EIGENVALUE ESTIMATES FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

ARI LAPTEV AND OLEG SAFRONOV

1. Introduction

Throughout the paper, $f_\pm$ denotes either the positive or the negative part of f, which is, in its turn, either a function or a selfadjoint operator. The symbols $\Re z$ and $\Im z$ denote the real and the imaginary part of z. Finally, if a is a function on $\mathbb{R}^d$, then $a(i\nabla)$ is the operator whose integral kernel is $(2\pi)^{-d}\int e^{i\xi(x-y)}a(\xi)d\xi$.

We consider the Schrödinger operator $H = -\Delta + V$ with a complex potential V and then we say something about the distribution of eigenvalues of H in the complex plane. Assume for simplicity that $\lim_{|x|\to\infty} V(x) = 0$.

The main result of [4] tells us, that for any $t > 0$, the eigenvalues z_j of H lying outside the sector $\{z : \ |\Im z| < t\,\Re z\}$ satisfy the estimate

$$\sum |z_j|^\gamma \le C\int |V(x)|^{\gamma+d/2}dx, \qquad \gamma \ge 1,$$

where the constant C depends on t, γ and d.

A natural question that appears in relation to this result is what estimates are valid for the eigenvalues situated inside the conical sector $\{z : \ |\Im z| < t\Re z\}$, where the eigenvalues might be close to the positive half-line? Our theorems provide some information about the rate of accumulation of eigenvalues to the set $\mathbb{R}_+ = [0,\infty)$. Namely, Theorems 1.1-1.4 give sufficient conditions on V that guarantee convergence of the sum

$$\sum_{a<\Re z_j<b} |\Im z_j|^\gamma < \infty$$

for $0 \le a < b < \infty$.

Theorem 1.1. *Let $H = -\Delta + V$ be the Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \ge 2$ and let $W = (|V|^2 + 4\Im V)_+$. Let z_j be the eigenvalues*

The authors would like to thank Grigori Rozenblioum, Rupert Frank and Robert Seiringer for their remarks.

of the operator H lying inside the semi-infinite strip $\Pi_b = \{z : \ 0 < \Re z < b,\ \Im z > 0\}$. Then for any $\gamma > 3/2$ and $r \in (\gamma - \frac{1}{2}, \gamma)$

$$\sum_{z_j \in \Pi_b} |\Im z_j|^\gamma \le C |\Psi_b(W)|^{\frac{2\gamma-1}{2r-1}} (b + |\Psi_b(W)|^{\frac{1}{2r-1}})$$

where

$$\Psi_b(W) = \int_{\mathbb{R}^d} W^{d/4-1/2+r}\,dx + b^{d/2-1} \int_{\mathbb{R}^d} W^r\,dx, \quad d \ge 2.$$

The constant in this inequality depends on d, γ and r.

Applying the same method one can prove the following statement.

Theorem 1.2. *Let z_j be the eigenvalues of the operator $H = -d^2/dx^2 + V$ lying inside the semi-infinite strip $\Pi = \{z : \ a < \Re z < b,\ \Im z > 0\}$ with $a > 0$. Let $W = (|V|^2 + 4\Im V)_+$. Then for any $\gamma > 3/2$ and $r \in (\gamma - \frac{1}{2}, \gamma)$*

$$\sum_{z_j \in \Pi} |\Im z_j|^\gamma \le C |\Psi_a(W)|^{\frac{2\gamma-1}{2r-1}} (b + |\Psi_a(W)|^{\frac{1}{2r-1}})$$

where

$$\Psi_a(W) = a^{-1/2} \int_{\mathbb{R}} W^r\,dx.$$

The constant in this inequality depends on γ and r.

An interesting property of Theorems 1.1 and 1.2 is that the lower borderline for γ is always $3/2$ in any dimension. Now we will try to formulate some results that are valid for smaller values of γ.

Theorem 1.3. *Let $\Re V \ge 0$ be a bounded function. Assume that $\Im V \in L^p(\mathbb{R}^d)$, where $p > d/2$ if $d \ge 2$ and $p \ge 1$ if $d = 1$. Then the eigenvalues λ_j of the operator $H = -\Delta + V$ satisfy the estimate*

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \le C \int_{\mathbb{R}^d} \Im V_+^p(x)\,dx. \tag{1.1}$$

The constant C in this inequality can be computed explicitly:

$$C = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{(\xi^2 + 1)^p}. \tag{1.2}$$

The right hand side of the estimate (1.1) does not contain the potential $\Re V$. This means that the conditions on $\Re V$ can be drastically relaxed. It is not the case when we try to obtain an estimate of the sum $\sum_j (\Im \lambda_j / (|\lambda_j + 1|^2 + 1))_+^p$ for $p \le d/2$. A certain regularity of $\Re V$ is required in this case because of an essential reason.

Theorem 1.4. *Let $\Re V \geq 0$ and $\Im V$ be two bounded real valued functions. Assume that $\Im V \in L^p(\mathbb{R}^d)$, where $p > d/4$ if $d \geq 4$ and $p \geq 1$ if $d \leq 3$. Then the eigenvalues λ_j of the operator $H = -\Delta + V$ satisfy the estimate*

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \leq (1 + ||V||_\infty)^{2p} C \int_{\mathbb{R}^d} \Im V_+^p(x)\, dx. \tag{1.3}$$

The constant C in this inequality can be computed explicitly:

$$C = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{((\xi^2 + 1)^2 + 1)^p}. \tag{1.4}$$

One should mention, that the paper [4] in its turn was motivated by the question of E.B. Davies about an integral estimate for eigenvalues of H (see [1] and [3]). If $d = 1$ then all eigenvalues λ of H which do not belong to $\mathbb{R}_+$ satisfy

$$|\lambda| \leq \frac{1}{4} \Big(\int |V(x)| dx \Big)^2.$$

The question is whether a similar integral estimate holds in dimension $d \geq 2$. By the word "similar", we mean an estimate by the L^p norm of the potential V with $p > d/2$. So, the problem can be formulated as a hypothesis in the following way:

Conjecture 1.1. Let $d \geq 2$ and let $\gamma > 0$ be given. There is a positive constant C such that

$$|\lambda|^\gamma \leq C \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma} dx, \tag{1.5}$$

for every complex valued potential $V \in C_0^\infty$ and every eigenvalue $\lambda \notin \mathbb{R}_+$ of the operator $-\Delta + V$.

So far, we are able to prove only the following result related to this conjecture:

Theorem 1.5. *Let V be a function from $L^p(\mathbb{R}^d)$, where $p \geq d/2$, if $d \geq 3>$; $p > 1$, if $d = 2$, and $p \geq 1$, if $d = 1$. Then every eigenvalue λ of the operator $H = -\Delta + V$ with the property $\Re \lambda > 0$ satisfies the estimate*

$$|\Im \lambda|^{p-1} \leq |\lambda|^{d/2-1} C \int_{\mathbb{R}^d} |V|^p dx. \tag{1.6}$$

The constant C in this inequality depends only on d and p. Moreover, $C = 1/2$ for $p = d = 1$.

The relation (1.6) was established in [1] in the case $d = p = 1$. We prove it in higher dimensions and in dimension $d = 1$ for $p > 1$.

We also know the elementary estimate (see Theorem 8.2)

$$|\Im\sqrt{\lambda}|^{2\gamma} \leq C \int_{\mathbb{R}^3} |V|^{3/2+\gamma} dx, \qquad \gamma > 0,\ d = 3,$$

however it is not quite the same as (1.5). While we do not prove Conjecture 1.1 directly, we find some interesting information about the location of eigenvalues of the operator $-\Delta + iV$ with a positive $V \geq 0$. In particular, in $d = 3$, we obtain that if $\int V dx$ is small and $\lambda \notin \mathbb{R}_+$ is an eigenvalue of $-\Delta + iV$, then $|\lambda|$ must be large. It might seem that eigenvalues do not exist at all for small values of $\int V dx$, however their presence in such cases can be easily established with the help of the following statement.

