ON THE BOUNDARY CONTROL APPROACH TO INVESE SPECTRAL AND SCATTERING THEORY FOR SCHRODINGER OPERATORS

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Abstract. We link boundary control theory and inverse spectral theory for the Schrödinger operator $H = -\partial_x^2 + q(x)$ on $L^2(0,\infty)$ with Dirichlet boundary condition at $x = 0$. This provides a shortcut to some results on inverse spectral theory due to Simon, Gesztesy-Simon and Remling. The approach also has a clear physical interpritation in terms of boundary control theory for the wave equation.

1. Introduction

This methodological paper links together a several developments of the late 1990s and earlier 2000s related to classical but still important issues of the inverse problems for the one dimensional wave and Schrodinger equations.

In 1986 Belishev put forward a very powerful approach to boundary value inverse problems (see his 2007 review [7] and the extensive literature therein). His approach is based upon deep connections between inverse problems and boundary control theory and is now referred to as the BC method. It is however much less known in the Schrodinger operator community (including inverse problems). Likewise, the boundary control community does not appear to have tested the BC method in the direct/inverse spectral/scattering theory. In [3] we showed that the BC method can be applied to the study of the Titchmarsh-Weyl m-function. In this short note we demonstrate yet another application of the BC method to inverse problems for the one-dimensional Schrodinger equation. In terms relevant to our situation, the main idea of the BC method is to study the (dynamic) Dirichlet-to-Neumann map $u(0,t) \to \partial_x u(0,t)$ for the wave equation

$$\partial_t^2 u - \partial_x^2 u + q(x)u = 0, x > 0, t > 0$$

with zero initial conditions. The map $u(0,t) \to \partial_x u(0,t)$ turns out to be the so-called response operator - a natural object available in physical experiments. The kernel of this operator, the response function $r(t)$, reconstructs the potential $q(x)$ on $(0,l)$ by $r(t)$ on $(0,l)$ through an elegant procedure; every step of which being physically motivated. Beside its transparency, this method is also essentially local, i.e. instead of studying the problem on $(0,\infty)$ at a time (as Gelfand-Levitan type methods require) one can solve it on $(0,l)$. The final integral equation (2.16),

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solving the inverse problem, is actually equivalent to (2.17) derived by Remling a few year ago in [15] and [16]. His concern was not inverse problems for the wave equation but inverse spectral theory for the half line Schrodinger equation

$$-\partial_x^2 u + q(x) u = zu, x > 0,$$

with Neumann or Dirichlet boundary conditions. The main idea of [15] and [16] is to employ the theory of de Brange spaces to develop a local (in the very same sense as in the BC method) approach to inverse spectral problems. The treatment,.as the author notices, "is neither short nor elementary" and physical motivations behind his arguments are not obvious. Yet, Remling was able to identify many important objects (including the final integral equation (2.17)) which in the BC method, in fact, have names. It is worth mentioning that Remling's work was in turn inspired by a new fundamental formalism of inverse spectral theory due to Simon [17] and Gesztesy-Simon [10] . Their approach is also local but leads to a non-linear integro-differential equation (3.11) (which could be reduced [16] to the non-linear integral equation (3.12)).

We believe that there is a point in offering a totally different physically transparent approach to the circle of questions treated in [17], [10], [15] and [16]. With some easily recoverable technical details omitted, it takes just a few pages to explain how the BC method works in inverse spectral theory. Applications of the BC method are not limited to the papers cited above but also to many other known inverse (not necessarily spectral) problems considered in [2], [1], [9], [12], [14] (to name just five). Some of these connections are worth considering but to keep our paper concise we have to leave them aside at this time. We plan to continue exploring this avenue in [4].

Through the paper we state seven seemingly assorted problems from boundary value and BC theories for the wave equation (direct and inverse), spectral theory for Schrodinger operators (direct and inverse) and inverse scattering theory. We show how the language of the BC theory links these problems together. The paper is organized as follows. In Section 2 we introduce the BC method in the situation pertinent to our setting. In Section 3, after a brief introduction to Titchmarsh-Weyl theory, we show how the BC method naturally appears in inverse spectral theory. In short Section 4 we consider an inverse problem which can be solved by the BC method.

