A PDE approach to finite time approximations in Ergodic Theory

Olga Bernardi, Franco Cardin,
Massimiliano Guzzo, Lorenzo Zanelli

Dipartimento di Matematica Pura ed Applicata
Via Trieste, 63 - 35121 Padova, Italy

e-mail: obern@math.unipd.it cardin@math.unipd.it
guzzo@math.unipd.it lzanelli@math.unipd.it

Abstract

For dynamical systems defined by vector fields over a compact invariant set, we introduce a new class of approximated first integrals based on finite-time averages and satisfying an explicit first order partial differential equation. Referring to the PDE framework, we detect their viscosity robustness with respect to stochastic perturbations of the vector field. We formulate this approximating device also in the Lyapunov exponents framework. In such a case, we compare the operative use of the introduced approximated first integrals to the common use of the Fast Lyapunov Indicators to detect the phase space structure of quasi-integrable systems.

1 Introduction

The existence of first integrals and their qualities, e.g. their number and smoothness, constrain the topological properties in the large of the paths of a dynamical system, see for example [17], [7] and [10]. Specifically, for the so-called integrable systems, a whole set of global first integrals determines completely the dynamics. On the opposite side with respect to integrable dynamical systems we find the so-called ergodic ones, for which non-trivial global first integrals are not allowed to exist. Both integrable and ergodic systems are very special extreme situations, which usually do not exist in nature, and the most typical case is represented by systems which are not integrable nor ergodic, but approximating one or the other situation, and sometimes both of them. Many examples of this kind can be found in Celestial Mechanics, where the systems typically exhibit some variables with integrable or quasi-integrable behavior and other variables with approximate ergodic behaviour; other examples arise in Statistical Physics and Plasma Physics, see for example [6], [23], [4] and [22]. The dynamics of such (non-integrable and non-ergodic) systems can be characterized by transient behaviours (such as temporary captures into resonances or stickiness phenomena) which are usually difficult to formally study with mathematical tools defined over the complete orbits.

During the last decades, in the numerical investigations of these systems, a very important role has been played by a practical use of approximated notion of first integrals, of time averages, and other popular indicators of the quality of the motion, like the Lyapunov exponents ([24], [5] and [25]). Since the eighties, important results have been achieved by finite time computations of dynamical indicators on discrete sets of points of the phase space, hereafter called grids, see [23] for a review on the subject. In particular, we recall the methods based on the Fourier analysis of the solutions, such as the frequency analysis method ([19], [20]), or on the Lyapunov exponents theory, such as the Fast Lyapunov Indicators (FLI), introduced in [11]. The finite time computation of a dynamical indicator on grids of the phase space provides effective criteria for determining integrability, quasi-integrability or stochasticity of motions (see [13] for the case of FLI).

In this paper, we introduce a new class of approximated global first integrals and we investigate around the mathematical robustness of their definition. Keeping in mind the Birkhoff-Khinchin Theorem [8] –which asserts that the time average of any function f is a global first integral– we specifically study the indicators based on finite-time averages of functions, with particular attention devoted to the FLI and their applications. We restrict ourselves to dynamical systems defined by the flow ϕ_X^t of smooth vector fields X over a compact invariant set $\Omega \subset \mathbb{R}^n$. Global smooth first integrals F of such a system –whenever they do exist– are solutions on Ω of the PDE equation:

$$\nabla F \cdot X(x) = 0. \tag{1}$$

Clearly, this is not generally the case of the finite-time average G_T of functions f on $[0, T]$, whose Lie derivative $\nabla G_T \cdot X(x) = \frac{1}{T}\left[f(\phi_X^T(x)) - f(x)\right]$ depends both on x and $\phi_X^T(x)$.

Our first contribution is framed in considering unusual "finite-time" approximations F_μ, where substantially μ plays the role of $1/T$:

$$F_\mu(x) := \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x)) d\tau. \tag{2}$$

The crucial advantage of F_μ with respect to the traditional finite-time averages G_T is represented by the fact that it satisfies an explicit PDE equation, precisely:

$$\nabla F_\mu \cdot X(x) = \mu(F_\mu - f)(x). \tag{3}$$

In view of the previous equation, we regard F_μ as an approximated first integral.

