On the Uniqueness of Weak Weyl Representations of the Canonical Commutation Relation

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Abstract

Let (T, H) be a weak Weyl representation of the canonical commutation relation (CCR) with one degree of freedom. Namely T is a symmetric operator and H is a self-adjoint operator on a complex Hilbert space $\mathcal{H}$ satisfying the weak Weyl relation: For all $t \in \mathbb{R}$ (the set of real numbers), $e^{-itH}D(T) \subset D(T)$ (i is the imaginary unit and $D(T)$ denotes the domain of T) and $Te^{-itH}\psi = e^{-itH}(T+t)\psi$, $\forall t \in \mathbb{R}, \forall \psi \in D(T)$. In the context of quantum theory where H is a Hamiltonian, T is called a strong time operator of H. In this paper we prove the following theorem on uniqueness of weak Weyl representations: Let $\mathcal{H}$ be separable. Assume that H is bounded below with $\varepsilon_0 := \inf \sigma(H)$ and $\sigma(T) = \{z \in \mathbb{C} | \mathrm{Im}\, z \geq 0\}$, where $\mathbb{C}$ is the set of complex numbers and, for a linear operator A on a Hilbert space, $\sigma(A)$ denotes the spectrum of A. Then $(\overline{T}, H)$ ($\overline{T}$ is the closure of T) is unitarily equivalent to a direct sum of the weak Weyl representation $(-\overline{p}_{\varepsilon_0,+}, q_{\varepsilon_0,+})$ on the Hilbert space $L^2((\varepsilon_0, \infty))$, where $q_{\varepsilon_0,+}$ is the multiplication operator by the variable $\lambda \in (\varepsilon_0, \infty)$ and $p_{\varepsilon_0,+} := -id/d\lambda$ with $D(d/d\lambda) = C_0^\infty((\varepsilon_0, \infty))$. Using this theorem, we construct a Weyl representation of the CCR from the weak Weyl representation $(\overline{T}, H)$.

Keywords: canonical commutation relation, Hamiltonian, weak Weyl representation, Weyl representation, spectrum, time operator.

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1 Introduction and Main Results

A pair (T, H) of a symmetric operator T and a self-adjoint operator H on a complex Hilbert space $\mathcal{H}$ is called a *weak Weyl representation* of the canonical commutation relation (CCR) with one degree of freedom if it obeys the *weak Weyl relation*: For all $t \in \mathbb{R}$ (the set of real numbers), $e^{-itH}D(T) \subset D(T)$ (i is the imaginary unit and $D(T)$ denotes the domain of T) and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \ \forall t \in \mathbb{R}, \forall \psi \in D(T). \tag{1.1}$$

This type of representations of the CCR was first discussed by Schmüdgen [13, 14] from a purely operator theoretical point of view and then by Miyamoto [8] in application to a theory of time operator in quantum theory. In the context of quantum theory where H is a Hamiltonian, T is called a *strong time operator* of H [3, 5]. A recent development on weak Weyl representations is found in [6]. Moreover a generalization of a weak Weyl relation was presented by the present author [2] to cover a wider range of applications to quantum physics including quantum field theory.

It is easy to see that, if (T, H) is a weak Weyl representation, then so are $(\overline{T}, H)$ and $(-T, -H)$, where $\overline{T}$ denotes the closure of T.

In this paper we are concerned with the problem on uniqueness of weak Weyl representations. Before stating the main results on this problem, however, we need some preliminaries.

We denote by $\mathrm{W}(\mathcal{H})$ the set of all the weak Weyl representations on $\mathcal{H}$:

$$\mathrm{W}(\mathcal{H}) := \{(T, H)|(T, H) \text{ is a weak Weyl representation on } \mathcal{H}\}. \tag{1.2}$$

For a linear operator A on a Hilbert space, $\sigma(A)$ (resp. $\rho(A)$) denotes the spectrum (resp. the resolvent set) of A (if A is closable, then $\sigma(A) = \sigma(\overline{A})$). Let $\mathbb{C}$ be the set of complex numbers and

$$\Pi_+ := \{z \in \mathbb{C}|\mathrm{Im}\, z > 0\}, \quad \Pi_- := \{z \in \mathbb{C}|\mathrm{Im}\, z < 0\}. \tag{1.3}$$

In the previous paper [4], we proved the following facts:

Theorem 1.1 [4] *Let $(T, H) \in \mathrm{W}(\mathcal{H})$. Then:*

(i) If H is bounded below, then either $\sigma(T) = \overline{\Pi}_+$ (the closure of Π_+) or $\sigma(T) = \mathbb{C}$.