2. Boundary Control Approach to an Inverse Problem for the Wave Equation

The material of this section is well-know in the BC community but much less known in the Schrodinger operator community. We follow Belishev [7] and Avdonin et al [5],[6] here but adjust notation to our setting. Associate with the second order differential operation

$$H = -d^2/dx^2 + q(x) \tag{2.1}$$

on $\mathbb{R}_+ := [0, \infty)$ with a real-valued locally integrable potential q, i.e.

$$q = \overline{q} \in L^1_{\text{loc}}(\mathbb{R}_+),$$

the following problem.

Problem 1 (Mixed Problem for the Wave Equation)**.** *Find the solution, which we call physical, to*

$$\begin{cases} Hu = -\partial_t^2 u \\ u(x,t) = 0, t < x \text{ (causality condition)} \\ u(x,0) = \partial_t u(x,0) = 0 \\ u(0,t) = f(t), \end{cases} \tag{2.2}$$

where f is a know $L^2_{\text{loc}}(\mathbb{R}_+)$ function referred in BC theory to as a <u>*boundary control*</u>*.*

It is well-know that in the distributional sense the solution to Problem 1 admits the representation

$$u(x,t) = \begin{cases} f(t-x) + \int_x^t w(x,s) f(t-s)\, ds, & x \le t, \\ 0, & x \geqslant t. \end{cases} \tag{2.3}$$

where $w(x,s)$ solves

Problem 2 (Goursat problem)**.** *Find the solution, usually referred to as Goursat, to*

$$\begin{cases} Hu = -\partial_t^2 u \\ u(0,t) = 0 \\ u(x,x) = -\frac{1}{2}\int_0^x q \end{cases} \tag{2.4}$$

It is a standard fact (see e.g. [9]) that Problem 2 is unequally solvable for any (even complex) locally integrable q and its solution

$$w \in AC^1_{\text{loc}} \tag{2.5}$$

(the partial derivatives are absolutely continuous). Observe that (2.3) relates Problems 1 and 2 and is nothing but the Duhamel principle. The advantage of the representation (2.3) is that it defines a linear transformation between boundary control functions f in Problem 1 and corresponding solutions. This suggests the following

Definition 1 (Control Operator)**.** *The operator W_t on $L^2(0,t)$ defined by*

$$(W_t f)(x) = f(t-x) + \int_x^t w(x,\tau) f(t-\tau)\, d\tau, \tag{2.6}$$

where $w(x,s)$ solves Problem 3, is called the <u>*control operator*</u>*.*

Note that t in (2.6) is a parameter and W_t represents a family of linear operators which, due to (2.5), are bounded on $L^2(0,t)$. Moreover $W_t - I$ is Volterra and hence W_t is invertible on $L^2(0,t)$. Due to (2.3), $(W_t f)(x)$ is a weak solution to Problem 1 and hence W_t establishes a one-to-one correspondence between boundary control functions f on $[0,t]$ and the solutions $u(x,t), x \in [0,t]$, to Problem 1 for every $t > 0$. This fact justifies its name - control operator.

The operator W_t may also be interpreted as a wave operator. Indeed, W_t transforms the boundary values $u(0,\tau), \tau \in [0,t]$, of the solution u to Problem 1 into the solution $u(x,t), x \in [0,t]$.

It is natural to pose

Problem 3 (BC Problem)**.** *Given fixed time $t > 0$ and a function $g \in L^2(0,t)$, find a boundary control $f \in L^2(0,t)$ such that*

$$(W_t f)(x) = g(x) \text{ for all } x \in [0,t]. \tag{2.7}$$

It is a fundamental but easy fact that Problem 3 is unequally solvable. Indeed, even for complex $q \in L^1_{\text{loc}}(\mathbb{R}_+)$, as we have seen, W_t is boundedly invertible and Problem 3 is unequally solvable for any $g \in L^2(0,t)$.The next important concept of BC theory is

Definition 2 (Response Operator)**.** *Let $q \in L^1(0,l), l < \infty$, and u be the solution to Problem 2. The operator R on $L^2(0,l)$ defined by*

$$(Rf)(t) = \partial_x u(0,t), t \in [0,l],$$

is called the response operator.