Referring to the PDE framework about existence and uniqueness results for (3), we recognize the viscosity robustness of the proposed approximated first integral F_μ. More precisely, this property consists in the fact that the solution of the Dirichlet's problem given by the following elliptic perturbation of equation (3):

$$\begin{cases} \frac{\nu^2}{2}\Delta F_\mu^\nu(x) + X \cdot \nabla F_\mu^\nu(x) = \mu(F_\mu^\nu - f)(x) \\ \\ F_\mu^\nu(x)|_{\partial\Omega} = \psi(x) \end{cases} \tag{4}$$

pointwise converges, for $\nu \to 0$, exactly to the above function F_μ. In particular, the proposed representation (2) of the solution for (3) is stable under stochastic perturbations. In fact, denoting by (X_τ^ν, P_x) the Markov process solving the stochastic differential equation related to (4), it arises the representation:

$$F_\mu^\nu(x) = M_x\left[\mu \int_0^{+\infty} e^{-\mu\tau} f(X_\tau^\nu) d\tau\right] \overset{\nu\to 0}{\longrightarrow} F_\mu(x), \qquad \forall x \in \Omega, \tag{5}$$

see [9] for some more detail. We note that as a consequence of the invariance of the compact set Ω, any boundary datum in (4) does not play any role, since all orbits with initial condition in Ω never reach the boundary $\partial\Omega$. The discussions above show that, among the all possible perturbations of (1), the equation (3) produces, for every $\mu > 0$, approximated first integrals with representation coming from a regularizing viscous technique.

This stable behaviour under viscous perturbations (and with arbitrary boundary data) prompts analogies with other important situations in which vanishing artificial viscosity is introduced in order to select a special solution: this occurs for example in the viscosity solutions theory to the Hamilton-Jacobi equation, we refer to [21], [3] and [2] for an exhaustive treatment of the matter. However, the approximate first integral F_μ here introduced behaves closer to Fokker-Planck –see for example [1] and [26]– than to Hamilton-Jacobi environment, especially by considering free-divergence systems, like the Hamiltonian ones. In fact, in such a case, searching first integrals is equivalent to searching (smooth) invariant measures; whenever this acts by adding (i) a μ-small relative *friction* (relative, to an assigned function f) and (ii) a ν-small *diffusion*, we are naturally leaded to the (stationary) Fokker-Planck equation:

$$\nabla F \cdot X(x) + \mu(f - F)(x) + \frac{\nu^2}{2}\Delta F(x) = 0.$$

In this order of ideas, we infer that the present F_μ is thus robust under the vanishing (i.e. for $\nu \to 0$) diffusion action. A previous use of relative friction and vanishing viscosity has been numerically implemented in [16].

We formulate this approximating device to define finite-time approximations of the usual Lyapunov exponents, which can be considered as time averages of suitable functions defined on the tangent space. This requires to rewrite the variational equation on a compact invariant set of the tangent space $\mathbb{R}^{2n}$. We will call exponentially dumped Lyapunov indicators these finite-time approximations of the Lyapunov exponents. Finally, in Section 5, we compare the operative use of the exponentially dumped Lyapunov indicators for the numerical detection of resonances and invariant tori of quasi-integrable Hamiltonian systems to the common use of the FLI.

2 Approximated Birkhoff-Khinchin first integrals

In this section, starting from the Birkhoff-Khinchin Theorem (see for example Chapter 1 in [8]), we propose a natural notion of approximated global first integral and we discuss it from a functional point of view.

Let X be a smooth (i.e. at least C^1) vector field defined over a compact invariant set $\Omega \subset \mathbb{R}^n$ and ϕ_X^t its flow. By the Birkhoff-Khinchin Theorem, the time average of every real-valued continuous function f:

$$\mathcal{F}(x) := \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(\phi_X^\tau(x))d\tau \tag{6}$$

exists a.e. and it is a first integral, that is $\mathcal{F}(\phi_X^t(x)) = \mathcal{F}(x)$ for all $t \in \mathbb{R}$. On the one hand, the function $\mathcal{F}(x)$ can be highly irregular and its operative use is limited. On the other hand, with reference to some possible applications to perturbation theory and allied topics, the finite time approximation of (6):

$$G_T(x) := \frac{1}{T}\int_0^T f(\phi_X^t(x))dt \tag{7}$$

is an approximated first integral, in the sense that its Lie derivative equals to:

$$X\cdot\nabla G_T(x) = \frac{d}{dt}G_T(\phi_X^t(x))\big|_{t=0} = \frac{1}{T}\left[f(\phi_X^T(x)) - f(x)\right]. \tag{8}$$

Let us denote $\mu := \frac{1}{T}$. Starting from this format, we consider below a different finite time approximation of (7), which offers a better notion of approximated global first integral.

Definition 2.1 *Let $\mu > 0$ and $f \in C^0(\Omega;\mathbb{R})$. The function*

$$F_\mu(x) := \mu\int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x))d\tau \tag{9}$$

is an approximated first integral in the sense that its Lie derivative equals to:

$$X\cdot\nabla F_\mu(x) = \mu(F_\mu - f)(x). \tag{10}$$

Remark 1 Let us consider the following change of the integral parameter:

$$[0, +\infty] \ni \tau \mapsto t(\tau) = (1 - e^{-\mu\tau})T \in [0, T].$$

We note that the function (9) occurs by considering a direct modification of the equivalent representation:

$$G_T(x) = \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^{t(\tau)}(x)) d\tau.$$

Remark 2 As a consequence of the dominated convergence Theorem, the approximated first integral (10) pointwise converges, for $\mu \to 0^+$, to the finite time average $G_{1/\mu}$ (see (7)):

$$\lim_{\mu \to 0^+} (F_\mu - G_{1/\mu})(x) = 0 \qquad \forall x \in \Omega.$$

The first plain difference of F_μ with respect to $G_{1/\mu}$ is that its Lie derivative does not depend on the flow ϕ_X^T.