(ii) If H is bounded above, then either $\sigma(T) = \overline{\Pi}_-$ or $\sigma(T) = \mathbb{C}$.

(iii) If H is bounded, then $\sigma(T) = \mathbb{C}$.

This theorem has to be taken into account in considering the uniqueness problem of weak Weyl representations.

A form of representations of the CCR stronger than weak Weyl representations is known as a *Weyl representation* of the CCR which is a pair (T, H) of *self-adjoint* operators on $\mathcal{H}$ obeying the *Weyl relation*

$$e^{itT}e^{isH} = e^{-its}e^{isH}e^{itT}, \quad \forall t, \forall s \in \mathbb{R}. \tag{1.4}$$

It is well known (the von Neumann uniqueness theorem [9]) that, every Weyl representation on a *separable* Hilbert space is unitarily equivalent to a direct sum of the Schrödinger representation (q, p) on $L^2(\mathbb{R})$, where q is the multiplication operator by the variable $x \in \mathbb{R}$ and $p = -iD_x$ with D_x being the generalized differential operator in x (cf. [3, §3.5], [10, Theorem 4.3.1], [11, Theorem VIII.14]).

It is easy to see that a Weyl representation is a weak Weyl representation (but the converse is not true). Therefore, as far as the Hilbert space under consideration is separable, the non-trivial case for the uniqueness problem of weak Weyl representations is the one where they are *not* Weyl representations. A general class of such weak Weyl representations (T, H) are given in the case where H is semi-bounded (bounded below or bounded above). In this case, T is not essentially self-adjoint [2, Theorem 2.8], implying Theorem 1.1.

Two simple examples in this class are constructed as follows:

Example 1.1 Let $a \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_a^+)$ with $\mathbb{R}_a^+ := (a, \infty)$. Let $q_{a,+}$ be the multiplication operator on $L^2(\mathbb{R}_a^+)$ by the variable $\lambda \in \mathbb{R}_a^+$:

$$D(q_{a,+}) := \left\{ f \in L^2(\mathbb{R}_a^+) | \int_a^\infty \lambda^2 |f(\lambda)|^2 d\lambda < \infty \right\}, \tag{1.5}$$

$$q_{a,+} f := \lambda f, \quad f \in D(q_{a,+}) \tag{1.6}$$

and

$$p_{a,+} := -i \frac{d}{d\lambda} \tag{1.7}$$

with $D(p_{a,+}) = C_0^\infty(\mathbb{R}_a^+)$, the set of infinitely differentiable functions on $\mathbb{R}_a^+$ with bounded support in $\mathbb{R}_a^+$. Then it is easy to see that $q_{a,+}$ is self-adjoint, bounded below with $\sigma(q_{a,+}) = [a, \infty)$ and $p_{a,+}$ is a symmetric operator. Moreover, $(-p_{a,+}, q_{a,+})$ is a weak Weyl representation of the CCR. Hence, as remarked above, $(-\overline{p}_{a,+}, q_{a,+})$ also is a weak Weyl representation.

Note that $p_{a,+}$ is not essentially self-adjoint and

$$\sigma(-p_{a,+}) = \sigma(-\overline{p}_{a,+}) = \overline{\Pi}_+. \tag{1.8}$$

In particular, $\pm\overline{p}_{a,+}$ are maximal symmetric, i.e., they have no non-trivial symmetric extensions (e.g., [12, §X.1, Corollary]).

