Spectral and scattering theory for some abstract QFT Hamiltonians

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Abstract

We introduce an abstract class of bosonic QFT Hamiltonians and study their spectral and scattering theories. These Hamiltonians are of the form $H = \mathrm{d}\Gamma(\omega) + V$ acting on the bosonic Fock space $\Gamma(\mathfrak{h})$, where ω is a massive one-particle Hamiltonian acting on $\mathfrak{h}$ and V is a Wick polynomial $\mathrm{Wick}(w)$ for a kernel w satisfying some decay properties at infinity.

We describe the essential spectrum of H, prove a Mourre estimate outside a set of thresholds and prove the existence of asymptotic fields. Our main result is the *asymptotic completeness* of the scattering theory, which means that the CCR representations given by the asymptotic fields are of Fock type, with the asymptotic vacua equal to the bound states of H. As a consequence H is unitarily equivalent to a collection of second quantized Hamiltonians.

1 Introduction

1.1 Introduction

In recent years a lot of effort was devoted to the spectral and scattering theory of various models of Quantum Field Theory like models of non-relativistic matter coupled to quantized radiation or self-interacting relativistic models in dimension 1+1 (see among many others the papers [AHH], [DG1], [DG2], [FGSch], [FGS], [LL], [P], [Sp] and references therein). Substantial progress was made by applying to these models methods originally developed in the study of $N-$particle Schroedinger operators, namely the Mourre positive commutator method and the method of propagation observables to study the behavior of the unitary group $\mathrm{e}^{-\mathrm{i}tH}$ for large times.

Up to now, the most complete results (valid for example for arbitrary coupling constants) on the spectral and scattering theory for these models are available only for massive models and for localized interactions. (For results on massless models see for example [FGS] and references therein).

It turns out that for this type of models, the details of the interaction are often irrelevant. The essential feature of the interaction is that it can be written as a *Wick polynomial*, with a symbol (see below) which decays sufficiently fast at infinity.

The conjugate operator (for the Mourre theory), or the propagation observables (for the proof of propagation estimates), are chosen as second quantizations of corresponding operators on the one-particle space $\mathfrak{h}$.

In applications the one-particle kinetic energy is usually the operator $(k^2+m^2)^{\frac{1}{2}}$ acting on $L^2(\mathbb{R}^d, \mathrm{d}k)$, which clearly has a nice spectral and scattering theory. Therefore the necessary one-particle operators are easy to construct.

Our goal in this paper is to describe an abstract class of bosonic QFT Hamiltonians to which the methods and results of [DG2], [DG1] can be naturally extended.

Let us first briefly describe this class of models. We consider Hamiltonians of the form:

$$H = H_0 + V, \text{ acting on the bosonic Fock space } \Gamma(\mathfrak{h}),$$

where $H_0 = \mathrm{d}\Gamma(\omega)$ is the second quantization of the one-particle kinetic energy ω and $V = \mathrm{Wick}(w)$ is a *Wick polynomial*. To define H without ambiguity, we assume that $H_0 + V$ is essentially selfdjoint and bounded below on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$.

The Hamiltonian H is assumed to be *massive*, namely we require that $\omega \geq m > 0$ and moreover that powers of the number operator N^p for $p \in \mathbb{N}$ are controlled by sufficiently high powers of the resolvent $(H+b)^{-m}$. These bounds are usually called *higher order estimates*.

The interaction V is supposed to be a *Wick polynomial*. If for example $\mathfrak{h} = L^2(\mathbb{R}^d,\ dk)$, this means that V is a finite sum $V = \sum_{p,q\in I} \mathrm{Wick}(w_{p,q})$ where $\mathrm{Wick}(w_{p,q})$ is formally defined as:

$$\mathrm{Wick}(w_{p,q}) = \int a^*(K)a(K')w_{p,q}(K,K')\mathrm{d}K\mathrm{d}K',$$

for

$$K = (k_1, \dots, k_p),\ K' = (k'_1, \dots, k'_q),\ a^*(K) = \Pi_{i=1}^p a^*(k_i), \quad a^*(K') = \Pi_{i=1}^q a(k'_i),$$

and $w_{p,q}(K,K')$ is a scalar function separately symmetric in K and K'. To define $\mathrm{Wick}(w)$ as an unbounded operator on $\Gamma(\mathfrak{h})$, the functions $w_{p,q}$ are supposed to be in $L^2(\mathbb{R}^{(p+q)d})$. The functions $w_{p,q}$ are then the distribution kernels of a Hilbert-Schmidt operator $w_{p,q}$ from $\otimes_{\mathrm{s}}^q\mathfrak{h}$ into $\otimes_{\mathrm{s}}^p\mathfrak{h}$. Putting together these operators we obtain a Hilbert-Schmidt operator w on $\Gamma(\mathfrak{h})$ which is called the *Wick symbol* of the interaction V.

In physical situations, this corresponds to an interaction which has both a space and an ultraviolet cutoff (in one space dimension, only a space cutoff is required).

As said above, it is necessary to assume that the one-particle energy ω has a nice spectral and scattering theory. It is possible to formulate the necessary properties of ω in a very abstract framework, based on the existence of only two auxiliary Hamiltonians on $\mathfrak{h}$. The first one is a *conjugate operator* a for ω, in the sense of the Mourre method. The second one is a *weight operator* $\langle x\rangle$, which is used both to control the 'order' of various operators on $\mathfrak{h}$ and as a way to localize bosons in $\mathfrak{h}$. Note that the one-particle energy ω may have bound states.

The first basic result on spectral theory that we obtain is the *HVZ theorem*, which describes the essential spectrum of H. If $\sigma_{\mathrm{ess}}(\omega) = [m_\infty, +\infty[$ for some $m_\infty \geq m > 0$, then we show that

$$\sigma_{\mathrm{ess}}(H) = [\inf \sigma(H) + m_\infty, +\infty[,$$

in particular H always has a ground state.

We then consider the Mourre theory and prove that the second quantized Hamiltonian $A = \mathrm{d}\Gamma(a)$ is a conjugate operator for H. In particular this proves the local finiteness of point spectrum outside of the set of *thresholds*, which is equal to

$$\tau(H) = \sigma_{\mathrm{pp}}(H) + \mathrm{d}\Gamma^{(1)}(\tau(\omega)),$$

where $\tau(\omega)$ is the set of thresholds of ω for a and $\mathrm{d}\Gamma^{(1)}(E)$ for $E \subset \mathbb{R}$ is the set of all finite sums of elements of E.

The scattering theory for our abstract Hamiltonians follows the standard approach based on the *asymptotic Weyl operators*. These are defined as the limits:

$$W^{\pm}(h) = \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} W(h_t) \mathrm{e}^{-\mathrm{i}tH}, \quad h \in \mathfrak{h}_{\mathrm{c}}(\omega),$$

where $\mathfrak{h}_{\mathrm{c}}(\omega)$ is the continuous spectral subspace for ω and $h_t = \mathrm{e}^{-\mathrm{i}t\omega}h$. The asymptotic Weyl operators define two CCR representations over $\mathfrak{h}_{\mathrm{c}}(\omega)$. Due to the fact that the theory is massive, it is rather easy to see that these representations are of Fock type. The main problem of scattering theory is to describe their *vacua*, i.e. the spaces of vectors annihilated by the asymptotic annihilation operators $a^{\pm}(h)$ for $h \in \mathfrak{h}_{\mathrm{c}}(\omega)$.

The main result of this paper is that the vacua coincide with the *bound states* of H. As a consequence one sees that H is unitarily equivalent to the asymptotic Hamiltonian:

$$H_{|\mathcal{H}_{\mathrm{pp}}(H)} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega), \text{ acting on } \mathcal{H}_{\mathrm{pp}}(H) \otimes \Gamma(\mathfrak{h}_{\mathrm{c}}(\omega)).$$

This result is usually called the *asymptotic completeness* of wave operators. It implies that H is unitarily equivalent to a direct sum of $E_i + \mathrm{d}\Gamma(\omega_{|\mathfrak{h}_{\mathrm{c}}(\omega)})$, where E_i are the eigenvalues of H. In more physical terms, asymptotic completeness means that for large times any initial state asymptotically splits into a bound state and a finite number of free bosons.

We conclude the introduction by describing the examples of abstract QFT Hamiltonians to which our results apply.

The first example is the space-cutoff $P(\varphi)_2$ model with a *variable metric*, which corresponds to the quantization of a non-linear Klein-Gordon equation with variable coefficients in one space dimension.

The one-particle space is $\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}x)$ and the usual relativistic kinetic energy $(D^2 + m^2)^{\frac{1}{2}}$ is replaced by the square root $h^{\frac{1}{2}}$ of a second order differential operator $h = Da(x)D + c(x)$, where $a(x) \to 1$ and $c(x) \to m_\infty^2$ for $m_\infty > 0$ when $x \to \infty$. (It is also possible to treat functions c having different limits $m_{\pm\infty}^2 > 0$ at $\pm\infty$).

The interaction is of the form:

$$V = \int_{\mathbb{R}} g(x) :P(x, \varphi(x)): \mathrm{d}x,$$

where $g \geq 0$ is a function on $\mathbb{R}$ decaying sufficiently fast at ∞, $P(x, \lambda)$ is a bounded below polynomial of even degree with variable coefficients, $\varphi(x) = \phi(\omega^{-\frac{1}{2}}\delta_x)$ is the relativistic field operator and $:\ :$ denotes the Wick ordering.

This model is considered in details in [GP], applying the abstract arguments in this paper. Note that some conditions on the eigenfunctions and generalized eigenfunctions of h are necessary in order to prove the higher order estimates.

The analogous model for constant coefficients was considered in [DG1]. Even in the constant coefficient case we improve the results in [DG1] by removing an unpleasant technical assumption on g, which excluded to take g compactly supported.

The second example is the generalization to higher dimensions. The one-particle energy ω is:

$$\omega = \Big(\sum_{1\leq i,j\leq d} D_i a_{ij}(x) D_j + c(x)\Big)^{\frac{1}{2}},$$

where $h = \sum_{1\leq i,j\leq d} D_i a_{ij}(x) D_j + c(x)$ is an elliptic second order differential operator converging to $D^2 + m_\infty^2$ when $x \to \infty$. The interaction is now

$$\int_{\mathbb{R}} g(x) P(x, \varphi_\kappa(x)) \mathrm{d}x,$$

where P is as before and $\varphi_\kappa(x) = \phi(\omega^{-\frac{1}{2}} F(\omega \leq \kappa)\delta_x)$ is now the UV-cutoff relativistic field. Here because of the UV cutoff, the Wick ordering is irrelevant. Again some conditions on eigenfunctions and generalized eigenfunctions of h are necessary.

We believe that our set of hypotheses should be sufficiently general to consider also Klein-Gordon equations on other Riemannian manifolds, like for example manifolds equal to the union of a compact piece and a cylinder $\mathbb{R}^+ \times M$, where the metric on $\mathbb{R}^+ \times M$ is of product type.

1.2 Plan of the paper

We now describe briefly the plan of the paper.

Section 2 is a collection of various auxiliary results needed in the rest of the paper. We first recall in Subsects. 2.1 and 2.2 some arguments connected with the abstract Mourre theory and a convenient functional calculus formula. In Subsect. 2.3 we fix some notation connected with one-particle operators. Standard results taken from [DG1], [DG2] on bosonic Fock spaces and Wick polynomials are recalled in Subsects. 2.4 and 2.6.

The class of abstract QFT Hamiltonians that we will consider in the paper is described in Sect. 3. The results of the paper are summarized in Sect. 4. In Sect. 5 we give examples of abstract QFT Hamiltonians to which all our results apply, namely the space-cutoff $P(\varphi)_2$ model with a variable metric, and the analogous models in higher dimensions, where now an ultraviolet cutoff is imposed on the polynomial interaction.

Sect. 6 is devoted to the proof of commutator estimates needed in various localization arguments. The spectral theory of abstract QFT Hamiltonians is studied in Sect. 7. The essential spectrum is described in Subsect. 7.1, the virial theorem and Mourre's positive commutator estimate are proved in Subsects. 7.2, 7.4 and 7.5. The results of Sect. 7 are related to those of [1], where abstract bosonic and fermionic QFT Hamiltonians are considered using a C^*-algebraic approach instead of the geometrical approach used in our paper. Our result on essential spectrum can certainly be deduced from the results in [1]. However the Mourre theory in [1] requires that the one-particle Hamiltonian ω has no eigenvalues and also that ω is affiliated to an abelian C^*-algebra $\mathcal{O}$ such that $\mathrm{e}^{\mathrm{i}ta}\mathcal{O}\mathrm{e}^{-\mathrm{i}ta} = \mathcal{O}$, where a is the one-particle conjugate operator. In concrete examples, this second assumption seems adapted to constant coefficients one-particle Hamiltonians and not satisfied by the examples we describe in Sect. 5.

In Sect. 8 we describe the scattering theory for abstract QFT Hamiltonians. The existence of asymptotic Weyl operators and asymptotic fields is shown in Subsect. 8.1. Other natural objects, like the wave operators and extended wave operators are defined in Subsects. 8.2, 8.3.

Propagation estimates are shown in Sect. 9. The most important are the phase-space propagation estimates in Subsect. 9.2, 9.3 and the minimal velocity estimate in Subsect. 9.4.

Finally asymptotic completeness is proved in Sect. 10. The two main steps is the proof of *geometric asymptotic completeness* in Subsect. 10.4, identifying the vacua with the states for which no bosons escape to infinity. The asymptotic completeness itself is shown in Subsect. 10.5.

Various technical proofs are collected in the Appendix.

2 Auxiliary results

In this section we collect various auxiliary results which will be used in the sequel.

2.1 Commutators

Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$. If $B \in B(\mathcal{H})$ on says that B is *of class* $C^1(A)$ [ABG] if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}tA} B \mathrm{e}^{-\mathrm{i}tA} \in B(\mathcal{H})$$

is C^1 for the strong topology.

If H is selfadjoint on $\mathcal{H}$, one says that H is *of class* $C^1(A)$ [ABG] if for some (and hence all) $z \in \mathbb{C}\backslash\sigma(H)$, $(H-z)^{-1}$ is of class $C^1(A)$. The classes $C^k(A)$ for $k \geq 2$ are defined similarly.

If H is of class $C^1(A)$, the commutator $[H, \mathrm{i}A]$ defined as a quadratic form on $\mathcal{D}(A) \cap \mathcal{D}(H)$ extends then uniquely as a bounded quadratic form on $\mathcal{D}(H)$. The corresponding operator in $B(\mathcal{D}(H), \mathcal{D}(H)^*)$ will be denoted by $[H, \mathrm{i}A]_0$.

If H is of class $C^1(A)$ then the *virial relation* holds (see [ABG]):

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]_0 \mathbb{1}_{\{\lambda\}}(H) = 0, \quad \lambda \in \mathbb{R}.$$

An estimate of the form

$$\mathbb{1}_I(H)[H, \mathrm{i}A]_0 \mathbb{1}_I(H) \geq c_0 \mathbb{1}_I(H) + K,$$

where $I \subset \mathbb{R}$ is a compact interval, $c_0 > 0$ and K a compact operator on $\mathcal{H}$, or:

$$\mathbb{1}_I(H)[H, \mathrm{i}A]_0 \mathbb{1}_I(H) \geq c_0 \mathbb{1}_I(H),$$

is called a (strict) *Mourre estimate* on I. An operator A such that the Mourre estimate holds on I is called a *conjugate operator* for H (on I). Under an additional regularity condition of H w.r.t. A (for example if H is of class $C^2(A)$), it has several important consequences like weighted estimates on $(H - \lambda \pm \mathrm{i}0)^{-1}$ for $\lambda \in I$ (see e.g. [ABG]) or abstract propagation estimates (see e.g. [HSS]).

We now recall some useful machinery from [ABG] related with the best constant c_0 in the Mourre estimate. Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and B be a quadratic form with domain $\mathcal{D}(H^M)$ for some $M \in \mathbb{N}$ such that the *virial relation*

$$\mathbb{1}_{\{\lambda\}}(H) B \mathbb{1}_{\{\lambda\}}(H) = 0, \quad \lambda \in \mathbb{R}, \tag{2.1}$$

is satisfied. We set

$$\rho_H^B(\lambda) := \sup\{a \in \mathbb{R} |\, \exists\, \chi \in C_0^\infty(\mathbb{R}),\ \chi(\lambda) \neq 0, \quad \chi(H) B \chi(H) \geq a\chi^2(H)\},$$

$$\tilde{\rho}_H^B(\lambda) := \sup\{a \in \mathbb{R} |\, \exists\, \chi \in C_0^\infty(\mathbb{R}),\ \chi(\lambda) \neq 0, \exists\, K \text{ compact}, \quad \chi(H) B \chi(H) \geq a\chi^2(H) + K\}.$$

The functions, ρ_H^B, $\tilde{\rho}_H^B$ are lower semi-continuous and it follows from the virial relation that $\rho_H^B(\lambda) < \infty$ iff $\lambda \in \sigma(H)$, $\tilde{\rho}_H^B(\lambda) < \infty$ iff $\lambda \in \sigma_{\mathrm{ess}}(H)$ (see [ABG, Sect. 7.2]). One sets:

$$\tau_B(H) := \{\lambda |\, \tilde{\rho}_H^B(\lambda) \leq 0\}, \quad \kappa_B(H) := \{\lambda |\, \rho_H^B(\lambda) \leq 0\},$$

which are closed subsets of $\mathbb{R}$, and

$$\mu_B(H) := \sigma_{\mathrm{pp}}(H) \backslash \tau_B(H).$$

The virial relation and the usual argument shows that the eigenvalues of H in $\mu_B(H)$ are of finite multiplicity and are not accumulation points of eigenvalues. In the next lemma we collect several abstract results adapted from [ABG], [BG].

Lemma 2.1 *i) if $\lambda \in \mu_B(H)$ then $\rho_H^B(\lambda) = 0$. If $\lambda \notin \mu_B(H)$ then $\rho_H^B(\lambda) = \tilde{\rho}_H^B(\lambda)$.*
ii) $\rho_H^B(\lambda) > 0$ iff $\tilde{\rho}_H^B(\lambda) > 0$ and $\lambda \notin \sigma_{\rm pp}(H)$, which implies that

$$\kappa_B(H) = \tau_B(H) \cup \sigma_{\rm pp}(H).$$

iii) Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $H = H_1 \oplus H_2$, $B = B_1 \oplus B_2$, where B_i, H, B are as above and satisfy (2.1). Then

$$\rho_H^B(\lambda) = \min(\rho_{H_1}^{B_1}(\lambda), \rho_{H_2}^{B_2}(\lambda)).$$

iv) Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2$, $B = B_1 \otimes \mathbb{1} + \mathbb{1} \otimes B_2$, where H_i, B_i, H, B are as above, satisfy (2.1) and H_i are bounded below. Then

$$\rho_H^B(\lambda) = \inf_{\lambda_1 + \lambda_2 = \lambda} \left(\rho_{H_1}^{B_1}(\lambda_1) + \rho_{H_2}^{B_2}(\lambda_2) \right).$$

Proof. *i), ii)* can be found in [ABG, Sect. 7.2], in the case $B = [H, \mathrm{i}A]$ for A a selfjadjoint operator such that $H \in C^1(A)$. This hypothesis is only needed to ensure the virial relation (2.1). *iii)* is easy and *iv)* can be found in [BG, Prop. Thm. 3.4] in the same framework. Again it is easy to see that the proof extends verbatim to our situation. □

Assume now that H, A are two selfadjoint operators on a Hilbert space $\mathcal{H}$ such that the quadratic form $[H, \mathrm{i}A]$ defined on $\mathcal{D}(H^M) \cap \mathcal{D}(A)$ for some M uniquely extends as a quadratic form B on $\mathcal{D}(H^M)$ and the virial relation (2.1) holds. Abusing notation we will in the rest of the paper denote by $\tilde{\rho}_H^A$, ρ_H^A, $\tau_A(H)$, $\kappa_A(H)$ the objects introduced above for $B = [H, \mathrm{i}A]$. The set $\tau_A(H)$ is usually called the set of *thresholds* of H for A.

2.2 Functional calculus

If $\chi \in C_0^\infty(\mathbb{R})$, we denote by $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ an almost analytic extension of χ, satisfying

$$\tilde{\chi}_{|\mathbb{R}} = \chi,$$
$$|\partial_{\overline{z}} \tilde{\chi}(z)| \leq C_n |\mathrm{Imz}|^n, \quad n \in \mathbb{N}.$$

We use the following functional calculus formula for $\chi \in C_0^\infty(\mathbb{R})$ and A selfadjoint:

$$\chi(A) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{\chi}(z)(z - A)^{-1} \mathrm{d}z \wedge \mathrm{d}\,\overline{z}. \tag{2.2}$$

2.3 Abstract operator classes

In this subsection we introduce a poor man's version of pseudodifferential calculus tailored to our abstract setup. It rests on two positive selfadjoint operators ω and $\langle x \rangle$ on the one-particle space $\mathfrak{h}$. Later ω will of course be the one-particle Hamiltonian. The operator $\langle x \rangle$ will have two purposes: first as a weight to control various operators, and second as an observable to localize particles in $\mathfrak{h}$.

We fix selfadjoint operators ω, $\langle x\rangle$ on $\mathfrak{h}$ such that:

$$\omega \geq m > 0, \ \langle x\rangle \geq 1,$$
$$\text{there exists a dense subspace } \mathcal{S} \subset \mathfrak{h} \text{ such that } \omega, \langle x\rangle : \mathcal{S} \to \mathcal{S}.$$

To understand the terminology below the reader familiar with the standard pseudodifferential calculus should think of the example

$$\mathfrak{h} = L^2(\mathbb{R}^d), \quad \omega = (D_x^2 + 1)^{\frac{1}{2}}, \quad \langle x\rangle = (x^2 + 1)^{\frac{1}{2}}, \text{ and } \mathcal{S} = \mathcal{S}(\mathbb{R}^d).$$

To control various commutators later it is convenient to introduce the following classes of operators on $\mathfrak{h}$. If $a, b : \mathcal{S} \to \mathcal{S}$ we set $\mathrm{ad}_a b = [a, b]$ as an operator on $\mathcal{S}$.

Definition 2.2 *For $m \in \mathbb{R}$, $0 \leq \delta < \frac{1}{2}$ and $k \in \mathbb{N}$ we set*

$$S^m_{(0)} = \{b : \mathcal{S} \to \mathfrak{h} \mid \langle x\rangle^s b \langle x\rangle^{-s-m} \in B(\mathfrak{h}), \quad s \in \mathbb{R}\},$$

and for $k \geq 1$:

$$S^m_{\delta,(k)} = \{b : \mathcal{S} \to \mathcal{S} \mid \langle x\rangle^{-s} \mathrm{ad}^{\alpha}_{\langle x\rangle} \mathrm{ad}^{\beta}_{\omega} b \langle x\rangle^{s-m+(1-\delta)\beta-\delta\alpha} \in B(\mathfrak{h}) \quad \alpha + \beta \leq k, \quad s \in \mathbb{R}\},$$

where the multicommutators are considered as operators on $\mathcal{S}$.

The parameter m control the "order" of the operator: roughly speaking an operator in $S^m_{\delta,(k)}$ is controlled by $\langle x\rangle^m$. The parameter k is the number of commutators of the operator with $\langle x\rangle$ and ω that are controlled. The lower index δ controls the behavior of multicommutators: one looses $\langle x\rangle^{\delta}$ for each commutator with $\langle x\rangle$ and gains $\langle x\rangle^{1-\delta}$ for each commutator with ω.

The operator norms of the (weighted) multicommutators above can be used as a family of seminorms on $S^m_{\delta,(k)}$.

The spaces $S^m_{\delta,(k)}$ for $\delta = 0$ will be denoted simply by $S^m_{(k)}$. We will use the following natural notation for operators depending on a parameter:

if $b = b(R)$ belongs to $S^m_{\delta,(k)}$ for all $R \geq 1$ we will say that

$$b \in O(R^{\mu}) S^m_{\delta,(k)},$$

if the seminorms of $R^{-\mu} b(R)$ in $S^m_{\delta,(k)}$ are uniformly bounded in R. The following lemma is easy.

Lemma 2.3 *i)*

$$S^{m_1}_{\delta,(k)} \times S^{m_2}_{\delta,(k)} \subset S^{m_1 m_2}_{\delta,(k)}.$$

ii) Let $b \in S^{(m)}_{(0)}$. Then $J(\frac{\langle x\rangle}{R}) b \langle x\rangle^s \in O(R^{m+s})$ for $m + s \geq 0$ if $J \in C_0^{\infty}(\mathbb{R})$ and for all $s \in \mathbb{R}$ if $J \in C_0^{\infty}(]0, +\infty[)$.

Proof. *i)* follows from Leibniz rule applied to the operators $\mathrm{ad}_{\langle x\rangle}$ and ad_{ω}. *ii)* is immediate. □

2.4 Fock spaces.

In this subsection we recall various definitions on bosonic Fock spaces. We will also collect some bounds needed later.

Bosonic Fock spaces.

If $\mathfrak{h}$ is a Hilbert space then

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathfrak{h},$$

is the *bosonic Fock space* over $\mathfrak{h}$. $\Omega \in \Gamma(\mathfrak{h})$ will denote the *vacuum vector*. The *number operator* N is defined as

$$N\Big|_{\bigotimes_{\mathrm{s}}^{n} \mathfrak{h}} = n\mathbb{1}.$$

We define the space of *finite particle vectors*:

$$\Gamma_{\mathrm{fin}}(\mathfrak{h}) := \{u \in \Gamma(\mathfrak{h}) \mid \text{for some} \ \ n \in \mathbb{N}, \ \ \mathbb{1}_{[0,n]}(N)u = u\},$$

The *creation-annihilation* operators on $\Gamma(\mathfrak{h})$ are denoted by $a^*(h)$ and $a(h)$. We denote by

$$\phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)), \ W(h) := \mathrm{e}^{\mathrm{i}\phi(h)},$$

the *field* and *Weyl operators*.

$\mathrm{d}\Gamma$ operators.

If $r : \mathfrak{h}_1 \to \mathfrak{h}_2$ is an operator one sets:

$$\begin{aligned} \mathrm{d}\Gamma(r) \quad &: \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\ \mathrm{d}\Gamma(r)\Big|_{\bigotimes_{\mathrm{s}}^{n} \mathfrak{h}_1} \quad &:= \sum_{j=1}^{n} \mathbb{1}^{\otimes(j-1)} \otimes r \otimes \mathbb{1}^{\otimes(n-j)}, \end{aligned}$$

with domain $\Gamma_{\mathrm{fin}}(\mathcal{D}(r))$. If r is closeable, so is $\mathrm{d}\Gamma(r)$.

Γ operators.

If $q : \mathfrak{h}_1 \mapsto \mathfrak{h}_2$ is bounded one sets:

$$\begin{aligned} &\Gamma(q) : \Gamma(\mathfrak{h}_1) \mapsto \Gamma(\mathfrak{h}_2) \\ &\Gamma(q)\Big|_{\bigotimes_{\mathrm{s}}^{n} \mathfrak{h}_1} = q \otimes \cdots \otimes q. \end{aligned}$$

$\Gamma(q)$ is bounded iff $\|q\| \leq 1$ and then $\|\Gamma(q)\| = 1$.

$\mathrm{d}\Gamma(r, q)$ operators.
If r, q are as above one sets:

$$\begin{aligned} \mathrm{d}\Gamma(q, r) \quad &: \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\ \mathrm{d}\Gamma(q, r)\Big|_{\bigotimes_{\mathrm{s}}^{n} \mathfrak{h}_1} \quad &:= \sum_{j=1}^{n} q^{\otimes(j-1)} \otimes r \otimes q^{\otimes(n-j)}, \end{aligned}$$

with domain $\Gamma_{\mathrm{fin}}(\mathcal{D}(r))$. We refer the reader to [DG1, Subsects 3.5, 3.6, 3.7] for more details.

Tensor products of Fock spaces.

If $\mathfrak{h}_1$, $\mathfrak{h}_2$ are two Hilbert spaces, one denote by $U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ the canonical unitary map (see e.g. [DG1, Subsect. 3.8] for details).

If $\mathcal{H} = \Gamma(\mathfrak{h})$, we set

$$\mathcal{H}^{\mathrm{ext}} := \mathcal{H} \otimes \mathcal{H} \simeq \Gamma(\mathfrak{h} \oplus \mathfrak{h}).$$

The second copy of $\mathcal{H}$ will be the state space for bosons *living near infinity* in the spectral theory of a Hamiltonian H acting on $\mathcal{H}$.