Remark 3 Another crucial advantage is that F_μ, viewed as a linear operator on $C^0(\Omega, \mathbb{R})$, provides also a precise characterization for exact global first integrals, in the sense stated by the following

Proposition 2.1 *Let $\mu > 0$ be fixed. A function $f \in C^0(\Omega, \mathbb{R})$ is a global first integral for the vector field X if and only if $F_\mu(x) = f(x)$, $\forall x \in \Omega$.*

Proof. Let us first suppose that f is a global first integral for X, that is: $f(\phi_X^t(x)) = f(x)$ $\forall t \in \mathbb{R}$. Therefore we have

$$F_\mu(x) = \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x)) d\tau = \mu \int_0^{+\infty} e^{-\mu\tau} f(x) d\tau = f(x).$$

Conversely, let $F_\mu(x) = f(x)$, $\forall x \in \Omega$. Then, accordingly to (10), the Lie derivative of f is equal to zero:

$$L_X f(x) = X \cdot \nabla F_\mu(x)(x) = \mu(F_\mu(x)(x) - f(x)) = 0.$$

Equivalently, f is a global first integral for the vector field X. $\square$

We finally underline that, in the previous proposition, the choice of the parameter $\mu > 0$ is arbitrary.

3 The PDE point of view

The aim of this section is to show the relevance of the previous notion of approximated first integral (see Definition 2.1) inside the robust PDE

framework and related viscosity techniques, see [9] and [1].
Simplifying the notation, in the sequel we refer to:

$$X \cdot \nabla F(x) = \mu(F - f)(x). \tag{11}$$

Now, considering the classical PDE theory for equations of elliptic type, we take into account the following regularization of (11), with vanishing viscosity $\nu > 0$ and a sort of friction $\mu > 0$:

$$\frac{\nu^2}{2} \Delta F^\nu(x) + X \cdot \nabla F^\nu(x) = \mu(F^\nu - f)(x). \tag{12}$$

The existence and the asymptotic behavior of the solutions for (12), namely the convergence of the functions F^ν for $\nu \to 0$, has been largely enquired: for the convenience to the reader, we briefly resume below the main results (see [9] for some more details).

Let us consider the following elliptic differential operator with a small parameter $\nu > 0$ at the derivatives of highest order:

$$L^\nu := \frac{\nu^2}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x^i}.$$

We are interested on the related Dirichlet's problem:

$$\begin{cases} L^\nu F^\nu(x) + c(x) F^\nu(x) = g(x) \\ \\ F^\nu(x)|_{\partial\Omega} = \psi(x) \end{cases}$$

that is,

$$\begin{cases} \frac{\nu^2}{2} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 F^\nu}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{n} b^i(x) \frac{\partial F^\nu}{\partial x^i}(x) + c(x) F^\nu(x) = g(x) \\ \\ F^\nu(x)|_{\partial\Omega} = \psi(x) \end{cases} \tag{13}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth connected boundary $\partial\Omega$ and ψ is supposed to be continuous.
We recall below the existence and uniqueness result.

Theorem 3.1 *(Existence and uniqueness) We assume that the following conditions are satisfied.*

1. *The function c is uniformly continuous, bounded and $c(x) \leq 0$ for all $x \in \mathbb{R}^n$.*

2. *The coefficients of L^ν satisfy a Lipschitz condition.*

3.

$$k^{-1}\sum_{i=1}^{n}\lambda_i^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\lambda_i\lambda_j \le k\sum_{i=1}^{n}\lambda_i^2$$

for every real $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $x \in \mathbb{R}^n$, where k is a positive constant.

Under these assumptions, for every $\nu > 0$ there exists a unique solution F^ν to the problem (13).

In order to investigate on the asymptotic behavior of such a solutions, we remind the following

Theorem 3.2 *(Pointwise limit) Suppose conditions 1., 2. and 3. are satisfied. In the case where $c(x) < 0$ for all $x \in \Omega$ and, for a given $x \in \Omega$, the trajectory $\phi_X^t(x),\ t \ge 0$ does not leave Ω, then*

$$\lim_{\nu\to 0} F^\nu(x) = F(x) = -\int_0^{+\infty} g(\phi_X^\tau(x))exp\Big[\int_0^\tau c(\phi_X^v(x))dv\Big]d\tau. \qquad (14)$$

It is now interesting to underline the following fact: the representation of the function F depends decisively on the behavior of the flow ϕ_X^t. In particular, since the vector field X admits the invariant bounded domain Ω, the pointwise limit $\lim_{\nu\to 0} F^\nu(x) = F(x)$, $\forall x \in \Omega$, does not depend on the boundary datum in (13).