Proposition 1.1. *Let $d \geq 3$. Then there is a sequence of positive functions $V_n \geq 0$ such that the "largest modulus" eigenvalue $\lambda_n \notin \mathbb{R}_+$ of the operator $-\Delta + iV_n$ satisfies $|\lambda_n| \to \infty$ as $n \to \infty$, while $\lim_{n\to\infty} \int V_n(x)dx = 0$.*

Proof. If λ is an eigenvalue of $-\Delta + iV(x)$, then $n^2\lambda$ is an eigenvalue of $-\Delta + n^2 iV(nx)$. It remains to note that $\int n^2 V(nx)dx = Cn^{2-d}$. The proof of existence of a non-real eigenvalue of $-\Delta + iV(x)$ at least for one $V \geq 0$ is left to the reader. □

Remark The proposition does not contradict Conjecture 1.1.

Note that our theorems imply also that the eigenvalues of $-\Delta + iV$ can not accumulate to zero in $d = 3$, if $V \geq 0$ is integrable.

2. Preliminaries

1. Consider a non self-adjoint operator $T = R + iB$ where R and B are self-adjoint operators. Suppose that R is semibounded from below and has a discrete negative spectrum. If $B(R - i)^{-1}$ is compact, then the operator T has only discrete set of eigenvalues in the left half of the plane $\mathbb{C}_{left} = \{z : \ \Re z < 0\}$. Moreover, suppose that $\lambda_j \in \mathbb{C}_{left}$ are eigenvalues of the operator T, and s_j are negative eigenvalues of R enumerated in the order of increasing real parts. Then

$$\Big|\Re \sum_1^n \lambda_j\Big| \leq \sum_1^n |s_j|$$

for all n. Indeed, let P be the orthogonal projection onto the span of eigenvectors corresponding to λ_j, $1 \le j \le n$. Then

$$\operatorname{tr} TP = \sum_1^n \lambda_j$$

Consequently,

$$\Re \sum_1^n \lambda_j = \operatorname{tr} RP \ge \min_P \operatorname{tr} RP \tag{2.1}$$

where the minimum is taken over all orthogonal projections of rank n. Thus,

$$\sum_1^n \Re\lambda_j \ge \sum_1^n s_j \tag{2.2}$$

since the minimum in the right hand side of (2.1) coincides with the sum in the right hand side of (2.2)

Corollary 2.1. *Let $\gamma > 0$. Then*

$$\sum_1^n (\Re\lambda_j + \gamma)_- \le \sum_1^n (s_j + \gamma)_-$$

2. Let T be a bounded operator in a Hilbert space, whose spectrum outside the unit circle $\{z: \ |z| > 1\}$ is discrete. Suppose also that the essential spectrum of the operator $(T^*T)^{1/2}$ is contained in $[0, 1]$ Let λ_j be the eigenvalues of the operator T lying outside of the unit circle, and let $s_j > 1$ be the eigenvalues of $(T^*T)^{1/2}$. If we enumerate the sequences $|\lambda_j|$ and s_j in the decreasing order, then

$$\prod_1^n |\lambda_j| \le \prod_1^n s_j \tag{2.3}$$

for all values of n. One should mention also that, if one of the sequences ends at $j = j_0$, we extend it for $j > j_0$ by 1-s.

This inequality has been discovered for compact operators by H. Weyl. Weyl's proof is carried over to the case of bounded operators without any difference. Indeed, let P be the orthogonal projection onto the span of eigenvectors corresponding to λ_j, $1 \le j \le n$. Then for any $\alpha > 0$

$$\det\left(I + P(\alpha T^*T - I)P\right) = \alpha^n \prod_1^n |\lambda_j|^2$$

Consequently,

$$\alpha^n \prod_1^n |\lambda_j|^2 \leq \det(I+P(\alpha T^*T-I)_+P) \leq \det(I+(\alpha T^*T-I)_+^{1/2}P(\alpha T^*T-I)_+^{1/2}).$$

Removing the orthogonal projection, we obtain

$$\alpha^n \prod_1^n |\lambda_j|^2 \leq \det(I+(\alpha T^*T-I)_+) = \prod_{\alpha s_j^2>1} \alpha s_j^2$$

It remains to choose $\alpha = s_n^{-1}$. Note that if the number of $s_j > 1$ is finite, we can take $\alpha = 1$ to obtain that

$$\prod_1^n |\lambda_j|^2 \leq \prod_{s_j^2>1} s_j^2$$

for all n.

Corollary 2.2. *Let $\gamma \geq 1$. Then*

$$\sum_1^n (|\lambda_j|^2 - 1)^\gamma \leq \sum_1^n (s_j^2 - 1)^\gamma$$

for all n.

Proof. Consider first the case $\gamma > 1$. As a consequence of (2.3), we obtain that

$$\sum_1^n \log|\lambda_j| \leq \sum_1^n \log s_j. \tag{2.4}$$

Moreover,

$$\sum_{j=1}^n (\log|\lambda_j| - \eta)_+ \leq \sum_{j=1}^n (\log s_j - \eta)_+ \tag{2.5}$$

for any $-\infty < \eta < \infty$. Note now that the function $\phi(t) = (e^{2t}-1)^\gamma$ is representable in the form

$$\phi(\lambda) = \int_0^\infty (\lambda - t)_+ \phi''(t)\,dt \quad \text{and} \quad \phi''(t) \geq 0 \ \text{ for } \ t \geq 0.$$

Since $\phi(\log\lambda) = (\lambda^2 - 1)^\gamma$, the statement of Corollary 2.2 for $\gamma > 1$ follows from (2.5).

If $\gamma = 1$, then one only needs to prove that

$$\sum_1^n |\lambda_j|^2 \leq \sum_1^n s_j^2$$

In order to do that we consider the function $\psi(t) = e^{2t}$, which is representable in the form

$$\psi(\lambda) = 1 + 2\lambda + \int_0^\infty (\lambda - t)_+ \psi''(t)\, dt \quad \text{with} \quad \psi''(t) > 0.$$

Since $\psi(\log \lambda) = \lambda^2$, the statement of Corollary 2.2 for $\gamma = 1$ follows from (2.4) and (2.5). $\square$

3. Let T be a compact operator in a Hilbert space. Then the square roots of eigenvalues of T^*T are called s-numbers. Let us introduce the distribution function $n(s, T)$ of s-numbers s_j of an operator T by the equality

$$n(s, T) = \text{card}\{j : \quad s_j > s\}, \quad s > 0.$$

Note that n satisfies the so called Ki-Fan inequality

$$n(s_1 + s_2, T_1 + T_2) \le n(s_1, T_1) + n(s_2, T_2),$$

for any pair of compact operators T_1 and T_2 and any pair of positive numbers s_1 and s_2. The class of operators T for which the following quantity

$$[T]_p^p := \sup_{s>0} s^p n(s, T) < \infty$$

is finite, is called the weak Neumann-Schatten class Σ_p.

