jost solutions in (1.11) with the so-called Weyl solution $\Psi_\pm$ which we can think of as a generalization of the Jost solution $\psi_\pm$. With the Titchmarsh-Weyl m-function in hand one, following Gesztesy-Nowell-Pötz [5], can define scattering quantities $R_\pm$ and T in terms of $m_\pm$ for very large classes of potentials. Due in part to this fact the Titchmarsh-Weyl m-function has recently enjoyed a spike of renewed interest (see, e.g. the recent paper [6] by Gesztesy-Zinchenko which has a excellent review of the Titchmarsh-Weyl m-function and extensive literature). However in the soliton community the advantage of the m-function approach has not been fully utilized.

[3] or harmonic breathers

[4] I.e. positive eigenvalues embedded into a.c. spetrum filling $(0,\infty)$.

[5] For the full-line case two m-functions are typically introduced m_- and m_+ associated with $-\infty$ and $+\infty$ respectively.

In Section 2 we therefore introduce this concept and describe its main properties pertinent to our consideration.

In Section 3 we study the time evolution of the so-called Weyl matrix, a 2×2 matrix made in a certain way of $m_\pm$, under the KdV flow. This consideration offers an inverse spectral formalism to solve a mixed type problem for the KdV on the full-line. While much more involved than the standard IST, each step in our formalism remains linear. This section was motivated by the 2001 book [3] by Freiling-Yurko where a similar algorithm was put forward for a mixed problem for the KdV equation on the domain $x \geq 0, t \geq 0$ under stronger (short-range) assumptions on q.

Section 4 is central and devoted to a derivation of the law of evolution of a suitably defined reflection coefficient.

The last Section 5 discusses how the Weyl matrix and reflection coefficient depend on a particular choice of the partition $(-\infty, \infty) = (-\infty, x_0] \cup [x_0, \infty)$.

Notation. A' stands for the transpose of a matrix A, $\mathbb{C}$ is the complex plane, $\mathbb{C}_\pm := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$, $\mathbb{R}^a_\pm := [a, \pm\infty)$, $L^p(S, d\mu), 1 \leq p \leq \infty$, is the usual Lebesgue space of measurable on a set $S \subseteq \mathbb{R}$ functions f:

$$\int_S |f|^p \, d\mu < \infty \text{ for } 1 \leq p < \infty, \operatorname{ess\,sup}_S |f| < \infty \text{ for } p = \infty.$$

$\sigma(H)$ stands for the spectrum of an operator H and $\sigma_{\text{ac}}(H)$ is its absolutely continuous (a.c.) component.

2. Weyl's Solution, Titchmarsh-Weyl's m-function and Weyl's Matrix

In this section we review Weyl-Titchmarsh theory for Schrodinger operators. The literature on the subject is very extensive and we refer the reader to [6] where this material is summarized in a nice concise form connivent for our purposes.

Consider the differential expression

$$L = -\partial_x^2 + q(x), x \in \mathbb{R},$$

with real-valued locally integrable potentials q, i.e.

$$q = \overline{q} \in L^1_{\text{loc}}(\mathbb{R}). \tag{2.1}$$

Pick up on the real line $\mathbb{R}$ an arbitrary point x_0, called a *reference point.* x_0 is commonly taken 0 but it is important to us to trace dependence of our formulas on x_0. Let $\theta(x, x_0, z)$ and $\phi(x, x_0, z)$ be the usual *fundamental solutions* to the Schrodinger equation

$$-\partial_x^2 + q(x) u = zu, x \in \mathbb{R}, z \in \mathbb{C}, \tag{2.2}$$

satisfying the following boundary conditions at the point $x = x_0$

$$\theta(x_0, x_0, z) = 1, \partial_x \theta(x_0, x_0, z) = 0 \tag{2.3}$$

$$\phi(x_0, x_0, z) = 0, \partial_x \phi(x_0, x_0, z) = 1. \tag{2.4}$$

Under conditions (2.1) $\theta(x, x_0, z)$ and $\phi(x, x_0, z)$ uniquely exist for any $x \in \mathbb{R}$, are entire functions of z and real for $z \in \mathbb{R}$. It is a cornerstone fact of Titchamarsh-Weyl theory that (2.2) has a solution [6] $\Psi_\pm$, called *Weyl*, subject to

$$\begin{aligned} \Psi_\pm (x_0, x_0, z) &= 1, & (2.5)\\ \Psi_\pm (x, x_0, z) &\in L^2\left(\mathbb{R}^{x_0}_\pm\right) \text{ for any } z \in \mathbb{C}_+ & (2.6)\end{aligned}$$

(here $\mathbb{R}^{x_0}_\pm$ abbreviates $[x_0, \pm\infty)$). $\Psi_\pm$ need not be unique. In the sequel we assume that q is in the so-called *limit point case* at $\pm\infty$. I.e., there is only one Weyl solution which we call the Weyl solution. It should be emphasized that the limit point condition is very mild and most of realistic potentials[7] q are limit point case. Necessary and sufficient conditions on q to be the limit point case at $\pm\infty$ are not known but loosely speaking the limit point case does not like q which go to $-\infty$ as $x \to \pm\infty$ too fast.

