FUNCTIONAL CLT FOR RANDOM WALK AMONG BOUNDED RANDOM CONDUCTANCES

MAREK BISKUP AND TIMOTHY M. PRESCOTT

Department of Mathematics, University of California at Los Angeles

ABSTRACT. We consider the nearest-neighbor simple random walk on $\mathbb{Z}^d$, $d \geq 2$, driven by a field of i.i.d. random nearest-neighbor conductances $\omega_{xy} \in [0,1]$. Apart from the requirement that the bonds with positive conductances percolate, we pose no restriction on the law of the ω's. We prove that, for a.e. realization of the environment, the path distribution of the walk converges weakly to that of non-degenerate, isotropic Brownian motion. The quenched functional CLT holds despite the fact that the local CLT may fail in $d \geq 5$ due to anomalously slow decay of the probability that the walk returns to the starting point at a given time.

1. INTRODUCTION

Let $\mathbb{B}_d$ denote the set of unordered nearest-neighbor pairs (i.e., edges) of $\mathbb{Z}^d$ and let $(\omega_b)_{b\in\mathbb{B}_d}$ be i.i.d. random variables with $\omega_b \in [0,1]$. We will refer to ω_b as the *conductance* of the edge b. Let $\mathbb{P}$ denote the law of the ω's and suppose that

$$\mathbb{P}(\omega_b > 0) > p_c(d), \tag{1.1}$$

where $p_c(d)$ is the threshold for bond percolation on $\mathbb{Z}^d$; in $d = 1$ we have $p_c(d) = 1$ so there we suppose $\omega_b > 0$ a.s. This condition ensures the existence of a unique infinite connected component $\mathscr{C}_\infty$ of edges with strictly positive conductances; we will typically restrict attention to ω's for which $\mathscr{C}_\infty$ contains a given site (e.g., the origin).

Each realization of $\mathscr{C}_\infty$ can be used to define a random walk $X = (X_n)$ which moves about $\mathscr{C}_\infty$ by picking, at each unit time, one of its $2d$ neighbors at random and moving to it with probability equal to the conductance of the corresponding edge. Technically, X is a Markov chain with state space $\mathscr{C}_\infty$ and the transition probabilities defined by

$$P_{\omega,z}(X_{n+1} = y | X_n = x) = \frac{\omega_{xy}}{2d} \tag{1.2}$$

if $x, y \in \mathscr{C}_\infty$ and $|x - y| = 1$, and

$$P_{\omega,z}(X_{n+1} = x | X_n = x) = 1 - \frac{1}{2d} \sum_{y\colon |y-x|=1} \omega_{xy}. \tag{1.3}$$

The second index on $P_{\omega,z}$ marks the initial position of the walk, i.e., $P_{\omega,z}(X_0 = z) = 1$. As is easy to check, the counting measure on $\mathscr{C}_\infty$ is invariant and reversible for this Markov chain.

The $d = 1$ walk is a simple, but instructive, exercise for harmonic analysis of reversible random walks in random environments. Let us quickly sketch the proof of the fact that, for a.e. conductance configuration sampled from a translation-invariant, ergodic law on $(0,\infty)^{\mathbb{B}_d}$ satisfying the moment conditions

$$\mathbb{E}(\omega_b) < \infty \quad \text{and} \quad \mathbb{E}\Bigl(\frac{1}{\omega_b}\Bigr) < \infty, \tag{1.4}$$

the walk scales to Brownian motion for the usual diffusive scaling of space and time. (Here and henceforth $\mathbb{E}$ denotes expectation with respect to the environment distribution.)

Abbreviate $C = \mathbb{E}(1/\omega_b)$. The key step of the proof is to realize that

$$\varphi_\omega(x) = x + \frac{1}{C}\sum_{n=0}^{x-1}\Bigl(\frac{1}{\omega_{n,n+1}} - C\Bigr) \tag{1.5}$$

is harmonic for the Markov chain. Hence $\varphi_\omega(X_n)$ is a martingale whose increments are, by (1.4) and a simple calculation, square integrable in the sense $\mathbb{E}E_{\omega,0}[\varphi_\omega(X_1)^2] < \infty$. Invoking the stationarity and ergodicity of the Markov chain on the space of environments "from the point of view of the particle"—we will discuss the specifics of this argument later—the martingale $(\varphi_\omega(X_n))$ satisfies the conditions of the Lindeberg-Feller martingale functional CLT and so the law of $t \mapsto \varphi_\omega(X_{\lfloor nt\rfloor})/\sqrt{n}$ tends weakly to that of a Brownian motion. By the Pointwise Ergodic Theorem and (1.4) we have $\varphi_\omega(x) - x = o(x)$ as $|x| \to \infty$. Thus the path $t \mapsto X_{\lfloor nt\rfloor}/\sqrt{n}$ scales, in the limit $n \to \infty$, to the same function as $t \mapsto \varphi_\omega(X_{\lfloor nt\rfloor})/\sqrt{n}$. In other words, a *quenched* functional CLT holds for almost every ω.