Example 1.2 Let $b \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_b^-)$ with $\mathbb{R}_b^- := (-\infty, b)$. Let $q_{b,-}$ be the multiplication operator on $L^2(\mathbb{R}_b^-)$ by the variable $\lambda \in \mathbb{R}_b^-$. and

$$p_{b,-} := -i \frac{d}{d\lambda} \tag{1.9}$$

with $D(p_{b,-}) = C_0^\infty(\mathbb{R}_b^-)$. Then $q_{b,-}$ is self-adjoint, bounded above with $\sigma(q_{b,-}) = (-\infty, b]$, $p_{b,-}$ is a symmetric operator, and $(-p_{b,-}, q_{b,-})$ is a weak Weyl representation of the CCR. As in the case of $p_{a,+}$, $p_{b,-}$ is not essentially self-adjoint and

$$\sigma(-p_{b,-}) = \overline{\Pi}_-. \tag{1.10}$$

A relation between $(-p_{a,+}, q_{a,+})$ and $(-p_{b,-}, q_{b,-})$ is given as follows. Let $U_{ab} : L^2(\mathbb{R}_a^+) \to L^2(\mathbb{R}_b^-)$ be a linear operator defined by

$$(U_{ab}f)(\lambda) := f(a+b-\lambda), \quad f \in L^2(\mathbb{R}_a^+), \text{ a.e.} \lambda \in \mathbb{R}_b^-.$$

Then U_{ab} is unitary and

$$U_{ab} q_{a,+} U_{ab}^{-1} = a + b - q_{b,-}, \quad U_{ab} p_{a,+} U_{ab}^{-1} = -p_{b,-}. \tag{1.11}$$

In view of the von Neumann uniqueness theorem for Weyl representations, the pair $(-\overline{p}_{a,+}, q_{a,+})$ (resp. $(-\overline{p}_{b,-}, q_{b,-})$) may be a reference pair in classifying weak Weyl representations (T, H) with H being bounded below (resp. bounded above).

By Theorem 1.1, we can define two subsets of $\mathrm{W}(\mathcal{H})$:

$$\mathrm{W}_+(\mathcal{H}) := \{(T, H) \in \mathrm{W}(\mathcal{H}) | H \text{ is bounded below and } \sigma(T) = \overline{\Pi}_+\}, \tag{1.12}$$
$$\mathrm{W}_-(\mathcal{H}) := \{(T, H) \in \mathrm{W}(\mathcal{H}) | H \text{ is bounded above and } \sigma(T) = \overline{\Pi}_-\}. \tag{1.13}$$

Then, as shown above, $(-p_{a,+}, q_{a,+}) \in \mathrm{W}_+(L^2(\mathbb{R}_a^+))$ and $(-p_{b,-}, q_{b,-}) \in \mathrm{W}_-(L^2(\mathbb{R}_b^-))$.

The main results of the present paper are as follows:

Theorem 1.2 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_+(\mathcal{H})$. Let $\varepsilon_0 := \inf \sigma(H)$. Then there exist mutually orthogonal closed subspaces $\mathcal{H}_\ell$, $\ell = 1, \cdots, N$ (N is a positive integer or ∞) such that the following (i)–(iii) hold:*

(i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$.

(ii) The operators $\overline{T}$ and H are reduced by each $\mathcal{H}_\ell$.

(iii) For each ℓ, there exists a unitary operator $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}_{\varepsilon_0}^+)$ such that

$$U_\ell \overline{T} U_\ell^{-1} = -\overline{p}_{\varepsilon_0,+}, \quad U_\ell H U_\ell^{-1} = q_{\varepsilon_0,+}. \tag{1.14}$$

In particular

$$\sigma(H) = [\varepsilon_0, \infty). \tag{1.15}$$

Remark 1.1 It is known that, for every weak Weyl representation $(T, H) \in \mathrm{W}(\mathcal{H})$ ($\mathcal{H}$ is not necessarily separable), H is purely absolutely continuous [8, 13].

As a corollary of Theorem 1.2, we have the following result:

Theorem 1.3 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_-(\mathcal{H})$. Let $b := \sup \sigma(H)$. Then there exist mutually orthogonal closed subspaces $\mathcal{H}_\ell$, $\ell = 1, \cdots, N$ (N is a positive integer or ∞) such that the following (i)–(iii) hold:*

(i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$.