Due to $(2.3), (2.4)$

$$\Psi_\pm (x, x_0, z) = \theta (x, x_0, z) + m_\pm \phi (x, x_0, z) \tag{2.7}$$

with a unique coefficient $m_\pm = m_\pm (x_0, z)$ commonly referred to as the Titchmarsh-Weyl m-function[8].

The following properties of the Titchmarsh-Weyl m-function will be particularly important in our consideration:

$$m_\pm (x_0, z) \text{ is analytic with respect to to } z \text{ on } \mathbb{C}\setminus\mathbb{R} \text{ and } m_\pm : \mathbb{C}_+ \to \mathbb{C}_+ \tag{2.8}$$

$$\overline{m_\pm (x_0, z)} = m_\pm (x_0, \overline{z}) . \tag{2.9}$$

Functions satisfying (2.8) are called Herglotz and by the Herglotz representation theorem

$$\pm m_\pm (x_0, z) = a_\pm (x_0) + b_\pm (x_0) z + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{d\mu_\pm (x_0, t)}{1 + t^2} \tag{2.10}$$

where $a_\pm \in \mathbb{R}$, $b_\pm \geq 0$, and $\mu_\pm$ is a non-negative measure on $\mathbb{R}$:

$$\int_{\mathbb{R}} \frac{d\mu_\pm (x_0, t)}{1 + t^2} < \infty.$$

The Titchmarsh-Weyl m-function plays a fundamental role in the spectral theory of the half-line Schrodinger operator $H^{x_0}_\pm$ with the Dirichlet boundary condition at x_0:

$$\begin{aligned} H^{x_0}_\pm &= -\partial_x^2 + q(x) \text{ on } L^2\left(\mathbb{R}^{x_0}_\pm\right) & (2.11)\\ \operatorname{Dom}\left(H^{x_0}_\pm\right) &= \left\{u, H^{x_0}_\pm u \in L^2\left(\mathbb{R}^{x_0}_\pm\right) : u(\pm x_0) = 0\right\}. \end{aligned}$$

Due to the limit point case condition, $H^{x_0}_\pm$ is self-adjoint and its spectral measure coincides with $\mu_\pm$ appearing in (2.10). By the Stieltjes inversion formula

$$\mu_\pm (x_0, (\lambda_1, \lambda_2)) + \frac{1}{2}\left(\mu_\pm (x_0, \{\lambda_1\}) + \mu_\pm (x_0, \{\lambda_2\})\right) = \pm\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \operatorname{Im} m_\pm (x_0, \lambda)\, d\lambda \tag{2.12}$$

and for its absolutely continuous part

$$d\mu_{\pm,\mathrm{ac}} (x_0, \lambda) = \pm\frac{1}{\pi} \operatorname{Im} m_\pm (x_0, \lambda)\, d\lambda.$$

[6] Recall our agreement on $\pm$ statements.

[7] Particularly the ones associated with the KdV equation.

[8] Also called the Titchmarsh-Weyl m-coefficient, Weyl function, m-function or any combination of these names.

Another reason why the Titchmarsh-Weyl m-function is fundamental in the spectral theory of the half-line Schrodinger operator, is the Borg-Marchenko theorem saying that $m_{\pm}(x_0, z)$ uniquely determines $q(x)$ on $x \gtrless x_0$ The actually procedure is rather involved and based on the Gelfand-Levitan inverse spectral algorithm, somewhat similar to the Marchenko inverse scattering procedure (1.8) – (1.9).

Note that it follows from (2.7) that

$$m_{\pm}(x_0, z) = \frac{\partial_x \Psi_{\pm}(x_0, x_0, z)}{\Psi_{\pm}(x_0, x_0, z)} = \partial_x \Psi_{\pm}(x_0, x_0, z). \tag{2.13}$$

Similarly to (2.13), we introduce now

$$\begin{aligned} m_{\pm}(x, z) &= \frac{\partial_x \Psi_{\pm}(x, x_0, z)}{\Psi_{\pm}(x, x_0, z)} \\ &= \frac{\partial_x \theta(x, x_0, z) + m_{\pm}(x_0, z)\, \partial_x \phi(x, x_0, z)}{\theta(x, x_0, z) + m_{\pm}(x_0, z)\, \phi(x, x_0, z)}. \end{aligned} \tag{2.14}$$

So defined $m_{\pm}(x, z)$ is the Titchmarsh-Weyl m-function associated with $H_{\pm}^{x}$. From (2.14) one has

$$\partial_x m_{\pm}(x, z) = -m_{\pm}(x, z)^2 + q(x) - z \tag{2.15}$$

which is a Riccati-type equation for $m_{\pm}$.

Note if q is short-range then $\psi_{\pm}(x_0, \sqrt{z})$ is in $L^2\left(\mathbb{R}_{\pm}^{x_0}\right)$ for any $\operatorname{Im} z > 0$ and for $m_{\pm}$ one has

$$m_{\pm}(x_0, z) = \frac{\partial_x \psi_{\pm}(x_0, \sqrt{z})}{\psi_{\pm}(x_0, \sqrt{z})}.$$

where $\psi_{\pm}$ satisfies (1.3).