It immediately follows from (2.3) that

$$(Rf)(t) = -f'(t) + \int_0^t r(t-\tau) f(\tau)\, d\tau, \tag{2.8}$$

where

$$r(\cdot) := \partial_x w(0,\cdot)$$

is called the response function.

Note that R transforms $u(0,t) \to \partial_x u(0,t), t \in [0,l]$, and by this reason can be called the (dynamic) Dirichlet-to-Newmann map. The representation (2.8) says that the operator R is convolution minus differentiation and hence unbounded on $L^2(0,l)$. The operator R does not play an important role in our consideration but the kernel r of its convolution part does. Due to (2.5), $r \in L^1_{\text{loc}}(\mathbb{R}_+)$.

Define now the last but crucially important operator.

Definition 3 (Connecting Operator)**.** *The operator on $L^2(0,t)$ defined by*

$$C_t = W_t^* W_t$$

is called the connecting operator.

Observe that since W_t is boundedly invertible and therefore

$$C_t > 0 \text{ on } L^2(0,t)$$

(note that the interval here is $(0,t)$ not $(0,l)$ as in Definition 2). The main benefit from introducing the connecting operator is that C_t admits the following convenient representation:

$$C_t = I + \mathbb{K}_t, \tag{2.9}$$

where $\mathbb{K}_t$ is the integral operator

$$(\mathbb{K}_t f)(x) = \int_0^t K_t(x,y) f(y)\, dy$$

with the kernel

$$\begin{aligned} K_t(x,y) \ &: = \frac{1}{2}\left[\phi(2t-x-y) - \phi(x-y)\right], \\ \phi(x) \ &: = \int_0^{|x|} r. \end{aligned} \tag{2.10}$$

The representation (2.9) is well-known in the boundary control community but typically assumes smoothness of the potential. A limiting argument however allows one to expand (2.9) to any $q \in L^1_{\text{loc}}(\mathbb{R}_+)$ in a straightforward way as the kernel $K_t(x,y)$ given by (2.10) remains absolutely continuous for any locally integrable potential.

Turn now to inverse problems - the main concern of this paper.

Problem 4 (Inverse BC Problem)**.** *Given response function r on $(0,l)$, associated with Problem 1, find the potential q on $(0,l)$.*

A marvelous solution to this problem was put forward by Belishev in the 80s which is outlined below.

Let y solves the Sturm-Liouville problem

$$\left\{\begin{array}{l} Hy=0 \\ y\left(0\right)=0, y'\left(0\right)=1. \end{array}\right. \tag{2.11}$$

Consider the boundary control problem (Problem 3): find the control function f such that

$$\left(W_t f\right)\left(x\right)=\left\{\begin{array}{l} y\left(x\right), x\leq t \\ 0, x>t. \end{array}\right. \tag{2.12}$$

In (2.12) both f and y are unknown. The following trick eliminates y. One first establishes the equation

$$\left(C_t f\right)\left(x\right)=t-x, \quad x\in\left[0,t\right]. \tag{2.13}$$

The proof comes from considering for an arbitrary $g\in C_0^\infty\left(0,t\right)$ (infinitely smooth functions vanishing at the endpoints)

$$\int_0^t C_t f\overline{g}=\int_0^t W_t f\overline{W_t g}=\int_0^t y\overline{W_t g},$$

using

$$\int_0^t\left(t-\tau\right)g''\left(\tau\right)d\tau=g\left(t\right)$$

and the fact that $W_t g$ solves Problem 1.