Let $H = \mathrm{d}\Gamma(\omega) + V$ be an abstract QFT Hamiltonian defined in Subsect. 3.1 Then we set:

$$\mathcal{H}^{\mathrm{scatt}} := \mathcal{H} \otimes \Gamma(\mathfrak{h}_{\mathrm{c}}(\omega)).$$

The Hilbert space $\Gamma(\mathfrak{h}_{\mathrm{c}}(\omega))$ will be the state space for *free bosons* in the scattering theory of a Hamiltonian H acting on $\mathcal{H}$. We will need also:

$$H^{\mathrm{ext}} := H \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega), \text{ acting on } \mathcal{H}^{\mathrm{ext}}.$$

Clearly $\mathcal{H}^{\mathrm{scatt}} \subset \mathcal{H}^{\mathrm{ext}}$ and H^{ext} preserves $\mathcal{H}^{\mathrm{scatt}}$. We will use the notation

$$N_0 := N \otimes \mathbb{1}, \quad N_\infty := \mathbb{1} \otimes N, \text{ as operators on } \mathcal{H}^{\mathrm{ext}} \text{ or } \mathcal{H}^{\mathrm{scatt}}.$$

Identification operators.

The *identification operator* is defined as

$$I : \mathcal{H}^{\mathrm{ext}} \to \mathcal{H},$$
$$I := \Gamma(i)U,$$

where U is defined as above for $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$ and:

$$i : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h},$$
$$(h_0, h_\infty) \mapsto h_0 + h_\infty.$$

We have:

$$I \prod_{i=1}^{n} a^*(h_i)\Omega \otimes \prod_{i=1}^{p} a^*(g_i)\Omega := \prod_{i=1}^{n} a^*(h_i) \prod_{i=1}^{p} a^*(g_i)\Omega, \quad h_i \in \mathfrak{h}, \ \ g_i \in \mathfrak{h}.$$

If ω is a selfadjoint operator as above, we denote by I^{scatt} the restriction of I to $\mathcal{H}^{\mathrm{scatt}}$.

Note that $\|i\| = \sqrt{2}$ so $\Gamma(i)$ and hence I, I^{scatt} are unbounded. As domain for I (resp. I^{scatt}) we can choose for example $\mathcal{D}(N^\infty) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$ (resp. $\mathcal{D}(N^\infty) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}_{\mathrm{c}}(\omega))$). We refer to [DG1, Subsect. 3.9] for details.

Operators $I(j)$ and $\mathrm{d}I(j,k)$.

Let $j_0, j_\infty \in B(\mathfrak{h})$ and set $j = (j_0, j_\infty)$. We define

$$I(j) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h})$$

$$I(j) := I\Gamma(j_0)\otimes\Gamma(j_\infty).$$

If we identify j with the operator

$$(2.3)\qquad \begin{array}{l} j:\mathfrak{h}\oplus\mathfrak{h}\to\mathfrak{h},\\ j(h_0\oplus h_\infty) := j_0h_0 + j_\infty h_\infty, \end{array}$$

then we have

$$I(j) = \Gamma(j)U.$$

We deduce from this identity that if $j_0j_0^* + j_\infty j_\infty^* = 1\!\!1$ (resp. $j_0j_0^* + j_\infty j_\infty^* \leq 1\!\!1$) then $I^*(j)$ is isometric (resp. is a contraction).

Let $j = (j_0, j_\infty)$, $k = (k_0, k_\infty)$ be pairs of maps from $\mathfrak{h}$ to $\mathfrak{h}$. We define

$$\mathrm{d}I(j,k) : \Gamma_{\mathrm{fin}}(\mathfrak{h})\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h})$$

as follows:

$$\mathrm{d}I(j,k) := I(\mathrm{d}\Gamma(j_0,k_0)\otimes\Gamma(j_\infty) + \Gamma(j_0)\otimes\mathrm{d}\Gamma(j_\infty,k_\infty)).$$

Equivalently, treating j and k as maps from $\mathfrak{h}\oplus\mathfrak{h}$ to $\mathfrak{h}$ as in (2.3), we can write

$$\mathrm{d}I(j,k) := \mathrm{d}\Gamma(j,k)U.$$

We refer to [DG1, Subsects. 3.10, 3.11] for details.

Various bounds.

Proposition 2.4 *i) let a, b two selfadjoint operators on $\mathfrak{h}$ with $b \geq 0$ and $a^2 \leq b^2$. Then*

$$\mathrm{d}\Gamma(a)^2 \leq \mathrm{d}\Gamma(b)^2.$$

ii) let $b \geq 0$, $1 \leq \alpha$. Then:

$$\mathrm{d}\Gamma(b)^\alpha \leq N^{\alpha-1}\mathrm{d}\Gamma(b^\alpha).$$

iii) let $0 \leq r$ and $0 \leq q \leq 1$. Then:

$$\mathrm{d}\Gamma(q,r) \leq \mathrm{d}\Gamma(r).$$

iv) Let $r, r_1, r_2 \in B(\mathfrak{h})$ and $\|q\| \leq 1$. Then:

$$|(u_2|\mathrm{d}\Gamma(q, r_2r_1)u_1)| \leq \|\mathrm{d}\Gamma(r_2r_2^*)^{\frac12}u_2\|\|\mathrm{d}\Gamma(r_1^*r_1)^{\frac12}u_1\|,$$

$$\|N^{-\frac12}\mathrm{d}\Gamma(q,r)u\| \leq \|\mathrm{d}\Gamma(r^*r)^{\frac12}u\|.$$

v) Let $j_0j_0^* + j_\infty j_\infty^* \leq 1$, k_0, k_∞ selfadjoint. Then:

$$\begin{array}{rl} |(u_2|\mathrm{d}I^*(j,k)u_1)| & \leq \|\mathrm{d}\Gamma(|k_0|)^{\frac12}\otimes 1\!\!1 u_2\|\|\mathrm{d}\Gamma(|k_0|)^{\frac12}u_1\| \\ & +\|1\!\!1\otimes\mathrm{d}\Gamma(|k_\infty|)^{\frac12}u_2\|\|\mathrm{d}\Gamma(|k_\infty|)^{\frac12}u_1\|,\ u_1\in\Gamma(\mathfrak{h}),\ u_2\in\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h}). \end{array}$$

$$\|(N_0+N_\infty)^{-\frac12}\mathrm{d}I^*(j,k)u\| \leq \|\mathrm{d}\Gamma(k_0k_0^* + k_\infty k_\infty^*)^{\frac12}u\|,\ u\in\Gamma(\mathfrak{h}).$$

Proof. *i)* is proved in [GGM, Prop. 3.4]. The other statements can be found in [DG1, Sect. 3].

2.5 Heisenberg derivatives

Let H be a selfadjoint operator on $\Gamma(\mathfrak{h})$ such that $H = \mathrm{d}\Gamma(\omega) + V$ on $\mathcal{D}(H^m)$ for some $m \in \mathbb{N}$ where ω is selfadjoint and V symmetric. We will use the following notations for various Heisenberg derivatives:

$$\mathbf{d}_0 = \frac{\partial}{\partial t} + [\omega, \mathrm{i}\cdot\,] \text{ acting on } B(\mathfrak{h}),$$

$$\mathbf{D}_0 = \frac{\partial}{\partial t} + [H_0, \mathrm{i}\cdot\,], \quad \mathbf{D} = \frac{\partial}{\partial t} + [H, \mathrm{i}\cdot\,], \text{ acting on } B(\Gamma(\mathfrak{h})),$$

where the commutators on the right hand sides are quadratic forms.

If $\mathbb{R} \ni t \mapsto M(t) \in B(\mathcal{D}(H), \mathcal{H})$ is of class C^1 then:

$$\mathbf{D}\chi(H)M(t)\chi(H) = \chi(H)\mathbf{D}_0 M(t)\chi(H) + \chi(H)[V, \mathrm{i}M(t)]\chi(H), \tag{2.4}$$

for $\chi \in C_0^\infty(\mathbb{R})$.

If $\mathbb{R} \ni m(t) \in B(\mathfrak{h})$ is of class C^1 and $H_0 = \mathrm{d}\Gamma(\omega)$ then:

$$\mathbf{D}_0 \mathrm{d}\Gamma(m(t)) = \mathrm{d}\Gamma(\mathbf{d}_0 m(t)).$$

2.6 Wick polynomials

In this subsection we recall some results from [DG1, Subsect. 3.12].

We set

$$B_{\mathrm{fin}}(\Gamma(\mathfrak{h})) := \{B \in B(\Gamma(\mathfrak{h})) \mid \text{for some } n \in \mathbb{N} \;\; \mathbb{1}_{[0,n]}(N) B \mathbb{1}_{[0,n]}(N) = B\}.$$

Let $w \in B(\otimes_{\mathrm{s}}^p \mathfrak{h}, \otimes_{\mathrm{s}}^q \mathfrak{h})$. We define the operator

$$\mathrm{Wick}(w) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h})$$

as follows:

$$\mathrm{Wick}(w)\Big|_{\otimes_{\mathrm{s}}^n \mathfrak{h}} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} w \otimes_{\mathrm{s}} \mathbb{1}^{\otimes(n-p)}. \tag{2.5}$$

The operator $\mathrm{Wick}(w)$ is called a *Wick monomial of order* (p, q). This definition extends to $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ by linearity. The operator $\mathrm{Wick}(w)$ is called a *Wick polynomial* and the operator w is called the *symbol* of the Wick polynomial $\mathrm{Wick}(w)$. If $w = \sum_{(p,q)\in I} w_{p,q}$ for $w_{p,q}$ of order (p, q) and $I \subset \mathbb{N}$ finite, then

$$\deg(w) := \sup_{(p,q)\in I} p + q$$

is called the *degree* of $\mathrm{Wick}(w)$. If $h_1, \dots, h_p, g_1, \dots, g_q \in \mathfrak{h}$ then:

$$\mathrm{Wick}\left(|\, g_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} g_q)(h_p \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} h_1)|\right) = a^*(q_1) \cdots a^*(g_q) a(h_p) \cdots a(h_1).$$

We recall some basic properties of Wick polynomials.

Lemma 2.5

$$i)\ \ \mathrm{Wick}(w)^* = \mathrm{Wick}(w^*) \ \textit{as a identity on } \Gamma_{\mathrm{fin}}(\mathfrak{h}).$$

ii) If $\text{s-}\lim w_s = w$*, for* w_s, w *of order* (p,q) *then for* $k+m \geq (p+q)/2$*:*

$$\text{s-}\lim_s (N+1)^{-k}\mathrm{Wick}(w_s)(N+1)^{-m} = (N+1)^{-k}\mathrm{Wick}(w)(N+1)^{-m}.$$

$$iii)\ \ \|(N+1)^{-k}\mathrm{Wick}(w)(N+1)^{-m}\| \leq C\|w\|_{B(\Gamma(\mathfrak{h}))},$$

uniformly for w *of degree less than* p *and* $k+m \geq p/2$.

Most of the time the symbols of Wick polynomials will be *Hilbert-Schmidt* operators. Let us introduce some more notation in this context: we set

$$B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h})) := B^2(\Gamma(\mathfrak{h})) \cap B_{\mathrm{fin}}(\Gamma(\mathfrak{h})),$$

where $B^2(\mathcal{H})$ is the set of Hilbert-Schmidt operators on the Hilbert space $\mathcal{H}$. Recall that extending the map:

$$B^2(\mathcal{H}) \ni |u)(v| \mapsto u \otimes \overline{v} \in \mathcal{H} \otimes \overline{\mathcal{H}}$$

by linearity and density allows to unitarily identify $B^2(\mathcal{H})$ with $\mathcal{H} \otimes \overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ is the Hilbert space conjugate to $\mathcal{H}$. Using this identification, $B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ is identified with $\Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\overline{\mathfrak{h}})$ or equivalently to $\Gamma_{\mathrm{fin}}(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. We will often use this identification in the sequel.

If $u \in \otimes_{\mathrm{s}}^m \mathfrak{h}$, $v \in \otimes_s^n \mathfrak{h}$, $w \in B(\otimes_{\mathrm{s}}^p \mathfrak{h}, \otimes_{\mathrm{s}}^q \mathfrak{h})$ with $m \leq p$, $n \leq q$, then one defines the contracted symbols:

$$\begin{aligned}
(v|w &:= \left((v| \otimes_{\mathrm{s}} \mathbb{1}^{\otimes(q-n)}\right) w \ \in B(\otimes_{\mathrm{s}}^p \mathfrak{h}, \otimes_{\mathrm{s}}^{q-n} \mathfrak{h}),\\
w|u) &:= w\left(|u) \otimes_s \mathbb{1}^{\otimes(p-m)}\right) \ \in B(\otimes_{\mathrm{s}}^{p-m} \mathfrak{h}, \otimes_{\mathrm{s}}^q \mathfrak{h}),\\
(v|w|u) &:= \left((v| \otimes_{\mathrm{s}} \mathbb{1}^{\otimes(q-n)}\right) w \left(|u) \otimes_s \mathbb{1}^{\otimes(p-m)}\right) \ \in B(\otimes_{\mathrm{s}}^{p-m} \mathfrak{h}, \otimes_{\mathrm{s}}^{q-n} \mathfrak{h}).
\end{aligned}$$

If a is selfadjoint on $\mathfrak{h}$ and $w \in B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$, we set

$$|||\mathrm{d}\Gamma(a)w||| = \sum_{1\leq i<\infty} \|(a)_i \otimes \mathbb{1}_{\Gamma(\overline{\mathfrak{h}})} w\|_{B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h}))} + \sum_{1\leq i<\infty} \|\mathbb{1}_{\Gamma(\mathfrak{h})} \otimes (\overline{a})_i w\|_{B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h}))},$$

where the sums are finite since $w \in B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h})) \simeq \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\overline{\mathfrak{h}})$ and one uses the convention $\|au\| = +\infty$ if $u \notin \mathcal{D}(a)$.

We collect now some bounds on various commutators with Wick polynomials.

Proposition 2.6 *i) Let* b *a selfadjoint operator on* $\mathfrak{h}$ *and* $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$*. Then:*

$$[\mathrm{d}\Gamma(b), \mathrm{Wick}(w)] = \mathrm{Wick}([\mathrm{d}\Gamma(b), w]),$$

as quadratic form on $\mathcal{D}(\mathrm{d}\Gamma(b)) \cap \mathcal{D}(N^{\deg(w)/2})$.

ii) Let q *a unitary operator on* $\mathfrak{h}$ *and* $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$*. Then*

$$\Gamma(q)\mathrm{Wick}(w)\Gamma(q)^{-1} = \mathrm{Wick}(\Gamma(q)w\Gamma(q)^{-1}).$$

iii) Let $w \in B_{\rm fin}(\Gamma(\mathfrak{h}))$ *of order* (p,q) *and* $h \in \mathfrak{h}$*. Then:*

$$[{\rm Wick}(w), a^*(h)] = p{\rm Wick}\big(w|h)\big), \quad [{\rm Wick}(w), a(h)] = q{\rm Wick}\big((h|w\big), \tag{2.6}$$

$$W(h){\rm Wick}(w)W(-h) = \sum_{s=0}^{p}\sum_{r=0}^{q} \frac{p!}{s!}\frac{q!}{r!}(\frac{i}{\sqrt{2}})^{p+q-r-s}{\rm Wick}(w_{s,r}), \tag{2.7}$$

where

$$w_{s,r} = (h^{\otimes(q-r)}|w|h^{\otimes(p-s)}). \tag{2.8}$$

Proposition 2.7 *i) Let* $q \in B(\mathfrak{h})$, $\|q\| \leq 1$ *and* $w \in B^2_{\rm fin}(\mathfrak{h})$*. Then for* $m + k \geq \deg(w)/2$*:*

$$\begin{array}{rl} & \|(N+1)^{-m}[\Gamma(q), {\rm Wick}(w)](N+1)^{-k}\| \\ \leq & C \,|||\, {\rm d}\Gamma(1\!\!1 - q)w \,|||\,. \end{array} \tag{2.9}$$

ii) Let $j = (j_0, j_\infty)$ *with* $j_0, j_\infty \in B(\mathfrak{h})$, $\|j_0^* j_0 + j_\infty^* j_\infty\| \leq 1$*. Then for* $m + k \geq \deg(w)/2$*:*

$$\begin{array}{rl} & \|(N_0 + N_\infty + 1)^{-m}\Big(I^*(j){\rm Wick}(w) - ({\rm Wick}(w)\otimes 1\!\!1)I^*(j)\Big)(N+1)^{-k}\| \\ \leq & C \,|||\, {\rm d}\Gamma(1\!\!1 - j_0)w \,|||\, + C \,|||\, {\rm d}\Gamma(j_\infty)w \,|||\,. \end{array} \tag{2.10}$$

3 Abstract QFT Hamiltonians

In this section we define the class of abstract QFT Hamiltonians that we will consider in this paper.

3.1 Hamiltonians

Let ω be a selfadjoint operator on $\mathfrak{h}$ and $w \in B^2_{\rm fin}(\Gamma(\mathfrak{h}))$ such that $w = w^*$. We set

$$H_0 := {\rm d}\Gamma(\omega), \quad V := {\rm Wick}(w).$$

Clearly H_0 is selfadjoint and V symmetric on $\mathcal{D}(N^n)$ for $n \geq \deg(w)/2$ by Lemma 2.5.

We assume:

$$(H1) \quad {\rm inf}\sigma(\omega) = m > 0,$$

$$(H2) \quad H_0 + V \text{ is essentially selfadjoint and bounded below on } \mathcal{D}(H_0)\cap\mathcal{D}(V).$$

We set

$$H := \overline{H_0 + V}.$$

In the sequel we fix $b > 0$ such that $H + b \geq 1$. We assume:

$$(H3) \quad \begin{array}{l} \forall n \in \mathbb{N}, \exists\, p \in \mathbb{N} \text{ such that } \|N^n H_0 (H+b)^{-p}\| < \infty, \\ \forall P \in \mathbb{N}, \ \ \exists\, P < M \in \mathbb{N} \text{ such that } \|N^M (H+b)^{-1}(N+1)^{-P}\| < \infty. \end{array}$$

The bounds in (H3) are often called *higher order estimates.*

Definition 3.1 *A Hamiltonian* H *on* $\Gamma(\mathfrak{h})$ *satisfying (Hi) for* $1 \leq i \leq 3$ *will be called an* abstract QFT Hamiltonian*.*

3.2 Hypotheses on the one-particle Hamiltonian

The study of the spectral and scattering theory of abstract QFT Hamiltonians relies heavily on corresponding statements for the one-particle Hamiltonian ω. The now standard approach to such results is through the proof of a Mourre estimate and suitable propagation estimates on the unitary group $\mathrm{e}^{-\mathrm{i}t\omega}$.

Many of these results can be formulated in a completely abstract way. A convenient setup is based on the introduction of only three selfadjoint operators on the one-particle space $\mathfrak{h}$, the Hamiltonian ω, a conjugate operator a for ω and a *weight operator* $\langle x\rangle$. In this subsection we describe the necessary abstract hypotheses and collect various technical results used in the sequel. We will use the abstract operator classes introduced in Subsect. 2.3.

Commutator estimates.

We assume that there exists a selfadjoint operator $\langle x\rangle \geq 1$ for ω such that:

(G1 i) there exists a subspace $\mathcal{S} \subset \mathfrak{h}$ such that $\mathcal{S}$ is a core for ω, ω^2 and the operators ω, $\langle x\rangle$ for $z \in \mathbb{C}\backslash\sigma(\langle x\rangle)$, $(\langle x\rangle - z)^{-1}$, $F(\langle x\rangle)$ for $F \in C_0^\infty(\mathbb{R})$ *preserve $\mathcal{S}$.*

(G1 ii) $[\langle x\rangle, \omega]$ belongs to $S^0_{(3)}$.

Definition 3.2 *An operator $\langle x\rangle$ satisfying (G1) will be called a* weight operator *for ω.*

Dynamical estimates.

Particles living at time t in $\langle x\rangle \geq ct$ for some $c > 0$ are interpreted as *free particles.* The following assumption says that states in $\mathfrak{h}_{\mathrm{c}}(\omega)$ describe free particles:

(S) there exists a subspace $\mathfrak{h}_0$ dense in $\mathfrak{h}_{\mathrm{c}}(\omega)$ such that for all $h \in \mathfrak{h}_0$ there exists $\epsilon > 0$ such that

$$\|\mathbb{1}_{[0,\epsilon]}(\frac{\langle x\rangle}{|t|})\mathrm{e}^{-\mathrm{i}t\omega}h\| \in O(t^{-\mu}), \qquad \mu > 1.$$

(We recall that $\mathfrak{h}_{\mathrm{c}}(\omega)$ is the continuous spectral subspace for ω).

Note that *(S)* can be deduced from *(G1)*, *(M1)* and *(G4)*, assuming that $\omega \in C^3(a)$. The standard way to see this is to prove first a *strong propagation estimate* (see e.g. [HSS]):

$$F(\frac{|a|}{|t|} \leq \epsilon)\chi(\omega)\mathrm{e}^{-\mathrm{i}t\omega}(a+i)^{-2} \in O(t^{-2}),$$

in norm if $\chi \in C_0^\infty(\mathbb{R})$ is supported away from $\kappa_a(\omega)$, and then to obtain a corresponding estimate with a replaced by $\langle x\rangle$ using *(G4)* and arguments similar to those in [GN, Lemma A.3].

The operators $[\omega, \mathrm{i}\langle x\rangle]$ and $[\omega, \mathrm{i}[\omega, \mathrm{i}\langle x\rangle]]$ are respectively the instantaneous *velocity* and *acceleration* for the weight $\langle x\rangle$. The following condition means roughly that the acceleration is positive:

(G2) there exists $0 < \epsilon < \frac{1}{2}$ such that

$$[\omega, \mathrm{i}[\omega, \mathrm{i}\langle x\rangle]] = \gamma^2 + r_{-1-\epsilon},$$

where $\gamma = \gamma^* \in S^{-\frac{1}{2}}_{\epsilon,(2)}$ and $r_{-1-\epsilon} \in S^{-1-\epsilon}_{(0)}$.

Mourre theory and local compactness.

We now state hypotheses about the conjugate operator a:

(M1 i) $\omega \in C^1(a)$, $[\omega, \mathrm{i}a]_0 \in B(\mathfrak{h})$.
(M1 ii) $\rho^a_\omega \geq 0$, $\tau^a(\omega)$ *is a closed countable set.*

We will also need the following condition which allows to localize the operator $[\omega, \mathrm{i}a]_0$ using the weight operator $\langle x\rangle$.

(G3) a preserves $\mathcal{S}$ and $[\langle x\rangle, [\omega, \mathrm{i}a]_0]$ belongs to $S^0_{(0)}$.

Note that if a preserves $\mathcal{S}$ then $[\omega, a]_0 = \omega a - a\omega$ on $\mathcal{S}$. Therefore $[\langle x\rangle, [\omega, a]_0]$ in *(G3)* is well defined as an operator on $\mathcal{S}$.

We will also need some conditions which roughly say that a is controlled by $\langle x\rangle$. This allows to translate propagation estimates for a into propagation estimates for $\langle x\rangle$.

(G4) a belongs to $S^1_{(0)}$.

Note that by Lemma 2.3 *i)*, $a^2 \in S^2_{(0)}$ hence $a\langle x\rangle^{-1}$ and $a^2\langle x\rangle^{-2}$ are bounded.
We state also an hypothesis on local compactness:

(G5) $\langle x\rangle^{-\epsilon}(\omega+1)^{-\epsilon}$ *is compact on* $\mathfrak{h}$ *for some* $0 < \epsilon \leq \frac{1}{2}$.

Comparison operator.

To get a sharp Mourre estimate for abstract QFT Hamiltonians, it is convenient to assume the existence of a *comparison operator* ω_∞ such that:
(C i) $C^{-1}\omega^2_\infty \leq \omega^2 \leq C\omega^2_\infty$, *for some* $C > 0$,

(C ii) ω_∞ satisfies *(G1)*, *(M1)*, *(G3)* for the same $\langle x\rangle$ and a and $\kappa^a_{\omega_\infty} \subset \tau^a_{\omega_\infty}$.

Note that the last condition in *(C ii)* is satisfied if ω_∞ has no eigenvalues.

(C iii) $\omega^{-\frac{1}{2}}(\omega - \omega_\infty)\omega^{-\frac{1}{2}}\langle x\rangle^\epsilon$ and $[\omega - \omega_\infty, \mathrm{i}a]_0\langle x\rangle^\epsilon$ are bounded for some $\epsilon > 0$.

Some consequences.

We now state some standard consequences of *(G1)*.

Lemma 3.3 *Assume (H1), (G1). Then for* $F \in C^\infty_0(\mathbb{R})$*:*

$$i)\quad [F(\frac{\langle x\rangle}{R}), \mathrm{ad}^k_{\langle x\rangle}\omega] = R^{-1}F'(\frac{\langle x\rangle}{R})[\langle x\rangle, \mathrm{ad}^k_{\langle x\rangle}\omega] + M(R), \quad k = 0, 1,$$

where $M(R) \in O(R^{-2})S^0_{(0)} \cap O(R^{-1})S^{-1}_{(0)}$.

$$ii)\quad F(\tfrac{\langle x\rangle}{R}) : \mathcal{D}(\omega) \to \mathcal{D}(\omega) \text{ and } \omega F(\tfrac{\langle x\rangle}{R})\omega^{-1} \in O(1),$$

$$iii)\quad [F(\tfrac{\langle x\rangle}{R}), [\omega, \langle x\rangle]] \in O(R^{-1}),$$

$$iv)\quad F(\tfrac{\langle x\rangle}{R})[\omega, \mathrm{i}\langle x\rangle](1 - F_1)(\tfrac{\langle x\rangle}{R}) \in O(R^{-2}),$$

if $F_1 \in C^\infty_0(\mathbb{R})$ *and* $FF_1 = F$.

Assume (H1), (M1 i), (G3). Then for $F \in C_0^\infty(\mathbb{R})$:

$$v) \quad [F(\frac{\langle x\rangle}{R}), [\omega, \mathrm{i}a]_0] \in O(R^{-1})$$

Assume (H1), (G1), (G2). Then for $F \in C_0^\infty(\mathbb{R})$:

$$vi) \quad F(\frac{\langle x\rangle}{R}) : \mathcal{D}(\omega^2) \mapsto \mathcal{D}(\omega^2) \text{ and } [\omega^2, F(\frac{\langle x\rangle}{R})]\omega^{-1} \in O(R^{-1}).$$

Let $b \in S^{-\mu}_{\delta,(1)}$ for $\mu \geq 0$ and $F \in C_0^\infty(\mathbb{R}\backslash\{0\})$. Then:

$$vii) \quad [F(\frac{\langle x\rangle}{R}), b] \in O(R^{-\mu-1+\delta}).$$

In i) for $k = 0$ the commutator on the l.h.s. is considered as a quadratic form on $\mathcal{D}(\omega)$.

Lemma 3.4 *Let ω_∞ be a comparison operator satisfying (C). Then for $F \in C^\infty(\mathbb{R})$ with $F \equiv 0$ near 0, $F \equiv 1$ near $+\infty$ we have:*

$$\omega^{-\frac{1}{2}}(\omega - \omega_\infty)F(\frac{\langle x\rangle}{R})\omega^{-\frac{1}{2}}, \quad [\omega - \omega_\infty, \mathrm{i}a]F(\frac{\langle x\rangle}{R}) \in o(R^0).$$

The proof of Lemmas 3.3, 3.4 will be given in the Appendix.

3.3 Hypotheses on the interaction

We now formulate the hypotheses on the interaction V. If $j \in C^\infty(\mathbb{R})$, we set for $R \geq 1$ $j^R = j(\frac{\langle x\rangle}{R})$.

For the scattering theory of abstract QFT Hamiltonians, we will need the following decay hypothesis on the symbol of V:

$$(Is) \quad \|\!|\, \mathrm{d}\Gamma(j^R)w\|\!| \in O(R^{-s}), \quad s > 0 \quad \text{if } j \equiv 0 \text{ near } 0, \quad j \equiv 1 \text{ near } \pm\infty.$$

Note that if $w \in B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ and j is as above then

$$\|\!|\mathrm{d}\Gamma(j^R)w\|\!| \in o(R^0), \text{ when } R \to \infty. \tag{3.1}$$

Another type of hypothesis concerns the Mourre theory. We fix a conjugate operator a for ω such that *(M1)* holds and set

$$A := \mathrm{d}\Gamma(a).$$

For the Mourre theory, we will impose:

$$(M2) \quad w \in \mathcal{D}(A \otimes \mathbb{1} - \mathbb{1} \otimes \overline{A}).$$

If hypothesis *(G4)* holds then $a\langle x\rangle^{-1}$ is bounded. It follows that the condition

$$(D) \quad \|\!|\, \mathrm{d}\Gamma(\langle x\rangle^s)w\|\!| < \infty, \text{ for some } s > 1$$

implies both *(Is)* for $s > 1$ and *(M2)*.