We now turn to the full line Schrodinger operator

$$\begin{aligned} H &= -\partial_x^2 + q(x) \text{ on } L^2(\mathbb{R}) \\ \operatorname{Dom}(H) &= \left\{u, Hu \in L^2(\mathbb{R})\right\}. \end{aligned}$$

Picking a reference point x_0, one introduces the matrix

$$M(x_0, z) = \begin{pmatrix} m_1 & m_{12} \\ m_{21} & m_2 \end{pmatrix}(x_0, z) \tag{2.16}$$

where

$$m_1 := \frac{m_- m_+}{m_- - m_+}, \quad m_2 := \frac{1}{m_- - m_+}$$

$$m_{12} = m_{21} := \frac{1}{2}\frac{m_- + m_+}{m_- - m_+}$$

and $m_{\pm}(x_0, z)$ is, as above, the Titchmarsh-Weyl m-function associated with $H_{\pm}^{x_0}$.

The matrix-valued function $M(x_0, z)$ is called the *Weyl matrix* or the spectral matrix associated with the full-line Schrodinger operator H. It has a Herglotz property (2.8) and hence admits a similar to (2.10) representation with a matrix-valued measure $\mu(x_0, t)$. It should be noticed that $M(x_0, z)$ depends on x_0 but the essential support of $\mu(x_0, t)$ does not and the spectrum of H can be expressed in terms of the elements of the matrix M. In particular,

$$\sigma_{\mathrm{ac}}(H) = \sigma_{\mathrm{ac}}\left(H_{+}^{x_0}\right) \cup \sigma_{\mathrm{ac}}\left(H_{-}^{x_0}\right) \tag{2.17}$$

3. Time Evolution of the Weyl Matrix under the KdV Flow

Throughout this section

$$L=-\partial_x^2+q(x,t),\ A=-4\partial_x^3+6q(x,t)\,\partial_x+3\partial_x q(x,t) \tag{3.1}$$

is the *Lax pair* associated with the KdV equation

$$\partial_t q-6q\partial_x q+\partial_x^3 q=0. \tag{3.2}$$

Theorem 1. *Assume that the Cauchy problem*

$$\begin{cases} \partial_t q-6q\partial_x q+\partial_x^3 q=0 \\ q(x,0)=q_0(x) \end{cases} \tag{3.3}$$

has a solution $q(x,t)$ *such that*

$$q,\partial_x q,\partial_t q\in L^\infty(\mathbb{R}\times[0,T]) \tag{3.4}$$

with some $T\in(0,\infty]$. *Then for any reference point* $x_0\in\mathbb{R}$ *the Weyl matrix* $M(x_0,t,z)$ *associated with the operator* $-\partial_x^2+q(x,t)$ *satisfies the linear evolution equation*

$$\begin{cases} \partial_t M=PM+MP' \\ M|_{t=0}=M_0 \end{cases} \tag{3.5}$$

for any $\operatorname{Im} z\geq 0$ *for which* $M(x_0,t,z)$ *exists*[9]. *In* (3.5)

$$\begin{aligned} P(x_0,t,z) &:= \begin{pmatrix} a & c \\ b & -a \end{pmatrix}(x_0,t,z) \\ a &:= \partial_x q(x_0,t) \\ b &:= 2\left(q(x_0,t)+2z\right) \\ c &:= 2\left(q(x_0,t)-z\right)\left(q(x_0,t)+2z\right)-\partial_x^2 q(x_0,t) \end{aligned} \tag{3.6}$$

and $M_0=M_0(x_0,z)$ *is the Weyl matrix associated with* $-\partial_x^2+q_0(x)$.

Proof. Assume first that $z\in\mathbb{C}_+$. Let, as in Section 2, $\Psi_\pm(x_0,t,z)$ be the Weyl solution associated with $H_\pm^{x_0}$. It is well-known that if q solves (3.2) and u solves $Lu=zu$ then $\partial_t u-Au$ also solves $Lu=zu$. So

$$\partial_t\Psi_\pm-A\Psi_\pm \tag{3.7}$$

is a solution. We now show that (3.7) is a Weyl solution. Note first that due to conditions (3.4)

$$q\Psi_\pm,\partial_x q\Psi_\pm,\partial_x q\Psi_\pm\in L^2\left(\mathbb{R}_\pm^{x_0},dx\right)\ \text{ if } z\in\mathbb{C}_+. \tag{3.8}$$

By a direct computation

$$\partial_t\Psi_\pm-A\Psi_\pm=\partial_t\Psi_\pm-(\partial_x q)\,\Psi_\pm-2(q+2z)\,\partial_x\Psi_\pm \tag{3.9}$$

and one needs to demonstrate that each term of the right hand side of (3.9) is in $L^2\left(\mathbb{R}_\pm^{x_0},dx\right)$ if $z\in\mathbb{C}_+$. From (3.8),

$$\partial_x^2\Psi_\pm=(q-z)\,\Psi_\pm\in L^2\left(\mathbb{R}_\pm^{x_0},dx\right). \tag{3.10}$$

The latter implies that $\partial_x\Psi_\pm$ is also $L^2\left(\mathbb{R}_\pm^{x_0},dx\right)$. Differentiating $(q-z)\,\Psi_\pm$ with respect to t yields

$$(H-z)\,\partial_t\Psi_\pm=-\partial_t q\Psi_\pm. \tag{3.11}$$

[9] I.e. all $\operatorname{Im} z>0$ and almost all real z.