(ii) The operators $\overline{T}$ and H are reduced by each $\mathcal{H}_\ell$.

(iii) For each ℓ, there exists a unitary operator $V_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}_b^-)$ such that

$$V_\ell \overline{T} V_\ell^{-1} = -\overline{p}_{b,-}, \quad V_\ell H V_\ell^{-1} = q_{b,-}. \tag{1.16}$$

In particular

$$\sigma(H) = (-\infty, b]. \tag{1.17}$$

Proof. As remarked in the second paragraph of this section, $(-T, -H) \in \mathrm{W}_+(\mathcal{H})$ with $a := \inf \sigma(-H) = -b$ and $\sigma(-T) = \overline{\Pi}_+$. Hence, we can apply Theorem 1.2 to conclude that there exist mutually orthogonal closed subspaces $\mathcal{H}_\ell$, $\ell = 1, \cdots, N$ (N is a positive integer or ∞) such that the following (i)–(iii) hold: (i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$; (ii) The operators $-\overline{T}$ and $-H$ are reduced by each $\mathcal{H}_\ell$; (iii) For each ℓ, there exists a unitary operator $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}_a^+)$ such that

$$U_\ell \overline{T} U_\ell^{-1} = \overline{p}_{a,+}, \quad U_\ell H U_\ell^{-1} = -q_{a,+}.$$

By Example 1.2, we have

$$U_{ab} \overline{p}_{a,+} U_{ab}^{-1} = -\overline{p}_{b,-}, \quad U_{ab} q_{a,+} U_{ab}^{-1} = -q_{b,-},$$

where we have used that $a + b = 0$. Hence, putting $V_\ell := U_{ab} U_\ell$, we obtain the desired result. ∎

Remark 1.2 In view of Theorems 1.2 and 1.3, it would be interesting to know when $\sigma(T) = \overline{\Pi}_+$ (resp. $\overline{\Pi}_-$) for $(T, H) \in \mathrm{W}(\mathcal{H})$ with H bounded below (resp. above). Concerning this problem, we have the following results [5]:

(i) Let $(T, H) \in \mathrm{W}(\mathcal{H})$ and H be bounded below. Suppose that, for some $\beta_0 > 0$, $\mathrm{Ran}(e^{-\beta_0 H} T)$ (the range of $e^{-\beta_0 H} T$) is dense in $\mathcal{H}$. Then $\sigma(T) = \overline{\Pi}_+$.

(ii) Let $(T, H) \in \mathrm{W}(\mathcal{H})$ and H be bounded above. Suppose that, for some $\beta_0 > 0$, $\mathrm{Ran}(e^{\beta_0 H} T)$ is dense in $\mathcal{H}$. Then $\sigma(T) = \overline{\Pi}_-$.

2 Some Facts and Proof of Theorem 1.2

To prove Theorem 1.2, we first present some key facts.

Lemma 2.1 *Let S be a closed symmetric operator on $\mathcal{H}$ such that $\sigma(S) = \overline{\Pi}_+$. Then there exists a unique strongly continuous one-parameter semi-group $\{Z(t)\}_{t \geq 0}$ whose generator is iS. Moreover, each $Z(t)$ is an isometry:*

$$Z(t)^* Z(t) = I, \quad \forall t \geq 0. \tag{2.1}$$

Proof. This fact is probably well known. But, for completeness, we give a proof. By the assumption $\sigma(S) = \overline{\Pi}_+$, we have $\sigma(iS) = \{z \in \mathbb{C} | \mathrm{Re}\, z \leq 0\}$. Therefore the positive real axis $(0,\infty)$ is included in the resolvent set $\rho(iS)$ of iS. Since S is symmetric, it follows that

$$\|(iS-\lambda)^{-1}\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Hence, by the Hille-Yosida theorem, iS generates a strongly continuous one-parameter semi-group $\{Z(t)\}_{t\geq 0}$ of contractions. For all $\psi \in D(iS) = D(S)$, $Z(t)\psi$ is in $D(S)$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt}Z(t)\psi = iSZ(t)\psi = Z(t)iS\psi.$$