One now solves (2.13), which is apparently unequally solvable, and finds the boundary control function $f=f_t\left(x\right)$ for $x\in[0,t]$. Setting in (2.12) $x=t-0$ yields

$$y\left(t\right)=\left(W_t f_t\right)\left(t-0\right). \tag{2.14}$$

Since y solves (2.11) and H is defined by (2.1) one has

$$q\left(t\right)=y''\left(t\right)/y\left(t\right).$$

By (2.6), the right hand side of (2.14) is $f_t\left(+0\right)$ and hence we finally have

$$q\left(t\right)=\frac{d^2}{dt^2}f_t\left(+0\right)/f_t\left(+0\right) \tag{2.15}$$

Since the kernel is locally absolutely continuous, the function $f_t\left(+0\right)$ is differentiable and its derivative is locally absolutely continuous which allows us to understand (2.15) almost everywhere on $(0,l)$.

For the reader's convenience we rewrite equation (2.13) in the form

$$f_t\left(x\right)+\frac{1}{2}\int_0^t\left[\phi\left(2t-x-y\right)-\phi\left(x-y\right)\right]f_t\left(y\right)dy \quad = \quad t-x, \tag{2.16}$$

$$x \quad \in \quad \left(0,t\right), \quad \phi\left(x\right)=\int_0^{|x|}r.$$

As it was established above, (2.16) is unequally solvable as long as the response function r comes from Problem 1 with a potential $q\in L^1\left(0,l\right)$.

Let us now make an important observation. It is straightforward to verify that

$$K_t(t-x,t-y)=\frac{1}{2}\left[\phi(x+y)-\phi(x-y)\right]$$

and the substitution $g_t(x)=f_t(t-x)$ immediately transforms (2.16) into the following equivalent form

$$g_t(x)+\frac{1}{2}\int_0^t\left[\phi(x+y)-\phi(x-y)\right]g_t(y)\,dy=x,\ \ x\in(0,t). \tag{2.17}$$

The potential q can then be recovered by $q(t)=\frac{d^2}{dt^2}g_t(t-0)/g_t(t-0)$ a.e. on $(0,l)$.

Note that in the above cited papers equation (2.16) is derived for smooth potentials. We do not assume this condition. Equation (2.17) looks more symmetric than (2.16). It was derived by Remling in [15] and [16] for a different from Problem 4 setting and by completely different methods. We discuss this in greater detail in the next section once some additional background information is introduced.

3. Inverse Spectral Problems for the Schrodinger equation

We start with reviewing Titchmarsh-Weyl theory. This material is standard and we follow a modern exposition given in [11]. Consider

Problem 5 (Weyl Problem). *Find the solution to*

$$\begin{cases} Hu=zu, \quad x\geq 0 \\ u(\cdot,z)\in L^2(\mathbb{R}_+)\ \text{ for any } z\in\mathbb{C}_+ \\ u(0,z)=1 \end{cases} \tag{3.1}$$

with a real-valued locally integrable potential q.

Note that Problem 5 is not local as it specifies the solution at $+\infty$. It is the central point of Titchmarsh-Weyl theory that Problem 5 has a unique solution $\Psi(x,z)$, called the Weyl solution, for a very broad class of potentials q. The case when Ψ is unique is referred to as the limit point case at $+\infty$ as opposed to the limit circle case at $+\infty$ when every solution to $Hu=zu, u(0,z)=1$, is in $L^2(\mathbb{R}_+)$ for any $z\in\mathbb{C}_+$. We assume that q is in the limit point case at $+\infty$.

The (principal or Dirichlet) Titchmarsh-Weyl m-function, $m(z)$, is defined for $z\in\mathbb{C}_+$ as

$$m(z):=\frac{\Psi'(0,z)}{\Psi(0,z)}. \tag{3.2}$$

Function $m(z)$ is analytic in $\mathbb{C}_+$ and satisfies the Herglotz property:

$$m:\mathbb{C}_+\to\mathbb{C}_+, \tag{3.3}$$

so m satisfies a Herglotz representation theorem,

$$m(z)=c+\int_{\mathbb{R}}\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)d\rho(\lambda), \tag{3.4}$$

where $c=\operatorname{Re}m(i)$ and ρ is a positive measure subject to

$$\int_{\mathbb{R}}\frac{d\rho(\lambda)}{1+\lambda^2}<\infty, \tag{3.5}$$

$$d\rho(\lambda)=w-\lim_{\varepsilon\to+0}\frac{1}{\pi}\operatorname{Im}m(\lambda+i\varepsilon)\,d\lambda. \tag{3.6}$$