Since $\Psi_\pm|_{x=x_0} = 1$ we have

$$\partial_t \Psi_\pm (x_0, x_0, t, z) = 0 \tag{3.12}$$

and (3.11) implies

$$\partial_t \Psi_\pm = -\left(H_\pm^{x_0} - z\right)^{-1} (\partial_t q \Psi_\pm) \in L^2 \left(\mathbb{R}_\pm^{x_0}, dx\right)$$

since $z \notin \operatorname{spec}\left(H_\pm^{x_0}\right)$.

Each term on the right hand side of (3.9) is in $L^2\left(\mathbb{R}_\pm^{x_0}, dx\right)$ and hence

$$\partial_t \Psi_\pm - A\Psi_\pm = \alpha \Psi_\pm \tag{3.13}$$

with some $\alpha = \alpha(t, z)$. Following [3] we derive the time evolution of the Titchmarsh-Weyl m-function under the KdV flow. Setting in (3.13) $x = x_0$ we get

$$\begin{aligned} \alpha &= -A\Psi_\pm|_{x=x_0} \\ &= (\partial_x q - 2(q + 2z)\partial_x \Psi_\pm)(x_0, t) \\ &= a - bm_\pm \end{aligned}$$

where a and b are defined in (3.6) and $m_\pm$ is defined by (2.13). Equation (3.13) now reads

$$\partial_t \Psi_\pm - A\Psi_\pm = (a - bm_\pm)\Psi_\pm. \tag{3.14}$$

Differentiate (3.14) with respect to x. Due to (3.1) and (3.10), for the left hand side we get

$$\begin{aligned} &\partial_t(\partial_x \Psi_\pm) + \partial_x^2 q \Psi_\pm + \partial_x q \partial_x \Psi_\pm - 2(q+2z)(q-z)\Psi_\pm \\ = \; &-\partial_t(\partial_x \Psi_\pm) - \partial_x q \partial_x \Psi_\pm + \left(\partial_x^2 q - 2(q+2z)(q-z)\right)\Psi_\pm. \end{aligned}$$

For the right hand side we have $(a - bm_\pm)\partial_x \Psi_\pm$ and hence

$$\begin{aligned} &-\partial_t(\partial_x \Psi_\pm) - \partial_x q \partial_x \Psi_\pm + \left(\partial_x^2 q - 2(q+2z)(q-z)\right)\Psi_\pm \\ = \; &(a - bm_\pm)\partial_x \Psi_\pm. \end{aligned} \tag{3.15}$$

Setting in (3.15) $x = x_0$ yields

$$\begin{aligned} &\partial_t m_\pm - \partial_x q(0,t) m_\pm + \partial_x^2 q(0,t) \\ &-2(q(0,t) + 2z)(q(0.t) - z) \\ = \; &\partial_x q(0,t) m_\pm - 2(q(0,t) + 2z) m_\pm^2. \end{aligned} \tag{3.16}$$

In the short-hand notation (3.6) equation (3.16) reads

$$\begin{cases} \partial_t m_\pm = 2am_\pm - bm_\pm^2 + c \\ m_\pm|_{t=0} = m_\pm^0 \end{cases} \tag{3.17}$$

where $m_\pm^0$ is the Titchmarsh-Weyl m-function associated with $H_\pm^{x_0}$. Equation (3.17) is Riccati and can be easily linearized:

$$\begin{cases} \partial_t \begin{pmatrix} \mu_\pm \\ \nu_\pm \end{pmatrix} = \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \begin{pmatrix} \mu_\pm \\ \nu_\pm \end{pmatrix} = P \begin{pmatrix} \mu_\pm \\ \nu_\pm \end{pmatrix} \\ \left.\begin{pmatrix} \mu_\pm \\ \nu_\pm \end{pmatrix}\right|_{t=0} = \begin{pmatrix} m_\pm^0 \\ 1 \end{pmatrix}. \end{cases} \tag{3.18}$$

For the solution to (3.17) one has

$$m_\pm = \frac{\mu_\pm}{\nu_\pm}. \tag{3.19}$$

Equation (3.18) was derived for $z \in \mathbb{C}_+$. One can easily extend it to all real λ for which $m_\pm^0(\lambda + i0)$ exists. Indeed, due to condition (3.4) matrix P is a bounded function of $t \in [0,T]$ and an entire function (even linear) of z and hence (3.18) has a unique solution $(\mu_\pm, \nu_\pm)$ continuously depending on initial condition. One therefore can then pass in (3.18) to the non-tangential limit $z \to \lambda + i0$ for every real λ for which $m_\pm^0(\lambda + i0)$ exists. Since the solution $m_\pm$ to (3.17) is related to $(\mu_\pm, \nu_\pm)$ by (3.19), one can also pass in (3.17) to the non-tangential limit $z \to \lambda + i0$ for every real λ for which $m_\pm^0(\lambda + i0)$ exists.