This equation and the symmetricity of S imply that $\|Z(t)\psi\|^2 = \|\psi\|^2, \forall t \geq 0$. Hence (2.1) follows. ∎

Lemma 2.2 *Let $(T,H) \in \mathrm{W}_+(\mathcal{H})$. Then there exists a unique strongly continuous one-parameter semi-group $\{U_T(t)\}_{t\geq 0}$ whose generator is $i\overline{T}$. Moreover, each $U_T(t)$ is an isometry and*

$$U_T(t)e^{-isH} = e^{its}e^{-isH}U_T(t), \quad t \geq 0, s \in \mathbb{R}. \tag{2.2}$$

Proof. We can apply Lemma 2.1 to $S = \overline{T}$ to conclude that $i\overline{T}$ generates a strongly continuous one-parameter semi-group $\{U_T(t)\}_{t\geq 0}$ of isometries on $\mathcal{H}$. For all $\psi \in D(\overline{T})$ and all $t \geq 0$, $U_T(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt}U_T(t)\psi = i\overline{T}U_T(t)\psi = U_T(t)i\overline{T}\psi.$$

Let $s \in \mathbb{R}$ be fixed and $V(t) := e^{its}e^{-isH}U_T(t)e^{isH}$. Then $\{V(t)\}_{t\geq 0}$ is a strongly continuous one-parameter semi-group of isometries. Let $\psi \in D(\overline{T})$. Then $e^{-isH}\psi \in D(\overline{T})$ and

$$\overline{T}e^{-isH}\psi = e^{-isH}\overline{T}\psi + se^{-isH}\psi.$$

Hence $V(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in t with

$$\frac{d}{dt}V(t)\psi = i\overline{T}V(t)\psi.$$

This implies that $V(t)\psi = U_T(t)\psi, \forall t \in \mathbb{R}$. Since $D(\overline{T})$ is dense, it follows that $V(t) = U_T(t), \forall t \in \mathbb{R}$, implying (2.2). ∎

Let $a \in \mathbb{R}$ be fixed. For each $t \geq 0$, we define a linear operator $U_a(t)$ on $L^2(\mathbb{R}_a^+)$ as follows: For each $f \in L^2(\mathbb{R}_a^+)$,

$$(U_a(t)f)(\lambda) := \begin{cases} f(\lambda - t) & \lambda > t + a \\ 0 & a < \lambda \leq t + a \end{cases} \tag{2.3}$$

Then it is easy to see that $\{U_a(t)\}_{t\geq 0}$ is a strongly continuous one-parameter semi-group of isometries on $L^2(\mathbb{R}_+^a)$.

Lemma 2.3 *The generator of $\{U_a(t)\}_{t\geq 0}$ is $-i\overline{p}_{a,+}$.*

Proof. Let iA be the generator of $\{U_a(t)\}_{t\geq 0}$. Then it follows from the isometry of $U_a(t)$ that A is a closed symmetric operator. It is easy to see that $-p_{a,+} \subset A$ and hence $-\overline{p}_{a,+} \subset A$. As already remarked in Example 1.1, $-\overline{p}_{a,+}$ is maximal symmetric. Hence $A = -\overline{p}_{a,+}$. ∎

We recall a result of Bracci and Picasso [7]. Let $\{U(\alpha)\}_{\alpha\geq 0}$ and $\{V(\beta)\}_{\beta\in\mathbb{R}}$ be a strongly continuous one-parameter semi-group and a strongly continuous one-parameter unitary group on $\mathcal{H}$ respectively, satisfying

$$U(\alpha)^* U(\alpha) = I, \quad \alpha \geq 0, \tag{2.4}$$

$$U(\alpha)V(\beta) = e^{i\alpha\beta} V(\beta)U(\alpha), \quad \alpha \geq 0, \beta \in \mathbb{R}. \tag{2.5}$$

Then, by the Stone theorem, there exists a unique self-adjoint operator P on $\mathcal{H}$ such that

$$V(\beta) = e^{-i\beta P}, \quad \beta \in \mathbb{R}. \tag{2.6}$$

Lemma 2.4 [7] *Let $\mathcal{H}$ be separable. Suppose that P is bounded below with $\nu := \inf \sigma(P)$. Then there exist mutually orthogonal closed subspaces $\mathcal{H}_\ell$, $\ell = 1, \cdots, N$ (N is a positive integer or ∞) such that the following (i)–(iii) hold:*

(i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$.