Now we are ready to come back to our original elliptic equation (12): with respect to the previous general setting, it corresponds to the coefficients $a^{ij}(x) = \delta^{ij}$, $b^i(x) = X^i(x)$, $c(x) = -\mu$ and $g(x) = -\mu f(x)$ and it trivially satisfies the three conditions of Theorems 3.1 and 3.2. In such a case, the pointwise limit for $\nu \to 0$ is *just* given by the above introduced function (9):

$$F(x) = \mu\int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x))d\tau, \qquad \forall x \in \Omega. \qquad (15)$$

This fact shows that, among the all possible perturbations of $X \cdot \nabla F(x) = 0$, the one proposed, that is $X \cdot \nabla F(x) = \mu(F - f)(x)$, exactly selects, for every $\mu > 0$, the approximated first integral coming from a regularizing viscous technique. This argument points out the robust viscosity motivation of (15) and –according to Definition 2.1– seems to mark a step towards the recognition of a good notion of approximated global first integral in Ergodic Theory.

4 Lyapunov exponents

Lyapunov exponents, first introduced at the beginning of the last century, provide a natural way to formalize the notion of exponential divergence for

the trajectories of a given dynamical system, see for example [18]. Subsequently, the use of such quantities in Ergodic Theory appeared in the fundamental paper of Oseledec (see [24] and [25]) and the numerical applications followed mainly after [5].
Here below, starting from the variational equation on the tangent space $\mathbb{R}^{2n}$, we make use of the formula (15) in order to obtain a finite-time approximation of Lyapunov exponents, which we will call exponentially dumped Lyapunov indicators. Moreover, we provide a viscous regularization of them, as we have done for the previously introduced approximated first integrals.

We start with the definition of Lyapunov exponent.

Definition 4.1 *Given a pair $(x, v) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, the Lyapunov exponent associated to (x, v) is defined as*

$$\chi(x, v) := \lim_{t \to +\infty} \frac{1}{t} \log \left(\frac{\|v_t\|}{\|v\|} \right), \tag{16}$$

where $v_t := D\phi_X^t(x)v$ is the tangent vector at the time $t > 0$ and $\|\cdot\|$ denotes a norm.

For convenience, in the sequel we use the Euclidean norm. We remark that, by an easy computation, the Lyapunov exponent χ admits also an integral representation as a time average:

$$\chi(x, v) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f_\chi(\phi_{\mathbf{X}}^\tau(x, v)) d\tau. \tag{17}$$

In the above formula, the function f_χ corresponds to:

$$f_\chi(x, v) = \frac{v \cdot DX(x)v}{\|v\|^2}. \tag{18}$$

Moreover, the variational vector field $\mathbf{X}(x, v) := (X(x), DX(x)v)$, where DX denotes the Jacobian matrix related to X, is defined on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and the corresponding flow $\phi_{\mathbf{X}}^t$ is given by:

$$\phi_{\mathbf{X}}^t(x, v) = \left(\phi_X^t(x), D\phi_X^t(x)v\right).$$

The time average representation formula (17) pushes to consider, in the light of the previous sections, the following

Definition 4.2 *(Exponentially dumped Lyapunov indicator) Let $\mu > 0$. Given a pair $(x, v) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, the exponentially dumped Lyapunov indicator associated to (x, v) is defined as*

$$\mathcal{K}_\mu(x, v) := \mu \int_0^{+\infty} e^{-\mu\tau} \ f_\chi(\phi_{\mathbf{X}}^\tau(x, v)) \ d\tau. \tag{19}$$

In the next section, we will show that $\mathcal{K}_\mu$ represents a powerful indicator for the integrability of a dynamical system.

We return now to the case of a smooth vector field X over a compact invariant set Ω. In this setting and by using Theorem 3.2, we will revisit below the formula (19) as a pointwise limit of functions $\mathcal{K}_\mu^\nu$, providing a viscosity regularization of $\mathcal{K}_\mu$. However, taking into account the hypothesis of Theorem 3.2, we observe that the flow of $\mathbf{X}$ is not defined on a compact connected invariant set because the norm of the tangent vectors can diverge to infinity. In order to solve this technical problem, we give alternative representations of χ and $\mathcal{K}_\mu$ through a reformulation of the variational dynamics as the flow of a vector field defined on a compact connected invariant subset of $\Omega \times \mathbb{R}^n$.
We proceed in two steps. We start by defining the following vector field $\mathbf{Y}$ on $\Omega \times (\mathbb{R}^n \setminus \{0\})$, which is substantially the v-orthogonal projection of $\mathbf{X}$:

$$\mathbf{Y}(x,v) := \left(X(x), DX(x)\frac{v}{\|v\|} - \frac{v}{\|v\|}\left[\frac{v}{\|v\|} \cdot DX(x)\frac{v}{\|v\|} \right] \right). \tag{20}$$

We prove now the following technical results.