It is a fundamental fact of the spectral theory of ordinary differential operators that the measure ρ is the spectral measure of the Schrodinger operator $-d^2/dx^2 + q(x)$ on $L^2(\mathbb{R}_+)$ with a Dirichlet boundary condition at $x = 0$. Another fundamental fact is the Borg-Marchenko uniqueness theorem stating

$$m_1 = m_2 \Longrightarrow q_1 = q_2. \tag{3.7}$$

In other words, the potential is recovered by the Titchmarsh-Weyl m-function uniquely. One can now pose natural

Problem 6 (Inverse Spectral Problem)**.** *Given Titchmarsh-Weyl m-function $m(z), z \in \mathbb{C}_+$, find the potential $q(x), x \in \mathbb{R}_+$.*

It should be emphasized that the only conditions imposed on q are: q is real, locally integrable and in the limit point case at ∞ which covers extremely general Sturm-Liouville problems.

Problem 6 was first solved by Gelfand-Levitan in the 1950s by means of a linear integral equation called now the Gelfand-Levitan equation. (see e.g. [12] or [9]). In our paper we concentrate on recent approaches to Problem 6.

To address Problem 6 Simon [17] and later Gesztesy-Simon [10] showed first that there is a real $L^1_{\text{loc}}(\mathbb{R}_+)$ function $A(t)$ such that for any finite positive a

$$m(z) = i\sqrt{z} - \int_0^a A(t)\, e^{2i\sqrt{z}t} dt + \widetilde{O}\left(e^{-2\alpha \operatorname{Im} \sqrt{z}}\right), \tag{3.8}$$

as $|z| \to \infty$ in the sector $\varepsilon < \arg z < \pi - \varepsilon$. Here $\widetilde{O}$ is defined as follows $f = \widetilde{O}(g)$ if $g \to 0$ and for any $\delta > 0$, $(f/g)\,|g|^\delta \to 0$ as $|z| \to \infty$.

The function A in (3.8) is referred in [17] as to the A-amplitude where it was also proved that a can be taken ∞ (and hence there is no the error term in (3.8) if $q \in L^1(\mathbb{R}_+)$ or $q \in L^\infty(\mathbb{R}_+)$. This was extended in [3] to all potentials q such that $\sup_{x \geq 0} \int_x^{x+1} |q| < \infty$.

Representation (3.8) implies (see e.g [8] for a short proof) the local analog of (3.7) which can loosely be stated

$$A_1 = A_2 \text{ on } (0, a) \Longrightarrow q_1 = q_2 \text{ on } (0, a). \tag{3.9}$$

In [3] we found that A and the response function r are related by the simple formula

$$A(t) = -2r(2t). \tag{3.10}$$

Note that the A-amplitude and response functions are objects of different nature: the former comes from spectral theory of the half-line Schrodinger operator and the latter comes from BC theory for the half-line wave equation. While the counterplay between the wave and Schrodinger equations is well-known and can be traced back to original work due to Levitan (see e.g. [13]), the relation (3.10) appears to be noteworthy. For instance, the local Borg-Marchenko uniqueness result (3.9) immediately follows from (3.10) and the considerations of the previous section.

In [17] and [10] there was introduced a family of functions $A(\cdot, x)$ $(A(\cdot, 0) = A(\cdot))$ corresponding to the $m-$functions $m(z, x)$ associated with $[x, \infty)$ which satisfy the following non-linear integro-differential equation

$$\frac{\partial A(t,x)}{\partial x} = \frac{\partial A(t,x)}{\partial t} + \int_0^t A(\tau, x)\, A(t-\tau, x)\, d\tau. \tag{3.11}$$

Since A shares the sanguinities with the potential q, equation (3.11) should in general be understood in the distributional sense. Remling [16] regularized equation (3.11) to read

$$B(t,x)=\int_0^x dy\int_0^{t-y} d\tau\,[B(y+\tau,y)+A(y+\tau)]\,[B(t-\tau,y)+A(t-\tau)]\,, \tag{3.12}$$

where

$$B(t,x)=A(t-x,x)-A(t)\,,\ x\in[0,t]$$

is continuous. Equation (3.12) holds pointwise.