Theorem 2.1 (M.Cwikel [2]). *Let Φ be the operator of Fourier transformation and let α and β be the operators of multiplication by the functions $\alpha(\xi)$ and $\beta(x)$. Suppose that β is in the space $L^q(\mathbb{R}^d)$ with $q > 2$ and*

$$[\alpha]_q^q = \sup_{t>0} t^q \text{meas}\{\xi \in (\mathbb{R}^d : \quad |\alpha(\xi)| > t\} < \infty.$$

Then the operator $T = \alpha \Phi \beta$ as (well as the operator $\beta \Phi^* \alpha$) is in Σ_p and

$$[T]_q^q \le C[\alpha]_q^q \int |\beta(x)|^q \, dx. \tag{2.6}$$

The main applications of Theorem 2.1 in mathematical physics are related to the following fact:

Proposition 2.1 (Birman-Schwinger principle). *Let A and V be two positive self-adjoint operators acting in the same Hilbert space. Suppose that V is bounded and the operator $\sqrt{V}(A + I)^{-1/2}$ is compact. Then the number $N(E)$ of eigenvalues of the operator $A - V$ lying to the left of a negative point $-E$ satisfies the relation*

$$N(E) = n(1, \sqrt{V}(A + E)^{-1}\sqrt{V}).$$

In applications, A is a differential operator and V is the operator of multiplication by a function.

4. Let T be a compact operator in a Hilbert space. Then the square roots of eigenvalues of T^*T are called s-numbers. The class of compact operators T for which the following quantity

$$||T||_p^p = \sum_j s_j^p < \infty, \qquad p \geq 1,$$

is finite, is called the Neumann-Schatten class $\mathfrak{S}_p$. It is easy to see that the functional $||T||_p = \big(\mathrm{tr}(T^*T)^{p/2}\big)^{1/p}$ has all properties of a norm on $\mathfrak{S}_p$.

The next theorem gives a sufficient condition guaranteeing that an operator of the form $b(x)a(i\nabla)$ belongs to the class $\mathfrak{S}_p$.

Theorem 2.2. *Let Φ be the operator of Fourier transformation and let a and b be the operators of multiplication by the functions $a(\xi)$ and $b(x)$. Suppose that a and b are in the space $L^p(\mathbb{R}^d)$ where $p \geq 2$ Then the operator $T = b\Phi^* a$ is in $\mathfrak{S}_p$ and*

$$||T||_p^p \leq (2\pi)^{-d} \int |a(\xi)|^p \, d\xi \int |b(x)|^p \, dx. \tag{2.7}$$

This theorem can be found in [6]. See also [5] and [7].

5. Below, we will need the following result about eigenvalue estimates for a certain operator with constant coefficients perturbed by a potential V. It is one of the consequences of the inequality (2.6).

Proposition 2.2. *Let $a(\xi) = (\xi^2 - \mu)^2$, $\xi \in \mathbb{R}^d$, and $V(x) \geq 0$ be two functions on $\mathbb{R}^d$. Suppose that $V \in C_0^\infty(\mathbb{R}^d)$. Let $N(E)$ be the number of eigenvalues of the operator $a(i\nabla) - V(x)$ lying to the left of the point $-E$ where $E > 0$. Then, for any $p > 1/2$,*

(2.8)

$$N(E) \leq \frac{C}{E^p}\Big(\int_{\mathbb{R}^d} V^{p+d/4} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} V^{p+1/2} dx\Big), \qquad \text{if} \quad d \geq 2;$$

$$N(E) \leq \frac{C}{E^p \mu^{1/2}} \int_{\mathbb{R}^d} V^{p+1/2} dx, \qquad \text{if} \quad d = 1. \tag{2.9}$$

Proof. The reasoning is based on the elementary application of the Cwikel estimate. Indeed, according to Birman-Schwinger principle

$$N(E) = n(1, X),$$

where X is the compact operator defined by the equality

$$X = \sqrt{V}(a(i\nabla) + E)^{-1}\sqrt{V}.$$

Let χ be the characteristic function of the ball $\{\xi \in \mathbb{R}^d : \quad \xi^2 \leq \mu\}$. Represent X in the form $X = X_1 + X_2$, where

$$X_1 = \sqrt{V}(a(i\nabla) + E)^{-1}\chi(i\nabla)\sqrt{V}.$$

According to the Ki-Fan inequality,

$$n(1, X) \leq n(1, 2X_1) + n(1, 2X_2). \tag{2.10}$$

Therefore it is sufficient to estimate each term in the right hand side of (2.10) separately. We begin with the first term. Set $\alpha = p + d/4$. Then, according to (2.6),

$$n(1, 2X_1) \leq C_0 \int V^\alpha dx \int_{\xi^2>\mu} \frac{d\xi}{((\xi^2 - \mu)^2 + E)^\alpha} \leq$$

$$\leq C_1 \int V^\alpha dx \int_\mu^\infty \frac{s^{d/2-1}ds}{((s-\mu)^2 + E)^\alpha} \leq C_2 \int V^\alpha dx \int_0^\infty \frac{s^{d/2-1}ds}{(s^2 + E)^\alpha} =$$

$$= \frac{C}{E^p} \int_{\mathbb{R}^d} V^{p+d/4} dx.$$

Let us now estimate the second term in (2.10). Set $\beta = p + 1/2$. According to (2.6),

$$n(1, 2X_2) \leq C_3 \int V^\beta dx \int_{\xi^2<\mu} \frac{d\xi}{((\xi^2 - \mu)^2 + E)^\beta} \leq$$

$$\leq C_4 \int V^\beta dx \int_0^\mu \frac{s^{d/2-1}ds}{((s-\mu)^2 + E)^\beta} \leq C_5 \int V^\beta dx \int_{-\infty}^\infty \frac{\mu^{d/2-1}ds}{(s^2 + E)^\beta} =$$

$$= \frac{C\mu^{d/2-1}}{E^p} \int_{\mathbb{R}^d} V^{p+1/2} dx.$$

Thus,

$$n(1, 2X_1) \leq \frac{C}{E^p} \int_{\mathbb{R}^d} V^{p+d/4} dx \quad \text{and} \quad n(1, 2X_2) \leq \frac{C\mu^{d/2-1}}{E^p} \int_{\mathbb{R}^d} V^{p+1/2} dx.$$

Therefore (2.10) implies the estimate (2.8) for $d \geq 2$.

Consider now the case $d = 1$. The arguments for $d = 1$ are different from the arguments for $d \geq 2$. First of all, note that $(\xi^2 - \mu)^2 = (|\xi| - \sqrt{\mu})^2(|\xi| + \sqrt{\mu})^2 \geq= (|\xi| - \sqrt{\mu})^2\mu$. Consequently, for any $\beta > 1$,

$$\int_\infty^\infty \frac{d\xi}{((\xi^2 - \mu)^2 + E)^\beta} \leq \int_\infty^\infty \frac{d\xi}{((\xi - \mu)^2\mu + E)^\beta} = \frac{C}{\sqrt{\mu}E^{\beta-1/2}}.$$

Set now $\beta = p + 1/2$. We obtain according to (2.6) that

$$N(E) \leq \frac{C \int V^\beta dx}{\sqrt{\mu}E^{\beta-1/2}} = \frac{C \int V^{p+1/2} dx}{\sqrt{\mu}E^p},$$

which means that (2.9) is also proven. $\square$

3. Proof of Theorem 1.1. Some relates results

Proof of Theorem 1.1. The central role in the proof is played by Corollary 2.1. The second trick which we apply in the proof is that we use the relation between some of the eigenvalues of the operator $-\Delta + V$ and the eigenvalues of the operator $(-\Delta + 2i - \mu + V)^2$, $\mu > 0$, situated to the left of the line $\Re z = -4$. Indeed, let z_j be eigenvalues of the operator $-\Delta + V$ lying in the hyperbolic domain $D_\mu = \{z : \ (\Im z + 2)^2 - (\Re z - \mu)^2 \geq 4, \ \ \Im z > 0\}$, then $(z_j - \mu + 2i)^2$ are eigenvalues of the operator $(-\Delta - \mu + 2i + V)^2$, and it is easy to see, that

$$\Re(z_j - \mu + 2i)^2 = (\Re z_j - \mu)^2 - (\Im z_j)^2 \leq -4, \qquad \forall z_j \in D_\mu.$$

Consequently, due to Corollary 2.1,

$$\sum_1^n \Big|\Re(z_j - \mu + 2i)^2 + 4\Big| \leq \Big|\sum_1^n s_j\Big|, \tag{3.1}$$

where s_j are eigenvalues of the operator

$$T_1 = (-\Delta - \mu)^2 + V_1(-\Delta - \mu) + (-\Delta - \mu)V_1 + V_1^2 - V_2^2 - 4V_2$$

where $V_1 = \Re V$ and $V_2 = \Im V$ are the real and the imaginary parts of the potential. The estimate (3.1) takes into account all eigenvalues from the domain D_μ. It turns out that we do not need all of them, we need only the eigenvalues z_j lying inside the domain $\Omega_\mu = \{z : (\Im z + 1)^2 - (\Re z - \mu)^2 \geq 1, \ \ \Im z > 0\}$. Note that the boundaries of both domains D_μ and Ω_μ touch the real line at the point $z = \mu$. Note also that $\Omega_\mu \subset D_\mu$. That might imply that the estimates for eigenvalues lying in Ω_μ are better than the estimates for the eigenvalues in D_μ.