Lemma 4.1 *For all $(x,v) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, it holds:*

$$\chi(x,v) = \lim_{t\to+\infty} \frac{1}{t} \int_0^t f_\chi\left(\phi_{\mathbf{Y}}^\tau\left(x, \frac{v}{\|v\|}\right)\right) d\tau,$$

where the function f_χ is given by formula (18). Moreover, every subset $\Omega \times \partial B^n(0,r)$, $r > 0$, is invariant under the flow $\phi_{\mathbf{Y}}^t$.

Proof. We start by introducing the retraction Π:

$$\Pi : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \times \mathbb{S}^{n-1}$$

$$(x,v) \longmapsto \Pi(x,v) = \left(x, \frac{v}{\|v\|}\right).$$

In view of the 0-homogeneity of the function f_χ, that is $f_\chi(x, \lambda v) = f_\chi(x,v)$ $\forall \lambda \neq 0$, we have that

$$f_\chi\left(\phi_{\mathbf{X}}^t(x,v)\right) = f_\chi\left(\Pi \circ \phi_{\mathbf{X}}^t(x,v)\right).$$

Therefore, we enquire the rectraction of the dynamics $\phi_{\mathbf{X}}^t(x,v)$, proving that it corresponds to the flow of the vector field $\mathbf{Y}$, that is:

$$\Pi \circ \phi_{\mathbf{X}}^t(x,v) = \phi_{\mathbf{Y}}^t \circ \Pi(x,v). \tag{21}$$

In order to do this, we denote $(x(t), v(t)) := \phi^t_{\mathbf{X}}(x, v)$ and compute:

$$
\begin{aligned}
\frac{d}{dt}\Pi \circ \phi^t_{\mathbf{X}}(x,v) &= \frac{d}{dt}\left(x(t), \frac{v(t)}{\|v(t)\|}\right) \\
&= \left(\dot{x}(t), \frac{\dot{v}(t)}{\|v(t)\|} - \frac{v(t)}{\|v(t)\|^2}\,\frac{v(t)\cdot \dot{v}(t)}{\|v(t)\|}\right) \\
&= \left(\dot{x}(t), \frac{DX(x(t))v(t)}{\|v(t)\|} - \frac{v(t)}{\|v(t)\|^2}\,\frac{v(t)\cdot DX(x(t))v(t)}{\|v(t)\|}\right) \\
&= \left(X(x(t)), DX(x(t))\frac{v(t)}{\|v(t)\|} - \frac{v(t)}{\|v(t)\|}\left[\frac{v(t)}{\|v(t)\|}\cdot DX(x(t))\frac{v(t)}{\|v(t)\|}\right]\right) \\
&= \mathbf{Y}\left(x(t), v(t)\right) = \mathbf{Y}\left(x(t), \frac{v(t)}{\|v(t)\|}\right).
\end{aligned}
$$

The use of the previous relation together with the initial condition $\Pi \circ \phi^0_{\mathbf{X}}(x,v) = \phi^0_{\mathbf{Y}} \circ \Pi\,(x,v)$ imply the relation (21). As a straightforward consequence $f_\chi\left(\Pi \circ \phi^t_{\mathbf{X}}(x,v)\right) = f_\chi\left(\phi^t_{\mathbf{Y}} \circ \Pi\,(x,v)\right)$, and the statement of the lemma follows. Finally, from the relation:

$$
v \cdot \mathbf{Y}^{(v)}(x,v) = v \cdot \left(DX(x)\frac{v}{\|v\|} - \frac{v}{\|v\|}\left[\frac{v}{\|v\|}\cdot DX(x)\frac{v}{\|v\|}\right]\right) = 0,
$$

we immediately gain the invariance of every subset $\Omega \times \partial B^n(0,r)$, $r > 0$, under the flow $\phi^t_{\mathbf{Y}}$. □

By using the same arguments of the previous proof, we gain the analogous result for the exponentially dumped Lyapunov indicators.

Lemma 4.2 *Let $\mu > 0$. For all $(x,v) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, it holds:*

$$
\mathcal{K}_\mu\,(x,v) = \mu \int_0^{+\infty} e^{-\mu\tau} f_\chi\left(\phi^\tau_{\mathbf{Y}}\left(x, \frac{v}{\|v\|}\right)\right) d\tau,
$$

where the function f_χ is given by formula (18).