4 Results

For the reader convenience, we summarize in this section the results of the paper. To simplify the situation we will assume that all the various hypotheses hold, i.e. we assume conditions *(Hi)*, $1 \leq i \leq 3$, *(Gi)*, $1 \leq i \leq 5$, *(S)*, *(M1)*, *(C)* and *(D)*. However various parts of Thm. 4.1 hold under smaller sets of hypotheses, we refer the reader to later sections for precise statements.

The notation $\mathrm{d}\Gamma^{(1)}(E)$ for a set $E \subset \mathbb{R}$ is defined in Subsect. 7.3.

Theorem 4.1 *Let H be an abstract QFT Hamiltonian. Then:*

1. *if* $\sigma_{\mathrm{ess}}(\omega) = [m_\infty, +\infty[$ *then*
$$\sigma_{\mathrm{ess}}(H) = [\inf \sigma(H) + m_\infty, +\infty[.$$
2. *The Mourre estimate holds for* $A = \mathrm{d}\Gamma(a)$ *on* $\mathbb{R}\backslash\tau$*, where*
$$\tau = \sigma_{\mathrm{pp}}(H) + \mathrm{d}\Gamma^{(1)}(\tau_a(\omega)),$$
where $\tau_a(\omega)$ *is the set of thresholds of* ω *for* a *and* $\mathrm{d}\Gamma^{(1)}(E)$ *for* $E \subset \mathbb{R}$ *is defined in (7.18).*
3. *The* asymptotic Weyl operators*:*
$$W^{\pm}(h) := \text{s-}\lim_{t \pm \infty} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH} \text{ exist for all } h \in \mathfrak{h}_{\mathrm{c}}(\omega),$$
and define two regular CCR representations over $\mathfrak{h}_{\mathrm{c}}(\omega)$*.*
4. *There exist unitary operators* $\Omega^{\pm}$*, called the* wave operators*:*
$$\Omega^{\pm} : \mathcal{H}_{\mathrm{pp}}(H) \otimes \Gamma(\mathfrak{h}_{\mathrm{c}}(\omega)) \to \Gamma(\mathfrak{h})$$
such that
$$W^{\pm}(h) = \Omega^{\pm} \mathbb{1} \otimes W(h) \Omega^{\pm *}, \quad h \in \mathfrak{h}_{\mathrm{c}}(\omega),$$
$$H = \Omega^{\pm}(H_{|\mathcal{H}_{\mathrm{pp}}(H)} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega))\Omega^{\pm *}.$$

Parts (1), (2), (3), (4) are proved respectively in Thms. 7.1, 7.10, 8.1 and 10.6.

Statement (1) is the familiar HVZ theorem, describing the essential spectrum of H.

Statement (2) is the well-known Mourre estimate. Under additional conditions, it is possible to deduce from it resolvent estimates which imply in particular that the singular continuous spectrum of H is empty. In our case this result follows from (4), provided we know that ω has no singular continuous spectrum.

Statement (3) is rather easy. Statement (4) is the most important result of this paper, namely the *asymptotic completeness* of wave operators.

Remark 4.2 *Assume that there exist another operator* ω_∞ *on* $\mathfrak{h}$ *such that* $\omega_{|\mathfrak{h}_{\mathrm{c}}(\omega)}$ *is unitarily equivalent to* ω_∞*. Typically this follows from the construction of a nice scattering theory for the pair* (ω, ω_∞)*. Then since* $\mathrm{d}\Gamma(\omega)$ *restricted to* $\Gamma(\mathfrak{h}_{\mathrm{c}}(\omega))$ *is unitarily equivalent to* $\mathrm{d}\Gamma(\omega_\infty)$*, we can replace* ω *by* ω_∞ *in statement (4) of Thm. 4.1.*

5 Examples

In this section we give examples of QFT Hamiltonians to which we can apply Thm. 4.1. Our two examples are space-cutoff $P(\varphi)_2$ Hamiltonians for a variable metric, and similar $P(\varphi)_{d+1}$ models for $d \geq 2$ if the interaction term has also an ultraviolet cutoff. For $\mu \in \mathbb{R}$ we denote by $S^{\mu}(\mathbb{R}^d)$ the space of C^{∞} functions on $\mathbb{R}^d$ such that:

$$\partial_x^{\alpha} f(x) \in O(\langle x\rangle^{-\mu-\alpha}) \ \ \alpha \in \mathbb{N}^d, \text{ where } \langle x\rangle = (1+x^2)^{\frac{1}{2}}.$$

5.1 Space-cutoff $P(\varphi)_2$ models with variable metric

We fix a second order differential operator on $\mathfrak{h} = L^2(\mathbb{R})$:

$$h := Da(x)D + c(x), \quad D = -\mathrm{i}\partial_x,$$

where $a(x) \geq c_0$, $c(x) \geq c_0$ for some $c_0 > 0$ and $a(x) - 1, c(x) - m_{\infty}^2 \in S^{-\mu}(\mathbb{R})$ for some $m_{\infty}, \mu > 0$. We set:

$$\omega := h^{\frac{1}{2}}$$

and consider the free Hamiltonian

$$H_0 = \mathrm{d}\Gamma(\omega), \text{ acting on } \Gamma(\mathfrak{h}).$$

To define the interaction, we fix a real polynomial with $x-$dependent coefficients:

$$P(x, \lambda) = \sum_{p=0}^{2n} a_p(x)\lambda^p, \quad a_{2n}(x) \equiv a_{2n} > 0, \tag{5.1}$$

and a function $g \in L^1(\mathbb{R})$ with $g \geq 0$. For $x \in \mathbb{R}$, one sets

$$\varphi(x) := \phi(\omega^{-\frac{1}{2}}\delta_x),$$

where δ_x is the Dirac distribution at x. The associated $P(\varphi)_2$ interaction is formally defined as:

$$V := \int_{\mathbb{R}} g(x) :P(x, \varphi(x)): \mathrm{d}x,$$

where : : denotes the Wick ordering.

In [GP] we prove the following theorem. Condition *(B3)* below is formulated in terms of a (generalized) basis of eigenfunctions of h. To be precise we say that the families $\{\psi_l(x)\}_{l\in I}$ and $\{\psi(x,k)\}_{k\in\mathbb{R}}$ form a generalized basis of eigenfunctions of h if:

$$\begin{aligned}
&\psi_l(\cdot) \in L^2(\mathbb{R}), \quad \psi(\cdot, k) \in \mathcal{S}'(\mathbb{R}),\\
&h\psi_l = \epsilon_l\psi_l, \quad \epsilon_l \leq m_{\infty}^2, \quad l \in I,\\
&h\psi(\cdot,k) = (k^2 + m_{\infty}^2)\psi(\cdot,k), \quad k \in \mathbb{R},\\
&\textstyle\sum_{l\in I} |\psi_l)(\psi_l| + \frac{1}{2\pi}\int_{\mathbb{R}} |\psi(\cdot,k))(\psi(\cdot,k)|\mathrm{d}k = \mathbb{1}.
\end{aligned}$$

Theorem 5.1 *Assume that:*

$$(B1)\ ga_p \in L^2(\mathbb{R}),\ 0 \le p \le 2n, \quad g \in L^1(\mathbb{R}), \quad g \ge 0, \quad g(a_p)^{2n/(2n-p)} \in L^1(\mathbb{R}),\ 0 \le p \le 2n-1,$$

$$(B2)\ \langle x\rangle^s ga_p \in L^2(\mathbb{R}) \quad \forall\, 0 \le p \le 2n, \text{ for some } s > 1.$$

Assume moreover that for a measurable function $M : \mathbb{R} \to \mathbb{R}^+$ *with* $M(x) \ge 1$ *there exists a generalized basis of eigenfunctions of* h *such that:*

$$(B3) \begin{cases} \sum_{l\in I} \|M^{-1}(\cdot)\psi_l(\cdot)\|_\infty^2 < \infty, \\ \|M^{-1}(\cdot)\psi(\cdot,k)\|_\infty \le C, \quad k \in \mathbb{R}. \end{cases}$$

$$(B4)\ ga_pM^s \in L^2(\mathbb{R}), \quad g(a_pM^s)^{2n/(2n-p+s)} \in L^1(\mathbb{R}), \quad \forall\, 0 \le s \le p \le 2n-1.$$

Then the Hamiltonian

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x) : P(x, \varphi(x)) : \mathrm{d}x$$

satisfies all the hypotheses of Thm. 4.1 for the weight operator $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$ *and conjugate operator* $a = \frac{1}{2}(x\langle D_x\rangle^{-1}D_x + \mathrm{hc})$.

Remark 5.2 *If* g *is compactly supported we can take* $M(x) = +\infty$ *outside* $\operatorname{supp} g$, *and the meaning of (B3) is that the sup norms* $\| \, \|_\infty$ *are taken only on* $\operatorname{supp} g$.

Remark 5.3 *Condition (B3) is discussed in details in [GP], where many sufficient conditions for its validity are given. As an example let us simply mention that if* $a(x) - 1$, $c(x) - m_\infty^2$ *and the coefficients* a_p *are in the Schwartz class* $\mathcal{S}(\mathbb{R})$, *then all conditions in Thm. 5.1 are satisfied.*

5.2 Higher dimensional examples

We work now on $L^2(\mathbb{R}^d)$ for $d \ge 2$ and consider

$$\omega = \Big(\sum_{1\le i,j\le d} D_i a_{ij}(x) D_j + c(x)\Big)^{\frac{1}{2}}$$

where a_{ij}, c are real, $[a_{ij}](x) \ge c_0 \mathbb{1}$, $c(x) \ge c_0$ for some $c_0 > 0$ and $[a_{ij}] - \mathbb{1} \in S^{-\mu}(\mathbb{R}^d)$, $c(x) - m_\infty^2 \in S^{-\mu}(\mathbb{R}^d)$ for some $m_\infty, \mu > 0$.

The free Hamiltonian is as above

$$H_0 = \mathrm{d}\Gamma(\omega),$$

acting on the Fock space $\Gamma(L^2(\mathbb{R}^d))$.

Since $d \ge 2$ it is necessary to add an ultraviolet cutoff to make sense out of the formal expression

$$\int_{\mathbb{R}^d} g(x) P(x, \varphi(x)) \mathrm{d}x.$$

We set

$$\varphi_\kappa(x) := \phi(\omega^{-\frac{1}{2}} \chi(\frac{\omega}{\kappa}) \delta_x),$$

where $\chi \in C_0^\infty([-1,1])$ is a cutoff function equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and $\kappa \gg 1$ is an ultraviolet cutoff parameter. Since $\omega^{-\frac{1}{2}}\chi(\frac{\omega}{\kappa})\delta_x \in L^2(\mathbb{R}^d)$, $\varphi_\kappa(x)$ is a well defined selfadjoint operator on $\Gamma(L^2(\mathbb{R}^d))$.

If $P(x,\lambda)$ is as in (5.1) and $g \in L^1(\mathbb{R}^d)$, then

$$V := \int_{\mathbb{R}^d} g(x) P(x, \varphi_\kappa(x)) \mathrm{d}x,$$

is a well defined selfadjoint operator on $\Gamma(L^2(\mathbb{R}^d))$. We have then the following theorem. As before we consider a generalized basis $\{\psi_l(x)\}_{l\in I}$ and $\{\psi(x,k)\}_{k\in\mathbb{R}^d}$ of eigenfunctions of h.

Theorem 5.4 *Assume that:*

$$(B1)\ ga_p \in L^2(\mathbb{R}^d),\ 0 \le p \le 2n,\quad g \in L^1(\mathbb{R}^d),\quad g \ge 0,\quad g(a_p)^{2n/(2n-p)} \in L^1(\mathbb{R}^d),\ 0 \le p \le 2n-1,$$

$$(B2)\ \langle x\rangle^s g a_p \in L^2(\mathbb{R}^d)\quad \forall\, 0 \le p \le 2n,\ \textit{for some}\ s > 1.$$

Assume moreover that for a measurable function $M : \mathbb{R}^d \to \mathbb{R}^+$ with $M(x) \ge 1$ there exists a generalized basis of eigenfunctions of h such that:

$$(B3)\ \left\{\begin{array}{l} \sum_{l\in I} \|M^{-1}(\cdot)\psi_l(\cdot)\|_\infty^2 < \infty, \\ \|M^{-1}(\cdot)\psi(\cdot,k)\|_\infty \le C,\quad k \in \mathbb{R}. \end{array}\right.$$

$$(B4)\ ga_pM^s \in L^2(\mathbb{R}^d),\quad g(a_pM^s)^{2n/(2n-p+s)} \in L^1(\mathbb{R}^d),\quad \forall\, 0 \le s \le p \le 2n-1.$$

Then the Hamiltonian

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}^d} g(x) P(x, \varphi_\kappa(x)) \mathrm{d}x$$

satisfies all the hypotheses of Thm. 4.1 for the weight operator $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$ and conjugate operator $a = \frac{1}{2}(x \cdot \langle D_x\rangle^{-1} D_x + \mathrm{hc})$.

Remark 5.5 *Sufficient conditions for (B3) to hold with $M(x) \equiv 1$ are given in [GP].*

6 Commutator estimates

In this section we collect various commutator estimates, needed in Sect. 7.

6.1 Number energy estimates

We recall first some notation from [DG1]: let an operator $B(t)$ depending on some parameter t map $\cap_n \mathcal{D}(N^n) \subset \mathcal{H}$ into itself. We will write

$$B(t) \in (N+1)^m O_N(t^p) \text{ for } m \in \mathbb{R} \text{ if} \tag{6.1}$$

$$\|(N+1)^{-m-k} B(t) (N+1)^k\| \le C_k \langle t\rangle^p, \qquad k \in \mathbb{Z}.$$

If (6.1) holds for any $m \in \mathbb{R}$, then we will write

$$B(t) \in (N+1)^{-\infty} O_N(t^p).$$

Likewise, for an operator $C(t)$ that maps $\cap_n \mathcal{D}(N^n) \subset \mathcal{H}$ into $\cap_n \mathcal{D}((N_0 + N_\infty)^n) \subset \mathcal{H}^{\mathrm{ext}}$ we will write

$$C(t) \in (N+1)^m \check{O}_N(t^p) \text{ for } m \in \mathbb{R} \text{ if} \tag{6.2}$$

$$\|(N_0+N_\infty)^{-m-k}C(t)(N+1)^k\| \leq C_k\langle t\rangle^p, \quad k\in\mathbb{Z}.$$

If (6.2) holds for any $m\in\mathbb{R}$, then we will write

$$B(t)\in(N+1)^{-\infty}\check{O}_N(t^p).$$

The notation $(N+1)o_N(t^p)$, $(N+1)^m\check{o}_N(t^p)$ are defined similarly.

Lemma 6.1 *Let H be an abstract QFT Hamiltonian. Then:*
i) for all $P\in\mathbb{N}$ there exists $\alpha>0$ such that for all $0\leq s\leq P$

$$N^{s+\alpha}(H-z)^{-1}N^{-s}\in O(|\mathrm{Im}z|^{-1}),\ \textit{uniformly for } z\in\mathbb{C}\backslash\mathbb{R}\cap\{|z|\leq R\}.$$

ii) for $\chi\in C_0^\infty(\mathbb{R})$ we have

$$\|N^m\chi(H)N^p\|<\infty, \quad m,p\in\mathbb{N}.$$

Proof. *ii)* follows directly from *(H3)*. It remains to prove *i)*. Let us fix $P\in\mathbb{N}$ and $M>P$ such that

$$N^M(H+b)^{-1}(N+1)^{-P}\in B(\mathcal{H}). \tag{6.3}$$

We deduce also from *(H3)* and interpolation that there exists $\alpha>0$ such that

$$N^\alpha(H+b)^{-1}\in B(\mathcal{H}). \tag{6.4}$$

We can choose $\alpha>0$ small enough such that $\delta=(M-\alpha)/P>1$. Interpolating between (6.3) and (6.4) we obtain first that $N^{\alpha+\delta x}(H+b)^{-1}(N+1)^{-x}$ is bounded for all $x\in[0,P]$. Since $\delta>1$, we get that

$$\|N^{\alpha(s+1)}(H+b)^{-1}(N+1)^{-s\alpha}\|<\infty, \quad s\in[0,P\alpha^{-1}]. \tag{6.5}$$

Without loss of generality we can assume that $\alpha^{-1}\in\mathbb{N}$, and we will prove by induction on $s\in\mathbb{N}$ that

$$N^{(s+1)\alpha}(H-z)^{-1}(N+1)^{-s\alpha}\in O(|\mathrm{Im}z|^{-1}), \tag{6.6}$$

uniformly for $z\in\mathbb{C}\backslash\mathbb{R}\cap\{|z|\leq R\}$ and $0\leq s\leq P\alpha^{-1}$.

For $s=0$ (6.6) follows from the fact that $N^\alpha(H+b)^{-1}$ is bounded. Let us assume that (6.6) holds for $s-1$. Then we write:

$$\begin{aligned}
& N^{(s+1)\alpha}(H-z)^{-1}(N+1)^{-s\alpha} \\
= \ & N^{(s+1)\alpha}(H+b)^{-1}N^{-s\alpha}N^{s\alpha}(H+b)(H-z)^{-1}(N+1)^{-s\alpha} \\
= \ & N^{(s+1)\alpha}(H+b)^{-1}N^{-s\alpha}N^{s\alpha}(\mathbb{1}+(b+z)(H-z)^{-1})(N+1)^{-s\alpha},
\end{aligned}$$

so (6.6) for s follows from (6.5) and the induction hypothesis. We extend then (6.6) from integer $s\in[0,P\alpha^{-1}]$ to all $s\in[0,P\alpha^{-1}]$ by interpolation. Denoting $s\alpha$ by s we obtain *i)*. □

6.2 Commutator estimates

Lemma 6.2 *Let H be an abstract QFT Hamiltonian and $\langle x\rangle$ a weight operator for ω. Let $q\in C_0^\infty(\mathbb{R})$, $0\leq q\leq 1$, $q\equiv 1$ near 0. Set for $R\geq 1$ $q^R=q(\frac{\langle x\rangle}{R})$. Then for $\chi\in C_0^\infty(\mathbb{R})$:*

$$[\Gamma(q^R),\chi(H)]\in\left\{\begin{array}{l}(N+1)^{-\infty}O_N(R^{-\inf(s,1)})\ \textit{under hypothesis } (Is),\\ (N+1)^{-\infty}o_N(R^0)\ \textit{otherwise.}\end{array}\right.$$

Proof. In all the proof M and P will denote integers chosen sufficiently large. We prove the lemma under hypothesis *(Is)* $s>0$, the general case being handled replacing hypothesis *(Is)* by the estimate (3.1). Clearly $\Gamma(q^R)$ preserves $\mathcal{D}(N^n)$. We have

$$[H_0,\Gamma(q^R)]=\mathrm{d}\Gamma(q^R,[\omega,q^R]),\tag{6.7}$$

By Lemma 3.3 *i)*, $[\omega,q^R]\in O(R^{-1})$ and hence $[H_0,\Gamma(q^R)](H_0+1)^{-1}$ is bounded. Therefore, $\Gamma(q^R)$ preserves $\mathcal{D}(H_0)$. As in [DG1, Lemma 7.11] the following identity is valid as a operator identity on $\mathcal{D}(H_0)\cap\mathcal{D}(N^P)$:

$$[H,\Gamma(q^R)]=[H_0,\Gamma(q^R)]+[V,\Gamma(q^R)]=:T.$$

From (6.7) and Prop. 2.4 *iv)* we get that

$$[\Gamma(q^R),H_0]\in(N+1)O_N(R^{-1}).$$

Using Prop. 2.7 *i)* and hypothesis *(Is)*, we get that

$$[\Gamma(q^R),V]\in(N+1)^nO_N(R^{-s}),\quad n\geq\deg(w)/2$$

which gives

$$T\in(N+1)^nO(R^{-\inf(s,1)}).\tag{6.8}$$

Let now

$$\begin{aligned}T(z)\ &:=[\Gamma(q^R),(z-H)^{-1}]\\ &=-(z-H)^{-1}[\Gamma(q^R),H](z-H)^{-1}.\end{aligned}$$

By *(H3)* $\mathcal{D}(H^M)\subset\mathcal{D}(H_0)\cap\mathcal{D}(N^P)$, so the following identity holds on $\mathcal{D}(H^M)$:

$$T(z)=(z-H)^{-1}T(z-H)^{-1}.$$

Let now $\chi_1\in C_0^\infty(\mathbb{R})$ with $\chi_1\chi=\chi$ and $\tilde{\chi}_1$, $\tilde{\chi}$ be almost analytic extensions of χ_1, χ. We write:

$$\begin{aligned}&N^m[\chi(H),\Gamma(q^R)]N^p\\ =\ &N^m\chi_1(H)[\chi(H),\Gamma(q^R)]N^p+N^m[\chi_1(H),\Gamma(q^R)]\chi(H)N^p\\ =\ &\tfrac{\mathrm{i}}{2\pi}\textstyle\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi}(z)N^m\chi_1(H)T(z)N^p\mathrm{d}z\wedge\mathrm{d}\,\overline{z}\\ &+\tfrac{\mathrm{i}}{2\pi}\textstyle\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi_1}(z)N^mT(z)\chi(H)N^p\mathrm{d}z\wedge\mathrm{d}\,\overline{z}.\end{aligned}$$

Using Lemma 6.1 *i)* and (6.8), we obtain that for all $n_1\in\mathbb{N}$ there exists $n_2\in\mathbb{N}$ such that

$$N^{n_1}T(z)(N+1)^{-n_2},\ \ (N+1)^{-n_2}T(z)N^{n_1}\in O(|\mathrm{Im}z|^{-2}),\ \text{uniformly for } z\in\mathbb{C}\backslash\mathbb{R}\cap\{|z|\leq R\}.$$

Using also Lemma 6.1 *ii)*, we obtain that

$$N^m[\chi(H), \Gamma(q^R)]N^p \in O(R^{-\inf(s,1)}),$$

which completes the proof of the Lemma. □

Let $j_0 \in C_0^\infty(\mathbb{R})$, $j_\infty \in C^\infty(\mathbb{R})$, $0 \le j_0$, $0 \le j_\infty$, $j_0^2 + j_\infty^2 \le 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). Set for $R \ge 1$ $j^R = (j_0(\frac{\langle x\rangle}{R}), j_\infty(\frac{\langle x\rangle}{R}))$.

Lemma 6.3 *Let H be an abstract QFT Hamiltonian and $\langle x\rangle$ a weight operator for ω. Then for $\chi \in C_0^\infty(\mathbb{R})$:*

$$\chi(H^{\rm ext})I^*(j^R) - I^*(j^R)\chi(H) \in \left\{ \begin{array}{l} (N+1)^{-\infty}\check{O}(R^{-\inf(s,1)}) \text{ under hypothesis } (Is), \\ (N+1)^{-\infty}\check{o}(R^0) \text{otherwise.} \end{array}\right.$$

Proof. Again we will only prove the lemma under hypothesis *(Is)*. As in [DG1, Lemma 7.12], we have:

$$H_0^{\rm ext}I^*(j^R) - I^*(j^R)H_0 \in (N+1)O(\|[\omega, j_0^R]\| + \|[\omega, j_\infty^R]\|).$$

Writing $[\omega, j_\infty^R] = [(1-j_\infty)^R, \omega]$, we obtain that $\|[\omega, j_0^R]\| + \|[\omega, j_\infty^R]\| \in O(R^{-1})$, hence:

$$H_0^{\rm ext}I^*(j^R) - I^*(j^R)H_0 \in (N+1)\check{O}_N(R^{-1}). \tag{6.9}$$

This implies that $I^*(j^R)$ sends $\mathcal{D}(H_0)$ into $\mathcal{D}(H_0^{\rm ext})$, and since $I^*(j^R)N = (N_0+N_\infty)I^*(j^R)$, $I^*(j^R)$ sends also $\mathcal{D}(N^n)$ into $\mathcal{D}((N_0+N_\infty)^n)$.

Next by Prop. 2.7 *ii)* and condition *(Is)* we have

$$(V\otimes 1\!\!1)I^*(j^R) - I^*(j^R)V \in (N+1)^n\check{O}_N(R^{-s}), \quad n \ge \deg(w)/2. \tag{6.10}$$

This and (6.9) show that as an operator identity on $\mathcal{D}(H_0)\cap\mathcal{D}(N^n)$ we have

$$H^{\rm ext}I^*(j^R) - I^*(j^R)H \in (N+1)^n\check{O}_N(R^{-\min(1,s)}). \tag{6.11}$$

Using then *(H3)* and the fact that $I^*(j^R)$ sends $\mathcal{D}(H_0)$ into $\mathcal{D}(H_0^{\rm ext})$ and $\mathcal{D}(N^n)$ into $\mathcal{D}((N_0 + N_\infty)^n)$, we obtain the following operator identity on $\mathcal{D}(H^M)$ for M large enough:

$$\begin{array}{rl} T(z) & := (z-H^{\rm ext})^{-1}I^*(j^R) - I^*(j^R)(z-H)^{-1} \\ & = (z-H^{\rm ext})^{-1}\Big(I^*(j^R)H - H^{\rm ext}I^*(j^R)\Big)(z-H)^{-1}, \end{array}$$

uniformly for $z \in \mathbb{C}\backslash\mathbb{R}\cap\{|z|\le R\}$.

Using then Lemma 6.1 *i)* (and its obvious extension for $H^{\rm ext}$), we obtain that for all $n_1 \in \mathbb{N}$ there exists $n_2 \in \mathbb{N}$ such that

$$(N_0+N_\infty)^{n_1}T(z)(N+1)^{-n_2}, \quad (N_0+N_\infty+1)^{-n_2}T(z)N^{n_1} \in O(|{\rm Im} z|^{-2})R^{-\inf(s,1)}. \tag{6.12}$$

Let us again pick $\chi_1 \in C_0^\infty(\mathbb{R})$ with $\chi_1\chi = \chi$. We have:

$$\begin{array}{rl} & (N_0+N_\infty)^m\left(\chi(H^{\rm ext})I^*(j^R) - I^*(j^R)\chi(H)\right)N^m \\ = & (N_0+N_\infty)^m\chi_1(H^{\rm ext})\Big(\chi(H^{\rm ext})I^*(j^R) - I^*(j^R)\chi(H)\Big)N^m \\ & +(N_0+N_\infty)^m\Big(\chi_1(H^{\rm ext})I^*(j^R) - I^*(j^R)\chi_1(H)\Big)\chi(H)N^m \\ = & \frac{\rm i}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi}(z)(N_0+N_\infty)^m\chi_1(H^{\rm ext})T(z)N^m{\rm d}z\wedge{\rm d}\,\overline{z} \\ & +\frac{\rm i}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi}_1(z)(N_0+N_\infty)^mT(z)\chi(H)N^m{\rm d}z\wedge{\rm d}\,\overline{z}. \end{array}$$

Using Lemma 6.1 *i)*, (6.12), the above operator is $O(R^{-\inf(s,1)})$ as claimed. $\Box$

7 Spectral analysis of abstract QFT Hamiltonians

In this section we study the spectral theory of our abstract QFT Hamiltonians. The essential spectrum is described in Subsect. 7.1. The Mourre estimate is proved in Subsect. 7.4. An improved version with a smaller threshold set is proved in Subsect. 7.5.