It turns out that the imaginary parts of eigenvalues in Ω_μ can be very well estimated in terms of real parts of eigenvalues of the operator $(H - \mu + 2i)^2 + 4$.

Let us investigate the relation between the spectra of operators H and $(H - \mu + 2i)^2$ in more detail. Assume that $z_j \in \Omega_\mu$ and $\Im z_j > s$. Then

$$2(\Im z_j - s) \leq (\Im z_j + 1)^2 - (\Re z - \mu)^2 - 1 + 2(\Im z - s) =$$

$$(\Im z_j + 2)^2 - (\Re z_j - \mu)^2 - 4 - 2s = -\Re(z_j - \mu + 2i)^2 - 4 - 2s$$

Due to Corollary2.1 it means that

$$2 \sum_{z_j \in \Omega_\mu} (\Im z_j - s)_+ \leq \operatorname{tr}\Big(\Re(H - \mu + 2i)^2 + 4 + 2s\Big)_- \leq \operatorname{tr}\Big(T_1 + 2s\Big)_-$$

Now, we represent the operator T_1 in the form

$$T_1 = \frac{1}{2}(-\Delta - \mu)^2 + (\frac{1}{\sqrt{2}}(-\Delta - \mu) + \sqrt{2}V_1)^2 - 4V_2 - V_1^2 - V_2^2$$

Since the operator

$$(\frac{1}{\sqrt{2}}(-\Delta - \mu) + \sqrt{2}V_1)^2 \geq 0$$

is positive, we obtain that the spectrum of the operator T_1 can be estimated by the spectrum of the operator

$$T_2 = \frac{1}{2}(-\Delta - \mu)^2 - |V|^2 - 4V_2.$$

Thus,

$$2 \sum_{z_j \in \Omega_\mu} (\Im z_j - s)_+ \leq \mathrm{tr}\Big(T_2 + 2s\Big)_- \tag{3.2}$$

It is clear now that we have to obtain an estimate for the negative eigenvalues λ_j of the operator T_2. In order to do that and in order to estimate the right hand side of (3.2) we need to apply Proposition 2.2 according to which the number $N(E)$ of eigenvalues of T_2 lying to the left of the point $-E$ satisfies the estimate

$$N(E) \leq \frac{C}{E^p}\Big(\int_{\mathbb{R}^d} W^{d/4+p} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} W^{1/2+p} dx\Big) \tag{3.3}$$

with $p > 1/2$ and $d \geq 2$.

If λ_j are negative eigenvalues of the operator T_2, then

$$\sum_j |\lambda_j|^q = q \int_0^\infty E^{q-1} N(E) dE \leq C\Big(\int_{\mathbb{R}^d} W^{d/4+p} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} W^{1/2+p} dx\Big) |\lambda_1|^{q-p}$$

for $q > p > 1/2$. Note that according to (3.3) the lowest eigenvalue λ_1 satisfies the relation

$$|\lambda_1|^{r-1/2} \leq C\Big(\int_{\mathbb{R}^d} W^{d/4+r-1/2} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} W^r dx\Big) = C\Psi(W).$$

Therefore

$$\sum_j |\lambda_j|^q \leq C\Big(\int_{\mathbb{R}^d} W^{d/4+p} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} W^{1/2+p} dx\Big) |\Psi_\mu(W)|^{2(q-p)/(2r-1)}$$

for $q > p > 1/2$ and $r > 1$.

Recall the inequality

$$2 \sum_{z_j \in \Omega_\mu} (\Im z_j - s)_+ \leq \sum_j (\lambda_j + 2s)_-.$$

It follows from this relation that

$$\sum_{z_j\in\Omega_\mu}(\Im z_j - s)_+ \le$$

(3.4)

$$C\Big(\int_{\mathbb{R}^d}(W-2s)_+^{d/4+p}dx + \mu^{d/2-1}\int_{\mathbb{R}^d}(W-2s)_+^{1/2+p}dx\Big)|\Psi_\mu(W)|^{\frac{2(1-p)}{2r-1}} =$$

$$=: F(s,\mu)$$

with $1/2 < p < 1$ and $r > 1$.

Let now Π_b be the strip $\{z : \quad 0 < \Re z < b\}$. Since the boundary of a domain Ω_μ touches the real line in the parabolical way, it is obvious, that for small values of $s < \varepsilon_0$, the set of all points $z \in \Pi_b$ whose $\Im z > s$ can be covered by not more than $m(b) = [Cb/\sqrt{s}] + 1$ sets of the form Ω_μ. Since Ω_μ contains the sector $\Im z > |\Re z - \mu|$, we obtain that the number of domains Ω_μ covering the strip Π_b can be also estimated by $[b/s] + 1$ for any $s > 0$. Finally, note that $1/\sqrt{s} \ge \sqrt{\varepsilon_0}/s$ for $s \ge \varepsilon_0$. Therefore without loss of generality one can assume that

$$m(b) = [Cb/\sqrt{s}] + 1, \qquad \forall s > 0.$$

Since there is no $z_j \in \Pi_b$ with the property $\Im z_j > C|\Psi_b(W)|^{\frac{2}{2r-1}}$, we obtain

$$\sum_{z_j\in\Pi_b}(\Im z_j - s)_+ \le \sum_{l=1}^{m(b)}\sum_{z_j\in\Omega_{\mu_l}}(\Im z_j - s)_+ \le C\frac{(b+|\Psi_b(W)|^{\frac{1}{2r-1}})}{\sqrt{s}}F(s,b)$$

Observe now that

$$\sum_{z_j\in\Pi_b}|\Im z_j|^\gamma = \gamma(\gamma-1)\sum_{z_j\in\Pi_b}\int_0^\infty(\Im z_j - s)_+ s^{\gamma-2}ds,$$

which leads to

$$\sum_{z_j\in\Pi_b}|\Im z_j|^\gamma \le (b+|\Psi_b(W)|^{\frac{1}{2r-1}})C\int_0^\infty s^{\gamma-5/2}F(s,b)ds$$

The integral in the right hand side converges only if $\gamma > 3/2$ and the previous relation means that

$$\sum_{z_j\in\Pi_b}|\Im z_j|^\gamma \le C|\Psi_b(W)|^{\frac{2\delta}{2r-1}}(b+|\Psi_b(W)|^{\frac{1}{2r-1}})\Big(\int_{\mathbb{R}^d}|W|^{d/4-1/2+\gamma-\delta}dx +$$

$$+ b^{d/2-1}\int_{\mathbb{R}^d}|W|^{\gamma-\delta}dx\Big)$$

with $0 < \delta < 1/2$. It remains to set $r = \gamma - \delta$ to complete the proof. □

In Theorem 1.1 and Theorem 1.2, we estimate the eigenvalue sum of the form $\sum |\Im z_j|^\gamma$ with $\gamma > 3/2$. If we restrict the set and consider not all eigenvalues but only those that belong to a certain domain, then one can estimate even the sum of the first powers.