(ii) For all $\alpha \geq 0$ and $\beta \in \mathbb{R}$, the operators $U(\alpha)$ and $V(\beta)$ leave $\mathcal{H}_\ell$ invariant for all $\ell \in \mathbb{N}$.

(iii) For each ℓ, there exists a unitary operator $S_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}_\nu^+)$ such that

$$S_\ell V(\beta) S_\ell^{-1} = e^{-i\beta q_{\nu,+}}, \beta \in \mathbb{R}, \tag{2.7}$$

$$S_\ell U(\alpha) S_\ell^{-1} = U_\nu(\alpha), \quad \alpha \geq 0. \tag{2.8}$$

Remark 2.1 This lemma is not the original form of a result in the paper [7], since they consider the case where the $*$-algebra $\mathcal{W}_+$ generated by $\{U(\alpha), V(\beta) | \alpha \geq 0, \beta \in \mathbb{R}\}$ is irreducible. But, if the Hilbert space under consideration is separable, then it is easy to see that the representation of $\mathcal{W}_+$ is decomposed into a direct sum of irreducible representations of it. In this way, Lemma 2.4 follows from a result in [7, §VII].

We denote the generator of $\{U(\alpha)\}_{\alpha\geq 0}$ by iQ. It follows that Q is closed and symmetric.

Lemma 2.5 *Let $\mathcal{H}_\ell$, S_ℓ and ν be as in Lemma 2.4. Then P and Q are reduced by each $\mathcal{H}_\ell$ and*

$$S_\ell P S_\ell^{-1} = q_{\nu,+}, \tag{2.9}$$

$$S_\ell Q S_\ell^{-1} = -\overline{p}_{\nu,+}. \tag{2.10}$$

In particular

$$\sigma(P) = [\nu, \infty). \tag{2.11}$$

Proof. Lemma 2.4-(ii) and (2.7) imply (2.9). Simlarly (2.10) follows from Lemma 2.4-(ii), (2.8) and Lemma 2.3. ∎

Proof of Theorem 1.2

By Lemma 2.2, we can apply Lemma 2.4 to the case where $V(\beta) = e^{-i\beta H}, \beta \in \mathbb{R}$ and $U(\alpha) = U_T(\alpha), \alpha \geq 0$. Then the desired results follow from Lemmas 2.4 and 2.5.

3 Examples

Example 3.1 Let $\mathbb{R}^d_{\mathbf{x}} = \{\mathbf{x} = (x_1, \cdots, x_d) | x_j \in \mathbb{R}, j = 1, \cdots, d\}$. We denote by q_j the j-th position operator on $L^2(\mathbb{R}^d_{\mathbf{x}})$ (the multiplication operator by the j-th variable x_j) and $p_j := -iD_j$ the j-th momentum operator, where D_j is the generalized partial differential operator in x_j. The free Hamiltonian for a non-relativistic quantum particle with mass $M > 0$ is given by

$$H_0 := -\frac{1}{2M}\Delta,$$

where $\Delta := \sum_{j=1}^d D_j^2$ is the generalized Laplacian on $L^2(\mathbb{R}^d_{\mathbf{x}})$. It is well known that H_0 is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^d_{\mathbf{x}})$ and absolutely continuous with $\sigma(H_0) = [0, \infty)$.