Note that the potential q is not present in (3.12) and hence given initial A-amplitude $A(t,0)=A(t)$ constructed from the $m-$ function by (3.8) one then solves equation (3.12) for $B(t,x)$ $x\in[0,t]$. The potential can then be recovered by

$$q(x)=A(x)+B(x,x)$$

producing a new inverse spectral problem formalism.

In [10] equation (3.11) is called a non-linear Gelfand-Levitan equation and it was established there that

$$A(\alpha)=-2\partial_y L(2\alpha,+0)\,, \tag{3.13}$$

where $L(x,y)$ is the Gelfand-Levitan kernel appearing in the Gelfand-Levitan equation.

A drawback of the outlined procedure is that equation (3.12) is non-linear. Remling followed [17], [10] up with [15], [16] where he puts forward yet another procedure to solve Problem 6 which is equivalent to that of Gesztesy-Simon but linear now. He essentially derives equation (2.17) basing upon a powerful machinery of de Branges spaces and his techniques are quite sophisticated. He also proves that equation (3.11) is uniquely solvable if and only if

$$I+\mathbb{K}_A>0, \tag{3.14}$$

where $\mathbb{K}_A$ is the integral operator with the kernel

$$\varphi(x-y)-\varphi(x+y)\,,\quad \varphi(x)=\frac{1}{2}\int_0^{|x|/2}A, \tag{3.15}$$

which solves an open problem from [10] on finding necessary and sufficient conditions for solubility of (3.11) in terms of the A-amplitude. Due to (3.10) one can easily see that condition (3.14) and our condition $C_t>0$ are equivalent.

Since (3.12) is actually derived from the Riccati equation on $m(z,x)$, which can be linearized, it is reasonable to believe that (3.12) could also be linearized. We can claim that the procedure leading to (2.16) (or equivalently (2.17)) does the job.

Remark that besides being linear equation (2.17) has another advantage over (3.12): it does not involve repeated integration.

An important part of solving Problem 6 is obtaining the A-amplitude by a given m-function. Formula (3.8) is not particularly convenient for this purpose. An alternative to (3.8) formula is put forward in [10]:

$$A(t)=-2\lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{-\varepsilon\lambda}\frac{\sin\left(2t\sqrt{\lambda}\right)}{\sqrt{\lambda}}d\rho(\lambda) \tag{3.16}$$

where ρ is found by (3.6). Formula (3.16) holds for any Lebesgue point of A, i.e. almost everywhere. Note that the Abelian regularization in (3.16) cannot be removed. Moreover the integral in (3.16) need not be even conditionally convergent (see [10] for a counterexample). Another warning regarding (3.16): Consider $\varphi(x)$ defined by (3.15). Assume that the spectrum is non-negative. If we now plug (3.16) into (3.15) then a totally formal computation immediately yields

$$\varphi(x) = -\int_0^\infty \frac{\sin^2\left(\frac{x}{2}\sqrt{\lambda}\right)}{\lambda} d\rho(\lambda) \tag{3.17}$$

which implies that $\varphi(x) < 0$ that looks counterintuitive. Moreover the integral in (3.17) does not actually converge under the assumption (3.5) only.

Note

$$r(t) = \frac{\partial}{\partial y} L(y, +0) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} e^{-\varepsilon\lambda} \frac{\sin\left(t\sqrt{\lambda}\right)}{\sqrt{\lambda}} d\rho(\lambda). \tag{3.18}$$

We were unable to locate formulas (3.18) in the literature on BC. Note that (3.18) looks slightly prettier than (3.16) and consequently ϕ is nicer than φ suggesting that the response function may be slightly easier to deal with.