Corollary 3.1. *Let z_j be the eigenvalues of the operator $-\Delta + V$ lying inside the domain $\{z : \ (\Im z_j + 1)^2 - (\Re z - \mu)^2 \geq 1, \ \ \Im z > 0\}$ and let $d \geq 2$. Then*

$$\sum_j |\Im z_j| \leq C\Big(\int_{\mathbb{R}^d} W^{d/4+p} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} W^{1/2+p} dx\Big) ||W||_\infty^{1-p}$$

for $1/2 < p < 1$.

Corollary 3.1 does not follow immediately from the Theorem 1.1, however it follows from a relation that is similar to (3.4).

In the same way, one can prove

Corollary 3.2. *Let $d = 1$ and let z_j be the eigenvalues of the operator $-d^2/dx^2 + V$ lying inside the domain $\{z : \ (\Im z_j + 1)^2 - (\Re z - \mu)^2 \geq 1, \ \ \Im z > 0\}$. Then*

$$\sum_j |\Im z_j| \leq C ||W||_\infty^{1-p} \mu^{-1/2} \int_{\mathbb{R}^d} W^{1/2+p} dx$$

for $1/2 < p < 1$.

There is an open gap in this theory in the case $d = 1$, where we are forced to keep away from the point $z = 0$ and deal with the strip $a < \Re z < b$ where $a > 0$. Probably, the reason why we have to do that is hidden in a special behaviour of the spectrum near zero. This approach is unable to say anything about the sums of the form $\sum_{0 < \Re z_j < b} |\Im z_j|^\gamma$ for γ close to $3/2$, nevertheless the situation changes as soon as $\gamma > 7/4$. Thus, if we study the behaviour of the eigenvalues in the semi-infinite strip $\{z : \ 0 < \Re z_j < b, \ \Im z > 0\}$ by this method, we lose the information about the sums $\sum_{0 < \Re z_j < b} |\Im z_j|^\gamma$ for $3/2 < \gamma \leq 7/4$.

Theorem 3.1. *Let z_j be the eigenvalues of the operator $H = -d^2/dx^2 + V$ lying inside the semi-infinite strip $\Pi_b = \{z : \ 0 < \Re z < b, \Im z > 0\}$. Then for any $\gamma > 7/4$ and $r \in (\gamma - \frac{1}{2}, \gamma)$*

$$\sum_{z_j \in \Pi_b} |\Im z_j|^\gamma \leq C ||W||_\infty^{\gamma - r} (b + ||W||_\infty^{1/2}) \Big(\int_{\mathbb{R}^d} |W|^{r-1/4} dx\Big),$$

where $W = (|V|^2 + 4\Im V)_+$.

Proof. We start with a modification of the relation (3.4) which can be easily changed and transformed into the form
(3.5)

$$\sum_{z_j\in\Omega_\mu}(\Im z_j - s) \le C\Big(\mu^{-1/2}\int_{\mathbb{R}^d}(W-2s)_+^{1/2+p}dx\Big)||W||_\infty^{1-p} =: F(s,\mu),$$

with $1/2 < p < 1$. The second factor in the middle part of (3.5) containing the norm of W in L^∞ appears when we estimate the lowest eigenvalue λ_1 of $T_2 = \frac{1}{2}(-d^2/dx^2-\mu)^2 - W$.

Now consider the part of the strip $\Pi_b = \{z : 0 < \Re z < b, \Im z > 0\}$ whose points satisfy the conditions $\Im z > s$ where $s > 0$. Let us cover this part by the sets Ω_μ, $\mu \in \mathbb{R}_+$. While doing this, we will avoid the value $\mu = 0$ and take μ as large as possible. It is obvious that it impossible to avoid the value $\mu = \mu_0$ satisfying $(s+1)^2 - \mu_0^2 = 1$, and if we choose the covering in the optimal way then μ_0 will be the minimal value of μ. Thus, without loss of generality, we can assume that $\mu \ge \sqrt{s^2+2s} = \mu_0$.

By the same argument as in the proof of Theorem 1.1, the set of all points $z \in \Pi_b$ whose $\Im z > s$ can be covered by not more than $m(b) = [Cb/\sqrt{s}] + 1$ sets of the form Ω_μ.

Since there is no $z_j \in \Pi_b$ with the property $\Im z_j > ||W||_\infty$, we obtain

$$\sum_{z_j\in\Pi_b}(\Im z_j - s)_+ \le \sum_{l=1}^{m(b)}\sum_{z_j\in\Omega_{\mu_l}}(\Im z_j - s)_+ \le C\frac{b+||W||_\infty^{1/2}}{\sqrt{s}}F(s,\mu_0)$$

Observe now that

$$\sum_{z_j\in\Pi_b}|\Im z_j|^\gamma = \gamma(\gamma-1)\sum_{z_j\in\Pi_b}\int_0^\infty(\Im z_j - s)_+ s^{\gamma-2}ds,$$

which leads to

$$\sum_{z_j\in\Pi_b}|\Im z_j|^\gamma \le (b+||W||_\infty^{1/2})C\int_0^\infty s^{\gamma-5/2}F(s,\sqrt{s})ds$$

The integral in the right hand side converges only if $\gamma > 7/4$ and the previous relation means that

$$\sum_{z_j\in\Pi_b}|\Im z_j|^\gamma \le C||W||_\infty^{1-p}(b+||W||_\infty^{1/2})\Big(\int_{\mathbb{R}^d}|W|^{\gamma-5/4+p}dx\Big)$$

with $1/2 < p < 1$. It remains to set $r = \gamma + p - 1$ to complete the proof.
□

4. Proof of Theorem 1.3

The main tool of the proof is the linear fractional mapping that takes the upper half-plane $\{z : \ \Im z > 0\}$ into the compliment of the unit disk $\{z : \ |z| > 1\}$. This transformation is given by the formula

$$z \mapsto \frac{z+i+1}{z-i+1}.$$

Insert the operator $H = -\Delta + V$ instead of z into this formula, i.e. consider the operator

$$U = (H+I+i)(H+I-i)^{-1} = I + 2i(H+I-i)^{-1}.$$

The number $z \notin \mathbb{R}$ is an eigenvalue of the operator H if and only if the point $(z+i+1)/(z-i+1)$ is an eigenvalue of U. Let us now find the operator U^*U. It is easy to see that

$$U^* = I - 2i(H^*+I+i)^{-1}.$$

Therefore

$$U^*U = I + 2i(H+I-i)^{-1} - 2i(H^*+I+i)^{-1} + 4(H^*+I+i)^{-1}(H+I-i)^{-1}.$$

Using the Hilbert identity, we obtain

$$U^*U = I + 2i(H^*+I+i)^{-1}(H^*-H)(H+I-i)^{-1}$$

or, put it differently,

$$U^*U = I + 4(H^*+I+i)^{-1}\Im V(H+I-i)^{-1}.$$

Thus, we obtain that

$$U^*U - I \leq 4Y^*Y,$$

where $Y = \sqrt{\Im V_+}(H+I-i)^{-1}$. According to Corollary 2.2, the eigenvalues λ_j of the operator H satisfy the estimate

$$\sum_j \Big(\Big|\frac{\lambda_j+1+i}{\lambda_j+1-i}\Big|^2 - 1\Big)_+^p \leq \mathrm{tr}\,(U^*U - I)_+^p \leq 4^p \mathrm{tr}\,(Y^*Y)^p = 4^p ||Y||_{2p}^{2p}.$$

It follows from this inequality that

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j+1|^2+1} \Big)_+^p \leq ||Y||_{2p}^{2p} \tag{4.1}$$

Indeed, denote $a = 2\Im\lambda_j/(|\lambda_j+1|^2+1)$ and suppose that $\Im\lambda_j > 0$. Then

$$\Big|\frac{\lambda_j+1+i}{\lambda_j+1-i}\Big|^2 - 1 = \Big(\frac{1+a}{1-a}\Big) - 1 \geq 2a.$$

We come to the conclusion that one needs to estimate the norm of the operator

$$Y = \sqrt{\Im V_+}(H+I-i)^{-1}$$

in the class $\mathfrak{S}_{2p}$. Let us represent this operator in the form

$$Y = \sqrt{\Im V_+}(-\Delta + I)^{-1/2}B, \qquad \text{where } B = (-\Delta + I)^{1/2}(H + I - i)^{-1}.$$

We will show that the operator B is bounded and its norm does not exceed 1. In other words, we will show that

$$||(-\Delta + I)^{1/2}(H + I - i)^{-1}f||^2 \le ||f||^2, \tag{4.2}$$

for all $f \in L^2$.