7.1 HVZ theorem and existence of a ground state

Theorem 7.1 *Let H be an abstract QFT Hamiltonian and let $\langle x\rangle$ be a weight operator for ω. Assume hypotheses (G1), (G5). Then*

i) if $\sigma_{\rm ess}(\omega)\subset[m_\infty,+\infty[$ then

$$\sigma_{\rm ess}(H)\subset[\inf\sigma(H)+m_\infty,+\infty[.$$

ii) if $\sigma_{\rm ess}(\omega)=[m_\infty,+\infty[$ then

$$\sigma_{\rm ess}(H)=[\inf\sigma(H)+m_\infty,+\infty[.$$

Proof. Let us pick functions $j_0,j_\infty\in C^\infty(\mathbb{R})$ with $0\leq j_0\leq 1$, $j_0\in C_0^\infty(\mathbb{R})$, $j_0\equiv 1$ near 0 and $j_0^2+j_\infty^2=1$. For $R\geq 1$, j^R is defined as in Subsect. 6.2 and we set $q^R=(j_0^R)^2$. From Subsect. 2.4 we know that

$$I(j^R)I^*(j^R)=\mathbb{1}.$$

We first prove *i)*. Let $\chi\in C_0^\infty(]-\infty,\inf\sigma(H)+m_\infty[)$. Using Lemma 6.3 we get:

$$\begin{array}{rl}\chi(H)= & \chi(H)I(j^R)I^*(j^R)\\ = & I(j^R)\chi(H^{\rm ext})I^*(j^R)+o(R^0)\\ = & \sum_{k=0}^{M}I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\chi(H^{\rm ext})I^*(j^R)+o(R^0),\end{array}\tag{7.1}$$

for some M, using the fact that H is bounded below and $\omega\geq m>0$. Using again Lemma 6.3, we have:

$$\begin{array}{rl} & I(j^R)\mathbb{1}_{\{0\}}(N_\infty)\chi(H^{\rm ext})I^*(j^R)\\ = & I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R)\chi(H)+o(R^0)\\ = & \Gamma(q^R)\chi(H)+o(R^0).\end{array}\tag{7.2}$$

It remains to treat the other terms in (7.1). Because of the support of χ and using again Lemma 6.3, we have:

$$\begin{array}{rl} & I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\chi(H^{\rm ext})I^*(j^R)\\ = & I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\mathbb{1}\otimes F({\rm d}\Gamma(\omega)<m_\infty)\chi(H^{\rm ext})I^*(j^R)\\ = & I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\mathbb{1}\otimes F({\rm d}\Gamma(\omega)<m_\infty)I^*(j^R)\chi(H)+o(R^0),\end{array}$$

where $F(\lambda < m_\infty)$ is a cutoff function supported in $]-\infty, m_\infty[$.

From hypothesis *(H3)*, it follows that $\mathbb{1}_{[P,+\infty[}(N)\chi(H)$ tends to 0 in norm when $P \to +\infty$. Since $I^*(j^R)$ is isometric, we obtain:

$$\begin{aligned} & I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\mathbb{1}\otimes F(\mathrm{d}\Gamma(\omega) < m_\infty)I^*(j^R)\chi(H) \\ = \;& I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\mathbb{1}\otimes F(\mathrm{d}\Gamma(\omega) < m_\infty)I^*(j^R)\mathbb{1}_{[0,P]}(N)\chi(H) + o(R^0) + o(P^0), \end{aligned}$$

where the error term $o(P^0)$ is uniform in R. Next we use the following identity from [DG2, Subsect. 2.13]:

$$\mathbb{1}_{\{k\}}(N_\infty)I^*(j^R)\mathbb{1}_{\{n\}}(N) = I_k\Big(\frac{n!}{(n-k)!k!}\Big)^{\frac{1}{2}}\underbrace{j_0^R\otimes\cdots\otimes j_0^R}_{n-k}\otimes\underbrace{j_\infty^R\otimes\cdots\otimes j_\infty^R}_{k},$$

where I_k is the natural isometry between $\bigotimes^n \mathfrak{h}$ and $\bigotimes^{n-k}\mathfrak{h}\otimes\bigotimes^k\mathfrak{h}$.

We note next that if $F \in C_0^\infty(\mathbb{R})$ is supported in $]-\infty, m_\infty[$, $F(\omega)$ is compact on $\mathfrak{h}$, so $F(\omega)j_\infty^R$ tends to 0 *in norm* when $R\to\infty$ since $\text{s-}\lim_{R\to\infty} j_\infty^R = 0$. It follows from this remark that for each $k \geq 1$ and $n \leq P$:

$$I(j^R)\mathbb{1}_{\{k\}}(N_\infty)\mathbb{1}\otimes F(\mathrm{d}\Gamma(\omega) < m_\infty)I^*(j^R)\mathbb{1}_{\{n\}}(N) = o_P(R^0),$$

and hence

$$I^*(j^R)\mathbb{1}_{\{k\}}(N_\infty)\chi(H^{\mathrm{ext}})I(j^R) = o(P^0) + o(R^0) + o_P(R^0) = o(R^0), \tag{7.3}$$

if we choose first P large enough and then R large enough. Collecting (7.1), (7.2) and (7.3) we finally get that

$$\chi(H) = \Gamma(q^R)\chi(H) + o(R^0).$$

We use now that for each R $\Gamma(q^R)(H_0+1)^{-\frac{1}{2}}$ is compact on $\Gamma(\mathfrak{h})$, which follows easily from *(H1)* and *(G5)* (see e.g. [DG2, Lemma 4.2]). We obtain that $\chi(H)$ is compact as a norm limit of compact operators. Therefore $\sigma_{\mathrm{ess}}(H) \subset [\inf\sigma(H) + m_\infty, +\infty[$.

Let us now prove *ii)*. Note that it follows from *i)* that H admits a ground state. Let $\lambda = \inf\sigma(H) + \varepsilon$ for $\varepsilon > m_\infty$. Since $\varepsilon \in \sigma_{\mathrm{ess}}(\omega)$, there exists unit vectors $h_n \in \mathcal{D}(\omega)$ such that $\lim_{n\to\infty}(\omega-\varepsilon)h_n = 0$ and $\mathrm{w}-\lim_{n\to\infty} h_n = 0$. Let $u\in\Gamma(\mathfrak{h})$ a normalized ground state of H and set

$$u_n = a^*(h_n)u.$$

Since $u\in\mathcal{D}(N)$ by *(H3)* u_n is well defined. Moreover since $\mathrm{w}-\lim h_n = 0$, we obtain that $\lim\|u_n\| = 1$ and $\mathrm{w}-\lim u_n = 0$. Since $u \in \mathcal{D}(H^\infty)$, we know from *(H3)* that $u, Hu \in \mathcal{D}(N^\infty)$ and hence the following identity is valid:

$$H_0a^*(h_n)u = a^*(h_n)H_0u + a^*(\omega h_n)u = a^*(h_n)Hu - a^*(h_n)Vu + a^*(\omega h_n)u,$$

which shows that $u_n = a^*(h_n)u \in \mathcal{D}(H_0)$. Clearly $u_n\in\mathcal{D}(N^\infty)$, so $u_n\in\mathcal{D}(H)$ and

$$\begin{aligned} (H-\lambda)u_n =\;& (H_0+V-\lambda)u_n \\ =\;& a^*(h_n)(H-\lambda)u + a^*(\omega h_n)u + [V, a^*(h_n)]u \\ =\;& a^*((\omega-\varepsilon)h_n)u + [V, a^*(h_n)]u. \end{aligned}$$

We can compute the Wick symbol of $[V, a^*(h_n)]$ using Prop. 2.6. Using the fact that h_n tends weakly to 0 and Lemma 2.5 *iii)* we obtain that $[V, a^*(h_n)]u$ tends to 0 in norm. Similarly the term $a^*((\omega-\varepsilon)h_n)u$ tends to 0 in norm. Therefore (u_n) is a Weyl sequence for λ. □

7.2 Virial theorem

Let H be an abstract QFT Hamiltonian. We fix a selfadjoint operator a on $\mathfrak{h}$ such that hypothesis *(M1 i)* holds and set

$$A := \mathrm{d}\Gamma(a).$$

On the interaction V we impose hypothesis *(M2)*.

Lemma 7.2 *Assume (M1 i) and set* $\omega_t = \mathrm{e}^{\mathrm{i}ta}\omega\mathrm{e}^{-\mathrm{i}ta}$. *Then:*
i) $\mathrm{e}^{\mathrm{i}ta}$ *induces a strongly continuous group on* $\mathcal{D}(\omega)$ *and*

$$\sup_{|t|\leq 1}\|\omega_t(\omega+1)^{-1}\| < \infty, \quad \sup_{|t|\leq 1}\|\omega(\omega_t+1)^{-1}\| < \infty.$$

$$ii) \quad \sup_{0<|t|\leq 1}|t|^{-1}\|(\omega-\omega_t)\| < \infty, \quad \text{s-}\lim_{t\to 0} t^{-1}(\omega-\omega_t) = -[\omega, \mathrm{i}a]_0.$$

Proof. The first statement of *i)* follows from [GG, Appendix]. This fact clearly implies the first bound in *i)*. The second follows from $\omega(\omega_t+1)^{-1} = \mathrm{e}^{-\mathrm{i}ta}\omega_t(\omega+1)^{-1}\mathrm{e}^{\mathrm{i}ta}$. We deduce then from *i)* that

$$\sup_{|t|,|s|\leq 1}\|\omega_s(\omega_t+1)^{-1}\| < \infty. \tag{7.4}$$

Since $\omega \in C^1(a)$ we have:

$$(\omega_t+1)^{-1} - (\omega+1)^{-1} = \int_0^t \mathrm{e}^{\mathrm{i}sa}(\omega+1)^{-1}[\omega, \mathrm{i}a]_0(\omega+1)^{-1}\mathrm{e}^{-\mathrm{i}sa}\mathrm{d}s,$$

as a strong integral, and hence:

$$\begin{aligned}(\omega-\omega_t) = & \ (\omega_t+1)\left((\omega_t+1)^{-1} - (\omega+1)^{-1}\right)(\omega+1)\\ = & \ \textstyle\int_0^t(\omega_t+1)(\omega_s+1)^{-1}\mathrm{e}^{\mathrm{i}sa}[\omega, \mathrm{i}a]_0\mathrm{e}^{-\mathrm{i}sa}(\omega_s+1)^{-1}(\omega+1)\mathrm{d}s.\end{aligned}$$

Using (7.4) we obtain *ii)*. □

We set now

$$A := \mathrm{d}\Gamma(a), \quad H_s = \mathrm{e}^{\mathrm{i}sA}H\mathrm{e}^{-\mathrm{i}sA}, \quad H_{0,s} = \mathrm{e}^{\mathrm{i}sA}H_0\mathrm{e}^{-\mathrm{i}sA}, \quad V_s = \mathrm{e}^{\mathrm{i}sA}V\mathrm{e}^{-\mathrm{i}sA},$$

and introduce the quadratic forms $[H_0, \mathrm{i}A]$, $[V, \mathrm{i}A]$, $[H, \mathrm{i}A]$ with domains $\mathcal{D}(H_0)\cap\mathcal{D}(A)$, $\mathcal{D}(N^n)\cap\mathcal{D}(A)$ and $\mathcal{D}(H^m)\cap\mathcal{D}(A)$ for $n \geq \deg w/2$ and m large enough.

Proposition 7.3 *Let H be an abstract QFT Hamiltonian such that (M1 i), (M2) hold. Then:*
i) $[H_0, \mathrm{i}A]$ *extends uniquely as a bounded operator from* $\mathcal{D}(N)$ *to* $\mathcal{H}$, *denoted by* $[H_0,\ iA]_0$,

ii) $[V, \mathrm{i}A]$ *extends uniquely as a bounded operator from* $\mathcal{D}(N^M)$ *to* $\mathcal{H}$ *for M large enough, denoted by* $[V, \mathrm{i}A]_0$,

iii) $[H, \mathrm{i}A]$ *extends uniquely as a bounded operator from* $\mathcal{D}(H^P)$ *to* $\mathcal{H}$ *for P large enough, denoted by* $[H, \mathrm{i}A]_0$ *and equal to* $[H_0, \mathrm{i}A]_0 + [V, \mathrm{i}A]_0$, *iv) for r large enough $(H+b)^{-r}$ is in $C^1(A)$ and the following identity is valid as a bounded operators identity from $\mathcal{D}(A)$ to $\mathcal{H}$:*

$$(7.5)\qquad A(H+b)^{-r} = (H+b)^{-r}A + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}(H_s+b)^{-r}_{|s=0},$$

where

$$(7.6)\qquad \frac{\mathrm{d}}{\mathrm{d}s}(H_s+b)^{-r}_{|s=0} = \sum_{j=0}^{r-1}(H+b)^{-r+j}([H_0,\mathrm{i}A]_0+[V,\mathrm{i}A]_0)(H+b)^{-j-1}$$

is a bounded operator on $\mathcal{H}$.

Proof. We have $[H_0,\mathrm{i}A] = \mathrm{d}\Gamma([\omega,\mathrm{i}a])$, which using hypothesis *(M1 i)* and Prop. 2.4 *i)* implies that $[H_0,\mathrm{i}A](N+1)^{-1}$ is bounded. The fact that the extension is unique follows from the fact that $\mathcal{D}(a)\cap\mathcal{D}(\omega)$ is dense in $\mathfrak{h}$ since $\omega\in C^1(a)$.

Let us now check *ii)*. Through the identification of $B^2_{\mathrm{fin}}(\mathfrak{h})$ with $\Gamma_{\mathrm{fin}}(\mathfrak{h})\otimes\Gamma_{\mathrm{fin}}(\overline{\mathfrak{h}})$, we get from Prop. 2.6 that

$$[V,\ iA] = [\mathrm{Wick}(w),\ iA] = \mathrm{Wick}(w^{(1)})$$

where $w^{(1)} = (\mathrm{d}\Gamma(a)\otimes\mathbb{1} - \mathbb{1}\otimes\mathrm{d}\Gamma(\overline{a}))\,w$. By *(M2)* $w^{(1)}\in B^2_{\mathrm{fin}}(\mathfrak{h})$ which implies that $[V,\ iA](N+1)^{-n}$ is bounded for $n\geq \deg w/2$ using Lemma 2.5. The fact that the extension is unique is obvious.

By the higher order estimates we have $[H,\mathrm{i}A] = [H_0,\mathrm{i}A]+[V,\mathrm{i}A]$ on $\mathcal{D}(A)\cap\mathcal{D}(H^M)$ for M large enough, so $[H,\mathrm{i}A]_0(H+b)^{-M}$ is bounded, again by the higher order estimates. To prove that the extension is unique we need to show that $\mathcal{D}(A)\cap\mathcal{D}(H^M)$ is dense in $\mathcal{D}(H^M)$ for M large enough. Let $u=(H+b)^{-M}v\in\mathcal{D}(H^M)$ and $u_\epsilon = (H+b)^{-M}(1+\mathrm{i}\epsilon A)^{-1}v$. Clearly $u_\epsilon\to u$ in $\mathcal{D}(H^M)$ when $\epsilon\to 0$. Next u_ϵ belongs to $\mathcal{D}(H^M)$ and to $\mathcal{D}(A)$ since $(H+b)^{-M}$ is in $C^1(A)$ by *iv)*. This completes the proof of *iii)*.

It remains to prove *iv)*. We start by proving some auxiliary properties of H_s. Since $H_{0,s} = \mathrm{d}\Gamma(\omega_s)$, we obtain using Lemma 7.2 *i)* and Prop. 2.4 *i)* that

$$(7.7)\qquad \sup_{|s|\leq 1}\|H_0(H_{0,s}+1)^{-1}\|<\infty.$$

The same arguments show also that $\mathcal{D}(H_0)=\mathcal{D}(H_{0,s})$ ie $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H_0)$. Since $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(N^n)$ we obtain from the higher order estimates that

$$(7.8)\qquad H_s = H_{0,s}+V_s \text{ on } \mathcal{D}(H^P).$$

Let us fix $n\geq \deg w/2$. Conjugating the bounds in *(H3)* by $\mathrm{e}^{\mathrm{i}sA}$, we obtain that there exists $p\in\mathbb{N}$ such that

$$N^{2n}H^2_{0,s}\leq C(H_s+b)^{2p},\ \text{uniformly in } |s|\leq 1.$$

Using also (7.7) we obtain

$$(7.9)\qquad N^{2n}H_0^2\leq C(H_s+b)^{2p},\ \text{uniformly in } |s|\leq 1.$$

Let us show that for r large enough:

$$(7.10)\qquad \|(H_s+b)^{-r}-(H+b)^{-r}\|\leq C|s|,\quad |s|\leq 1.$$

Using (7.8), we can write for P large enough:

$$
\begin{array}{rcl}
 & & ((H_s+b)^{-r}-(H+b)^{-r})\,(H+b)^{-P} \\
(7.11) & = & \sum_{j=0}^{r-1}(H_s+b)^{-r+j}(H-H_s)(H+b)^{-j-1}(H+b)^{-P} \\
 & = & \sum_{j=0}^{r-1}(H_s+b)^{-r+j}(H_0-H_{0,s}+V-V_s)(H+b)^{-j-1}(H+b)^{-P}.
\end{array}
$$

Using that $H_{0,s}-H_0=\mathrm{d}\Gamma(\omega_s-\omega)$, Lemma 7.2 *ii)* and Prop. 2.4 *i)* we obtain that

$$
\|(H_{0,s}-H_0)(N+1)^{-1}\|\leq C|s|,\quad |s|\leq 1. \tag{7.12}
$$

If $r\geq 2p$ then for $0\leq j\leq r-1$ then either $j+1\geq p$ or $r-j\geq p$. Using (7.9) and (7.12) we deduce that

$$
\|(H_s+b)^{-r+j}(H_{0,s}-H_0)(H+b)^{-j-1}\|\leq C|s|,\quad |s|\leq 1. \tag{7.13}
$$

Next from Prop. 2.6, we have:

$$
V_s=\mathrm{Wick}(\mathrm{e}^{\mathrm{i}sA}w\mathrm{e}^{-\mathrm{i}sA}).
$$

Through the identification of $B^2_{\mathrm{fin}}(\mathfrak{h})$ with $\Gamma_{\mathrm{fin}}(\mathfrak{h})\otimes\Gamma_{\mathrm{fin}}(\overline{\mathfrak{h}})$, the symbol $\mathrm{e}^{\mathrm{i}sA}w\mathrm{e}^{-\mathrm{i}sA}$ is identified with $\mathrm{e}^{\mathrm{i}sA}\otimes\mathrm{e}^{-\mathrm{i}s\overline{A}}w$. From hypothesis *(M2)* and Prop. 2.5, we obtain that for $M\geq\deg(w)/2$:

$$
\|(V_s-V)(N+1)^{-M}\|\leq C|s|,\quad |s|\leq 1. \tag{7.14}
$$

By the same argument as above we obtain:

$$
\|(H_s+b)^{-r+j}(V-V_s)(H+b)^{-j-1}\|\leq C|s|,\quad |s|\leq 1. \tag{7.15}
$$

Combining (7.11), (7.15) and (7.13), we obtain (7.10).

Next from (7.12) we obtain by considering first finite particle vectors that

$$
\mathrm{s}\text{-}\lim_{s\to 0}s^{-1}(H_{0,s}-H_0)(N+1)^{-1}\ \text{exists.}
$$

We note next that by hypothesis *(M2)* we know that $s^{-1}(\mathrm{e}^{\mathrm{i}sA}w\mathrm{e}^{-\mathrm{i}sA}-w)$ converges in $B^2_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ when $s\to 0$. Using then Lemma 2.5 *ii)*, we obtain also that

$$
\mathrm{s}\text{-}\lim_{s\to 0}s^{-1}(V_s-V)(N+1)^{-n}\ \text{exists.}
$$

From (7.11) we obtain that for $r\geq p$ and P large enough:

$$
\mathrm{s}\text{-}\lim_{s\to 0}s^{-1}\left((H_s+b)^{-r}-(H+b)^{-r}\right)\ \text{exists on }\mathcal{D}(H^P)
$$

By (7.10) the strong limit exists on $\Gamma(\mathfrak{h})$, which shows that $(H+b)^{-r}$ is in $C^1(A)$. □

Remark 7.4 *The same proof as in Prop. 7.3 iv) shows that for r large enough and* $z_i\in\mathbb{C}\backslash\mathbb{R}$, *the operator* $\prod_{i=1}^{r}(z_i-H)^{-1}$ *is in* $C^1(A)$. *Using the functional calculus formula (2.2), it is easy to deduce from this fact that* $\chi(H)$ *is in* $C^1(A)$ *for all* $\chi\in C_0^\infty(\mathbb{R})$.

The following proposition is the main consequence of Prop. 7.3.

Proposition 7.5 *Let H be an abstract QFT Hamiltonian such that (M1 i), (M2) hold. Then the* virial relation *holds:*

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]_0 \mathbb{1}_{\{\lambda\}}(H) = 0, \ \ \lambda \in \mathbb{R}. \tag{7.16}$$

Proof. Let us fix r large enough such that $(H+b)^{-r} \in C^1(A)$ so that $(H+b)^{-r} : \mathcal{D}(A) \to \mathcal{D}(A)$ and $[(H+b)^{-r}, \mathrm{i}A]$ extends as a bounded operator on $\mathcal{H}$ denoted by $[(H+b)^{-r}, \mathrm{i}A]_0$. Moreover from Prop. 7.3 *iv)* we have:

$$[(H+b)^{-r}, \mathrm{i}A]_0 = -\sum_{j=0}^{r-1} (H+b)^{-r+j}[H, \mathrm{i}A]_0 (H+b)^{-j-1}.$$

Let now $u_1, u_2 \in \mathcal{H}$ such that $Hu_i = \lambda u_i$. Since $(H+b)^{-r} \in C^1(A)$ and u_i is an eigenvector of $(H+b)^{-r}$, we have the virial relation:

$$\begin{aligned} 0 = & \ \ (u_1, [(H+b)^{-r}, \mathrm{i}A]_0 u_2) \\ = & \ \ -\textstyle\sum_{j=0}^{r-1} (u_1, (H+b)^{-r+j}[H, \mathrm{i}A]_0 (H+b)^{-j-1} u_2) \\ = & \ \ -\textstyle\sum_{j=0}^{r-1} (\lambda+b)^{-r-1} (u_1, [H, \mathrm{i}A]_0 u_2) \\ = & \ \ -r(\lambda+b)^{-r-1}(u_1, [H, \mathrm{i}A]_0 u_2), \end{aligned}$$

which proves the lemma. □

7.3 Mourre estimate for second quantized Hamiltonians

In this subsection we will apply the abstract results in Subsect. 2.1 to second quantized Hamiltonians.

Let ω, a be two selfadjoint operators on $\mathfrak{h}$ such that *(H1)*, *(M1)* hold. Note that it follows from Lemma 2.1 and the results recalled above it that *(M1)* imply also that

$$\kappa_a(\omega) \text{ is a closed countable set.} \tag{7.17}$$

Clearly $\mathrm{d}\Gamma(\omega) \in C^1(\mathrm{d}\Gamma(a))$ and $[\mathrm{d}\Gamma(\omega), \mathrm{i}\mathrm{d}\Gamma(a)]_0 = \mathrm{d}\Gamma([\omega, \mathrm{i}a]_0)$. Since $\mathrm{d}\Gamma(\omega)$ and $\mathrm{d}\Gamma([\omega, \mathrm{i}a]_0)$ commute with N, we can restrict them to each $n-$particle sector $\otimes_{\mathrm{s}}^n \mathfrak{h}$. We denote by

$$\rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(A)\,(1)}$$

the corresponding restriction of $\rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(A)}$ to the range of $\mathbb{1}_{[1,+\infty[}(N)$.

Finally we introduce the following natural notation for $E \subset \mathbb{R}$:

$$\mathrm{d}\Gamma^{(1)}(E) = \bigcup_{n=0}^{+\infty} \underbrace{E + \cdots + E}_{n}, \ \mathrm{d}\Gamma(E) = \{0\} \cup \mathrm{d}\Gamma^{(1)}(E). \tag{7.18}$$

Remark 7.6 *As an example of use of this notation, note that if b is a selfadjoint operator on $\mathfrak{h}$, then:*

$$\sigma(\mathrm{d}\Gamma(b)) = \{0\} \cup \mathrm{d}\Gamma(\sigma(b)).$$

Note also that if E is a closed countable set included in $[m, +\infty[$ for some $m > 0$, $\mathrm{d}\Gamma^{(1)}(E)$ is a closed countable set.

Lemma 7.7 *Let ω, a be two selfadjoint operators on $\mathfrak{h}$ such that (M1) holds. Then:*

$$i)\ \ \rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(a)} \geq 0,$$

$$ii)\ \ \rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(a)\,(1)}(\lambda) = 0 \ \Rightarrow \lambda \in \mathrm{d}\Gamma^{(1)}(\kappa_a(\omega)).$$

Proof. We have $[\mathrm{d}\Gamma(\omega), \mathrm{i}\mathrm{d}\Gamma(a)] = \mathrm{d}\Gamma([\omega, \mathrm{i}a])$. Since $\mathrm{d}\Gamma(\omega) \in C^1(\mathrm{d}\Gamma(a))$ the virial relation is satisfied. Denote by ρ_n the restriction of $\rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(a)}$ to $\otimes_{\mathrm{s}}^n \mathfrak{h}$. Applying Lemma 2.1 *iv)* we obtain

$$\rho_0(\lambda) = \left\{ \begin{array}{ll} 0, & \lambda = 0, \\ +\infty, & \lambda \neq 0 \end{array} \right. ,$$

$$\rho_n(\lambda) = \inf_{\lambda_1 + \cdots \lambda_n = \lambda} (\rho_\omega^a(\lambda_1) + \cdots + \rho_\omega^a(\lambda_n))$$

for $n \geq 1$. We note next that since $\omega \geq m > 0$, $\chi(\mathrm{d}\Gamma(\omega))\mathbb{1}_{[n,+\infty[}(N) = 0$ if n is large enough, where $\chi \in C_0^\infty(\mathbb{R})$. Therefore only a finite number of $n-$particle sectors contribute to the computation of $\rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(a)}$ near an energy level λ. We can hence apply Lemma 2.1 *iii)* and obtain that $\rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(a)} \geq 0$.

Let us now prove the second statement of the lemma. Since $\rho_\omega^a(\lambda) = +\infty$ if $\lambda \notin \sigma(\omega)$, we have $\rho_\omega^a(\lambda) = +\infty$ for $\lambda < 0$. Therefore

$$\rho_n(\lambda) = \inf_{I_n(\lambda)} (\rho_\omega^a(\lambda_1) + \cdots + \rho_\omega^a(\lambda_n)),$$

for $I_n(\lambda) = \{(\lambda_1, \ldots, \lambda_n) |\ \ \lambda_1 + \cdots \lambda_n = \lambda,\ \ \lambda_i \geq 0\}$. The function $\rho_\omega^a(\lambda_1) + \cdots + \rho_\omega^a(\lambda_n)$ is lower semicontinuous on $\mathbb{R}^n$, hence attains its minimum on the compact set $I_n(\lambda)$. Therefore using also that $\rho_\omega^a \geq 0$, we see that $\rho_n(\lambda) = 0$ iff $\lambda \in \kappa_a(\omega) + \cdots + \kappa_a(\omega)$ (n factors). Using Lemma 2.1 *iii)* as above, we obtain that $\rho_{\mathrm{d}\Gamma(\omega)}^{\mathrm{d}\Gamma(A)\,(1)}(\lambda) = 0$ implies that $\lambda \in \mathrm{d}\Gamma^{(1)}(\kappa_a(\omega))$, which proves *ii)*. □

7.4 Mourre estimate for abstract QFT Hamiltonians

In this subsection we prove the Mourre estimate for abstract QFT Hamiltonians. Let H be an abstract QFT Hamiltonian and a a selfadjoint operator on $\mathfrak{h}$ such that *(M1)* holds. Let also $\langle x \rangle$ be a weight operator for ω.

Theorem 7.8 *Let H be an abstract QFT Hamiltonian and a a selfadjoint operator on $\mathfrak{h}$ such that (M1) and (M2) hold. Let $\langle x \rangle$ be a weight operator for ω such that conditions (G1), (G3), (G5) hold. Set*

$$\tau := \sigma_{\mathrm{pp}}(H) + \mathrm{d}\Gamma^{(1)}(\kappa_a(\omega))$$

and $A = \mathrm{d}\Gamma(a)$. Then:

i) Let $\lambda \in \mathbb{R}\backslash\tau$. Then there exists $\epsilon > 0$, $c_0 > 0$ and a compact operator K such that

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,\mathrm{i}A]_0\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq c_0\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H) + K.$$

ii) for all $\lambda_1 \leq \lambda_2$ such that $[\lambda_1, \lambda_2] \cap \tau = \emptyset$ one has:

$$\dim \mathbb{1}_{[\lambda_1,\lambda_2]}(H) < \infty.$$

Consequently $\sigma_{\rm pp}(H)$ can accumulate only at τ, which is a closed countable set.

iii) Let $\lambda \in \mathbb{R}\backslash(\tau \cup \sigma_{\rm pp}(H))$. Then there exists $\epsilon > 0$ and $c_0 > 0$ such that

$$\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,\mathrm{i}A]_0\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq c_0\mathbb{1}_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

Proof. We note first that $[H,\mathrm{i}A]_0$ satisfies the virial relation by Prop. 7.5. Therefore we will be able to apply the abstract results in Lemma 2.1 in our situation. Recall that $H^{\rm ext} = H\otimes\mathbb{1} + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega)$ and set

$$A^{\rm ext} = A\otimes\mathbb{1} + \mathbb{1}\otimes A.$$

By Prop. 7.3 $[H,\mathrm{i}A]_0$ considered as an operator on $\mathcal{H}$ with domain $\mathcal{D}(H^M)$ is equal to H_1+V_1, where $H_1 = \mathrm{d}\Gamma([\omega,\mathrm{i}a]_0)$, $V_1 = [V,\mathrm{i}A]_0$. Note that by *(M2)* V_1 is a Wick polynomial with a symbol in $B^2_{\rm fin}(\mathfrak{h})$, and by *(G3)*, $[\langle x\rangle, [\omega, \mathrm{i}a]]$ is bounded on $\mathfrak{h}$. Therefore using Lemma 3.3 *v)* we see that the analog of (6.11) holds for $[H,\mathrm{i}A]_0$. We obtain:

$$I^*(j^R)[H,\mathrm{i}A]_0 = [H^{\rm ext},\mathrm{i}A^{\rm ext}]_0 I^*(j^R) + (N+1)^n\check{O}_N(R^0),$$

for some n. We recall (7.2):

$$\chi(H) = \Gamma(q^R)\chi(H) + I(j^R)\chi(H^{\rm ext})\mathbb{1}_{[1,+\infty[}(N_\infty)I^*(j^R) + o(R^0), \tag{7.19}$$

for $q^R = (j_0^R)^2$.