We denote by $\mathcal{F} : L^2(\mathbb{R}^d_{\mathbf{x}}) \to L^2(\mathbb{R}^d_{\mathbf{k}})$ the Fourier transform:

$$(\mathcal{F}f)(\mathbf{k}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d_{\mathbf{x}}} e^{-i\mathbf{k}\mathbf{x}} f(\mathbf{x}) d\mathbf{x}, \quad f \in L^2(\mathbb{R}^d_{\mathbf{x}})$$

in the L^2 sense. Let

$$\mathbb{M}_j := \left\{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{R}^d_{\mathbf{k}} | k_j \neq 0\right\} \subset \mathbb{R}^d_{\mathbf{k}}$$

For each $j = 1, \cdots, d$, we define

$$T_j^{\mathrm{AB}} := \frac{M}{2}\left(q_j p_j^{-1} + p_j^{-1} q_j\right)$$

with $D(T_j^{\mathrm{AB}}) := \mathcal{F}^{-1} C_0^\infty(\mathbb{M}_j)$. It is easy to see that (T_j^{AB}, H_0) is a weak Weyl representation of the CCR [2, 8]. The operator T_j^{AB} is called the *Aharonov-Bohm time operator* [1]. In the previous paper [5], we proved that $\sigma(T_j^{\mathrm{AB}}) = \overline{\Pi}_+$. Hence $(T_j^{\mathrm{AB}}, H_0) \in \mathrm{W}_+(L^2(\mathbb{R}^d_{\mathbf{x}}))$. Note that $\inf \sigma(H_0) = 0$. Thus we can apply Theorem 1.2 to conclude that $(\overline{T}_j^{\mathrm{AB}}, H_0)$ is unitarily equivalent to a direct sum of the weak Weyl representation $(-\overline{p}_{0,+}, q_{0,+})$ on $L^2((0, \infty))$.

Example 3.2 (A relativistic time operator [2]) The free Hamiltonian for a relativistic quantum particle with mass $m \geq 0$ and spin 0 is given by

$$H_{\mathrm{rel}} := \sqrt{-\Delta + m^2}$$

acting in $L^2(\mathbb{R}^d_{\mathbf{x}})$. For each $j = 1, \cdots, d$, we define

$$T_j^{\mathrm{rel}} := \frac{1}{2}\left(H_{\mathrm{rel}} p_j^{-1} q_j + q_j p_j^{-1} H_{\mathrm{rel}}\right)$$

with $D(T_j^{\rm rel}) := \mathcal{F}^{-1}C_0^\infty(\mathbb{M}_j)$. As is shown in [2], $(T_j^{\rm rel}, H_{\rm rel})$ is a weak Weyl representtion. Moreover $\sigma(T_j^{\rm rel}) = \overline{\Pi}_+$ [4]. Hence $(T_j^{\rm rel}, H_{\rm rel}) \in {\rm W}_+(L^2(\mathbb{R}^d_{\mathbf{x}}))$. Note that $\inf \sigma(H_{\rm rel}) = m$. Thus we can apply Theorem 1.2 to conclude that $(\overline{T}_j^{\rm rel}, H_0)$ is unitarily equivalent to a direct sum of the weak Weyl representation $(-\overline{p}_{m,+}, q_{m,+})$ on $L^2((m,\infty))$.

4 Construction of a Weyl representation from a weak Weyl representation

In the previous paper [6], a general structure was found to construct a Weyl representation from a weak Weyl representation. Here we recall it.

Theorem 4.1 [6, Corollary 2.6] *Let (T,H) be a weak Weyl representation on a Hilbert space $\mathcal{H}$. Then the operator*

$$L := \log |H| \tag{4.1}$$

is well-defined, self-adjoint and the operator

$$D := \frac{1}{2}(TH + \overline{HT}) \tag{4.2}$$

is a symmetric operator. Moreover, if D is essentially self-adjoint, then $(\overline{D}, L)$ is a Weyl representation of the CCR and $\sigma(|H|) = [0,\infty)$.

To apply this theorem, we need a lemma.

Lemma 4.2 *Let $a \in \mathbb{R}$ and*

$$d_a := -\frac{1}{2}(p_{a,+}q_{a,+} + \overline{q_{a,+}p_{a,+}}) \tag{4.3}$$

acting in $L^2(\mathbb{R}_a^+)$. Then d_a is essentially self-adjoint if and only if $a \leq 0$.