With (3.17) in hand we are able to derive (3.5). Differentiate the elements of matrix M:

$$\begin{aligned} \partial_t m_1 &= \frac{\partial_t m_+ m_-^2 - \partial_t m_- m_+^2}{(m_- - m_+)^2} \\ &= 2a \frac{m_+ m_-}{m_- - m_+} + c \frac{m_- + m_+}{m_- - m_+}. \end{aligned}$$

I.e.,

$$\partial_t m_1 = 2a m_1 + 2c m_{12}. \tag{3.20}$$

Similarly

$$\partial_t m_{12} = b m_1 + c m_2 \tag{3.21}$$

$$\partial_t m_2 = -2a m_2 + 2b m_{12}. \tag{3.22}$$

Equations (3.20) – (3.22) can then be combined as

$$\partial_t M = \begin{pmatrix} a & c \\ b & -a \end{pmatrix} M + M \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

and the evolution equation (3.5) follows. □

Remark 1. *If q_0 in* (3.3) *is from the Schwarz class then conditions* (3.4) *are automatically satisfied with $T = \infty$.*

Remark 2. *Conditions* (3.4) *are far from being optimal. Indeed,* (3.4) *were used only for proving that the right hand side of* (3.9) *is in $L^2(\mathbb{R}_\pm^{x_0}, dx)$ for any $z \in \mathbb{C}_+$. It is clear that the weaker conditions $\partial_t q(x,t), \partial_x q(x,t) \in L^2_{\text{loc}}(\mathbb{R})$ for any $t \in [0,T]$ and $\partial_t q(x,t), \partial_x q(x,t)$ are bounded at infinity would also do it.*

Theorem 1 offers an inverse spectral procedure for solving a mixed problem for the KdV equation on the domain $\mathbb{R} \times \mathbb{R}_+$.

Algorithm 1. *Assume that the mixed problem (x_0 is fixed)*

$$\begin{cases} \partial_t q - 6q\partial_x q + \partial_x^3 q = 0 \\ q(x,0) = q_0(x) \\ q(x_0,t) = q_1(t) \\ \partial_x q(x_0,t) = q_2(t) \\ \partial_x^2 q(x_0,t) = q_3(t) \end{cases} \tag{3.23}$$

has a unique solution $q(x,t)$ subject to

$$q, \partial_x q, \partial_t q \in L^\infty(\mathbb{R} \times [0,T]) \text{ for some } T \in (0,\infty] \tag{3.24}$$

Then $q(x,t)$ on $\mathbb{R} \times [0,T]$ can be obtained by the following procedure:

1. Find the Titchmarsh-Weyl m-functions $m_\pm^0(z) := m_\pm(x_0,z), z \in \mathbb{C}_+$, associated with $-\partial_x^2 + q_0(x)$

2. Compose the Weyl matrix

$$M_0(z) = \begin{pmatrix} m_1^0 & m_{12}^0 \\ m_{12}^0 & m_2^0 \end{pmatrix}(z)$$

where

$$\begin{aligned} m_1^0 &= \frac{m_-^0 m_+^0}{m_-^0 - m_+^0}, m_2^0 = \frac{1}{m_-^0 - m_+^0} \\ m_{12}^0 &= \frac{1}{2}\frac{m_-^0 + m_+^0}{m_-^0 - m_+^0} \end{aligned}$$

and the matrix

$$P(t,z) = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}(t,z)$$

where

$$\begin{aligned} a &= 2q_2(t), b = 2(q_1(t) + 2z), \\ c &= 2(q_1(t) - z)(q_1(t) + 2z) - q_3(t) \end{aligned}$$

3. Solve the linear equation (3.6) *or, equivalently, the linear initial value problem*

$$\partial_t \begin{pmatrix} m_1 \\ m_{12} \\ m_2 \end{pmatrix} = \begin{pmatrix} 2a & 2c & 0 \\ b & 0 & c \\ 0 & 2b & -2a \end{pmatrix} \begin{pmatrix} m_1 \\ m_{12} \\ m_2 \end{pmatrix}, \quad \left. \begin{pmatrix} m_1 \\ m_{12} \\ m_2 \end{pmatrix} \right|_{t=0} = \begin{pmatrix} m_1^0 \\ m_{12}^0 \\ m_2^0 \end{pmatrix}$$

4. Find the time evolved $m_\pm(x_0, t, z)$ *by*

$$m_\pm = m_2^{-1}(m_{12} \mp 1/2)$$

5. Find $q(x,t)$ *for* $x \leq x_0$ *by* $m_-(x_0,t,z)$ *and* $q(x,t)$ *for* $x \geq x_0$ *by* $m_+(x_0,t,z)$ *using the standard Gelfand-Levitan inverse spectral procedure. Namely, by* (2.12) *find the spectral measures* $\mu_\pm$ *of* $H_\pm^{x_0}$ *and then construct and solve the Gelfand-Levitan equation (somewhat similar to* (1.8) *integral equation).*

It should be emphasized that the inverse spectral formalism put forward in Algorithm 1 does not assume any decay of the initial profile q_0 at $\pm\infty$. However it is unclear how to find T in (3.24) for which the problem (3.23)-(3.24) is well-posed. This question does not appear to have a satisfactory answer (weaker conditions given in Remark 2 do not help much either). The main issue here is that one can construct smooth decaying initial profiles $q_0(x)$ but not satisfying (1.10) such that $q(x,t)$ has strong local singularities. A relevant open problem is stated by Matveev in [11]: does there exist a bounded $q_0(x)$ subject to the conditions[10]

$$q_0(x) \sim \frac{\alpha_\pm \sin(\beta_\pm x + \delta_\pm)}{x}, x \to \pm\infty,$$

such that $q(x,t)$ remains bounded? In other words, does there exist a bounded positon?