Using then Lemma 6.3 and the higher order estimates (which hold also for $H^{\rm ext}$ with the obvious modifications), we obtain that:

$$\begin{aligned}\chi(H)[H,\mathrm{i}A]_0\chi(H) = \;& \Gamma(q^R)\chi(H)[H,\mathrm{i}A]_0\chi(H)\\ &+I(j^R)\chi(H^{\rm ext})[H^{\rm ext},\mathrm{i}A^{\rm ext}]_0\chi(H^{\rm ext})\mathbb{1}_{[1,+\infty[}(N_\infty)I^*(j^R) + o(R^0).\end{aligned} \tag{7.20}$$

We will now prove by induction on $n \in \mathbb{N}$ the following statement:

$$H(n)\quad \left\{\begin{array}{ll} i) & \rho_H^A(\lambda) \geq 0, \text{ for } \lambda \in]-\infty, \inf\sigma(H) + nm[,\\ ii) & \tau^A(H)\cap]-\infty,\inf\sigma(H)+nm[\subset \sigma_{\rm pp}(H) + \mathrm{d}\Gamma(\kappa_a(\omega)).\end{array}\right.$$

Statement *H(0)* is clearly true since $\rho_H^A(\lambda) = +\infty$ for $\lambda < \inf\sigma(H)$.

Let us assume that *H(n-1)* holds. Let us denote by $\rho^{{\rm ext}\,(1)}$ the restriction of $\rho^{A^{\rm ext}}_{H^{\rm ext}}$ to the range of $\mathbb{1}_{[1,+\infty[}(N_\infty)$. This function is well defined since $H^{\rm ext}$ and $[H^{\rm ext},\mathrm{i}A^{\rm ext}]_0$ commute with N_∞.

Let $\lambda \in]-\infty, \inf\sigma(H) + nm[$. Using Lemma 2.1 *iv)* and the fact that $\omega \geq m$ we obtain:

$$\rho^{{\rm ext}\,(1)}(\lambda) = \inf_{(\lambda_1,\lambda_2)\in I^{(n)}(\lambda)}\left(\rho_H^A(\lambda_1) + \rho_{H_0}^{A\,(1)}(\lambda_2)\right),$$

where

$$I^{(n)}(\lambda) = \{(\lambda_1, \lambda_2) |\ \lambda_1 + \lambda_2 = \lambda,\ \inf \sigma(H) \le \lambda_1 \le \inf \sigma(H) + (n-1)m,\quad 0 \le \lambda_2 \le -\inf \sigma(H)\},$$

and the function $\rho_{H_0}^{A\,(1)}$ is defined in Subsect. 7.3. Note that by *H(n-1) i)* and Lemma 7.7 *i)* the two functions $\rho_H^A(\lambda_1)$ and $\rho_{H_0}^{A\,(1)}(\lambda_2)$ are positive for $(\lambda_1, \lambda_2) \in I^{(n)}(\lambda)$. We deduce first from this fact that:

$$\rho^{\text{ext}\,(1)}(\lambda) \ge 0 \text{ for } \lambda \in]-\infty, \inf \sigma(H) + nm[. \tag{7.21}$$

Moreover using that the lower semicontinuous function $\rho_H^A(\lambda_1) + \rho_{H_0}^{A\,(1)}(\lambda_2)$ attains its minimum on the compact set $I^{(n)}(\lambda) \subset \mathbb{R}^2$, we obtain that

$$\begin{aligned} &\rho^{\text{ext}\,(1)}(\lambda) = 0, \quad \lambda \in]-\infty, \inf \sigma(H) + nm[\Rightarrow \\ &\lambda = \lambda_1 + \lambda_2,\ \text{ where } (\lambda_1, \lambda_2) \in I^{(n)}(\lambda), \quad \rho_H^A(\lambda_1) = \rho_{H_0}^{A\,(1)}(\lambda_2) = 0. \end{aligned} \tag{7.22}$$

From *H(n-1) ii)* and Lemma 2.1 *ii)* we get that

$$\rho_H^A(\lambda_1) = 0, \quad \lambda_1 \in]-\infty, \inf \sigma(H) + (n-1)m[\Rightarrow \lambda_1 \in \sigma_{\text{pp}}(H) + \mathrm{d}\Gamma(\kappa_a(\omega)).$$

From Lemma 7.7 *ii)* we know that

$$\rho_{H_0}^{A\,(1)}(\lambda_2) = 0 \ \Rightarrow\ \lambda_2 \in \mathrm{d}\Gamma^{(1)}(\kappa_a(\omega)).$$

Using (7.22) we get that

$$\rho^{\text{ext}\,(1)}(\lambda) = 0, \quad \lambda \in]-\infty, \inf \sigma(H) + nm[\Rightarrow \lambda \in \sigma_{\text{pp}}(H) + \mathrm{d}\Gamma^{(1)}(\kappa_a(\omega)). \tag{7.23}$$

The operators $\Gamma(q^R)\chi(H)$ and hence $\Gamma(q^R)\chi(H)[H, \mathrm{i}A]_0\chi(H)$ are compact on $\mathcal{H}$. Choosing hence R large enough in (7.20) we obtain using (7.19) and the fact that $I(j^R)I^*(j^R) = \mathbb{1}$ that

$$\tilde{\rho}_H^A(\lambda) \ge \rho^{\text{ext}\,(1)}(\lambda), \quad \lambda \in]-\infty, \inf \sigma(H) + nm[. \tag{7.24}$$

By Lemma 2.1 *i)* this implies first that $\rho_H^A \ge 0$ on $]-\infty, \inf \sigma(H) + nm[$, i.e. *H(n) i)* holds. Using then (7.23) we obtain that

$$\tilde{\rho}_H^A(\lambda) = 0, \quad \lambda \in]-\infty, \inf \sigma(H) + nm[\Rightarrow \lambda \in \sigma_{\text{pp}}(H) + \mathrm{d}\Gamma^{(1)}(\kappa_a(\omega)),$$

which proves *H(n) ii)*. Since *H(n)* holds for any n we obtain statement *i)* of the theorem. The fact that $\dim \mathbb{1}_{[\lambda_1, \lambda_2]}(H) < \infty$ if $[\lambda_1, \lambda_2] \cap \tau = \emptyset$ follows from the abstract results recalled in Subsect. 2.1. We saw in (7.17) that $\kappa_a(\omega)$ is a closed countable set. Using also Remark 7.6, this implies by induction on n that $\tau \cap]-\infty, \inf \sigma(H) + nm[$ is a closed countable set for any n. Finally statement *iii)* follows from Lemma 2.1. This completes the proof of the theorem. □

7.5 Improved Mourre estimate

Thm. 7.8 can be rephrased as:

$$\tau_A(H) \subset \sigma_{\rm pp}(H) + {\rm d}\Gamma^{(1)}(\kappa_a(\omega)),$$

which is sufficient for our purposes. Nevertheless a little attention shows that one should expect a better result, namely:

$$\tau_A(H) \subset \sigma_{\rm pp}(H) + {\rm d}\Gamma^{(1)}(\tau_a(\omega)),$$

i.e. eigenvalues of ω away from $\tau_a(\omega)$ should not contribute to the set of thresholds of H. In this subsection we prove this result if there exists a comparison operator ω_∞ such that hypothesis *(C)* holds.

We fix a function $q \in C^\infty(\mathbb{R})$ such that

$$0 \leq q \leq 1, \quad q \equiv 0 \text{ near } 0, \quad q \equiv 1 \text{ near } 1. \tag{7.25}$$

Lemma 7.9 *Assume (H1), (G1), (G3), (M1) for ω and ω_∞ and (C). Set $H_0 = {\rm d}\Gamma(\omega)$, $H_\infty = {\rm d}\Gamma(\omega_\infty)$. Let q as in (7.25) and $\chi \in C_0^\infty(\mathbb{R})$. Then:*

$$(\chi^2(H_0) - \chi^2(H_\infty))\Gamma(q^R) \in o(R^0), \tag{7.26}$$

$$\begin{aligned} &\chi(H_0)[H_0, {\rm i}A]_0\chi(H_0)\Gamma(q^R) \\ = \ &\chi(H_\infty)[H_\infty, {\rm i}A]_0\chi(H_\infty)\Gamma(q^R) + o(R^0), \end{aligned} \tag{7.27}$$

Assume additionally (G5). Then

$$\tilde{\rho}^a_\omega = \tilde{\rho}^a_{\omega_\infty}. \tag{7.28}$$

Proof. We will first prove the following estimates:

$$[\chi(H_\epsilon), \Gamma(q^R)], \quad (\chi(H_0) - \chi(H_\infty))\,\Gamma(q^R) \in o(R^0), \tag{7.29}$$

$$\begin{aligned} &(H_{\epsilon_1} + {\rm i})^{-1}[H_0 - H_\infty, {\rm i}A]_0\Gamma(q^R)(H_{\epsilon_2} + {\rm i})^{-1} \in o(R^0) \\ &(H_{\epsilon_1} + {\rm i})^{-1}[[H_\infty, {\rm i}A]_0, \Gamma(q^R)](H_{\epsilon_2} + {\rm i})^{-1} \in o(R^0), \end{aligned} \tag{7.30}$$

for $\epsilon, \epsilon_1, \epsilon_2 \in \{0, \infty\}$. If we use the identities

$$[{\rm d}\Gamma(b_i), \Gamma(q^R)] = {\rm d}\Gamma(q^R, [b_i, q^R]), \quad {\rm d}\Gamma(b_1 - b_2)\Gamma(q^R) = {\rm d}\Gamma(q^R, (b_1 - b_2)q^R),$$

for $b_1 = \omega$, $b_2 = \omega_\infty$, Lemma 3.4, Lemma 3.3 *(i)* and the bounds in Prop. 2.4, it is easy to see that uniformly in $z \in \mathbb{C}\backslash\mathbb{R} \cap \{|z| \leq R\}$:

$$\begin{aligned} &[(z - H_\epsilon)^{-1}, \Gamma(q^R)] \in O(R^{-1})|{\rm Im} z|^{-2}, \\ &(z - H_{\epsilon_1})^{-1}(H_0 - H_\infty)\Gamma(q^R)(z - H_{\epsilon_2})^{-1} \in o(R^0)|{\rm Im} z|^{-2}. \end{aligned}$$

Using the functional calculus formula (2.2) this implies (7.29). The proof of (7.30) is similar using Lemma 3.4 and Lemma 3.3 *(v)*. The proof of (7.27) is now easy: we move the operator

$\Gamma(q^R)$ to the left, changing H_0 into H_∞ along the way, and then move $\Gamma(q^R)$ back to the right. All errors terms are $o(R^0)$, by (7.29), (7.30). (7.26) follows from (7.29). If we restrict (7.26), (7.27) to the one-particle sector we obtain that

$$\begin{aligned}&(\chi^2(\omega)-\chi^2(\omega_\infty))q^R \in o(R^0),\\ &\chi(\omega)[\omega,\mathrm{i}a]_0\chi(\omega)q^R = \chi(\omega_\infty)[\omega_\infty,\mathrm{i}a]_0\chi(\omega_\infty)q^R + o(R^0).\end{aligned}$$

Using *(G5)* and the fact that $(1-q)\in C_0^\infty(\mathbb{R})$ we see that $\chi(H_\epsilon)(1-q)^R$ is compact for $\epsilon=0,\infty$. Writing $1=(1-q)^R+q^R$, we easily obtain (7.28). □

Theorem 7.10 *Let H be an abstract QFT Hamiltonian satisfying the hypotheses of Thm. 7.8. Let ω_∞ be a comparison Hamiltonian on $\mathfrak{h}$ such that (C1) holds. Then the conclusions of Thm. 7.8 hold for*

$$\tau := \sigma_{\rm pp}(H) + \mathrm{d}\Gamma^{(1)}(\tau_a(\omega)).$$

Proof. We use the notation in the proof of Thm. 7.8. We pick a function q_1 satisfying (7.25) such that $q_1 j_\infty = j_\infty$, so that

$$I^*(j^R) = \mathbb{1}\otimes\Gamma(q_1^R)I^*(j^R).$$

Therefore in (7.20) we can insert $\mathbb{1}\otimes\Gamma(q_1^R)$ to the left of $I^*(j^R)$. If we set

$$H_\infty^{\rm ext} := H\otimes\mathbb{1} + \mathbb{1}\otimes H_\infty,$$

then using the obvious extension of Lemma 7.9 to $H^{\rm ext}$ and $H_\infty^{\rm ext}$, we obtain instead of (7.20):

$$\begin{aligned}&\chi(H)[H,\mathrm{i}A]_0\chi(H)\\ (7.31)\qquad =\ &\Gamma(q^R)\chi(H)[H,\mathrm{i}A]_0\chi(H)\\ &+I(j^R)\chi(H_\infty^{\rm ext})[H_\infty^{\rm ext},\mathrm{i}A^{\rm ext}]_0\chi(H_\infty^{\rm ext})\mathbb{1}_{[1,+\infty[}(N_\infty)I^*(j^R)+o(R^0).\end{aligned}$$

Therefore in the later steps of the proof we can replace ω by ω_∞. By assumption $\kappa_a(\omega_\infty) = \tau_a(\omega_\infty)$ and by Lemma 7.9 $\tau_a(\omega_\infty)=\tau_a(\omega)$. This completes the proof of the theorem. □

8 Scattering theory for abstract QFT Hamiltonians

In this section we consider the scattering theory for our abstract QFT Hamiltonians. This theory is formulated in terms of *asymptotic Weyl operators*, (see Thm. 8.1) which form regular CCR representations over $\mathfrak{h}_{\rm c}(\omega)$. Using the fact that the theory is massive, it is rather easy to show that this representation is of Fock type (see Thm. 8.5). The basic question of scattering theory, namely the asymptotic completeness of wave operators, amounts then to prove that the space of vacua for the two asymptotic CCR representations coincide with the space of bound states for H. This will be shown in Thm. 10.6, using the propagation estimates of Sect. 9.

In all this section we only consider objects with superscript $+$, corresponding to $t\to+\infty$. The corresponding objects with superscript $-$ corresponding to $t\to-\infty$ have the same properties.

8.1 Asymptotic fields

For $h \in \mathfrak{h}$ we set $h_t := \mathrm{e}^{-\mathrm{i}t\omega}h$. Recall that $\mathfrak{h}_{\mathrm{c}}(\omega) \subset \mathfrak{h}$ is the continuous spectral subspace for ω and that by hypothesis *(S)* there exists a subspace $\mathfrak{h}_0$ dense in $\mathfrak{h}_{\mathrm{c}}(\omega)$ such that for all $h \in \mathfrak{h}_0$ there exists $\epsilon > 0$ such that

$$\|\mathbb{1}_{[0,\epsilon]}(\frac{\langle x \rangle}{|t|})\mathrm{e}^{-\mathrm{i}t\omega}h\| \in O(t^{-\mu}), \qquad \mu > 1.$$

Theorem 8.1 *Let H be an abstract QFT Hamiltonian such that hypotheses (Is) for $s > 1$ and (S) hold. Then: i) For all $h \in \mathfrak{h}_{\mathrm{c}}(\omega)$ the strong limits*

$$W^+(h) := \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} \tag{8.1}$$

exist. They are called the asymptotic Weyl operators. *The asymptotic Weyl operators can be also defined using the norm limit:*

$$W^+(h)(H+b)^{-n} = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)(H+b)^{-n}\mathrm{e}^{-\mathrm{i}tH}, \tag{8.2}$$

for n large enough.

ii) The map

$$\mathfrak{h}_{\mathrm{c}}(\omega) \ni h \mapsto W^+(h) \tag{8.3}$$

is strongly continuous and for n large enough, the map

$$\mathfrak{h} \ni h_{\mathrm{c}}(\omega) \mapsto W^+(h)(H+b)^{-n} \tag{8.4}$$

is norm continuous.

iii) The operators $W^+(h)$ satisfy the Weyl commutation relations:

$$W^+(h)W^+(g) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\mathrm{Im}(h|g)}W^+(h+g).$$

iv) The Hamiltonian preserves the asymptotic Weyl operators:

$$\mathrm{e}^{\mathrm{i}tH}W^+(h)\mathrm{e}^{-\mathrm{i}tH} = W^+(h_{-t}). \tag{8.5}$$

Proof. The proof is almost identical to the proof of [DG1, Thm. 10.1], therefore we will only sketch it. We have:

$$W(h_t) = \mathrm{e}^{-\mathrm{i}tH_0}W(h)\mathrm{e}^{\mathrm{i}tH_0},$$

which implies that, as a quadratic form on $\mathcal{D}(H_0)$, one has

$$\partial_t W(h_t) = -[H_0, \mathrm{i}W(h_t)]. \tag{8.6}$$

Using (8.6) and the fact that for n large enough $\mathcal{D}(H^n) \subset \mathcal{D}(H_0) \cap \mathcal{D}(V)$, we have, as quadratic forms on $\mathcal{D}(H^n)$:

$$\partial_t \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}tH}[V, \mathrm{i}W(h_t)]\mathrm{e}^{-\mathrm{i}tH}.$$

Integrating this relation we have as a quadratic form identity on $\mathcal{D}(H^n)$

$$\mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} - W(h) = \int_0^t \mathrm{e}^{\mathrm{i}t'H}[V, \mathrm{i}W(h_{t'})]\mathrm{e}^{-\mathrm{i}t'H}\mathrm{d}t'. \tag{8.7}$$

We claim that for $h \in \mathfrak{h}_0$ (see hypothesis *(S)*), and $p \geq \mathrm{deg}w/2$:

$$\|[V, W(h_t)](N+1)^{-p}\| \in L^1(\mathrm{d}t). \tag{8.8}$$

In fact writing w as $\sum_{p+q\leq \mathrm{deg}(w)} w_{p,q}$, where $w_{p,q}$ is of order (p,q) and using Prop. 2.6, we obtain that

$$[\mathrm{Wick}(w_{p,q}), W(h_t)] = W(h_t)\mathrm{Wick}(w_{p,q}(t)),$$

where $w_{p,q}(t)$ is the sum of the symbols in the r.h.s. of (2.7) for $(s,r) \neq (p,q)$. Using *(Is)* and *(S)* we obtain writing $\mathbb{1} = \mathbb{1}_{[0,\epsilon]}(\frac{\langle x\rangle}{\langle t\rangle}) + \mathbb{1}_{]\epsilon,+\infty[}(\frac{\langle x\rangle}{\langle t\rangle})$ that

$$\|w_{p,q}(t)\|_{B^2(\mathfrak{h})} \in L^1(\mathrm{d}t),$$

which proves (8.8) using Lemma 2.5.

Using then the higher order estimates, we obtain that the identity (8.7) makes sense as an identity between bounded operators from $\mathcal{D}(H^n)$ to $\mathcal{H}$ for n large enough. It also proves that the norm limit (8.2) exists for $h \in \mathfrak{h}_0$. The rest of the proof is identical to [DG1, Thm. 10.1]. It relies on the bound

$$\begin{aligned} & \|\left(\mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH}W(g_t)\mathrm{e}^{-\mathrm{i}tH}\right)(H+b)^{-n}\| \\ \leq\ & \|\left(W(h) - W(g)\right)(N+1)^{-1}\|\|(N+1)(H+b)^{-n}\| \\ \leq\ & C\|h-g\|(\|h\|^2 + \|g\|^2 + 1). \end{aligned}$$

□

Theorem 8.2 *i) For any $h \in \mathfrak{h}_{\mathrm{c}}(\omega)$:*

$$\phi^+(h) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}W^+(sh)_{|s=0}$$

defines a self-adjoint operator, called the asymptotic field, *such that*

$$W^+(h) = \mathrm{e}^{\mathrm{i}\phi^+(h)}.$$

ii) The operators $\phi^+(h)$ satisfy in the sense of quadratic forms on $\mathcal{D}(\phi^+(h_1)) \cap \mathcal{D}(\phi^+(h_2))$ the canonical commutation relations

$$[\phi^+(h_2), \phi^+(h_1)] = \mathrm{i}\mathrm{Im}(h_2|h_1). \tag{8.9}$$

iii)

$$\mathrm{e}^{\mathrm{i}tH}\phi^+(h)\mathrm{e}^{-\mathrm{i}tH} = \phi^+(h_{-t}).$$

iv) For $p \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for $h_i \in \mathfrak{h}_{\mathrm{c}}(\omega), 1 \leq i \leq p$, $\mathcal{D}(H^n) \subset \mathcal{D}(\Pi_1^p\phi^+(h_i))$,

$$\prod_{i=1}^p \phi^+(h_i)(H+\mathrm{i})^{-n} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\prod_{i=1}^p \phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-n},$$

and the map

$$\mathfrak{h}_{\mathrm{c}}(\omega)^p \ni (h_1, \dots, h_p) \mapsto \prod_{i=1}^p \phi^+(h_i)(H+\mathrm{i})^{-n} \in B(\mathcal{H})$$

is norm continuous.

Proof. The proof is very similar to [DG1, Thm. 10.2] so we will only sketch it. Properties *i)* and *ii)* are standard consequences of the fact that the asymptotic Weyl operators define a regular CCR representation (see e.g. [DG1, Sect. 2]). Property *iii)* follows from Thm. 8.1 *iv)*. It remains to prove *iv)*. For fixed p we pick $n \in \mathbb{N}$ such that $N^{p/2}(H+b)^{-n}$ is bounded. It follows that

$$\sup_{t\in\mathbb{R}} \|\mathrm{e}^{\mathrm{i}tH}\Pi_1^p\phi(h_{i,t})(H+b)^{-n}\mathrm{e}^{-\mathrm{i}tH}\| < \infty. \tag{8.10}$$

Let us first establish the existence of the strong limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_1^p\phi(h_{i,t})(H+b)^{-n}\mathrm{e}^{-\mathrm{i}tH} =: R(h_1,\dots,h_p), \ \text{ for } h_i \in \mathfrak{h}. \tag{8.11}$$

If m is large enough such that $H = H_0 + V$ on $\mathcal{D}(H^m)$, then as quadratic form on $\mathcal{D}(H^m)$ we have:

$$\mathbf{D}\Pi_1^p\phi(h_{i,t})(H+b)^{-n} = [V, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-n},$$

where the Heisenberg derivative $\mathbf{D}$ is defined in Subsect. 2.5. Next:

$$[V, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-n} = \sum_{j=1}^{p} \Pi_1^{j-1}\phi(h_{i,t})[V, i\phi(h_{j,t})]\Pi_{j+1}^p\phi(h_{i,t})(H+b)^{-n},$$

as an operator identity on $\mathcal{D}(H^m)$. The term $[V, \mathrm{i}\phi(h_t)]$ is by Prop. 2.6 a sum of Wick monomials with kernels of the form $w_{p,q}|h_t)$ or $(h_t|w_{p,q}$.

Arguing as in the proof of Thm. 8.1 we see from hypotheses *(S)* and *(Is)* for $s > 1$ that for $h \in \mathfrak{h}_0$

$$\|[V, \mathrm{i}\phi(h_t)](H+b)^{-n}\| \in L^1(\mathrm{d}t). \tag{8.12}$$

This proves the existence of the limit (8.11) for $u \in \mathcal{D}(H^m), h_i \in \mathfrak{h}_0$. The fact that the map

$$\mathfrak{h}^p \ni (h_1,\dots,h_p) \mapsto \Pi_{j=1}^p\phi(h_j)(H+b)^{-n} \in B(\mathcal{H}) \tag{8.13}$$

is norm continuous implies the existence of the limit for $u \in \mathcal{D}(H^m)$ and $h_i \in \mathfrak{h}_{\mathrm{c}}(\omega)$. The estimate (8.10) shows the existence of (8.11) for all $u \in \mathcal{H}$.

We prove now *iv)*. We recall that

$$\sup_{|s|\leq 1, \|h\|\leq C} \left\|\Big(\frac{W(sh)-\mathbb{1}}{s}\Big)(N+1)^{-1}\right\| < \infty, \tag{8.14}$$

and

$$\lim_{s\to 0}\sup_{\|h\|\leq C} \left\|\Big(\frac{W(sh)-\mathbb{1}}{s} - \mathrm{i}\phi(h)\Big)(N+1)^{-1}\right\| = 0. \tag{8.15}$$

We fix $P \in \mathbb{N}$ and M large enough so that $N^{P+1}(H+b)^{-M}$ is bounded and prove *iv)* by induction on $1 \leq p \leq P$.

We have to show that $\mathcal{D}(H^M) \subset \mathcal{D}(\Pi_1^p\phi^+(h_i))$ and that $R(h_1,\dots,h_p) = \Pi_1^p\phi^+(h_i)(H+b)^{-M}$. This amounts to show that

$$R(h_1,\dots,h_p) = \text{s-}\lim_{s\to 0}(\mathrm{i}s)^{-1}(W^+(sh_1)-\mathbb{1})\Pi_2^p\phi^+(h_i)(H+b)^{-M}.$$

Note that by the induction assumption $\mathcal{D}(H^M)\subset\mathcal{D}(\Pi_2^p\phi^+(h_i))$ and

$$\Pi_2^p\phi^+(h_i)(H+b)^{-M} = \text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}\Pi_2^p\phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+b)^{-M}. \tag{8.16}$$

Using (8.16) and the fact that $\mathrm{e}^{\mathrm{i}tH}W(h_{1,t})\mathrm{e}^{-\mathrm{i}tH}$ is uniformly bounded in t, we have:

$$\begin{aligned}&(is)^{-1}(W^+(sh_1)-\mathbb{1})\Pi_2^p\phi^+(h_i)(H+b)^{-M}\\&=\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}(\mathrm{i}s)^{-1}(W(sh_{1,t})-\mathbb{1})\Pi_2^p\phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+b)^{-M}.\end{aligned}$$

So to prove *iv)*, it suffices to check that

$$\text{s-}\lim_{s\to0}\text{s-}\lim_{t\to\infty}\mathrm{e}^{\mathrm{i}tH}R(s,t)\mathrm{e}^{-\mathrm{i}tH}=0, \tag{8.17}$$

for

$$R(s,t)=\Big(\frac{W(sh_{1,t})-\mathbb{1}}{s}-\mathrm{i}\phi(h_{1,t})\Big)\Pi_2^p\phi(h_{i,t})(H+b)^{-M}.$$

Using (8.14) and the higher order estimates, we see that $R(s,t)$ is uniformly bounded for $|s|\leq 1, t\in\mathbb{R}$, and using then (8.15) we see that $\lim_{s\to0}\sup_{t\in\mathbb{R}}\|R(s,t)u\|=0$, for $u\in\mathcal{D}(H^M)$. This shows (8.17). The norm continuity result in *iv)* follows from the norm continuity of the map (8.13). □

Finally the following theorem follows from Thm. 8.2 as in [DG1, Subsect. 10.1].