Proof. Let $a > 0$. Then the function u on $\mathbb{R}_a^+$ defined by $u(\lambda) = 1/\lambda^{3/2}$, $\lambda > a$ is in $C^\infty(\mathbb{R}_a^+) \cap L^2(\mathbb{R}_a^+)$ with $\lambda u'(\lambda) = -(3/2)u(\lambda)$. In the present case, we have $D(p_{a,+}q_{a,+}) = C_0^\infty(\mathbb{R}_a^+) = D(p_{a,+})$. Hence $D(d_a) = C_0^\infty(\mathbb{R}_a^+)$. It follows that, for all $f \in D(d_a)$, $\langle u, (d_a - i)f\rangle = 0$. This implies that $u \in \ker(d_a^* + i)$ and hence $\ker(d_a^* + i) \neq \{0\}$. Therefore d_a is not essentially self-adjoint. Thus, if d_a is essentially self-adjoint, then $a \leq 0$.

Conversely, let $a \leq 0$ and $v \in \ker(d_a^* + i)$. Then, for all $f \in C_0^\infty(\mathbb{R}_a^+)$, $\langle v, (d_a - i)f\rangle = 0$. This implies the distribution equation $\lambda D_\lambda v(\lambda) = -(3/2)v(\lambda)$ on $\mathbb{R}_a^+$. Hence $v(\lambda) = c_1/|\lambda|^{3/2}$ for a.e. $\lambda \in \mathbb{R}_a^+$ with a constant c_1. Since v is in $L^2(\mathbb{R}_a^+)$, it follows that $c_1 = 0$ and hence $v = 0$. Thus $\ker(d_a^* + i) = \{0\}$.

Next, let $w \in \ker(d_a^* - i)$. Then, in the same way as in the preceding case, we have $w(\lambda) = c_2|\lambda|^{1/2}$ with a constant c_2. Since w is in $L^2(\mathbb{R}_a^+)$, it follows that $c_2 = 0$ and hence $w = 0$. Thus $\ker(d_a^* - i) = \{0\}$. By a general criterion on essential self-adjointness, we conclude that d_a is essentially self-adjoint. ∎

Now we can prove the following theorem.

Theorem 4.3 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_+(\mathcal{H})$ with $\varepsilon_0 = \inf \sigma(H)$. Let L and D be as in (4.1) and (4.2) respectively. Then:*

(i) D is essentially self-adjoint if and only if $\varepsilon_0 \leq 0$.

(ii) $(\overline{D}, L)$ is a Weyl representation of the CCR if and only if $\varepsilon_0 \leq 0$.

Proof. (i) By Theorem 1.2, $\overline{D}$ is unitarily equivalent to a direct sum of $\overline{d}_{\varepsilon_0}$. Hence, by Lemma 4.2, D is essentially self-adjoint if and only if $\varepsilon_0 \leq 0$.

(ii) Let $(\overline{D}, L)$ be a Weyl representation of the CCR. This means that D is essentially self-adjoint. Hence, by part (i), $\varepsilon_0 \leq 0$.

Conversely let $\varepsilon_0 \leq 0$. Then, by part (i), D is essentially self-adjoint. Hence, by Theorem 4.1, $(\overline{D}, L)$ is a Weyl representation of the CCR. ∎

Finally we remark on the case where $(T, H) \in \mathrm{W}_-(\mathcal{H})$:

Corollary 4.4 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_-(\mathcal{H})$ with $\mu = \sup \sigma(H)$. Let L and D be as in (4.1) and (4.2) respectively. Then*

(i) D is essentially self-adjoint if and only if $\mu \geq 0$.

(ii) $(\overline{D}, L)$ is a Weyl representation of the CCR if and only if $\mu \geq 0$.

Proof. We have $(-T, -H) \in \mathrm{W}_+(\mathcal{H})$ with $\inf \sigma(-H) = -\mu$. The operator D (resp. L) for $(-T, -H)$ is the same as that for (T, H). Hence the conclusions (i) and (ii) follow from Theorem 4.3. ∎

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