In the context of the KdV on the quarter-plane $\mathbb{R}_+ \times \mathbb{R}_+$ a similar to Algorithm 1 procedure was treated in great detail by Freiling-Yurko [3] under additional short-range assumptions

$$q, \partial_t q, \partial_x q, \partial_x^2 q, \partial_x^3 q \in L^1(\mathbb{R}_+, (1+x)\,dx)) \text{ for all } t \in [0,T]. \tag{3.25}$$

[10]commonly called a Wigner-von Neumann potential

Note that, being both inverse spectral, Algorithm 4.2.1. from [3] and our Algorithm 1 target different objectives. Algorithm 4.2.1 is concerned with the KdV on a half-line but the assumptions on the decay at ∞ are generous whereas Algorithm 1 is concerned with the KdV on the full-line but without a priori decay at infinity. Our proof of Theorem 1 can actually be used to relax conditions (3.25) in Algorithm 4.2.1 of [3].

An equation for time evolution of certain spectral data for a matrix valued KdV on $\mathbb{R}_+ \times \mathbb{R}_+$ was also put forward in the short note [12] by Sakhnovich. No inverse spectral procedure was discussed in [12].

It should be mentioned that both Algorithm 1 and 4.2.1 have a serious drawback: they require too much data $\{q_k(t)\}_{k=0}^3$ at one point.

4. Time Evolution of a Reflection Coefficient

The simple law of the time evolution of the reflection coefficient for short-range initial profiles is in the core of the IST. Our more general setting is far beyond the scattering theoretical situation but, following [5], scattering quantities can actually be defined as long as the underlying Schrodinger operator has some a.c. spectrum, i.e. there is a non-trivial transmission.

Throughout this section we assume that the a.c. spectrum of $H_+^{x_0}$ is non-empty for some reference point x_0 (and hence for any $x_0 \in \mathbb{R}$). By (2.17), $\sigma_{\mathrm{ac}}(H) \neq \varnothing$ but need not be of multiplicity 2.

For almost every $\lambda \in \sigma_{\mathrm{ac}}\left(H_+^{x_0}\right)$, the pair $\Psi_+(x, x_0, \lambda + i0), \overline{\Psi_+}(x, x_0, \lambda + i0)$ is linearly independent and hence the Weyl solution $\Psi_-(x, x_0, \lambda + i0)$ is a linear combination of $\Psi_+(x, x_0, \lambda + i0), \overline{\Psi_+}(x, x_0, \lambda + i0)$. I.e.,

$$\begin{aligned} & c_+(x_0, \lambda)\, \Psi_-(x, x_0, \lambda + i0) \\ = \; & \overline{\Psi_+}(x, x_0, \lambda + i0) + R_+(x_0, \lambda)\, \Psi_+(x, x_0, \lambda + i0) \end{aligned} \tag{4.1}$$

with some coefficients $c_+(x_0, \lambda)$ and $R_+(x_0, \lambda)$.

If one interprets $\overline{\Psi_+}(x, x_0, \lambda + i0)$ as a plane wave incident from $+\infty$ with momentum $\sqrt{\lambda}$ then $\overline{\Psi_+}(x, x_0, \lambda + i0) + R_+(x_0, \lambda)\, \Psi_+(x, x_0, \lambda + i0), x \to \infty$, is the superposition of the on-coming $\overline{\Psi_+}$ and the reflected $R_+\Psi_+$ waves. By this reason $R_+(x_0, \lambda)$ can be referred to as a right reflection coefficient with respect to reference point x_0 corresponding to momentum $\sqrt{\lambda}$.

Setting in (4.1) $x = x_0$ one gets

$$c_+(x_0, \lambda) = 1 + R_+(x_0, \lambda) \tag{4.2}$$

Differentiating (4.1) and then setting $x = x_0$ one gets

$$\begin{aligned} & c_+(x_0, \lambda)\, \partial_x \Psi_-(x_0, x_0, \lambda + i0) \\ = \; & \partial_x \overline{\Psi_+}(x_0, x_0, \lambda + i0) + R_+(x_0, \lambda)\, \partial_x \Psi_+(x_0, x_0, \lambda + i0) \end{aligned}$$

Recalling $m_\pm(x_0, \lambda) = \partial_x \Psi_\pm(x_0, x_0, \lambda + i0)$ we have

$$c_+(x_0, \lambda)\, m_-(x_0, \lambda + i0) = \overline{m_+}(x_0, \lambda + i0) + R_+(x_0, \lambda)\, m_+(x_0, \lambda + i0) \tag{4.3}$$