Theorem 8.3 *i) For any $h\in\mathfrak{h}_{\mathrm{c}}(\omega)$, the* asymptotic creation and annihilation operators *defined on $\mathcal{D}(a^{+\sharp}(h)):=\mathcal{D}(\phi^+(h))\cap\mathcal{D}(\phi^+(\mathrm{i}h))$ by*

$$\begin{aligned}a^{+*}(h)&:=\tfrac{1}{\sqrt2}\left(\phi^+(h)-\mathrm{i}\phi^+(\mathrm{i}h)\right),\\a^{+}(h)&:=\tfrac{1}{\sqrt2}\left(\phi^+(h)+\mathrm{i}\phi^+(\mathrm{i}h)\right).\end{aligned}$$

are closed.
ii) The operators $a^{+\sharp}$ satisfy in the sense of quadratic forms on $\mathcal{D}(a^{+\#}(h_1))\cap\mathcal{D}(a^{+\#}(h_2))$ the canonical commutation relations

$$\begin{aligned}&[a^+(h_1),a^{+*}(h_2)]=(h_1|h_2)\mathbb{1},\\&[a^+(h_2),a^+(h_1)]=[a^{+*}(h_2),a^{+*}(h_1)]=0.\end{aligned}$$

iii)

$$\mathrm{e}^{\mathrm{i}tH}a^{+\sharp}(h)\mathrm{e}^{-\mathrm{i}tH}=a^{+\sharp}(h_{-t}). \tag{8.18}$$

iv) For $p\in\mathbb{N}$, there exists $n\in\mathbb{N}$ such that for $h_i\in\mathfrak{h}_{\mathrm{c}}(\omega), 1\leq i\leq p$, $\mathcal{D}((H+\mathrm{i})^n)\subset\mathcal{D}(\Pi_1^pa^{+\sharp}(h_i))$ and

$$\Pi_1^pa^{+\sharp}(h_i)(H+b)^{-n}=\text{s-}\lim_{t\to\infty}\mathrm{e}^{\mathrm{i}tH}\Pi_1^pa^{\sharp}(h_{i,t})(H+b)^{-n}\mathrm{e}^{-\mathrm{i}tH}.$$

8.2 Asymptotic spaces and wave operators

In this subsection we recall the construction of asymptotic vacuum spaces and wave operators taken from [DG1, Subsect. 10.2] and adapted to our setup.

We define the *asymptotic vacuum space*:

$$\mathcal{K}^{+} := \{u \in \mathcal{H} \mid a^{+}(h)u = 0,\ h \in \mathfrak{h}_{\mathrm{c}}(\omega)\}.$$

The *asymptotic space* is defined as

$$\mathcal{H}^{+} := \mathcal{K}^{+} \otimes \Gamma(\mathfrak{h}_{\mathrm{c}}(\omega)).$$

The proof of the following proposition is completely analogous to [DG1, Prop. 10.4].

Proposition 8.4 *i)* $\mathcal{K}^{+}$ *is a closed* $H-$*invariant space.*
ii) $\mathcal{K}^{+}$ *is included in the domain of* $\Pi_1^p a^{+\sharp}(h_i)$ *for* $h_i \in \mathfrak{h}_{\mathrm{c}}(\omega)$.
iii)

$$\mathcal{H}_{\mathrm{pp}}(H) \subset \mathcal{K}^{+}.$$

The *asymptotic Hamiltonian* is defined by

$$H^{+} := K^{+} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega),\ \text{for } K^{+} := H\Big|_{\mathcal{K}^{+}}.$$

We also define

$$\text{(8.19)} \qquad \begin{aligned} &\Omega^{+} : \mathcal{H}^{+} \to \mathcal{H}, \\ &\Omega^{+}\psi \otimes a^{*}(h_1)\cdots a^{*}(h_p)\Omega := a^{+*}(h_1)\cdots a^{+*}(h_p)\psi, \quad h_1, \dots, h_p \in \mathfrak{h}_{\mathrm{c}}(\omega), \quad \psi \in \mathcal{K}^{+}. \end{aligned}$$

The map Ω^{+} is called the *wave operator*. The following theorem is analogous to [DG1, Thm. 10.5]

Theorem 8.5 Ω^{+} *is a unitary map from* $\mathcal{H}^{+}$ *to* $\mathcal{H}$ *such that:*

$$\begin{aligned} &a^{+\sharp}(h)\Omega^{+} = \Omega^{+} 1\!\!1 \otimes a^{\sharp}(h), \quad h \in \mathfrak{h}_{\mathrm{c}}(\omega), \\ &H\Omega^{+} = \Omega^{+}H^{+}. \end{aligned}$$

Proof. By general properties of regular CCR representations, (see [DG1, Prop. 4.2]) the operator Ω^{+} is well defined and isometric. To prove that it is unitary, it suffices to show that the CCR representation $\mathfrak{h}_{\mathrm{c}}(\omega) \ni h \mapsto W^{+}(h)$ admits a densely defined number operator (see e.g. [DG1, Subsect. 4.2]).

Let n^{+} be the quadratic form associated to the CCR representation W^{+}. Let us show that $\mathcal{D}(n^{+})$ is dense in $\mathcal{H}$. We fix $n \in \mathbb{N}$ such that

$$a^{+}(h)(H+b)^{-n} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} a(h_t) \mathrm{e}^{-\mathrm{i}tH}(H+b)^{-n}, \quad h \in \mathfrak{h}_{\mathrm{c}}(\omega).$$

For each finite dimensional space $\mathfrak{f} \subset \mathfrak{h}_{\rm c}(\omega)$ set:

$$n^+_{\mathfrak{f}}(u) = \sum_{i=1}^{\dim\mathfrak{f}} \|a^+(h_i)u\|^2,$$

for $\{h_i\}$ an orthonormal base of $\mathfrak{f}$. We have for $u \in \mathcal{D}(H^n)$:

$$\begin{array}{rl} n^+_{\mathfrak{f}}(u) & = \lim\limits_{t\to+\infty} \sum\limits_{i=1}^{\dim\mathfrak{f}} \|a(h_{i,t}){\rm e}^{-{\rm i}tH}u\|^2 \\ & = \lim\limits_{t\to+\infty} ({\rm e}^{-{\rm i}tH}u|{\rm d}\Gamma(P_{\mathfrak{f},t}){\rm e}^{-{\rm i}tH}u), \end{array}$$

if $P_{\mathfrak{f},t}$ is the orthogonal projection on ${\rm e}^{-{\rm i}t\omega}\mathfrak{f}$. But ${\rm d}\Gamma(P_{\mathfrak{f},t}) \leq N$, so

$$n^+_{\mathfrak{f}}(u) \leq \sup_t \|N^{\frac{1}{2}}{\rm e}^{-{\rm i}tH}u\|^2 \leq C\|(H+b)^p u\|^2,$$

for some p, by the higher order estimates. Therefore

$$\mathcal{D}(H^p) \subset \mathcal{D}(n^+),$$

which for p large enough, which implies that $\mathcal{D}(n^+)$ is densely defined. □

8.3 Extended wave operator

In Subsect. 2.4 we introduced the scattering Hilbert space $\mathcal{H}^{\rm scatt} \subset \mathcal{H}^{\rm ext}$. Clearly $\mathcal{H}^{\rm scatt}$ is preserved by $H^{\rm ext}$. We see that $\mathcal{H}^+$ is a subspace of $\mathcal{H}^{\rm scatt}$ and

$$H^+ = H^{\rm ext}_{|\mathcal{H}^+}.$$

We define the *extended wave operator* $\Omega^{\rm ext,+} : \mathcal{D}(\Omega^{\rm ext,+}) \to \mathcal{H}$ by:

$$\mathcal{D}(\Omega^{\rm ext,+}) = D(H^\infty) \otimes \Gamma_{\rm fin}(\mathfrak{h}_{\rm c}(\omega)),$$

and

$$\Omega^{\rm ext,+}\psi \otimes a^*(h_1)\cdots a^*(h_p)\Omega := a^{*+}(h_1)\cdots a^{*+}(h_p)\psi, \quad \psi \in D(H^\infty), \quad h_i \in \mathfrak{h}_{\rm c}(\omega).$$

Note that $\Omega^{\rm ext,+} : \mathcal{H}^{\rm scatt} \to \mathcal{H}$ is unbounded and:

$$\Omega^+ = \Omega^{\rm ext,+}_{|\mathcal{H}^+}.$$

Considering Ω^+ as a partial isometry equal to 0 on $\mathcal{H}^{\rm scatt} \ominus \mathcal{H}^+$, we can rewrite this identity as:

$$\Omega^+ = \Omega^{\rm ext,+}1\!\!1_{\mathcal{H}^+}, \qquad (8.20)$$

where $1\!\!1_{\mathcal{H}^+}$ denotes the projection onto $\mathcal{H}^+$ inside the space $\mathcal{H}^{\rm scatt}$.

Moreover using Thm. 8.3 *iv)*, we obtain as in [DG1, Thm. 10.7] the following alternative expression for $\Omega^{\rm ext,+}$.

Theorem 8.6 *i) Let $u \in \mathcal{D}(\Omega^{\mathrm{ext},+})$. Then the limit*

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} u$$

exists and equals $\Omega^{\mathrm{ext},+}u$.
ii) Let $\chi \in C_0^\infty(\mathbb{R})$. Then $\mathrm{Ran}\chi(H^{\mathrm{ext}}) \subset \mathcal{D}(\Omega^{\mathrm{ext},+})$, $I\chi(H^{\mathrm{ext}})$ and $\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})$ are bounded operators and

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} \chi(H^{\mathrm{ext}}) = \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}). \tag{8.21}$$

9 Propagation estimates

In this section we consider an abstract QFT Hamiltonian H and fix a weight operator $\langle x\rangle$. We will prove various propagation estimates for H. The proof of the phase-space estimates will be more involved than in [DG1], [DG2]. In fact the operator playing the role of the acceleration $[\omega, \mathrm{i}[\omega, \mathrm{i}\langle x\rangle]]$ vanishes in the situation considered in these papers.

9.1 Maximal velocity estimates

The following proposition shows that bosons cannot propagate in the region $\langle x\rangle > v_{\max}t$ where

$$v_{\max} := \|[\omega, \mathrm{i}\langle x\rangle]\|.$$

Proposition 9.1 *Assume hypotheses (G1), (Is) for $s > 1$. Let $\chi \in C_0^\infty(\mathbb{R})$. Then for $R' > R > v_{\max}$, one has:*

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(1\!\!1_{[R,R']}(\frac{|x|}{t})\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u\right\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. The proof is almost identical to [DG1, Prop. 11.2] so we will only sketch it. We fix $G \in C_0^\infty(]v_{\max}, +\infty[)$ with $G \geq 1\!\!1_{[R,R']}$ and set $F(s) = \int_s^{+\infty} G^2(t)\mathrm{d}t$. We use the propagation observable $\Phi(t) = \chi(H)\mathrm{d}\Gamma(F(\frac{\langle x\rangle}{t}))\chi(H)$. We use that

$$\begin{aligned}\mathbf{d}_0 F(\tfrac{\langle x\rangle}{t}) &= t^{-1}G(\tfrac{\langle x\rangle}{t})([\omega, \mathrm{i}\langle x\rangle] - \tfrac{\langle x\rangle}{t})G(\tfrac{\langle x\rangle}{t}) + O(t^{-2})\\ &\leq -\tfrac{C_0}{t}G^2(\tfrac{\langle x\rangle}{t}) + O(t^{-2})\end{aligned}$$

by Lemma 3.3. The term $\chi(H)[V, \mathrm{i}\mathrm{d}\Gamma(F(\frac{\langle x\rangle}{t}))]\chi(H)$ is $O(t^{-s})$ in norm by hypothesis *(Is)*, Lemma 2.5 and the higher order estimates. □

9.2 Phase space propagation estimates

Set

$$v := [\omega, \mathrm{i}\langle x\rangle],$$

and recall from hypothesis *(G2)* that

$$[\omega, \mathrm{i}v] = \gamma^2 + r_{-1-\epsilon},$$

where $\gamma \in S^{-\frac{1}{2}}_{\epsilon,(1)}$, $r_{-1-\epsilon} \in S^{-1-\epsilon}_{(0)}$ for some $\epsilon > 0$.

We will show that for free bosons the instantaneous velocity v and the average velocity $\frac{\langle x\rangle}{t}$ converge to each other when $t \to \pm\infty$.

Proposition 9.2 *Assume (G1), (G2) and (Is) for $s > 1$ and let $\chi \in C_0^\infty(\mathbb{R})$ and $0 < c_0 < c_1$. Then*

$$i) \quad \int_1^{+\infty} \|\mathrm{d}\Gamma\left((\frac{\langle x\rangle}{t} - v)1\!\!1_{[c_0,c_1]}(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v)\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2 \mathrm{d}t \leq C\|u\|^2,$$

$$ii) \quad \int_1^{+\infty} \|\mathrm{d}\Gamma\left(\gamma 1\!\!1_{[c_0,c_1]}(\frac{\langle x\rangle}{t})\gamma\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. We follow the proof of [DG1, Prop. 11.3], [DG2, Prop. 6.2] with some modifications due to our abstract setting.

It clearly suffices to prove Prop. 9.2 for $c_1 > v_{\max} + 1$, which we will assume in what follows. We fix a function $F \in C^\infty(\mathbb{R})$, with $F, F' \geq 0$, $F(s) = 0$ for $s \leq c_0/2$, $F(s), F'(s) \geq d_1 > 0$ for $s \in [c_0, c_1]$. We set

$$R_0(s) = \int_0^s F^2(t)\mathrm{d}t,$$

so that $R_0(s) = 0$ for $s \leq c_0/2$, $R_0'(s), R_0''(s) \geq d_2 > 0$ for $s \in [c_0, c_1]$. Finally we fix another function $G \in C^\infty(\mathbb{R})$ with $G(s) = 1$ for $s \leq c_1 + 1$, $G(s) = 0$ for $s \geq c_1 + 2$, and set:

$$R(s) := G(s)R_0(s).$$

The function R belongs to $C_0^\infty(\mathbb{R})$ and satisfies:

$$(9.1) \qquad R(s) = 0 \text{ in } [0, c_0/2], \quad R'(s) \geq d_3 1\!\!1_{[c_0,c_1]}(s) + \chi_1(s), \quad R''(s) \geq d_3 1\!\!1_{[c_0,c_1]}(s) + \chi_2(s),$$

for $\chi_1, \chi_2 \in C_0^\infty(]v_{\max}, +\infty[)$ and $d_3 > 0$. We set

$$b(t) := R(\frac{\langle x\rangle}{t}) - \frac{1}{2}\left(R'(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \text{h.c.}\right),$$

which satisfies $b(t) \in O(1)$ and use the propagation observable

$$\Phi(t) = \chi(H)\mathrm{d}\Gamma(b(t))\chi(H).$$

Using Lemma 3.3 we obtain that:

$$(9.2) \qquad \partial_t b(t) = \frac{1}{t}\left(R''(\frac{\langle x\rangle}{t})\frac{\langle x\rangle^2}{t^2} - \frac{1}{2}\frac{\langle x\rangle}{t}R''(\frac{\langle x\rangle}{t})v - \frac{1}{2}vR''(\frac{\langle x\rangle}{t})\frac{\langle x\rangle}{t}\right) + O(t^{-2}),$$

and

$$(9.3) \qquad \begin{aligned}[\omega, \mathrm{i}b(t)] = {} & \tfrac{1}{t}\left(vR''(\tfrac{\langle x\rangle}{t})v - \tfrac{1}{2}\tfrac{\langle x\rangle}{t}R''(\tfrac{\langle x\rangle}{t})v - \tfrac{1}{2}vR''(\tfrac{\langle x\rangle}{t})\tfrac{\langle x\rangle}{t}\right) \\ & + \tfrac{1}{2}\left(R'(\tfrac{\langle x\rangle}{t})[\omega, \mathrm{i}v] + \text{h.c.}\right) + O(t^{-2}).\end{aligned}$$

Adding (9.2) and (9.3) we obtain:

$$\mathbf{d}_0 b(t) = \frac{1}{t}(\frac{\langle x\rangle}{t} - v)R"(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \frac{1}{2}\left(R'(\frac{\langle x\rangle}{t})[\omega, \mathrm{i}v] + \text{h.c.}\right) + O(t^{-2}).$$

By hypothesis *(G2)*, we have:

$$[\omega,\ iv] = \gamma^2 + r_{-1-\epsilon},$$

for $\gamma \in S^{-\frac{1}{2}}_{\epsilon,(1)}$, $r_{-1-\epsilon} \in S^{-1-\epsilon}_{(0)}$. Since $0 \notin \operatorname{supp} R'$, we know by Lemma 2.3 that

$$R'(\frac{\langle x\rangle}{t})r_{-1-\epsilon} \in O(t^{-1-\epsilon}).$$

Using that $\gamma \in S^{-\frac{1}{2}}_{\epsilon,(1)}$, we get by Lemma 3.3 *vii)* that:

$$\frac{1}{2}\left(R'(\frac{\langle x\rangle}{t})\gamma^2 + \text{h.c.}\right) = \gamma R'(\frac{\langle x\rangle}{t})\gamma + O(t^{-3/2+\epsilon}).$$

Finally this gives:

$$\mathbf{d}_0 b(t) = \frac{1}{t}(\frac{\langle x\rangle}{t} - v)R"(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \gamma R'(\frac{\langle x\rangle}{t})\gamma + O(t^{-1-\epsilon_1}),$$

for some $\epsilon_1 > 0$.

We note that R' and $R"$ are positive, except for the error terms due to χ_1, χ_2 in (9.1). To handle these terms we pick $\chi_3 \in C_0^\infty(]v_{\max}, +\infty[)$ such that $\chi_3\chi_i = \chi_i$, $i = 1, 2$. Then $[\frac{\langle x\rangle}{t} - v, \chi_3(\frac{\langle x\rangle}{t})] \in O(t^{-1})$ and $[\gamma, \chi_3(\frac{\langle x\rangle}{t})] \in O(t^{-3/2+\epsilon})$ by Lemma 3.3 *i)* and *vii)*. This yields:

$$\begin{aligned}\pm\tfrac{1}{t}(\tfrac{\langle x\rangle}{t} - v)\chi_2(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t} - v) = &\ \pm\tfrac{1}{t}\chi_3(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t} - v)\chi_2(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t} - v)\chi_3(\tfrac{\langle x\rangle}{t}) + O(t^{-2}) \\ \leq &\ \tfrac{C}{t}\chi_3^2(\tfrac{\langle x\rangle}{t}) + O(t^{-2}),\end{aligned}$$

$$\begin{aligned}\pm\gamma\chi_1(\tfrac{\langle x\rangle}{t})\gamma = &\ \pm\chi_3(\tfrac{\langle x\rangle}{t})\gamma\chi_1(\tfrac{\langle x\rangle}{t})\gamma\chi_3(\tfrac{\langle x\rangle}{t}) + O(t^{-3/2+\epsilon}) \\ \leq &\ \tfrac{C}{t}\chi_3^2(\tfrac{\langle x\rangle}{t}) + O(t^{-3/2+\epsilon}),\end{aligned}$$

using that $\gamma \in S^{-\frac{1}{2}}_{(0)}$ and Lemma 2.3. Using again (9.1), we finally get:

$$\begin{aligned}\mathbf{d}_0 b(t) \geq &\ \tfrac{C_1}{t}(\tfrac{\langle x\rangle}{t} - v)\mathbb{1}_{[c_0,c_1]}(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t} - v) + C_1\gamma\mathbb{1}_{[c_0,c_1]}(\tfrac{\langle x\rangle}{t})\gamma \\ &-\tfrac{C_2}{t}\chi_3^2(\tfrac{\langle x\rangle}{t}) + O(t^{-1-\epsilon_1}),\end{aligned} \tag{9.4}$$

for some $C_1, \epsilon_1 > 0$.

To handle the commutator $[V, \mathrm{id}\Gamma(b(t))]$ we note that using Lemma 3.3 *iv)* and the fact that $0 \notin \operatorname{supp} R$, we have

$$b(t) = \mathbb{1}_{[\epsilon,+\infty[}(\frac{\langle x\rangle}{t})b(t)\mathbb{1}_{[\epsilon,+\infty[}(\frac{\langle x\rangle}{t}) + O(t^{-2})$$

for some $\epsilon > 0$. Using also hypothesis *(Is)* for $s > 1$, this implies that if $V = \mathrm{Wick}(w)$ then $|||\mathrm{d}\Gamma(b(t))w||| \in L^1(\mathrm{d}t)$. Using the higher order estimates this implies that

$$\|\chi(H)[V, \mathrm{id}\Gamma(b(t))\chi(H)]\| \in L^1(\mathrm{d}t).$$

The rest of the proof is as in [DG1, Prop. 11.3]. □

9.3 Improved phase space propagation estimates

In this subsection we will prove improved propagation estimates. We will use the following lemma which is an analog of [DG2, Lemma 6.4] in our abstract setting. Its proof will be given in the Appendix.

Lemma 9.3 *Assume (H1), (G1), (G2) and set* $v = [\omega, \mathrm{i}\langle x\rangle]$ *which is a bounded operator on* $\mathfrak{h}$. *Let* $c = (\frac{\langle x\rangle}{t} - v)^2 + t^{-\delta}$, $\delta > 0$ *and set* $\epsilon_0 = \inf(\delta, 1-\delta/2)$. *If* $J \in C_0^\infty(\mathbb{R})$ *then:*

$$i)\quad J(\frac{\langle x\rangle}{t})c^{\frac{1}{2}} \in O(1),$$

$$ii)\quad [c^{\frac{1}{2}}, J(\frac{\langle x\rangle}{t})] \in O(t^{-1+\delta/2}).$$

If $J \in C_0^\infty(\mathbb{R}\backslash\{0\})$ *then for* δ *small enough:*

$$iii)\quad \begin{array}{rl} & J(\frac{\langle x\rangle}{t})\mathbf{d}_0 c^{\frac{1}{2}} J(\frac{\langle x\rangle}{t}) \\ = & -\frac{1}{t}J(\frac{\langle x\rangle}{t})c^{\frac{1}{2}}J(\frac{\langle x\rangle}{t}) + \gamma J(\frac{\langle x\rangle}{t})M(t)J(\frac{\langle x\rangle}{t})\gamma + O(t^{-1-\epsilon_1}), \end{array}$$

where $\epsilon_1 > 0$ *and* $M(t) \in O(1)$.
If $J, J_1 \in C_0^\infty(\mathbb{R})$ *and* $J_1 \equiv 1$ *on* $\operatorname{supp} J$, *then:*

$$iv)\quad |J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \mathrm{h.c.}| \leq CJ_1(\frac{\langle x\rangle}{t})c^{\frac{1}{2}}J_1(\frac{\langle x\rangle}{t}) + O(t^{-\epsilon_0}).$$

If $J, J_1, J_2 \in C_0^\infty(\mathbb{R})$ *with* $J_2 \equiv 1$ *on* $\operatorname{supp} J$ *and* $\operatorname{supp} J_1$, *then:*

$$v)\quad \pm(J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v)c^{\frac{1}{2}}J_1(\frac{\langle x\rangle}{t}) + \mathrm{h.c.}) \leq C(\frac{\langle x\rangle}{t} - v)J_2^2(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + O(t^{-\epsilon_0}).$$

Proposition 9.4 *Assume (G1), (G2), (Is) for* $s > 1$. *Let* $J \in C_0^\infty(]c_0, c_1[)$ *for* $0 < c_0 < c_1$ *and* $\chi \in C_0^\infty(\mathbb{R})$. *Then:*

$$\int_1^{+\infty} \|\mathrm{d}\Gamma(\left|J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \mathrm{h.c.}\right|)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2\frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. We fix $J_1 \in C_0^\infty(]c_0, c_1[)$ with $J_1 \equiv 1$ on $\operatorname{supp} J$ and set

$$b(t) = J_1(\frac{\langle x\rangle}{t})c^{\frac{1}{2}}J_1(\frac{\langle x\rangle}{t}),\ \text{for } c = (\frac{\langle x\rangle}{t} - v)^2 + t^{-\delta},$$

and $\delta > 0$ will be chosen small enough later. We will use the propagation observable

$$\Phi(t) = \chi(H)\mathrm{d}\Gamma(b(t))\chi(H).$$

Note that by Lemma 9.3 *i)* and the higher order estimates $b(t)$, $\Phi(t) \in O(1)$. We first note that

$$\chi(H)[V, \mathrm{i}\mathrm{d}\Gamma(b(t))]\chi(H) \in O(t^{-s}),$$

using hypothesis *(Is)* and Lemma 9.3 *i)*. Next

$$\mathbf{D}_0\mathrm{d}\Gamma(b(t)) = \mathrm{d}\Gamma(\mathbf{d}_0 b(t)),$$

$$\mathbf{d}_0 b(t) = \left(\mathbf{d}_0 J_1(\frac{\langle x\rangle}{t})\right) c^{\frac{1}{2}} J_1(\frac{\langle x\rangle}{t}) + \text{h.c.} + J_1(\frac{\langle x\rangle}{t})(\mathbf{d}_0 c^{\frac{1}{2}}) J_1(\frac{\langle x\rangle}{t}).$$

By Lemma 9.3 *iii)* we know that choosing δ small enough:

$$\begin{array}{rl} & J_1(\frac{\langle x\rangle}{t})(\mathbf{d}_0 c^{\frac{1}{2}}) J_1 \frac{\langle x\rangle}{t}) \\ = & -J_1(\frac{\langle x\rangle}{t})\frac{c^{\frac{1}{2}}}{t} J_1(\frac{\langle x\rangle}{t}) + \gamma J_1(\frac{\langle x\rangle}{t}) M(t) J_1(\frac{\langle x\rangle}{t})\gamma + O(t^{-1-\epsilon_1}), \end{array}$$

for some $\epsilon_1 > 0$ and $M(t) \in O(1)$. By Lemma 9.3 *iv)* we get then that

$$\begin{array}{rl} & -J_1(\frac{\langle x\rangle}{t})(\mathbf{d}_0 c^{\frac{1}{2}}) J_1(\frac{\langle x\rangle}{t}) \\ \geq & \frac{C}{t}\left|J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \text{h.c.}\right| - C\gamma J_1^2(\frac{\langle x\rangle}{t})\gamma - Ct^{-1-\epsilon_1} \end{array}$$

for some $\epsilon_1 > 0$. Next by Lemma 3.3:

$$\mathbf{d}_0 J_1(\frac{\langle x\rangle}{t}) = -\frac{1}{2t} J_1'(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + O(t^{-2}),$$

which by Lemma 9.3 *v)* gives for $J_2 \in C_0^\infty(]c_0, c_1[)$ and $J_2 \equiv 1$ on $\operatorname{supp} J_1$:

$$\left(\mathbf{d}_0 J_1(\frac{\langle x\rangle}{t})\right) c^{\frac{1}{2}} J_1(\frac{\langle x\rangle}{t}) + \text{h.c.} \geq -\frac{C}{t}(\frac{\langle x\rangle}{t} - v) J_2^2(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + O(t^{-1-\epsilon_1})$$

for some $\epsilon_1 > 0$. Collecting the various estimates, we obtain finally

$$-\mathbf{D}\Phi(t) \geq \frac{C}{t}\chi(H)\mathrm{d}\Gamma(\left|J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) + \text{h.c.}\right|)\chi(H) - CR_1(t) - CR_2(t) + O(t^{-1-\epsilon_1}),$$

where

$$R_1(t) = \chi(H)\mathrm{d}\Gamma(\gamma J_1^2(\frac{\langle x\rangle}{t})\gamma)\chi(H), \quad R_2(t) = \frac{1}{t}\chi(H)\mathrm{d}\Gamma((\frac{\langle x\rangle}{t} - v) J_2^2(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v))\chi(H)$$

are integrable along the evolution by Prop. 9.2. We can then complete the proof as in [DG2, Prop. 6.3]. □

9.4 Minimal velocity estimate

In this subsection we prove the minimal velocity estimate. It says that for states with energy away from thresholds and eigenvalues of H, at least one boson should escape to infinity. We recall that as in Subsect. 7.4, $A = \mathrm{d}\Gamma(a)$.

Lemma 9.5 *Let H be an abstract QFT Hamiltonian. Assume (G4). Let $k \in \mathbb{N}$, $m = 1, 2$ and $\chi \in C_0^\infty(\mathbb{R})$. Then there exists C such that for any $\epsilon > 0$ and $q \in C_0^\infty([-2\epsilon, 2\epsilon])$ with $0 \leq q \leq 1$ one has:*

$$\|N^k \frac{A^m}{t^m}\Gamma(q^t)\chi(H)\| \leq C\epsilon^m.$$

where $q^t = q(\frac{\langle x\rangle}{t})$.