We remark now that, although the flow $\phi^t_{\mathbf{Y}}$ leaves invariant $\Omega \times \partial B^n(0,r)$, $r > 0$, we cannot apply Theorem 3.2 yet, essentially because $\mathbf{Y}$ cannot be extended by continuity in $v = 0$ and thus it does not admit a compact connected invariant set. To solve this problem, the second step consists in modifying the vector field $\mathbf{Y}$ in a small neighborhood of $0 \in B^n(0,1)$. More precisely, given $h \in C^\infty(B^n(0,1), \mathbb{R})$, such that $h(v) = 1$ for $\|v\| \geq \varepsilon$ and $h(v) = 0$ for $\|v\| < \frac{\varepsilon}{2}$, we introduce the following vector field $\widehat{\mathbf{Y}}$ on $\Omega \times B^n(0,1)$:

$$
\widehat{\mathbf{Y}}(x,v) = \begin{cases} \mathbf{Y}(x,v) & \text{if } \|v\| \geq \varepsilon \\ \\ h(v)\mathbf{Y}(x,v) & \text{if } \|v\| < \varepsilon \end{cases} \tag{22}
$$

A direct application of previous Lemma 4.2 and Theorem 3.2 allows us now to take into account the elliptic Dirichlet's problem:

$$\begin{cases} \frac{\nu^2}{2}\Delta\mathcal{K}^\nu_\mu(x,v) + \widehat{\mathbf{Y}}^{(x)}\nabla_x\mathcal{K}^\nu_\mu(x,v) + \widehat{\mathbf{Y}}^{(v)}\nabla_v\mathcal{K}^\nu_\mu(x,v) = \mu(\mathcal{K}^\nu_\mu - f_X)(x,v) \\ \\ \mathcal{K}^\nu_\mu(x,v)|_{\partial D} = \psi(x,v) \end{cases} \tag{23}$$

where $D := \Omega \times B^n(0,1)$ and ψ is supposed to be continuous.
We finally denote by $(\widehat{\mathbf{Y}}^\nu_\tau, P_{(x,v)})$ the Markov process solving the stochastic differential equation (now on the phase space $\mathbb{R}^{2n}$) related to (23). As explained in the previous section, the solution $\mathcal{K}^\nu_\mu$ admits the representation:

$$\mathcal{K}^\nu_\mu(x,v) = M_{(x,v)}\left[\mu\int_0^{+\infty} e^{-\mu\tau} f_X(\widehat{\mathbf{Y}}^\nu_\tau)d\tau\right], \tag{24}$$

and can be considered as a viscous regularization of the exponentially dumped Lyapunov indicator $\mathcal{K}_\mu$. In fact, the following convergence result holds:

Proposition 4.1 *Let $\nu, \mu > 0$. The corresponding solution (24), when restricted to $\Omega\times(B^n(0,1)\setminus \overset{\circ}{B^n}(0,\varepsilon))$, is independent on the above regularizing function $h(v)$, and the following pointwise limit holds:*

$$\lim_{\nu\to 0}\mathcal{K}^\nu_\mu(x,v) = \mathcal{K}_\mu(x,v).$$

5 Fast and exponentially dumped Lyapunov indicators

In the last years, the so called Fast Lyapunov Indicators [11] have been extensively used to numerically detect the phase space structure, i.e. the distribution of KAM tori and resonances, of quasi–integrable systems, see [12] and [13]. For the equation $\dot{x} = X(x)$, the simplest definition of Fast Lyapunov Indicator of a point x and of a tangent vector v, at time T, is:

$$\mathrm{FLI}_T(x,v) = \log\Big(\frac{\|v_T\|}{\|v\|}\Big), \tag{25}$$

where $v_T = D\phi^T_X(x)v$. In [13] it is proved that, for Hamiltonian vector fields, if T is suitably long (precisely of some inverse power of the perturbing parameter, see [13] for precise statements) and v is generic, the value of $\mathrm{FLI}_T(x,v)$ is different, at order 0 in ε, in the case x belongs to an invariant KAM torus from the case x belongs to a resonant elliptic torus. Therefore, the computation of the FLI on grids of initial conditions in the phase space allows one to detect the distribution of invariant tori and resonances in relatively short CPU times. Let us remark that if at a first glance the function FLI_T seems a crude way of estimating Lyapunov exponents from finite time computations, in [13] it is proved that it provides informations on the dynamics of x that cannot be obtained with the largest Lyapunov exponent, which in fact is equal to zero for all KAM tori and resonant elliptic tori. Moreover, resonances and KAM tori are detected by the FLI on times T which are much smaller than the times required to compute finite-time approximations of the largest Lyapunov exponent. These computational advantages allows one to use the FLI for extensive dynamical analysis of dynamical systems representing accurate models of real systems, such as the dynamical model for the outer solar system ([14] and [15]).