It follows now from (4.2) and (4.3) that

$$R_+(x_0, \lambda) = -\frac{m_- - \overline{m}_+}{m_- - m_+}(x_0, \lambda), \text{ for almost every } \lambda \in \sigma_{\mathrm{ac}}\left(H_+^{x_0}\right).$$

Similarly, assuming $\sigma_{\mathrm{ac}}\left(H_{-}^{x_0}\right) \neq \varnothing$, one defines a left reflection coefficient R_-

$$R_-\left(x_0, \lambda\right) = \frac{m_+ - \overline{m}_-}{m_- - m_+}\left(x_0, \lambda\right), \text{ for almost every } \lambda \in \sigma_{\mathrm{ac}}\left(H_{-}^{x_0}\right).$$

Thus, we have

$$R_\pm\left(x_0, \lambda\right) = \mp\frac{m_\mp - \overline{m}_\pm}{m_- - m_+}\left(x_0, \lambda\right), \text{ for almost every } \lambda \in \sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right). \tag{4.4}$$

Remark 3. *Definition of $R_\pm$ given by* (4.4) *is extremely general and applies to any full line Schrodinger operator H having some a.c. spectrum. Moreover both $R_\pm\left(x_0, \lambda\right)$ exist if $\lambda \in \sigma_{\mathrm{ac}}\left(H_{+}^{x_0}\right) \cap \sigma_{\mathrm{ac}}\left(H_{-}^{x_0}\right)$. However $R_\pm$ is consistent with the usual definition accepted in the short-range scattering theory only for $L^1(\mathbb{R})$ potentials equal to zero on $x \gtrless x_0$. If $\sigma_{\mathrm{ac}}\left(H_{+}^{x_0}\right) \cap \sigma_{\mathrm{ac}}\left(H_{-}^{x_0}\right) \neq \varnothing$ then a different definition of $R_\pm$ can be considered (see* [5]*) which has better consistency with the short-range scattering. However all reasonable definitions produce reflection coefficients different by a unimodular factor.*

Remark 4. *Definition* (4.4) *depends on the reference point x_0 which will be discussed in the next section.*

Let $R_\pm\left(x_0, t, \lambda\right)$ denote the reflection coefficient corresponding to $q(x,t)$.

Theorem 2. *Let $q(x,t)$ satisfy the conditions of Theorem 1 and $\sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right) \neq \varnothing$. Then for almost every $\lambda \in \sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right)$*

$$\begin{aligned} & R_\pm\left(x_0, t, \lambda\right) \\ = \; & R_\pm\left(x_0, \lambda\right) \exp\left\{4i\int_0^t \left(q\left(x_0, s\right) + 2\lambda\right) \operatorname{Im} m_\pm\left(x_0, s, \lambda + i0\right) ds\right\}, t \in [0, T]. \end{aligned} \tag{4.5}$$

Proof. Differentiating (4.4) with respect to t and using (3.17) yields

$$\partial_t R_\pm = 2ib \operatorname{Im} m_\pm \cdot R_\pm, \text{ for almost every } \lambda \in \sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right). \tag{4.6}$$

Since $b = 2\left(q + 2\lambda\right)$, (4.6) immediately implies (4.5). □

Corollary 1. *$\left|R_\pm\left(x_0, t, \lambda\right)\right|$ is an invariant. That is for $t \in [0, T]$*

$$\left|R_\pm\left(x_0, t, \lambda\right)\right| = \left|R_\pm\left(x_0, \lambda\right)\right| \quad \textit{for almost every } \lambda \in \sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right).$$

Corollary 2. *Assume that $q(x,t)$ decays as $x \to \pm\infty$ sufficiently fast for any $t \in [0, T]$. Then*

$$\lim_{x_0 \to \pm\infty} \frac{R_\pm\left(x_0, t, \lambda\right)}{R_\pm\left(x_0, \lambda\right)} = e^{\pm 8i\lambda^{3/2}t} \textit{ for almost every } \lambda \in \sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right). \tag{4.7}$$

To see (4.7) *it is enough to notice that* $\lim_{x_0 \to \pm\infty} m_\pm\left(x_0, t, \lambda\right) = \pm i\sqrt{\lambda}$.

Equation (4.5) admits a different derivation. For almost every $\lambda \in \sigma_{\mathrm{ac}}\left(H_{\pm}^{x_0}\right)$

$$R_\pm\left(x_0, t, \lambda\right) = \frac{\nu_\pm\left(x_0, t, \lambda\right)}{\overline{\nu}_\pm\left(x_0, \lambda\right)} R_\pm\left(x_0, \lambda\right), t \in [0, T]. \tag{4.8}$$

where $\nu_\pm$ solves (3.18). Indeed, by a direct computation one verifies

$$b\left(x_0, t, \lambda\right) m_\pm\left(x_0, t, \lambda\right) = \frac{\partial_t \nu_\pm\left(x_0, t, \lambda\right)}{\overline{\nu}_\pm\left(x_0, \lambda\right)} + a\left(x_0, \lambda\right), t \in [0, T]. \tag{4.9}$$

Plugging (4.9) into (4.5) and observing that a is real and $\nu_{\pm}(x_0, 0, \lambda) = 1$, yields (4.8).