Proof. Applying Prop. 2.4 *ii)* we get

$$(9.5)\qquad (\mathrm{d}\Gamma(a))^{2m} \leq N^{2m-1}\mathrm{d}\Gamma(a^{2m}).$$

Next

$$(9.6)\qquad \Gamma(q^t)\mathrm{d}\Gamma(a^{2m})\Gamma(q^t) = \mathrm{d}\Gamma((q^t)^2, q^t a^{2m} q^t) \leq \mathrm{d}\Gamma(q^t a^{2m} q^t),$$

by Prop. 2.4 *iv)*. We write using *(G4)*:

$$q^t a^{2m} q^t = G^t \langle x\rangle^{-m} a^{2m} \langle x\rangle^{-m} G^t \leq C t^{2m} (G^t)^2, \quad m = 1, 2,$$

for $G^t = G(\frac{\langle x\rangle}{t})$ and $G(s) = s^m q(s)$. Using that $|G(s)| \leq C\epsilon^m$ we obtain that

$$(9.7)\qquad q^t a^{2m} q^t \leq C\epsilon^{2m} t^{2m}, \quad m = 1, 2.$$

From (9.7) and (9.5), (9.6) we obtain

$$(9.8)\qquad \Gamma(q^t) N^{2k} \mathrm{d}\Gamma(a)^{2m} \Gamma(q^t) \leq C\epsilon^{2m} t^{2m} N^{2k+2m}.$$

This implies the Lemma using the higher order estimates. □

Proposition 9.6 *Let H be an abstract QFT Hamiltonian. Assume hypotheses (Gi), for $1 \leq i \leq 5$, (M1), (M2), (Is) for $s > 1$. Let $\chi \in C_0^\infty(\mathbb{R})$ be supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. Then there exists $\epsilon > 0$ such that:*

$$\int_1^{+\infty} \left\| \Gamma\left(1\!\!1_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right) \chi(H) \mathrm{e}^{-\mathrm{i}tH} u \right\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. Let us first prove the proposition for χ supported near an energy level $\lambda \in \mathbb{R}\backslash\tau\cup\sigma_{\mathrm{pp}}(H)$. By Thm. 7.8, we can find $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 near λ such that for some $c_0 > 0$:

$$(9.9)\qquad \chi(H)[H, \mathrm{i}A]_0 \chi(H) \geq c_0 \chi^2(H).$$

Let $\epsilon > 0$ be a parameter which will be fixed later. Let $q \in C_0^\infty(|s| \leq 2\epsilon)$, $0 \leq q \leq 1$, $q = 1$ near $\{|s| \leq \epsilon\}$ and let $q^t = q(\frac{\langle x\rangle}{t})$.

We use the propagation observable

$$\Phi(t) := \chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H).$$

We fix cutoff functions $\tilde{q} \in C_0^\infty(\mathbb{R})$, $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that

$$\mathrm{supp}\, \tilde{q} \subset [-4\epsilon, 4\epsilon], \ 0 \leq \tilde{q} \leq 1, \ \tilde{q}q = q, \ \tilde{\chi}\chi = \chi.$$

By Lemma 9.5 for $m = 1$ the observable $\Phi(t)$ is uniformly bounded. We have:

$$(9.10)\qquad \begin{array}{rl} \mathbf{D}\Phi(t) & = \chi(H)\mathrm{d}\Gamma(q^t, \mathbf{d}_0 q^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{h.c.} \\ & +\chi(H)[V, \mathrm{i}\Gamma(q^t)]\frac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{h.c.} \\ & +t^{-1}\chi(H)\Gamma(q^t)[H, \mathrm{i}A]\Gamma(q^t)\chi(H) \\ & -t^{-1}\chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H) \\ & =: R_1(t) + R_2(t) + R_3(t) + R_4(t). \end{array}$$

We have used the fact, shown in the proof of Lemma 6.2, that $\Gamma(q^t)$ preserves $\mathcal{D}(H_0)$ and $\mathcal{D}(N^n)$ to expand the commutator $[H, \mathrm{i}\Phi(t)]$ in (9.10).

Let us first estimate $R_2(t)$. By Prop. 2.7 and hypothesis *(Is)*

$$[V, \mathrm{i}\Gamma(q^t)] \in (N+1)^n O_N(t^{-s}),\ s > 1,$$

for some n. Therefore by the higher order estimates and Lemma 9.5 for $m = 1$:

$$R_2(t) \in O(t^{-s}),\ s > 1. \tag{9.11}$$

We estimate now $R_1(t)$. By Lemma 3.3 *i)*:

$$\mathbf{d}_0 q^t = -\frac{1}{2t}\left((\frac{\langle x\rangle}{t} - v)q'(\frac{\langle x\rangle}{t}) + \text{h.c.}\right) + r^t =: \frac{1}{t}g^t + r^t,$$

where $r^t \in O(t^{-2})$. By the higher order estimates $\|\chi(H)d\Gamma(q^t, r^t)\| \in O(t^{-2})$, which using Lemma 9.5 for $m = 1$ yields

$$\|\chi(H)d\Gamma(q^t, r^t)\frac{A}{t}\Gamma(q^t)\chi(H)\| \in O(t^{-2}).$$

Then we set

$$B_1 := \chi(H)\mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}}, \quad B_2^* := (N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(q^t)\chi(H),$$

and use the inequality

$$\begin{aligned}\chi(H)\mathrm{d}\Gamma(q^t, g^t)\tfrac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.} \ &= B_1B_2^* + B_2B_1^* \\ &\geq -B_1B_1^* - B_2B_2^*.\end{aligned} \tag{9.12}$$

We can write:

$$\begin{aligned}-B_2B_2^* \ &= -\chi(H)\tilde\chi(H)\Gamma(q^t)\Gamma(\tilde q^t)\tfrac{A^2}{t^2}(N+1)\Gamma(\tilde q^t)\Gamma(q^t)\tilde\chi(H)\chi(H) \\ &= \chi(H)\Gamma(q^t)\tilde\chi(H)\Gamma(\tilde q^t)\tfrac{A^2}{t^2}(N+1)\Gamma(\tilde q^t)\tilde\chi(H)\Gamma(q^t)\chi(H) + O(t^{-1}) \\ &\geq -\epsilon^2 C_1\chi(H)\Gamma^2(q^t)\chi(H) + O(t^{-1}).\end{aligned} \tag{9.13}$$

In the first step we use that $[\tilde\chi(H), \Gamma(q^t)] \in O(t^{-1})$ by Lemma 6.2 and that $\frac{A^2}{t^2}(N+1)\Gamma(q^t)\chi(H) \in O(1)$ by Lemma 9.5 for $m = 2$. In the second step we use the following estimate analogous to (9.8):

$$\tilde\chi(H)\Gamma(\tilde q^t)\frac{A^2}{t^2}(N+1)\Gamma(\tilde q^t)\tilde\chi(H) \leq C_1\epsilon^2.$$

Next we use Prop. 2.4 *iv)* to obtain:

$$\begin{aligned}B_1^*B_1 = \ &\chi(H)\mathrm{d}\Gamma(q^t, g^t)^2(N+1)^{-1}\chi(H) \\ \leq \ &\chi(H)\mathrm{d}\Gamma((g^t)^2)\chi(H).\end{aligned}$$

By Prop. 9.2, we obtain

$$\int_1^{+\infty} \|B_1\mathrm{e}^{-\mathrm{i}tH}u\|^2\frac{\mathrm{d}t}{t} \leq C\|u\|^2. \tag{9.14}$$

To handle $R_3(t)$, we write using Lemma 6.2:

$$
\begin{aligned}
R_3(t) &= t^{-1}\Gamma(q^t)\chi(H)[H,\mathrm{i}A]\chi(H)\Gamma(q^t) + O(t^{-2}) \\
&\geq C_0 t^{-1}\Gamma(q^t)\chi^2(H)\Gamma(q^t) - Ct^{-2} \\
&\geq C_0 t^{-1}\chi(H)\Gamma^2(q^t)\chi(H) - Ct^{-2}.
\end{aligned} \tag{9.15}
$$

It remains to estimate $R_4(t)$. We write using Lemma 9.5:

$$
\begin{aligned}
R_4(t) &= -t^{-1}\chi(H)\Gamma(q^t)\tfrac{A}{t}\Gamma(q^t)\chi(H) \\
&= -t^{-1}\chi(H)\Gamma(q^t)\tilde{\chi}(H)\Gamma(\tilde{q}^t)\tfrac{A}{t}\Gamma(\tilde{q}^t)\tilde{\chi}(H)\Gamma(q^t)\chi(H) + O(t^{-2}) \\
&\geq -\epsilon C_2 t^{-1}\chi(H)\Gamma(q^t)^2\chi(H) + O(t^{-2}).
\end{aligned} \tag{9.16}
$$

Collecting (9.13), (9.15) and (9.16), we obtain

$$
\begin{aligned}
&-t^{-1}B_2^*(t)B_2(t) + R_3(t) + R_4(t) \\
&\geq (-\epsilon^2 C_1 + C_0 - \epsilon C_2)t^{-1}\chi(H)\Gamma(q^t)^2\chi(H) + O(t^{-2}).
\end{aligned} \tag{9.17}
$$

We pick now ϵ small enough so that $\tilde{C}_0 = -\epsilon^2 C_1 + C_0 - \epsilon C_2 > 0$. Using (9.11), (9.14) and (9.17) we conclude that

$$
\mathbf{D}\Phi(t) \geq \frac{\tilde{C}_0}{t}\chi(H)\Gamma^2(q^t)\chi(H) - R(t) - Ct^{-s},\ s > 1.
$$

where $R(t)$ is integrable along the evolution. We finish the proof as in [DG1, Prop. 11.5]. □

10 Asymptotic Completeness

In this section we prove the asymptotic completeness of wave operators. The first step is the *geometric asymptotic completeness*, identifying the asymptotic vacua with the subspace of states living at large times t in $\langle x\rangle \leq \epsilon t$ for arbitrarily small $\epsilon > 0$. In the second step, using the minimal velocity estimate, one shows that these states have to be bound states of H.

10.1 Existence of asymptotic localizations

Theorem 10.1 *Let H be an abstract QFT Hamiltonian. Assume hypotheses (G1), (G2), (Is) for $s > 1$. Let $q \in C_0^\infty(\mathbb{R})$, $0 \leq q \leq 1$, $q = 1$ on a neighborhood of zero. Set $q^t = q(\frac{\langle x\rangle}{t})$. Then there exists*

$$
\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH} =: \Gamma^+(q). \tag{10.1}
$$

We have

$$
\Gamma^+(q\tilde{q}) = \Gamma^+(q)\Gamma^+(\tilde{q}), \tag{10.2}
$$

$$
0 \leq \Gamma^+(q) \leq \Gamma^+(\tilde{q}) \leq \mathbb{1},\ \textit{if}\ 0 \leq q \leq \tilde{q} \leq 1, \tag{10.3}
$$

$$
[H, \Gamma^+(q)] = 0. \tag{10.4}
$$

The proof is completely similar to the proof of [DG1, Thm. 12.1], using Prop. 9.4. An analogous result is true for the free Hamiltonian H_0.

Proposition 10.2 *Assume hypotheses (H1), (G1), (G2). Let $q \in C^\infty(\mathbb{R})$, $0 \leq q \leq 1$, $q \equiv 1$ near ∞. Then there exists*

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_0}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH_0} =: \Gamma^+_{\mathrm{free}}(q). \tag{10.5}$$

Moreover if additionally $q \equiv 0$ near 0 then:

$$\Gamma^+_{\mathrm{free}}(q) = \Gamma^+_{\mathrm{free}}(q)\Gamma(1\!\!1_{\mathrm{c}}(\omega)), \tag{10.6}$$

where $1\!\!1_{\mathrm{c}}(\omega)$ is the projection on the continuous spectral subspace of ω.

Proof. By density it suffices to the existence of the limit (10.5) on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$.

Using the identity (see e.g. [DG1, Lemma 3.4]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(r_t) = \mathrm{d}\Gamma(r_t, r_t'),$$

we obtain for $a, b \in B(\mathfrak{h})$:

$$\Gamma(a) - \Gamma(b) = \int_0^1 \mathrm{d}\Gamma(ta + (1-t)b, a-b)\mathrm{d}t.$$

It follows then from Prop.2.4 that

$$B(\mathfrak{h}) \ni a \mapsto \Gamma(a)(N+1)^{-1} \in B(\Gamma(\mathfrak{h}))$$

is norm continuous. This implies that it suffices to prove the existence of the limit for $q \in C^\infty(\mathbb{R})$ $0 \leq q \leq 1$ and $q \equiv 1$ near ∞, $q \equiv Cst$ near 0. In particular $q' \in C_0^\infty(\mathbb{R}\backslash\{0\})$. We can then repeat the proof of [DG1, Thm. 12.1], noting that the only place where $q \equiv 1$ near 0 is needed is to control the commutator $[V, \mathrm{i}\Gamma(q^t)]$ which is absent in our case. This proves (10.5). Restricting (10.5) to the one-particle sector we obtain the existence of

$$q^+ := \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\omega}q^t\mathrm{e}^{-\mathrm{i}t\omega}. \tag{10.7}$$

By Lemma 3.3 *i)*, we see that $[\chi(\omega), q^+] = 0$ for each $\chi \in C_0^\infty(\mathbb{R})$ hence q^+ commutes with ω.

If $q \equiv 0$ near 0 then clearly

$$1\!\!1_{\mathrm{pp}}(\omega)q^+ = q^+1\!\!1_{\mathrm{pp}}(\omega) = 0, \text{ and hence } q^+ = q^+1\!\!1_{\mathrm{c}}(\omega) = 1\!\!1_{\mathrm{c}}(\omega)q^+.$$

We note now that

$$\Gamma^+_{\mathrm{free}}(q) = \Gamma(q^+),$$

which implies (10.6). □

10.2 The projection P_0^+.

Theorem 10.3 *Let H be an abstract QFT Hamiltonian. Assume hypotheses (G1), (G2), (Is) for $s > 1$. Let $\{q_n\} \in C_0^\infty(\mathbb{R})$ be a decreasing sequence of functions such that $0 \le q_n \le 1$, $q_n = \equiv 1$ on a neighborhood of 0 and $\cap_{n=1}^\infty \mathrm{supp}\, q_n = \{0\}$. Then*

$$P_0^+ := \text{s-}\lim_{n\to\infty} \Gamma^+(q_n) \text{ exists.} \tag{10.8}$$

P_0^+ is an orthogonal projection independent on the choice of the sequence $\{q_n\}$. Moreover:

$$[H, P_0^+] = 0.$$

Moreover if (S) holds:

$$\mathrm{Ran} P_0^+ \subset \mathcal{K}^+. \tag{10.9}$$

The range of P_0^+ can be interpreted as the space of states asymptotically containing no bosons away from the origin.
Proof. The proof is analogous to [DG1, Thm. 12.3]. We will only detail (10.9). Let $n \in \mathbb{N}$ such that $\mathcal{D}(H^n) \subset \mathcal{D}(a^{+*}(h))$ for all $h \in \mathfrak{h}_c(\omega)$. We will show that for $u \in \mathrm{Ran} P_0^+$:

$$(H+b)^{-n} a^+(h) u = 0, \quad h \in \mathfrak{h}_c(\omega).$$

Since $h \mapsto (H+b)^{-n} a^+(h)$ is norm continuous by Thm. 8.2, we can assume that $h \in \mathfrak{h}_0$. By *(S)* and the fact that $u \in \mathrm{Ran} P_0^+$ we can choose $q \in C_0^\infty(\mathbb{R})$ with $0 \le q \le 1$ such that:

$$u = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} \Gamma(q^t) \mathrm{e}^{-\mathrm{i}tH} u, \quad q^t h_t \in o(1).$$

Then:

$$\begin{aligned}(H+b)^{-n} a^+(h) u = &\ \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} (H+b)^{-n} a(h_t) \Gamma(q^t) \mathrm{e}^{-\mathrm{i}tH} u \\ = &\ \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} (H+b)^{-n} \Gamma(q^t) a(q^t h_t) \mathrm{e}^{-\mathrm{i}tH} u \\ = &\ 0,\end{aligned}$$

using that $(N+1)^{-1} a(q^t h_t) \in o(1)$ and the higher order estimates. □

10.3 Geometric inverse wave operators

Let $j_0 \in C_0^\infty(\mathbb{R})$, $j_\infty \in C^\infty(\mathbb{R})$, $0 \le j_0, j_\infty$, $j_0^2 + j_\infty^2 \le 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). Set $j := (j_0, j_\infty)$, $j^t = (j_0^t, j_\infty^t)$.

As in Subsect. 2.4, we introduce the operator $I(j^t) : \mathcal{H}^{\mathrm{ext}} \to \mathcal{H}$.

Theorem 10.4 *Assume (G1), (G2), (Is) for $s > 1$. Then:*

i) The following limits exist:

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}} I^*(j^t) \mathrm{e}^{-\mathrm{i}tH}, \tag{10.10}$$

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I(j^t) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}. \tag{10.11}$$

If we denote (10.10) by $W^+(j)$, then (10.11) equals $W^+(j)^*$ and $\|W^+(j)\| \le 1$.
ii) For any bounded Borel function F one has

$$W^+(j)F(H) = F(H^{\mathrm{ext}})W^+(j).$$

iii) Let $q_0, q_\infty \in C^\infty(\mathbb{R})$, $\nabla q_0, \nabla q_\infty \in C_0^\infty(\mathbb{R})$, $0 \le q_0, q_\infty \le 1$, $q_0 \equiv 1$ near 0 and $q_\infty \equiv 1$ near ∞. Set $\tilde{j} := (\tilde{j}_0, \tilde{j}_\infty) := (q_0 j_0, q_\infty j_\infty)$. Then

$$\Gamma^+(q_0) \otimes \Gamma^+_{\mathrm{free}}(q_\infty) W^+(j) = W^+(\tilde{j}).$$

iv) Assume additionally that $j_0 + j_\infty = 1$. Then $\mathrm{Ran}W^+(j) \subset \mathcal{H}^{\mathrm{scatt}}$ and if $\chi \in C_0^\infty(\mathbb{R})$:

$$\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})W^+(j) = \chi(H).$$

Note that statement *iv)* of Thm. 10.4 makes sense since $\mathrm{Ran}W^+(j) \subset \mathcal{H}^{\mathrm{scatt}}$ and $\chi(H^{\mathrm{ext}})$ preserves $\mathcal{H}^{\mathrm{scatt}}$.
Proof. Statements *i)*, *ii)*, *iii)* are proved exactly as in [DG1, Thm. 12.4], we detail only *iv)*.

We pick $q_\infty \in C^\infty(\mathbb{R})$ with $q_\infty \equiv 1$ near ∞, $q_\infty \equiv 0$ near 0 and $q_\infty j_\infty = j_\infty$. Applying *iii)* for $q_0 \equiv 1$, we obtain by *iii)* that $\mathbb{1} \otimes \Gamma^+_{\mathrm{free}}(q_\infty)W^+(j) = W^+(j)$. Applying then (10.6) we get that $\mathbb{1} \otimes \Gamma(\mathbb{1}_{\mathrm{c}}(\omega))W^+(j) = W^+(j)$ i.e. $\mathrm{Ran}W^+(j) \subset \mathcal{H}^{\mathrm{scatt}}$. The rest of the proof of *iv)* is as in [DG1, Thm. 12.4]. □

10.4 Geometric asymptotic completeness

In this subsection we will show that

$$\mathrm{Ran}P_0^+ = \mathcal{K}^+.$$

We call this property *geometric asymptotic completeness.* It will be convenient to work in the scattering space $\mathcal{H}^{\mathrm{scatt}}$ and to treat Ω^+ as a partial isometry $\Omega^+ : \mathcal{H}^{\mathrm{scatt}} \to \mathcal{H}$, as explained in Subsect. 8.3.

Theorem 10.5 *Assume (G1), (G2), (S), (Is) for $s > 1$. Let $j_n = (j_{0,n}, j_{\infty,n})$ satisfy the conditions of Subsect. 10.3. Additionally, assume that $j_{0,n} + j_{\infty,n} = 1$ and that for any $\epsilon > 0$, there exists m such that, for $n > m$, $\mathrm{supp}\, j_{0,n} \subset [-\epsilon, \epsilon]$. Then*

$$\Omega^{+*} = \mathrm{w} - \lim_{n\to\infty} W^+(j_n).$$

Besides

$$\mathcal{K}^+ = \mathrm{Ran}P_0^+.$$

Proof. The proof is analogous to [DG1, Thm. 12.5]. Since it is in important step, we will give some details. If $q \in C_0^\infty(\mathbb{R})$ is such that $q = 1$ in a neighborhood of 0, $0 \le q \le 1$ then for sufficiently big n we have $q j_{0,n} = j_{0,n}$. Therefore, for sufficiently big n by Thm. 10.4 *iii)*

$$(\Gamma^+(q) \otimes \mathbb{1})W^+(j_n) - W^+(j_n) = 0.$$

Hence

$$\mathrm{w}-\lim_{n\to\infty}\Big(P_0^+\otimes 1\!\!1 W^+(j_n)-W^+(j_n)\Big)=0. \tag{10.12}$$

Let $\chi\in C_0^\infty(\mathbb{R})$. We have

$$\begin{aligned}
\Omega^{+*}\chi(H) &= \Omega^{+*}\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})W^+(j_n) && (1)\\
&= \mathrm{w}-\lim_{n\to\infty}\Omega^{+*}\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})W^+(j_n) && (2)\\
&= \mathrm{w}-\lim_{n\to\infty}\Omega^{+*}\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})P_0^+\otimes 1\!\!1 W^+(j_n) && (3)\\
&= \mathrm{w}-\lim_{n\to\infty}P_0^+\otimes 1\!\!1\chi(H^{\mathrm{ext}})W^+(j_n) && (4)\\
&= \mathrm{w}-\lim_{n\to\infty}P_0^+\otimes 1\!\!1 W^+(j_n)\chi(H) && (5)\\
&= \mathrm{w}-\lim_{n\to\infty}W^+(j_n)\chi(H) && (6).
\end{aligned}$$

We use Thm. 10.4 in step (1), (10.12) in step (3), $\mathrm{Ran}P_0^+\subset\mathcal{K}^+$ in step (4), Thm. 10.4 *ii)* in step (5) and (10.12) again in step (6). Clearly this implies that:

$$\Omega^{+*}=\mathrm{w}-\lim_{n\to\infty}W^+(j_n).$$

Therefore by (10.12)

$$\mathrm{Ran}\Omega^{+*}\subset\mathrm{Ran}P_0^+\otimes\Gamma(\mathfrak{h})\subset\mathcal{K}^+\otimes\Gamma(\mathfrak{h}).$$

But by construction

$$\mathrm{Ran}\Omega^{+*}=\mathcal{K}^+\otimes\Gamma(\mathfrak{h}).$$

Hence $\mathcal{K}^+\otimes\Gamma(\mathfrak{h})=\mathrm{Ran}P_0^+\otimes\Gamma(\mathfrak{h})$, and therefore $\mathcal{K}^+=\mathrm{Ran}P_0^+$. $\Box$

10.5 Asymptotic completeness

In this subsection, we will prove asymptotic completeness.

Theorem 10.6 *Assume hypotheses (Hi), $1\le i\le 3$, (Gi), $1\le i\le 5$, (Mi) $i=1,2$, (Is) for $s>1$ and (S). Then:*

$$\mathcal{K}^+=\mathcal{H}_{\mathrm{pp}}(H).$$

Proof. By Proposition 8.4 and geometric asymptotic completeness we already know that

$$\mathcal{H}_{\mathrm{pp}}(H)\subset\mathcal{K}^+=\mathrm{Ran}P_0^+.$$

It remains to prove that $P_0^+\le 1\!\!1_{\mathrm{pp}}(H)$. Let $\chi\in C_0^\infty(\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H)))$. We deduce from Prop. 9.6 in Subsect. 9.4 that there exists $\epsilon>0$ such that for $q\in C_0^\infty([-\epsilon,\epsilon])$ with $q(x)=1$ for $|x|<\epsilon/2$ we have

$$\int_1^{+\infty}\|\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2\frac{\mathrm{d}t}{t}\le c\|u\|^2.$$

Since $\|\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|\to\|\Gamma^+(q)\chi(H)u\|$, we have $\Gamma^+(q)\chi(H)=0$. This implies that

$$P_0^+\le 1\!\!1_{\tau\cup\sigma_{\mathrm{pp}}}(H).$$

Since τ is a closed countable set and $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, we see that $1\!\!1_{\mathrm{pp}}(H)=1\!\!1_{\tau\cup\sigma_{\mathrm{pp}}}(H)$. This completes the proof of the theorem. $\Box$

A Appendix

A.1 Proof of Lemma 3.3

To prove *i)* we restrict the quadratic form $[F(\frac{\langle x\rangle}{R}),\omega]$ to $\mathcal{S}$. Using (2.2), we get

$$
\begin{aligned}
[F(\tfrac{\langle x\rangle}{R}),\omega] = &\ \tfrac{\mathrm{i}}{2\pi R}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)(z-\tfrac{\langle x\rangle}{R})^{-1}[\langle x\rangle,\omega](z-\tfrac{\langle x\rangle}{R})^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z},\\
(A.1)\qquad = &\ \tfrac{\mathrm{i}}{2\pi R}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)(z-\tfrac{\langle x\rangle}{R})^{-2}[\langle x\rangle,\omega]\mathrm{d}z\wedge\mathrm{d}\,\overline{z}\\
&+\tfrac{\mathrm{i}}{2\pi R^2}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)(z-\tfrac{\langle x\rangle}{R})^{-2}\mathrm{ad}^2_{\langle x\rangle}\omega(z-\tfrac{\langle x\rangle}{R})^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z}
\end{aligned}
$$

where the right hand sides are operators on $\mathcal{S}$. Since $\mathrm{ad}^2_{\langle x\rangle}\omega\in S^0_{(0)}$, we see that the last term belongs to $R^{-2}S^0_{(0)}$. Using the bound $\frac{\langle x\rangle}{R}(z-\frac{\langle x\rangle}{R})^{-1}=O(|\mathrm{Im}z|^{-1})$ for $z\in\mathrm{supp}\,\tilde{F}$, we see that the last term belongs also to $R^{-1}S^{-1}_{(0)}$. This proves *i)* for $k=0$.

Replacing ω by $[\omega,\langle x\rangle]$ and using that $\mathrm{ad}^2_{\langle x\rangle}[\omega,\langle x\rangle]\in S^{(0)}_{(0)}$ we get also *i)* for $k=1$.

ii) follows from *i)* for $k=0$ since $\mathcal{S}$ is a core for ω. *iii)* and *iv)* are proved similarly. *v)* is proved as *i)*, replacing ω by $[\omega,\mathrm{i}a]_0$ and using only the first line of (A.1). To prove *vi)* we restrict again the quadratic form $[F(\frac{\langle x\rangle}{R}),\omega^2]$ to $\mathcal{S}$ and get:

$$
\begin{aligned}
&[F(\tfrac{\langle x\rangle}{R}),\omega^2]\\
(A.2)\qquad = &\ \tfrac{\mathrm{i}}{2\pi R}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)(z-\tfrac{\langle x\rangle}{R})^{-1}[\langle x\rangle,\omega^2](z-\tfrac{\langle x\rangle}{R})^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z},\\
= &\ \tfrac{\mathrm{i}}{2\pi R}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)(z-\tfrac{\langle x\rangle}{R})^{-1}\left(2[\langle x\rangle,\omega]\omega+[\omega,[\langle x\rangle,\omega]]\right)(z-\tfrac{\langle x\rangle}{R})^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z},
\end{aligned}
$$

where the right hand sides are operators on $\mathcal{S}$. Note that $[\omega,[\langle x\rangle,\omega]]$ is bounded by *(G2)*. We use next that $\omega(z-\frac{\langle x\rangle}{R})^{-1}\omega^{-1}\in O(|\mathrm{Im}z)|^{-2}$ uniformly in $R\geq 1$ to obtain *vi)*.

To prove *vii)*, we pick another function $F_1\in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $F_1F=F$ and note that

$$[F(\frac{\langle x\rangle}{R}),b]=F(\frac{\langle x\rangle}{R})[F_1(\frac{\langle x\rangle}{R}),b]+[F(\frac{\langle x\rangle}{R}),b]F_1(\frac{\langle x\rangle}{R}).$$

Applying again (2.2), we get

$$[F(\frac{\langle x\rangle}{R}),b]=\frac{\mathrm{i}}{2\pi R}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)(z-\frac{\langle x\rangle}{R})^{-1}[\langle x\rangle,b](z-\frac{\langle x\rangle}{R})^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z},$$

and the analogous formula for $[F_1(\frac{\langle x\rangle}{R}),b]$. We use then that $[\langle x\rangle,b]\in S^{-\mu+\delta}_{(0)}$ and Lemma 2.3, moving powers of $\langle x\rangle$ through the resolvents either to the left or to the right to obtain *vii)*. □

A.2 Proof of Lemma 3.4.