In this section we propose a practical use of the exponentially dumped Lyapunov indicators, which we have defined in section 4, as a global definition of Fast Lyapunov Indicators, in the sense specified in the introduction. In fact, on the one hand the definition (25) of Fast Lyapunov Indicators is a pointwise definition, on the other hand the definition (19) with $\mu = 1/T$ provides almost the same information on the dynamics as the FLI.
We compare the results provided by the two indicators on the quasi-integrable Hamiltonian system defined in [12]:

$$H = \frac{I_1^2}{2} + \frac{I_1^2}{2} + I_3 + \varepsilon f(\varphi_1, \varphi_2, \varphi_3), \tag{26}$$

where $I_1, I_2, I_3 \in \mathbb{R}$, $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{T}^1$, the underlying symplectic structure is $dI \wedge d\varphi$, $\varepsilon > 0$ is the perturbing parameter and the perturbation f is given by:

$$f(\varphi_1, \varphi_2, \varphi_3) = \frac{1}{\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) + 4}. \tag{27}$$

Hamiltonian system (26) is particularly suited for the detection of the KAM tori and web of resonances (see [12]), in fact: for $\varepsilon > 0$ suitable small, the KAM Theorem applies to (26); each KAM torus of the system intersects transversely in only one point the section of phase space:

$$S := \{(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3) \ \text{ with } \ (\varphi_1, \varphi_2, \varphi_3) = (0, 0, 0)\},$$

which we call action space; the perturbation (27) is non-generic in the sense of the Poincaré Theorem about the non-integrability of quasi-integrable systems.

We now compute the exponentially dumped Lyapunov indicators defined by $\mu = \varepsilon$ for a grid of equally spaced initial conditions on the section S. The practical computation of the integral in (19) is done by restricting the integration interval up to a total time T_1 such that the exponential dump $\exp(-\mu T_1)$ is smaller than the numerical precision adopted for the computation. For example, we set T_1 such that: $\exp(-\mu T_1) < 10^{-16}$. The initial vector for any initial condition was chosen as: $(v_{I_1}, v_{I_2}, v_{I_3}, v_{\varphi_1}, v_{\varphi_2}, v_{\varphi_3}) = (1/\sqrt{5}, \sqrt{2/5}, 0, 1/\sqrt{5}, 1/\sqrt{5}, 0)$. The result of the computation is reported

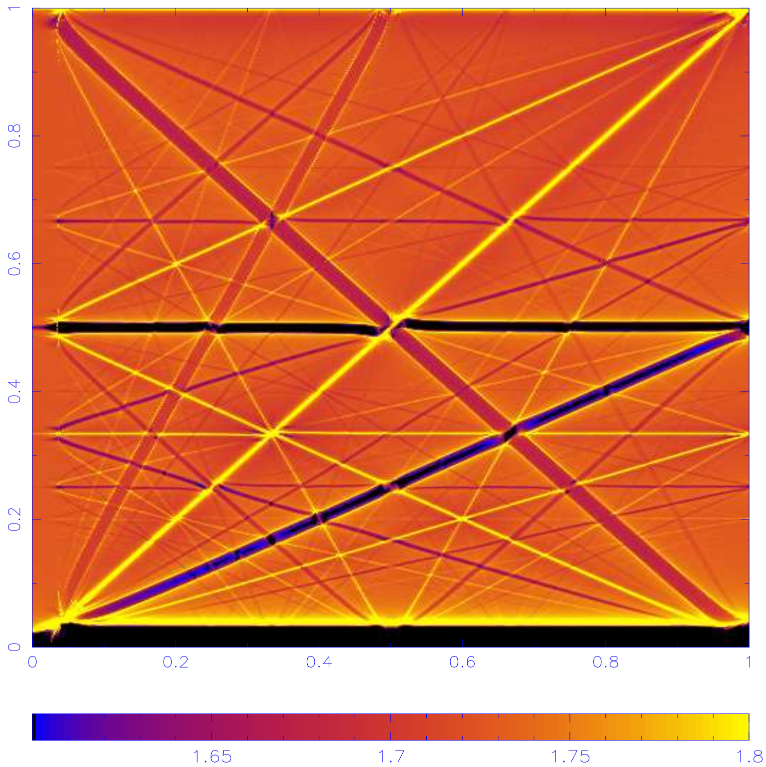

Figure 1: Computation of the exponentially dumped Lyapunov indicators for initial conditions $(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3)$ on the section S, $\varepsilon = 0.004$ and $\mu = \sqrt{\varepsilon}/10$. The x-axis corresponds to the value of I_1, the y-axis corresponds to the value of I_2; for each initial condition the value of the exponentially dumped Lyapunov indicator is reported using the color scale reported below the picture. The well known structure of resonances of this system is clearly detected by the highest and lowest values of the exponentially dumped Lyapunov indicator, see [12] and [13].

in Figure 1, where we report for any initial actions (I_1, I_2) the value of the computed exponentially dumped Lyapunov indicator using a color scale[1] such that dark gray corresponds to the lowest values of the indicator and light gray corresponds to the highest values of the indicator. Following [13], the KAM tori are characterized by intermediate gray, hyperbolic motions by light gray and resonant elliptic tori by dark gray. It is clear that the distribution of the values of the exponentially dumped Lyapunov indicator shown in Figure 1 corresponds to the distribution of resonances and KAM tori as it is described in [12].

[1]The color version of the figure can be found on the electronic version of the paper so that light gray corresponds there to yellow and darker gray corresponds there to darker orange.