We owe the following comment to Remling. Taking Fourier transforms of (2.3) and then formally changing the order of integration in the double integral, we obtain

$$\widehat{u}(x.k) = \widehat{f}(k)\left(e^{ikx} + \int_x^\infty e^{iks} w(x,s)\, ds\right)$$

which looks exactly like the Gelfand-Levitan-Marchenko type representation of the L^2 solution, and, indeed, the corresponding kernel can be obtained as the solution to Problem 2 (see e.g. [12] or [14]). Thus it appears that since Problem 1 naturally leads to Problem 2, the BC method essentially starts out by introducing the Gelfand-Levitan-Marchenko kernel. This argument is totally formal but explains in part that different approaches to inverse spectral theory use essentially the same objects.

It should be noticed that since the data in Problem 6 is not readily available from an experiment and one needs to come up with some observable data which allows us to find the Titchamarsh-Weyl m-function. It is discussed in the last section.

4. An Inverse Scattering Method

Titchmarsh-Weyl m-function does not have an explicit physical meaning but does appear explicitly in certain scattering quantities. In this section we consider an inverse scattering problem of practical interest. To state the problem we need one concept.

Consider the full line Schrodinger equation

$$Hu = -u'' + q(x)u = k^2 u, x \in \mathbb{R}, k \in \mathbb{R}, \tag{4.1}$$

with a potential supported on $\mathbb{R}_+$. Since there is enough free space on the line for plane waves e^{ikx} to propagate it is intuitively clear that scattering theory should be in order even without any decay assumption on $q(x)$ as $x \to \infty$. Following Faddeev[1], one finds the scattering solution $\psi(x,k)$ to the equation (4.1) subject to

$$\begin{aligned} \psi(x,k) &= C(k)\Psi\left(x,k^2\right), x > 0, \\ \psi(x,k) &= e^{ikx} + R(k)e^{-ikx}, x \le 0, \end{aligned} \tag{4.2}$$

[1] We were unable to locate a paper where Faddeev introduce this first

where Ψ is the Weyl solution. The requirement of continuity of $\Psi(x,k)$ and $\Psi'(x,k)$ at $x=0$ implies

$$R(k)=\frac{ik-m\left(k^2\right)}{ik+m\left(k^2\right)}. \tag{4.3}$$

If q is short range then Ψ can be chosen as the Jost solution and hence ψ defined by (4.2) coincides with the standard reflection coefficient. $R(k)$ can therefore in general be interpreted as the reflection coefficient from the left incident and it exists for all arbitrary potentials q which are at the limit point case at $+\infty$.

Problem 7 (Inverse Scattering Problem)**.** *Let q be supported on $(0,\infty)$, real and in the limit point case at $+\infty$. Given scattering data $\mathcal{S}=\{R(k),k\geq 0\}$ find the potential q on $(0,\infty)$.*

Observe that due to the Borg-Marchenko uniqueness theorem (3.7) and (4.3)

$$R_1=R_2\Longrightarrow q_1=q_2. \tag{4.4}$$

and hence the inverse problem

$$\mathcal{S}\Longrightarrow q \tag{4.5}$$

is well-posed and can be easily solved. From (4.3) one finds

$$m(k^2)=ik\frac{1-R(k)}{1+R(k)},k\geq 0,$$

which transforms Problem 7 into Problem 6.

Note that if the spectrum is positive and purely absolutely continuous then the response function r can be computed by

$$r(t)=\frac{2}{\pi}\lim_{\varepsilon\to 0}\int_0^\infty e^{-\varepsilon k^2}\frac{1-|R(k)|^2}{|1+R(k)|^2}k\sin(kt)\,dk.$$

There are some well known inverse scattering procedures for (4.5) (see, e.g. Aktosun-Klaus [1]) but our particular derivation (using (2.17) in place of the Gelfand-Levitan or Marchenko equations) appears to be new. In addition, some extra assumptions on the potential are usually imposed. Among such assumptions are: absence of bound states, or either $q(x)\to 0$ or $q(x)\to c$ sufficiently fast as $x\to\infty$.

Note in the conclusion that, due to (4.3) , $R(k)$ is analytic in the domain $k^2\in\mathbb{C}_+$ and therefore the knowledge of $R(k)$ on a certain subset of $\mathbb{R}$ is only required to recover the potential q uniquely.

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