We use the identity:

$$\omega^{-\frac{1}{2}}=c_0\int_0^{+\infty}s^{-\frac{1}{2}}(\omega+s)^{-1}\mathrm{d}s,$$

to get:

$$\omega^{\frac{1}{2}}[F(\frac{\langle x\rangle}{R}),\omega^{-\frac{1}{2}}]=c_0\int_0^{+\infty}s^{-\frac{1}{2}}\omega^{\frac{1}{2}}(\omega+s)^{-1}[F(\frac{\langle x\rangle}{R}),\omega](\omega+s)^{-1}\mathrm{d}s\in O(R^{-1}),$$

since $\omega \geq m > 0$. Hence

$$\begin{aligned}
& \omega^{-\frac{1}{2}}(\omega-\omega_\infty)F(\tfrac{\langle x\rangle}{R})\omega^{-\frac{1}{2}} \\
= \; & \omega^{-\frac{1}{2}}(\omega-\omega_\infty)\omega^{-\frac{1}{2}}F(\tfrac{\langle x\rangle}{R}) + \omega^{-\frac{1}{2}}(\omega-\omega_\infty)\omega^{-\frac{1}{2}}\omega^{\frac{1}{2}}[F(\tfrac{\langle x\rangle}{R}), \omega^{-\frac{1}{2}}] \\
= \; & \omega^{-\frac{1}{2}}(\omega-\omega_\infty)\omega^{-\frac{1}{2}}\langle x\rangle^{\epsilon}\langle x\rangle^{-\epsilon}F(\tfrac{\langle x\rangle}{R}) + O(R^{-1}) \\
= \; & O(R^{-\epsilon}) + O(R^{-1}).
\end{aligned}$$

The second statement of the lemma is obvious. □

A.3 Proof of Lemma 9.3.

Since by *(G1)* $[v, \langle x\rangle]$ extends from $\mathcal{S}$ as a bounded operator on $\mathfrak{h}$ and $\mathcal{S}$ is a core for $\langle x\rangle$, we get that v preserves $\mathcal{D}(\langle x\rangle)$. Since $\frac{\langle x\rangle}{t} - v$ is selfadjoint on $\mathcal{D}(\langle x\rangle)$ we get

$$\mathcal{D}(c) = \mathcal{D}((\frac{\langle x\rangle}{t} - v)^2) = \{u \in \mathcal{D}(\langle x\rangle) | \; (\frac{\langle x\rangle}{t} - v)u \in \mathcal{D}(\langle x\rangle)\} = \mathcal{D}(\langle x\rangle^2),$$

so c is selfadjoint on $\mathcal{D}(\langle x\rangle^2)$. Since $v \in S^0_{(0)}$ we get by Lemma 2.3 that $J(\frac{\langle x\rangle}{t})cJ(\frac{\langle x\rangle}{t}) \in O(1)$ which proves *i)*.

Let us now prove *ii)*. We first consider the commutator $[c, J(\frac{\langle x\rangle}{t})]$ for $J \in C_0^\infty(\mathbb{R})$. We have

$$\begin{aligned}
[c, J(\tfrac{\langle x\rangle}{t})] = \; & (\tfrac{\langle x\rangle}{t} - v)[v, J(\tfrac{\langle x\rangle}{t})] + [v, J(\tfrac{\langle x\rangle}{t})](\tfrac{\langle x\rangle}{t} - v) \\
= \; & t^{-1}(\tfrac{\langle x\rangle}{t} - v)J'(\tfrac{\langle x\rangle}{t})[v, \langle x\rangle] + t^{-1}J'(\tfrac{\langle x\rangle}{t})[v, \langle x\rangle](\tfrac{\langle x\rangle}{t} - v) \\
+ \; & (\tfrac{\langle x\rangle}{t} - v)M(t) + M(t)(\tfrac{\langle x\rangle}{t} - v),
\end{aligned}$$

where $M(t) \in t^{-2}S^0_{(0)} \cap t^{-1}S^{-1}_{(0)}$ by Lemma 3.3 *i)*. This implies that the last two terms in the r.h.s. are $O(t^{-2})$. Using then that $[v, J'(\frac{\langle x\rangle}{t})] \in O(t^{-1})$ and $[[v, \langle x\rangle], \frac{\langle x\rangle}{t}] \in O(t^{-1})$ since $v \in S^0_{(3)}$, we see that

$$\begin{aligned}
(\tfrac{\langle x\rangle}{t} - v)J'(\tfrac{\langle x\rangle}{t})[v, \langle x\rangle] = J'(\tfrac{\langle x\rangle}{t})M_1(t) + O(t^{-1}), \\
J'(\tfrac{\langle x\rangle}{t})[v, \langle x\rangle](\tfrac{\langle x\rangle}{t} - v) = J'(\tfrac{\langle x\rangle}{t})M_2(t) + O(t^{-1}),
\end{aligned}$$

where $M_i(t) \in O(1)$. This shows that:

$$[c, J(\frac{\langle x\rangle}{t})] = \frac{1}{t}J'(\frac{\langle x\rangle}{t})O(1) + O(t^{-2}). \tag{A.3}$$

We will use the following identities valid for $\lambda > 0$:

$$\lambda^{-\frac{1}{2}} = c_0 \int_0^{+\infty} s^{-\frac{1}{2}}(\lambda+s)^{-1}\mathrm{d}s, \;\; \lambda^{\frac{1}{2}} = c_0 \int_0^{+\infty} s^{-\frac{1}{2}}\lambda(\lambda+s)^{-1}\mathrm{d}s, \tag{A.4}$$

and

$$\lambda^{-\frac{3}{2}} = 2c_0 \int_0^{+\infty} s^{-\frac{1}{2}}(\lambda+s)^{-2}\mathrm{d}s, \tag{A.5}$$

which follows by differentiating the first identity of (A.4) w.r.t. λ. A related obvious bound is:

$$\int_0^{+\infty} s^{-\frac12}(t^{-\delta}+s)^{-n}\mathrm{d}s = O(t^{(n-\frac12)\delta}),\quad n\geq 1. \tag{A.6}$$

From (A.4) we obtain that

$$c^{\frac12} = c_0\int_0^{+\infty} s^{-\frac12}c(c+s)^{-1}\mathrm{d}s,\ \text{as a strong integral on } \mathcal{D}(c). \tag{A.7}$$

Therefore

$$[c^{\frac12}, J(\frac{\langle x\rangle}{t})] = c_0\int_0^{+\infty} s^{-\frac12}\left([c, J(\frac{\langle x\rangle}{t})](c+s)^{-1} - c(c+s)^{-1}[c, J(\frac{\langle x\rangle}{t})](c+s)^{-1}\right)\mathrm{d}s$$

We use the bounds

$$\|c(c+s)^{-1}\|\leq 1,\quad \|(c+s)^{-1}\|\leq (t^{-\delta}+s)^{-1}, \tag{A.8}$$

and (A.3) to obtain

$$\|[c^{\frac12}, J(\frac{\langle x\rangle}{t})]\|\leq Ct^{-1}\int_0^{+\infty} s^{-\frac12}(t^{-\delta}+s)^{-1}\mathrm{d}s = O(t^{-1+\delta/2}),$$

by (A.4), which proves *ii)*.

To prove *iii)* we first compute

$$\mathbf{d}_0 c = -\frac{2}{t}(\frac{\langle x\rangle}{t}-v)^2 - \left([\omega, \mathrm{i}v](\frac{\langle x\rangle}{t}-v) + \mathrm{h.c.}\right) - \delta t^{-\delta-1}. \tag{A.9}$$

We first rewrite the second term in the r.h.s. in a convenient way:
by *(G2)*, we have

$$[\omega, \mathrm{i}v] = \gamma^2 + r_{-1-\epsilon},\quad \gamma\in S^{-\frac12}_{\epsilon,(1)},\quad r_{-1-\epsilon}\in S^{-1-\epsilon}_{(0)}.$$

Since $v\in S^0_{(0)}$, we get first that:

$$(\frac{\langle x\rangle}{t}-v)r_{-1-\epsilon}\in O(t^{-1})S^{-\epsilon}_{(0)} + S^{-1-\epsilon}_{(0)}. \tag{A.10}$$

We claim also that

$$[\gamma, \frac{\langle x\rangle}{t}-v]\in O(t^{-1})S^{-\frac12+\epsilon}_{(0)} + S^{-3/2+2\epsilon}_{(0)}. \tag{A.11}$$

Clearly $[\gamma, \langle x\rangle]\in S^{-\frac12+\epsilon}_{(0)}$. To handle $[\gamma, v]$ we use the Lie identity and write:

$$\mathrm{i}\,[\gamma, v] = -[\gamma, [\omega, \langle x\rangle]] = [\omega, [\langle x\rangle, \gamma]] + [\langle x\rangle, [\omega, \gamma]]\in S^{-3/2+2\epsilon}_{(0)}, \tag{A.12}$$

which proves (A.11). By Lemma 2.3 *i)*, we get that

$$\gamma[\gamma, \frac{\langle x\rangle}{t}-v],\quad [\gamma, \frac{\langle x\rangle}{t}-v]\gamma\in t^{-1}S^{-1+\epsilon}_{(0)} + S^{-2+2\epsilon}_{(0)},$$

and hence using that $0<\epsilon<\frac{1}{2}$:

$$[\omega, \mathrm{i}v](\frac{\langle x\rangle}{t}-v)+\text{h.c..}=2\gamma(\frac{\langle x\rangle}{t}-v)\gamma+R_2(t),$$

where $R_2(t)\in O(t^{-1})S^{-\epsilon_1}_{(0)}+S^{-1-\epsilon_1}_{(0)}$, for some $\epsilon_1>0$. We set now:

$$R_0(t)=-\frac{2c}{t},\quad R_1(t)=-(\delta-2)t^{-\delta-1},\quad R_3(t)=-2\gamma(\frac{\langle x\rangle}{t}-v)\gamma,$$

and rewrite (A.9) as

$$\mathbf{d}_0 c=\sum_{i=0}^{3}R_i(t).$$

Using (A.7), we obtain as a strong integral on $\mathcal{D}(c)$:

$$\begin{aligned}
\mathbf{d}_0 c^{\frac{1}{2}} &= c_0\int_0^{+\infty}s^{-\frac{1}{2}}\left(\mathbf{d}_0 c(c+s)^{-1}-c(c+s)^{-1}\mathbf{d}_0 c(c+s)^{-1}\right)\mathrm{d}s\\
&= \sum_{i=0}^{3}c_0\int_0^{+\infty}s^{-\frac{1}{2}}\left(R_i(t)(c+s)^{-1}-c(c+s)^{-1}R_i(t)(c+s)^{-1}\right)\mathrm{d}s\\
&=: \sum_{i=0}^{3}I_i(t).
\end{aligned}$$

Using (A.4) we obtain

$$I_0(t)=-\frac{1}{t}c^{\frac{1}{2}},\quad I_1(t)=Ct^{-\delta-1}c^{-\frac{1}{2}}=O(t^{-\delta/2-1}).$$

It remains to handle the terms $J(\frac{\langle x\rangle}{t})I_i(t)J(\frac{\langle x\rangle}{t})$ for $i=2,3$. We write them as:

$$\begin{aligned}
J(\tfrac{\langle x\rangle}{t})I_i(t)J(\tfrac{\langle x\rangle}{t}) =\ & c_0\int_0^{+\infty}s^{-\frac{1}{2}}J(\tfrac{\langle x\rangle}{t})R_i(t)(c+s)^{-1}J(\tfrac{\langle x\rangle}{t})\mathrm{d}s\\
& -c_0\int_0^{+\infty}s^{-\frac{1}{2}}J(\tfrac{\langle x\rangle}{t})c(c+s)^{-1}R_i(t)(c+s)^{-1}J(\tfrac{\langle x\rangle}{t})\mathrm{d}s.
\end{aligned}$$

We will need to use the fact that $O\not\in\operatorname{supp}J$. To do this we claim that if $J,J_1\in C_0^\infty(\mathbb{R})$ with $J_1\equiv 1$ near $\operatorname{supp}J$ then:

$$J(\frac{\langle x\rangle}{t})(c+s)^{-1}(1-J_1)(\frac{\langle x\rangle}{t})\in O(t^{-2}(t^{-\delta}+s)^{-2})+O(t^{-2}(t^{-\delta}+s)^{-3}),\tag{A.13}$$

$$J(\frac{\langle x\rangle}{t})c(c+s)^{-1}(1-J_1)(\frac{\langle x\rangle}{t})\in O(t^{-2}(t^{-\delta}+s)^{-1})+O(t^{-2}(t^{-\delta}+s)^{-2}).\tag{A.14}$$

We pick $T_1\in C_0^\infty(\mathbb{R})$, $T_1\equiv 1$ on $\operatorname{supp}J_1'$, $T_1\equiv 0$ on $\operatorname{supp}J$. We write using (A.3):

$$\begin{aligned}
& J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}(1-J_1)(\tfrac{\langle x\rangle}{t})\\
=\ & J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}[c,J_1(\tfrac{\langle x\rangle}{t})](c+s)^{-1}\\
=\ & J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}T_1(\tfrac{\langle x\rangle}{t})O(t^{-1})(c+s)^{-1}+J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}O(t^{-2})(c+s)^{-1}\\
=\ & J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}[T_1(\tfrac{\langle x\rangle}{t},c)](c+s)^{-1}O(t^{-1})(c+s)^{-1}+J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}O(t^{-2})(c+s)^{-1}\\
=\ & J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}O(t^{-1})(c+s)^{-1}O(t^{-1})(c+s)^{-1}+J(\tfrac{\langle x\rangle}{t})(c+s)^{-1}O(t^{-2})(c+s)^{-1}.
\end{aligned}$$

We obtain (A.13) using the bound $\|(c+s)^{-1}\| \le (t^{-\delta}+s)^{-1}$. (A.14) follows from (A.13) using that $c(c+s)^{-1} = \mathbb{1} - s(c+s)^{-1}$.

We hence fix a cutoff $J_1 \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $J_1 \equiv 1$ on $\operatorname{supp} J$ and set

$$\tilde{R}_i(t) = J_1(\frac{\langle x\rangle}{t})R_i(t)J_1(\frac{\langle x\rangle}{t}),$$

and denote by $\tilde{I}_i(t)$ the analogs of $I_i(t)$ for $R_i(t)$ replaced by $\tilde{R}_i(t)$.

We claim that:

$$J(\frac{\langle x\rangle}{t})\left(I_i(t) - \tilde{I}_i(t)\right)J(\frac{\langle x\rangle}{t}) \in O(t^{-2+5\delta/2}), \quad i = 2,3. \tag{A.15}$$

To prove (A.15), we note that $\tilde{I}_i(t)$ is obtained from $I_i(t)$ by inserting $J_1(\frac{\langle x\rangle}{t})$ to the left and right of $R_i(t)$ under the integral sign. The error terms under the integral sign coming from this insertion are estimated using (A.13), (A.14) and the fact that $R_i(t) \in O(1)$ for $i = 2,3$, since $\gamma \in S_{(0)}^{-\frac{1}{2}}$. The integrals of these error terms are estimated using (A.6), which by a painful but straightforward computation gives (A.15).

By Lemma 2.3 *ii)*, we know that $\tilde{R}_2(t) \in O(t^{-1-\epsilon_1})$ for some $\epsilon_1 > 0$ small enough, hence using the bounds (A.8) and (A.6), we obtain that for $\delta > 0$ small enough

$$\tilde{I}_2(t) \text{ and hence } J(\frac{\langle x\rangle}{t})I_2(t)J(\frac{\langle x\rangle}{t}) \in O(t^{-1-\epsilon_2}), \quad \epsilon_2 > 0.$$

To treat $\tilde{I}_3(t)$, we use that

$$\tilde{R}_3(t) = \gamma_t^*(\frac{\langle x\rangle}{t} - v)\gamma_t, \text{ for } \gamma_t = \gamma J_1(\frac{\langle x\rangle}{t}).$$

We claim that

$$[\gamma_t, c] \in O(t^{-3/2+\epsilon}). \tag{A.16}$$

Let us prove this claim. We write:

$$[\gamma_t, c] = (\frac{\langle x\rangle}{t} - v)[\gamma_t, \frac{\langle x\rangle}{t} - v] + [\gamma_t, \frac{\langle x\rangle}{t} - v](\frac{\langle x\rangle}{t} - v),$$

and

$$[\gamma_t, \langle x\rangle] = [\gamma, \langle x\rangle]J_1(\frac{\langle x\rangle}{t}), \quad [\gamma_t, v] = [\gamma, v]J_1(\frac{\langle x\rangle}{t}) + \gamma[J_1(\frac{\langle x\rangle}{t}), v].$$

Now

$$(\frac{\langle x\rangle}{t} - v)[\gamma, \langle x\rangle]J_1(\frac{\langle x\rangle}{t}), \quad [\gamma, \langle x\rangle]J_1(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) \in O(t^{-\frac{1}{2}+\epsilon}).$$

This follows from the fact that $[\gamma, \langle x\rangle] \in S_{(0)}^{-\frac{1}{2}+\epsilon}$, $0 \notin \operatorname{supp} J_1$ and Lemma 2.3 *ii)*. Similarly we saw in (A.12) that $[\gamma, v] \in S_{(0)}^{-3/2+2\epsilon}$, which implies that:

$$(\frac{\langle x\rangle}{t} - v)[\gamma, v]J_1(\frac{\langle x\rangle}{t}), \quad [\gamma, v]J_1(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t} - v) \in O(t^{-3/2+2\epsilon}).$$

Finally using Lemma 3.3 *i)* we write:

$$[J_1(\frac{\langle x\rangle}{t},v)]=\frac{1}{t}J_1'(\frac{\langle x\rangle}{t})[\langle x\rangle,v]+M(t),\ \ M(t)\in O(t^{-2})S^0_{(0)}\cap O(t^{-1})S^{-1}_{(0)}.$$

Since $\gamma\in S^{-\frac{1}{2}}_{(0)}$ and $[\langle x\rangle,v]\in S^0_{(0)}$, we get that

$$(\frac{\langle x\rangle}{t}-v)\gamma J_1'(\frac{\langle x\rangle}{t})[\langle x\rangle,v],\ \ \gamma J_1'(\frac{\langle x\rangle}{t})[\langle x\rangle,v](\frac{\langle x\rangle}{t}-v)\in O(t^{-\frac{1}{2}}),$$

and since $M(t)\in O(t^{-2})S^0_{(0)}\cap O(t^{-1})S^{-1}_{(0)}$:

$$(\frac{\langle x\rangle}{t}-v)\gamma M(t),\ \ \gamma M(t)(\frac{\langle x\rangle}{t}-v)\in O(t^{-2}).$$

Collecting the various estimates we obtain (A.16).

From the estimate of $[\gamma_t,c]$ we obtain:

$$[\gamma_t,(c+s)^{-1}]\in O(t^{-3/2+\epsilon}(t^{-\delta}+s)^{-2}), \tag{A.17}$$

$$[\gamma_t,c(c+s)^{-1}]\in O(t^{-3/2+\epsilon}(t^{-\delta}+s)^{-1}). \tag{A.18}$$

We now write:

$$\begin{aligned}\tilde{I}_3(t)= &\ c_0\int_0^{+\infty}s^{-\frac{1}{2}}\gamma_t^*(\tfrac{\langle x\rangle}{t}-v)\gamma_t(c+s)^{-1}\mathrm{d}s\\ &-c_0\int_0^{+\infty}s^{-\frac{1}{2}}c(c+s)^{-1}\gamma_t^*(\tfrac{\langle x\rangle}{t}-v)\gamma_t(c+s)^{-1}\mathrm{d}s\end{aligned}$$

We first move γ_t to the right in the two integrals using (A.17) and the fact that

$$\gamma_t^*(\frac{\langle x\rangle}{t}-v)=J_1(\frac{\langle x\rangle}{t})\gamma(\frac{\langle x\rangle}{t}-v)\in O(1),$$

since $\gamma\in S^{-\frac{1}{2}}_{(0)}$. We obtain errors terms of size $O(t^{-3/2+\epsilon+5\delta/2})$ using (A.6). We then move γ_t^* to the left in the second integral using (A.18) and the fact that

$$\|(\frac{\langle x\rangle}{t}-v)(c+s)^{-1}\|\leq\|c^{\frac{1}{2}}(c+s)^{-1}\|\leq t^{\delta/2}.$$

We obtain error terms of size $O(t^{-3/2+\epsilon+\delta})$ using again (A.6). Hence for $\delta>0$ small enough, we get:

$$\begin{aligned}\tilde{I}_3(t)= &\ c_0\int_0^{+\infty}\gamma_t^*s^{-\frac{1}{2}}(\tfrac{\langle x\rangle}{t}-v)(c+s)^{-1}\gamma_t\mathrm{d}s\\ &-c_0\int_0^{+\infty}\gamma_t^*s^{-\frac{1}{2}}c(c+s)^{-1}(\tfrac{\langle x\rangle}{t}-v)(c+s)^{-1}\gamma_t\mathrm{d}s\\ &+O(t^{-1-\epsilon_1})\end{aligned}$$

for some $\epsilon_1>0$. The integrals can be computed exactly since $\frac{\langle x\rangle}{t}-v$ commutes with c and are equal to $C_1(\frac{\langle x\rangle}{t}-v)c^{-\frac{1}{2}}$ for some constant C_1 and hence $O(1)$. This yields:

$$\begin{aligned}J(\tfrac{\langle x\rangle}{t})\tilde{I}_3(t)J(\tfrac{\langle x\rangle}{t})= &\ J(\tfrac{\langle x\rangle}{t})\gamma_t^*M(t)\gamma_tJ(\tfrac{\langle x\rangle}{t})+O(t^{-1-\epsilon_1})\\ = &\ J(\tfrac{\langle x\rangle}{t})\gamma M(t)\gamma J(\tfrac{\langle x\rangle}{t})+O(t^{-1-\epsilon_1}),\end{aligned}$$

for $M(t)\in O(1)$. Using also (A.15), the same equality holds for $J(\frac{\langle x\rangle}{t})I_3(t)J(\frac{\langle x\rangle}{t})$. Finally we use that $\gamma J(\frac{\langle x\rangle}{t})\in O(t^{-\frac{1}{2}})$, by Lemma 2.3 *ii)* and $[\gamma, J(\frac{\langle x\rangle}{t})]\in O(t^{-3/2+\epsilon})$, to get:

$$J(\frac{\langle x\rangle}{t})\gamma M(t)\gamma J(\frac{\langle x\rangle}{t}) = \gamma J(\frac{\langle x\rangle}{t})M(t)J(\frac{\langle x\rangle}{t})\gamma + O(t^{-2+\epsilon}).$$

Hence

$$J(\frac{\langle x\rangle}{t})I_3(t)J(\frac{\langle x\rangle}{t}) = \gamma J(\frac{\langle x\rangle}{t})M(t)J(\frac{\langle x\rangle}{t})\gamma + O(t^{-1-\epsilon_1}),$$

which completes the proof of *iii)*.

Let us now prove *iv)*. Set

$$B_0 = J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t}-v) + \text{h.c.}, \quad B_1 = J_1(\frac{\langle x\rangle}{t})c^{\frac{1}{2}}J_1(\frac{\langle x\rangle}{t}).$$

By Lemma 3.3 we have:

$$\begin{aligned}
B_0^2 = &\ 4(\tfrac{\langle x\rangle}{t}-v)J^2(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t}-v) + O(t^{-1})\\
\le &\ C(\tfrac{\langle x\rangle}{t}-v)J_1^4(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t}-v) + O(t^{-1})\\
= &\ CJ_1^2(\tfrac{\langle x\rangle}{t})(\tfrac{\langle x\rangle}{t}-v)^2J_1(\tfrac{\langle x\rangle}{t}) + O(t^{-1})\\
= &\ CJ_1^2(\tfrac{\langle x\rangle}{t})cJ_1^2(\tfrac{\langle x\rangle}{t}) + O(t^{-\delta})\\
= &\ CJ_1(\tfrac{\langle x\rangle}{t})c^{\frac{1}{2}}J_1^2(\tfrac{\langle x\rangle}{t})c^{\frac{1}{2}}J_1(\tfrac{\langle x\rangle}{t}) + O(t^{-\epsilon_0})\\
= &\ CB_1^2 + O(t^{-\epsilon_0}),
\end{aligned}$$

where we used *ii)* in the last step. Applying then Heinz theorem we obtain that

$$|B_0| \le C(B_1^2 + t^{-\epsilon_0})^{\frac{1}{2}} \le CB_1 + Ct^{-\epsilon_0/2},$$

which proves *iv)*.

To prove *v)* we set

$$B_2 = J(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t}-v)c^{\frac{1}{2}}J_1(\frac{\langle x\rangle}{t}) + \text{h.c.}.$$

Using *ii)* and Lemma 3.3, we get:

$$\begin{aligned}
\pm B_2 = &\ \pm\Big((\tfrac{\langle x\rangle}{t}-v)JJ_1(\tfrac{\langle x\rangle}{t})c^{\frac{1}{2}} + \text{h.c.}\Big) + O(t^{-1+\delta/2})\\
= &\ \pm\Big(c^{\frac{1}{2}}(\tfrac{\langle x\rangle}{t}-v)c^{-\frac{1}{2}}JJ_1(\tfrac{\langle x\rangle}{t})c^{\frac{1}{2}} + \text{h.c.}\Big) + O(t^{-1+\delta/2})\\
\le &\ Cc + O(t^{-1+\delta/2})\\
\le &\ C(\tfrac{\langle x\rangle}{t}-v)^2 + O(t^{-\epsilon_0}),
\end{aligned}$$

since $(\frac{\langle x\rangle}{t}-v)c^{-\frac{1}{2}}$ is bounded with norm $O(1)$. Since $B_2 = J_2(\frac{\langle x\rangle}{t})B_2J_2(\frac{\langle x\rangle}{t})$ we get

$$\pm B_2 \le CJ_2(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t}-v)^2J_2(\frac{\langle x\rangle}{t}) + O(t^{-\epsilon_0}) = C(\frac{\langle x\rangle}{t}-v)J_2^2(\frac{\langle x\rangle}{t})(\frac{\langle x\rangle}{t}-v) + O(t^{-\epsilon_0}),$$

by Lemma 3.3. □

References

[ABG] Amrein, W., Boutet de Monvel, A., Georgescu, W.: *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[AHH] Arai A., Hirokawa M., Hiroshima F.: On the absence of eigenvectors of Hamiltonian in a class of massless quantum field model without infrared cutoff, J.Functional Analysis **168** (1999) 470-497.

[BG] Boutet de Monvel, A., Georgescu, V.: Graded C^*-algebras and many-body perturbation theory II: the Mourre estimate, in *Colloque "Méthodes semiclassiques" Nantes 1991 Vol. 2* Astérisque, **210** (1992) 75-97.

[DG1] Dereziński, J., Gérard, C.: Spectral and scattering theory of spatially cut-off $P(\varphi)_2$ Hamiltonians, Comm. Math. Phys. **213** (2000) 39-125.

[DG2] Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, Rev. Math. Phys. **11** (1999) 383-450.

[FGSch] Fröhlich J., Griesemer M., Schlein B.: Rayleigh Scattering at atoms with dynamical nuclei, Comm. Math. Phys. **271** (2007) 387-430.

[FGS] Fröhlich J., Griesemer M., Sigal I.M.: Spectral Theory for the Standard Model of non-relativistic QED, preprint arXiv:math-ph/0611013v2

[1] Georgescu, V.: On the spectral analysis of Quantum Field Hamiltonians, J. Funct. Anal. **245** (2007) 89-143.

[GG] Georgescu, V., Gérard, C.: On the virial theorem in quantum mechanics, Comm. Math. Phys. **208** (1999) 275-281.

[GGM] Georgescu, V., Gérard, C., Moeller, J.:Spectral theory of massless Nelson models, Comm. Math. Phys. **249** (2004), 29-78.

[GN] Gérard, C., Nier, F.: Scattering theory for pertubations of periodic Schödinger operators, J. Math. Kyoto Univ. **38** (1998) 595-634.

[GP] Gérard, C., Panati, A.: Spectral and scattering theory for the space-cutoff $P(\varphi)_2$ model with variable metric, in preparation.

[Ho] Hörmander, L.: *The analysis of linear partial differential operators*, vol. III Springer Verlag Berlin Heidelberg New York. (1985).

[HSS] Hunziker, W., Sigal, I. M., Soffer, A.: Minimal escape velocities. Comm. P.D.E. **24** (1999), 2279-2295.

[LL] Lieb E., Loss M. Existence of Atoms and Molecules in Non-Relativistic Quantum Electrodynamics, Adv.Theor.Math.Phys. **7** (2003) 667-710

[P] Pizzo, A.: One-particle (improper) states and scattering states in Nelson's massless model, Ann. Henri Poincaré, **4** (2003) 439-483.

[R] Robert, D.: Propriétés spectrales d'opérateurs pseudo-différentiels, Comm. P.D.E. **3** (1978), 755-826.

[Ro] Rosen, L.: The $(\phi^{2n})_2$ Quantum Field Theory: Higher Order Estimates, Comm. Pure Appl. Math. **24** (1971), 417-457.

[SHK] Simon, B., Høgh-Krohn, R.: Hypercontractive Semigroups and Two dimensional Self-Coupled Bose Fields, J. Funct. Anal. **9** (1972) 121-180.

[Sp] Spohn, H.:*Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, 2004.