Acknowledgements O. Bernardi, M. Guzzo and L. Zanelli have been supported by the project "Tecniche variazionali e PDE in topologia simplettica e applicazioni fisico-matematiche" of Gruppo Nazionale di Fisica Matematica GNFM. M. Guzzo, F. Cardin and O. Bernardi have been supported by the project CPDA063945/06 of the University of Padova.

References

[1] V. I. Arnol'd, B. A. Khesin, *Topological Methods in Hydrodynamics.* Applied Mathematical Sciences, 125. Springer-Verlag, New York, (1998).

[2] M. Bardi, I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Systems and Control: Foundations and Applications. Boston, MA: Birkhauser. xvii, 570 pp. (1997).

[3] G. Barles, *Solutions de viscosité des équations de HamiltonJacobi.* Springer, Paris, (1994).

[4] *Hamiltonian Systems and Fourier Analysis, new prospects for gravitational dynamics.* D. Benest, C. Froeschlé, E. Lega editors. Advances in Astronomy and Astrophysics. Cambridge Scientific Publishers, (2005).

[5] G. Benettin, L. Galgani, A. Giorgilli, J. M. Strelcyn, *Tous les nombres caractristiques de Lyapounov sont effectivement calculables.* C. R. Acad. Sci. Paris Sr. A-B 286, (1978).

[6] G. Benettin, *Physical applications of Nekhoroshev theorem and exponential estimates.* Hamiltonian dynamics theory and applications, 1-76, Lecture Notes in Math., 1861, Springer, Berlin, (2005).

[7] G. Contopoulos, *A classification of the integrals of motion.* Astrophys. J. 138, 1297-1305, (1963).

[8] I. P. Cornfeld, S.V. Fomin, Ya. G. Sinaĭ, *Ergodic theory.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 245. Springer-Verlag, New York, x+486 pp. (1982).

[9] M. I. Freidlin, A. D. Wentzell, *Random perturbations of dynamical systems*, Grundlehren der mathematischen Wissenschaften 260, Springer-Verlag, New York, (1984).

[10] C. Froeschlé, *On the number of isolating integrals in systems with three degrees of freedom.* Astrophysics and Space Research 14, 110-117, (1971).

[11] C. Froeschlé, E. Lega, and R. Gonczi, *Fast Lyapunov indicators. Application to asteroidal motion.* Celest. Mech. and Dynam. Astron., Vol. 67, 41-62, (1997).

[12] C. Froeschlé, M. Guzzo, E. Lega, *Graphical Evolution of the Arnold Web: From Order to Chaos.* Science, Volume 289, n. 5487, (2000).

[13] M. Guzzo, E. Lega, C. Froeschlé, *On the numerical detection of the effective stability of chaotic motions in quasi-integrable systems.* Physica D, Volume 163, Issues 1-2, 1-25, (2002).

[14] M. Guzzo, *The web of three-planets resonances in the outer Solar System.* Icarus, vol. 174, n. 1, pag. 273-284, (2005).

[15] M. Guzzo, *The web of three-planet resonances in the outer solar system II: a source of orbital instability for Uranus and Neptune.* Icarus, vol. 181, 475-485, (2006).

[16] M. Guzzo, O. Bernardi, F. Cardin, *The experimental localization of Aubry-Mather sets using regularization techniques inspired by viscosity theory.* Chaos 17, no. 3, 033107, 9 pp. (2007).

[17] M. Hénon, C. Heiles, *The applicability of the third integral of motion: Some numerical experiments.* Astronom. J. 69, 73-79, (1964).

[18] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems.* With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, (1995).

[19] J. Laskar, *The chaotic motion of the solar system. A numerical estimate of the size of the chaotic zones.* Icarus, 88, 266-29, (1990).

[20] J. Laskar, C. Froeschlé, A. Celletti, *The measure of chaos by the numerical analysis of the fundamental frequencies. Application to the standard mapping.* Physica D, 56, 253, (1992).

[21] P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations.* Research Notes in Mathematics, 69. Boston - London - Melbourne: Pitman Advanced Publishing Program. 317 pp. (1982).

[22] M. Month, J. C. Herrera, *Nonlinear Dynamics and the Beam-Beam Interaction.* American Institute of Physics, New York, (1979).

[23] A. Morbidelli, *Modern Celestial Mechanics: aspects of Solar System dynamics.* In "Advances in Astronomy and Astrophysics", Taylor & Francis, London, (2002).

[24] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Lyapunov exponents of dynamical systems.* (Russian) Trudy Moskov. Mat. Obšč. 19, 179-210, (1968).

[25] Y. B. Pesin, *Characteristic Lyapunov Exponents and Smooth Ergodic Theory.* Russian Math. Surveys, 32, 4, 55-114, (1977).

[26] H. Risken, *Fokker-Planck Equation: Methods of Solution and Applications.* Second edition. Springer Series in Synergetics, 18. Springer-Verlag, Berlin, (1989).