Denote $u = (H + I - i)^{-1}f$. It is obvious that

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + (1 + \Re V(x))|u|^2)\, dx = \Re \int_{\mathbb{R}^d} f\bar{u}\, dx.$$

Due to the condition $\Re V \ge 0$, we obtain from this relation that

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2)\, dx \le \frac{1}{2} \int_{\mathbb{R}^d} (|f|^2 + |u|^2)\, dx.$$

The latter inequality can be written in the form

$$\int_{\mathbb{R}^d} (2|\nabla u|^2 + |u|^2)\, dx \le \int_{\mathbb{R}^d} |f|^2\, dx.$$

Replacing 2 by a smaller number we will make the inequality weaker. As a result we obtain the estimate

$$||(-\Delta + I)^{1/2}u||^2 \le ||f||^2. \tag{4.3}$$

It remains to note that (4.3) is equivalent to (4.2).

Let us summarize the results. Since

$$Y = \sqrt{\Im V_+}(H + I - i)^{-1} = \sqrt{\Im V_+}(-\Delta + I)^{-1/2}B \quad \text{and} \quad ||B|| = 1,$$

we have the relation

$$||Y||_{2p} \le ||\sqrt{\Im V_+}(-\Delta + I)^{-1/2}||_{2p}. \tag{4.4}$$

On the other side, according to Theorem 2.2,

$$||\sqrt{\Im V_+}(-\Delta + I)^{-1/2}||_{2p}^{2p} \le (2\pi)^{-d} C_0 \int \Im V_+^p dx,$$

where $C_0 = \int_{\mathbb{R}^d} (\xi^2 + 1)^{-p}\, d\xi$. Combining (4.4) with (4.1), we obtain (1.1).

5. Proof of Theorem 1.4

The arguments repeat the proof of Theorem 1.3 with the only exception that the estimate for the norm $||Y||_{2p}$ is carried out differently. Recall that

$$\sum_j \Big(\frac{\Im\lambda_j}{|\lambda_j+1|^2+1}\Big)_+^p \le ||Y||_{2p}^{2p} \tag{5.1}$$

where $Y=\sqrt{\Im V_+}(H+I-i)^{-1}$.

In order to estimate the s-numbers of the operator Y we represent it in the form

$$Y=\sqrt{\Im V_+}(-\Delta+I-i)^{-1}(I-V(H+I-i)^{-1})$$

In the previous proof we have shown that

$$(H+I-i)^{-1}=(-\Delta+I)^{-1/2}B \qquad \text{and} \quad ||B||\le 1.$$

Consequently,

$$||(H+I-i)^{-1}||\le 1,$$

and this means that

$$||Y||_{2p}\le ||\sqrt{\Im V_+}(-\Delta+I-i)^{-1}||_{2p}(1+||V||_\infty).$$

An estimate for the norm of $\sqrt{\Im V_+}(-\Delta+I-i)^{-1}$ does not represent any difficulty. According to Theorem 2.2,

$$||\sqrt{\Im V_+}(-\Delta+I-i)^{-1}||_{2p}^{2p}\le (2\pi)^{-d}C_0\int \Im V_+^p dx, \qquad C_0=\int\frac{d\xi}{((\xi^2+1)^2+1)^p},$$

for any $p>d/4$. Consequently,

$$||Y||_{2p}^{2p}\le (2\pi)^{-d}(1+||V||_\infty)^{2p}C_0\int \Im V_+^p dx. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain (1.3).

6. Proof of Theorem 1.5

Theorem 1.5 was proven before for $d=p=1$ (see[1], [3]). Consider first the case when $p>\max\{1,d/2\}$. The reader can easily check that $\lambda\notin\mathbb{R}_+$ is an eigenvalue of the operator H if and only if 1 is an eigenvalue of the operator

$$X=|V|^{1/2}(-\Delta-\lambda)^{-1}|V|^{-1/2}V.$$

But then $||X||\ge 1$. Note now that

$$||X||\le ||X||_p\le ||Q||_{2p}^2$$

where

$$Q=|V|^{1/2}|-\Delta-\lambda|^{-1/2}.$$

Therefore we can use Theorem 2.2 to obtain that

$$1 \leq ||Q||_{2p}^{2p} \leq (2\pi)^{-d} \int_{\mathbb{R}^d} |V|^p dx \int_{\mathbb{R}^d} \frac{d\xi}{|\xi^2 - \lambda|^p}.$$

It remains to estimate the integral

$$J = \int_{\mathbb{R}^d} \frac{d\xi}{|\xi^2 - \lambda|^p}, \qquad p > d/2.$$

Using homogeneity one can easily obtain that

$$J = |\lambda|^{d/2-p} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi^2 - e^{i\phi}|^p},$$

where $\phi = \arg \lambda$. Therefore the question about an estimate of J is reduced to the problem of estimating the integral $\int_{\mathbb{R}^d} |\xi^2 - e^{i\phi}|^{-p} d\xi$, which behaves as

$$\int_{\mathbb{R}^d} \frac{d\xi}{|(\xi^2-1)^2 + \phi^2|^{p/2}} \sim C \int_{\mathbb{R}} \frac{dt}{(t^2 + \phi^2)^{p/2}} \sim C\phi^{1-p}.$$

In other words,

$$\int_{\mathbb{R}^d} \frac{d\xi}{|\xi^2 - e^{i\phi}|^p} \leq C |\sin \phi|^{1-p}.$$

Consequently,

$$J \leq C|\lambda|^{d/2-p} |\sin \phi|^{1-p} = C|\Im \lambda|^{1-p} |\lambda|^{d/2-1}.$$

It remains to note that

$$1 \leq (2\pi)^{-d} J \int_{\mathbb{R}^d} |V|^p dx.$$

The proof in the case $p = d/2 > 1$ is similar to the proof in the case $p > d/2$ with the only exception that instead of Theorem 2.2 one needs to use Theorem 2.1.

Indeed, let

$$a(\xi) = \frac{1}{|\xi^2 - \lambda|} \qquad \text{and} \quad p = d/2 > 1.$$

Then, using homogeneity, one can easily obtain that

$$[a]_p^p = [a_0]_p^p, \qquad \text{where} \quad a_0 = \frac{1}{|\xi^2 - e^{i\phi}|}$$

and $\phi = \arg \lambda$. Therefore the question about an estimate of $[a]_p^p$ is reduced to the problem of estimating the quantity $[a_0]_p^p$ which is not bigger than

$$C_1 \int_{|\xi|<2} \frac{d\xi}{|(\xi^2-1)^2 + \phi^2|^{p/2}} + C_2 \sim C \int_{\mathbb{R}} \frac{dt}{(t^2+\phi^2)^{p/2}} \sim C\phi^{1-p}.$$

In other words,

$$[a]_p^p = [a_0]_p^p \leq C|\sin\phi|^{1-p} = C\Big|\frac{\Im\lambda}{\lambda}\Big|^{1-p}.$$

It remains to note that, if λ is an eigenvalue of H, then

$$1 \leq C[a]_p^p \int_{\mathbb{R}^d} |V|^p dx \qquad p = d/2.$$

7. Individual eigenvalue estimates

Consider the operator $H = -\Delta + iV(x)$ with $V \geq 0$. Assume for simplicity that $\lim_{|x|\to\infty} V(x) = 0$.