Note that it follows from (3.18) and (3.19) that $\nu_{\pm}$ in (4.9) is the solution to the initial value problem

$$\partial_t^2 \nu_{\pm} - \frac{\partial_t b}{b} \partial_t \nu_{\pm} - \left(a \frac{\partial_t b}{b} + bc + a^2 - \partial_t a \right) \nu_{\pm} = 0$$

$$\nu_{\pm}(0) = 1, \partial_t \nu_{\pm}(0) = bm_{\pm} - a$$

for a second order linear ordinary differential equation with real coefficients.

We took definition (4.4) of the reflection coefficient for simplicity. We plan to investigate more suitable definitions elsewhere.

5. Transformation of the Weyl matrix and reflection coefficient under the change of the reference point

Most of equations in the previous sections crucially depend on the reference point x_0. In this section we derive the differentiate equations describing this dependence. We shall start with the Weyl matrix.

Theorem 3. *Let q be real valued and $q \in L^1_{\text{loc}}(\mathbb{R})$ and $-\partial_x^2 + q(x)$ be in the limit point case at $+\infty$ and $-\infty$. Then*

$$\partial_x M = QM + MQ' \tag{5.1}$$

where

$$Q(x, z) = \begin{pmatrix} 0 & q(x) - z \\ 1 & 0 \end{pmatrix}.$$

Proof. Differentiate the coefficients of M taking into account (2.11) one gets

$$\begin{aligned} \partial_x m_1 &= 2(q - z) m_{12} \\ \partial_x m_{12} &= m_1 + (q - z) m_2 \\ \partial_x m_2 &= 2m_{12} \end{aligned}$$

which can be written in the matrix form

$$\partial_x M = \begin{pmatrix} 0 & q - z \\ 1 & 0 \end{pmatrix} M + M \begin{pmatrix} 0 & 1 \\ q - z & 0 \end{pmatrix}.$$

□

Remark 5. *Combine* (3.5) *and* (5.1)

$$\begin{cases} \partial_t M = PM + MP' \\ \partial_x M = QM + MQ' \end{cases} \tag{5.2}$$

It is straightforward to show that the compatibility of the equations in (5.2) *implies*

$$\partial_x P - \partial_t Q + [P, Q] = 0$$

and we arrive at the famous compatibility condition which is equivalent to the KdV (3.2) *equation. The pair P, Q is, of course, the Ablowitz-Kaup-Newell-Segur (AKNS) pair* [1].

Remark 6. *The solution to* (5.1) *can be represented in terms of the fundamental solutions* θ *and* ϕ (2.2) − (2.4) *by*

$$\begin{pmatrix} m_1 \\ m_{12} \\ m_2 \end{pmatrix}(x,z) \tag{5.3}$$

$$= \begin{pmatrix} (\partial_x\phi)^2 & 2\partial_x\phi 2\partial_x\theta & (\partial_x\theta)^2 \\ \frac{1}{2}\partial_x\phi^2 & \partial_x(\phi\theta) & \frac{1}{2}\partial_x\theta^2 \\ \phi^2 & 2\phi\theta & \theta^2 \end{pmatrix}(x,x_0,z)\begin{pmatrix} m_1 \\ m_{12} \\ m_2 \end{pmatrix}(x_0,z) \tag{5.4}$$

To demonstrate (5.3) *one substitutes* (2.14) *into* (2.16).

Equation (5.3) shows that Weyl matrixes $M(x,z)$ and $M(x_0,z)$ corresponding to different reference points are not in general similar.

Turn now to the reflection coefficient

Theorem 4. *Let* q *be real valued and* $q \in L^1_{\text{loc}}(\mathbb{R})$ *and* $-\partial_x^2 + q(x)$ *be in the limit point case at* $+\infty$ *and* $-\infty$ *and such that* $\sigma_{\text{ac}}\left(H_\pm^{x_0}\right) \neq \varnothing$. *Then for almost every* $\lambda \in \sigma_{\text{ac}}\left(H_\pm^{x_0}\right)$

$$R_\pm(x,\lambda) = \frac{\Psi_\pm(x,x_0,\lambda)}{\overline{\Psi}_\pm(x,x_0,\lambda)} R_\pm(x_0,\lambda). \tag{5.5}$$

Proof. Differentiating (4.4) with respect to x and using (2.15) we have: for almost every $\lambda \in \sigma_{\text{ac}}\left(H_\pm^{x_0}\right)$

$$\partial_x R_\pm = 2i \operatorname{Im} m_\pm \cdot R_\pm. \tag{5.6}$$

Integrating (5.6)

$$R_\pm(x,\lambda) = R_\pm(x_0,\lambda)\exp\left\{2i\int_{x_0}^{x} \operatorname{Im} m_\pm(s,\lambda+i0)\,ds\right\}$$

and recalling (2.14), we get (5.5). □

Corollary 3. *For almost every* $\lambda \in \sigma_{\text{ac}}\left(H_\pm^{x_0}\right)$ *and any* x, x_0

$$|R_\pm(x,\lambda)| = |R_\pm(x_0,\lambda)|.$$

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Department of Math and Statistics, University of Alaska Fairbanks, Fairbanks, AK 99709
E-mail address: ffavr@uaf.edu