Our first result is devoted to the case $d = 3$. It shows, in particular, that if the integral of V is small, then the real part of the square root of the eigenvalue of H is large. That implies that the non-real eigenvalues of $-\Delta + itV$ escape any compact subset of $\mathbb{C}$, as $t \to 0$, in the sense that the compact subset will not contain non-real eigenvalues for small values of t. It does not necessary imply that the eigenvalues tend to infinity as $t \to 0$, because they might simply reach the positive real line for some $t > 0$ (see Theorem 8.1 for that matter).

Theorem 7.1. *Let $d = 3$ and $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of H. Then*

$$\frac{\Re k}{4\pi} \int V(x)\, dx \geq 1. \tag{7.1}$$

Proposition 7.1. *A number $z = k^2 \notin \mathbb{R}_+$ is an eigenvalue of the operator H if and only if the point 1 is the eigenvalue of the operator*

$$X = -i\sqrt{V}(-\Delta - z)^{-1}\sqrt{V}.$$

The proof of this proposition is left to the reader.

Proof of the theorem. Suppose that $\Im z > 0$. Note that the real part of the operator X is positive. Consequently, the spectrum of this operator lies in the right half plane. Therefore whenever z is the eigenvalue of H the sum of the real parts of the eigenvalues ζ_j of X is greater than or equal 1, i.e.

$$\sum_j \Re\zeta_j \geq 1.$$

On the other side,

$$\sum_j \Re\zeta_j \leq \mathrm{tr}\Re X = \int \tau(x,x)\, dx,$$

where $\tau(x, y)$ is the integral kernel of the operator $\Re X$.

Since the kernel of the operator $(-\Delta - z)^{-1}$ is

$$g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x - y|},$$

we obtain that the operator $\Im(-\Delta - z)^{-1}$ has a kernel $g_0(x, y) = (2i)^{-1}(g(x, y) - \overline{g(y, x)})$ whose value on the diagonal is

$$g_0(x, x) = \frac{k + \bar{k}}{8\pi} = \frac{\Re k}{4\pi}.$$

Therefore

$$\mathrm{tr}\Re X = \int V(x) g_0(x, x)\, dx = \frac{\Re k}{4\pi} \int V(x)\, dx$$

Thus, (7.1) follows. $\square$

Corollary 7.1. *Let $d = 3$ and let $V \in L^1(\mathbb{R}^3)$ be a positive function. Then non-real eigenvalues of $-\Delta + iV$ do not accumulate to zero.*

The same arguments work in $d = 1$ and in $d = 2$. We begin with the case $d = 1$

Theorem 7.2. *Let $d = 1$. Let $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of H. Then*

$$\frac{\Re k}{2|k|^2} \int V(x)\, dx \geq 1,$$

which means that k lies inside the circle of radius $4^{-1} \int V(x)\, dx$ around the point $4^{-1} \int V(x)\, dx$

Unlike the previous cases the eigenvalues in $d = 2$ do not appear at all if the integral of V is small

Theorem 7.3. *Let $d = 2$. Let $z \notin \mathbb{R}_+$ be an eigenvalue of H. Then*

$$\frac{1}{2}\Big(\frac{\pi}{2} + \arctan(\Re z / \Im z)\Big) \int V(x)\, dx \geq 1.$$

In particular, the spectrum of $-\Delta + iV$ is real if

$$\frac{\pi}{2} \int V(x)\, dx < 1.$$

Indeed,

$$\mathrm{tr}\, \Re X = \int V(x)\, dx \int_{\mathbb{R}^2} \Im\Big[\frac{1}{2\pi(\xi^2 - z)}\Big] d\xi.$$

The case $d \geq 4$ is similar to the case $d = 3$. We illustrate it with the help of the following

Theorem 7.4. *Let $d \geq 4$ and let $z \notin \mathbb{R}_+$ be an eigenvalue of $H = -\Delta + iV$ with $V \geq 0$. Then*

$$(2\pi)^{-d+1}\omega_d \Big|\Re z + 2||V||_\infty\Big|^{(d-2)/2} \int V(x)\,dx \geq 2, \tag{7.2}$$

where ω_d is the area of the unit sphere.

Proof. As before, we denote

$$X = -i\sqrt{V}(-\Delta - z)^{-1}\sqrt{V}.$$

Suppose that z is an eigenvalue of the operator H. Then 1 is an eigenvalue of the operator X. It follows trivially from this that $1/2$ is an eigenvalue of the operator $X - 1/2$. Consequently,

$$\mathrm{tr}\,(\Re X - 1/2)_+ \geq 1/2.$$

Indeed, for the eigenvalues λ_j of the operator X we have

$$\sum (\Re \lambda_j - 1/2)_+ \leq \mathrm{tr}\,(\Re X - 1/2)_+.$$

It remains to note that the eigenvalue sum in the left hand side is not less than $1/2$. Obviously,

$$(\Re X - 1/2)_+ \leq \Big(\Re X - \frac{V}{2||V||_\infty}\Big)_+,$$

since $\frac{V}{||V||_\infty} \leq 1$.

On the other hand,

$$\Big(\Re X - \frac{V}{2||V||_\infty}\Big)_+ \leq \sqrt{V}\Big(\Im(-\Delta - z)^{-1} - \frac{1}{2||V||_\infty}\Big)_+\sqrt{V}.$$

Consequently,

$$1/2 \leq (2\pi)^{-d} \int V(x)\,dx \int_{\mathbb{R}^d} \Big(\frac{\Im z}{(\xi^2 - \Re z)^2 + (\Im z)^2} - \frac{1}{2||V||_\infty}\Big)_+ d\xi.$$

The integration in the last integral is done over the domain where

$$\xi^2 \leq \Re z + \sqrt{(2||V|| - \Im z)_+ \Im z} \leq \Re z + 2||V||_\infty.$$

Thus, we obtain (7.2), since

$$\omega_d^{-1} \int_{\mathbb{R}^d} \Big(\frac{\Im z}{(\xi^2 - \Re z)^2 + (\Im z)^2} - \frac{1}{2||V||_\infty}\Big)_+ d\xi \leq 2^{-1}\pi \Big|\Re z + 2||V||_\infty\Big|^{(d-2)/2}. \tag{7.3}$$

□

In all theorems formulated above, we work with the L^1 norm of the potential V. Actually, this restriction is excessive and it does not completely agree with Conjecture 1.1. It is better to work with integrals of the form

$$\int V^{d/2+\gamma} dx,$$

and this is exactly what we do in the following statement.

Theorem 7.5. *Let $d \geq 3$ and let $V \geq 0$. Suppose that $z \notin \mathbb{R}$ is an eigenvalue of $H = -\Delta + iV$. Then there are positive constants C_1 and C_2 depending only on d and $\gamma \geq 0$ such that*

$$|\Im z|^\gamma \leq (C_1 + C_2(\frac{\Re z}{\Im z})^{d/2-1}) \int V^{d/2+\gamma} dx \tag{7.4}$$

Proof. Set, as before,

$$X = -i\sqrt{V}(-\Delta - z)^{-1}\sqrt{V}.$$

If z is an eigenvalue of H, then 1 is an eigenvalue of X. Consequently, one can find at least one eigenvalue of the operator $\Re X$ that is not less than 1. It means in particular that supremum

$$\sup_{s>0} s^{-(d/2+\gamma)} \operatorname{card}\{j: \quad s_j > s, \text{ where } s_j \text{ are eigenvalues of } \Re X\}$$

is not less then 1. This supremum is related to the norm in the weak Neumann-Schatten class $\Sigma_{d/2+\gamma}$ and, due to Theorem **??**, can be estimated by the expression

$$\int V^{d/2+\gamma} dx \int_{\mathbb{R}^d} \Big(\frac{\Im z}{(\xi^2 - \Re z)^2 + \Im z^2}\Big)^{d/2+\gamma} d\xi. \tag{7.5}$$

Indeed, $\Re X$ is representable in the form $\Re X = Q^* Q$ where

$$Q = \Big[\frac{\Im z}{(\xi^2 - \Re z)^2 + \Im z^2}\Big]^{1/2} \Phi \sqrt{V}$$

(here $[a]$ denotes the operator of multiplication by a). Consequently, $n(s, \Re X) = n(\sqrt{s}, Q)$ which leads to the fact that $[\Re X]^{d/2+\gamma}_{d/2+\gamma}$ is less than the expression (7.5) multiplied by a constant depending only on d and γ.

It remains to estimate the integral

$$\int_{\mathbb{R}^d} \Big(\frac{\Im z}{(\xi^2 - \Re z)^2 + \Im z^2}\Big)^{d/2+\gamma} d\xi \leq$$

$$C \int_{-\infty}^{\infty} \Big(\frac{\Im z}{s^2 + \Im z^2}\Big)^{d/2+\gamma} s^{d/2-1} ds + C \int_{-\infty}^{\infty} \Big(\frac{\Im z}{s^2 + \Im z^2}\Big)^{d/2+\gamma} |\Re z|^{d/2-1} ds$$

$$\leq (C_1 + C_2 \Big|\frac{\Re z}{\Im z}\Big|^{d/2-1}) |\Im z|^{-\gamma}.$$

The relation (7.4) follows. □

Corollary 7.2. *Let $d \geq 3$ and let C_1 be the constant in* (7.4) *written for $\gamma = 0$. If $C_1 \int V^{d/2} dx < 1$, then the eigenvalues of $-\Delta + iV$ are situated in the conical sector $\{z : \quad 0 \leq \arg z \leq \alpha\}$, which opens with the angle α satisfying*

$$(C_1 + C_2(\cot \alpha)^{d/2-1}) \int V^{d/2} dx = 1.$$

The proof of the next statement is exactly the same as the one of Theorem 7.5.

Corollary 7.3. *Let $d = 2$ and let $V \geq 0$. Suppose that $z \notin \mathbb{R}$ is an eigenvalue of $H = -\Delta + iV$. Then there is a positive constant C depending only on $\gamma > 0$ such that*

$$|\Im z|^\gamma \leq C \int V^{1+\gamma} dx, \quad \gamma > 0.$$

8. Additional remarks

In conclusion of this paper, we mention two rather obvious facts, that are valid for an arbitrary complex potential V. For the sake of simplicity, we restrict our study to the case $d = 3$. As before, $H = -\Delta + V$ is the Schrödinger operator and ω_3 is the area of the unit sphere.

Theorem 8.1. *Let $d = 3$. If $V \in L^\infty \cap L^1$ and $\omega_3 ||V||_\infty + 2||V||_1 < 8\pi$, then the spectrum of the operator H is real. The same is true under the condition that*

$$\sup_x \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|} dy < 4\pi.$$

Theorem 8.2. *Let $d = 3$ and let $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of the operator $H = -\Delta + V$, $\Im k > 0$. Then there is a positive constant C depending only on $\gamma > 0$, such that*

$$(\Im k)^{2\gamma} \leq C \int_{\mathbb{R}^3} |V|^{3/2+\gamma} dx.$$

Proof of both theorems. Suppose that $z = k^2$ is an eigenvalue of the operator H. Then the norm of the operator $X = |V|^{1/2}(-\Delta - z)^{-1}|V|^{1/2}$ is not smaller then 1. It remains to estimate the norm of X in terms of z and V. In order to do that we use the Schur estimate. It says that the square of the norm of an integral operator with the kernel $g(x, y)$ does not exceed the product of the quantities

$$m_1 = \sup_x \int |g(x, y)| \frac{dy}{\rho(x, y)}$$

and

$$m_2 = \sup_y \int |g(x,y)|\rho(x,y)\, dx$$

where ρ is a positive weight. Since the kernel of the operator X equals

$$|V(x)|^{1/2} \frac{e^{ik|x-y|}}{4\pi|x-y|} |V(y)|^{1/2},$$

applying the Schur estimate with the weight $\rho = \sqrt{V(x)/V(y)}$, we obtain that

$$||X|| \le \frac{1}{4\pi} \sup_x \int \frac{e^{-\Im k|x-y|}}{|x-y|} |V(y)|\, dy$$

The assertion of the first theorem follows from the trivial estimate

$$1 \le ||X|| \le \frac{1}{4\pi} \sup_x \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|} dy \le \frac{1}{8\pi}(\omega_3 ||V||_\infty + 2||V||_1).$$

The assertion of the second theorem follows from the Hölder inequality

$$1 \le \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|} e^{-\Im k|x-y|} dy \le C_0 ||V||_p \Big(\int_{\mathbb{R}^3} \frac{e^{-q\Im k|y|}}{|y|^q} dy \Big)^{1/q} =$$

$$= C \frac{||V||_p}{(\Im k)^{2\gamma/p}}, \quad \text{where } \ p = 3/2 + \gamma \ \text{ and } \ q = p/(p-1). \qquad \square$$

Remark. By the same means, one can show that

$$\Big| \frac{\sqrt{z}}{\Im \sqrt{z}} \Big|^{\gamma+1/2} |\Im \sqrt{z}|^{2\gamma} \le C \int_{\mathbb{R}} |V|^{1/2+\gamma} dx, \quad \gamma \ge 1/2,$$

for all eigenvalues $z \notin \mathbb{R}_+$ of the one-dimensional Schrödinger operator $H = -d^2/dx^2 + V$. The constant C in this inequality can be computed explicitly:

$$C = \frac{1}{2} \Big(\frac{2\gamma - 1}{2\gamma + 1} \Big)^{\gamma - 1/2}.$$

References

[1] Abramov, A.A., Aslanyan, A., Davies, E.B.: *Bounds on complex eigenvalues and resonances.* J. Phys. A **34**, 57–72 (2001)

[2] Cwikel, M.: *Weak type estimates for singular values and the number bound states of Schr¨odinger operators*, Ann. of Math. **106**, (1977)

[3] Davies, E.B., Nath, J.: *Schrödinger operators with slowly decaying potentials.* J. Comput. Appl. Math. **148** (1), 1–28 (2002)

[4] Frank, Rupert L.; Laptev, Ari; Lieb, Elliott H.; Seiringer, Robert *Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials.* Lett. Math. Phys. **77** (2006), no. 3, 309–316.

[5] Lieb, E. H. Thirring, W.:*Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, in Studies in Mathematical Physics (Essays in Honor of Valentine Bargmann), 269–303. Princeton Univ. Press, Princeton, NJ, 1976.

[6] Seiler, E. and Simon, B.: *Bounds in the Yukawa2 quantum field theory: upper bound on the pressure, Hamiltonian bound and linear lower bound*, Commun. Math. Phys., **45** (1975), 99-114.

[7] B. Simon: *Trace ideals and their applications.* London Mathematical Society Lecture Note Series **35**, Cambidge University Press, 1979.