While the main ideas of the above $d = 1$ solution work in all dimensions, the situation in $d \ge 2$ is, even for i.i.d. conductances, significantly more complicated. Progress has been made under additional conditions on the environment law. One such condition is *strong ellipticity*,

$$\exists \alpha > 0: \quad \mathbb{P}(\alpha \le \omega_b \le {}^1\!/\!_\alpha) = 1. \tag{1.6}$$

Here an annealed invariance principle was proved by Kipnis and Varadhan [17] and its queneched counterpart by Sidoravicius and Sznitman [25]. Another natural family of environments are those of supercritical *bond percolation* on $\mathbb{Z}^d$; i.e., $\omega_b \in \{0,1\}$ with $\mathbb{P}(\omega_b = 1) > p_c(d)$. For these cases an annealed invariance principle was proved by De Masi, Ferrari, Goldstein and Wick [9, 10] and the quenched case was established in $d \ge 4$ by Sidoravicius and Sznitman [25], and in all $d \ge 2$ by Berger and Biskup [6] and Mathieu and Piatnitski [21] .

A significant conceptual deficiency of the latter proofs is that, in $d \ge 3$, they require the use of heat-kernel upper bounds of the form

$$P_{\omega,x}(X_n = y) \le \frac{c_1}{n^{d/2}} \exp\Bigl\{-c_2\frac{|x-y|^2}{n}\Bigr\}, \qquad x, y \in \mathscr{C}_\infty, \tag{1.7}$$

where c_1, c_2 are absolute constants and n is assumed to exceed a random quantity depending on the environment in the vicinity of x and y. These were deduced by Barlow [2] using sophisticated arguments that involve isoperimetry, regular volume growth and comparison of graph and Euclidean distances for the percolation cluster.

Apart from the conceptual difficulties—need of local-CLT type estimates to establish a plain CLT—the use of heat-kernel bounds suffers from another significant problem: The bound (1.7) may actually *fail* once the conductance law has sufficiently heavy tails at zero. This was noted to happen by Fontes and Mathieu [12] for the heat-kernel averaged over the environment; the more relevant quenched situation was analyzed recently by Berger, Biskup, Hoffman and Kozma [7]. The main conclusion of [7] is that the diagonal (i.e., $x = y$) bound in (1.7) holds in $d = 2, 3$ but can be as bad as $o(n^{-2})$ in $d \geq 5$ and, presumably, $o(n^{-2} \log n)$ in $d = 4$. This is caused by the existence of *traps* that may capture the walk for a long time and thus, paradoxically, increase its chances to arrive back to the starting point.

A natural question arises at this point: In the absence of heat-kernel estimates, does the quenched CLT still hold? Our answer to this question is affirmative and constitutes the main result of this note. Another interesting question is what happens when the conductances are unbounded from above; this is currently being studied by Barlow and Deuschel [3].

Note: While this paper was in the process of writing, we received a preprint from P. Mathieu [20] in which he proves a result that is a continuous-time version of our main theorem. As for [6] and [21], the proofs differ in many subtle aspects—e.g., homogenization arguments vs. computations based on geometry of the infinite cluster; the principal ideas—the use of the corrector and the martingale CLT—are, of course, more or less the same. Our approach streamlines considerably the proof of [6] in $d \geq 3$ in that it limits the use of "heat-kernel technology" to a uniform bound on the heat-kernel decay (implied by isoperimetry) and a diffusive bound on the expected distance of the walk from its initial position (implied by regular volume growth).

2. Main results and outline

Let $\Omega = [0, 1]^{\mathbb{B}_d}$ be the set of all admissible random environments and let $\mathbb{P}$ be an i.i.d. law on Ω. Assuming (1.1), let $\mathscr{C}_\infty$ denote the a.s. unique infinite connected component of edges with positive conductances and introduce the conditional measure

$$\mathbb{P}_0(-) = \mathbb{P}(-|0 \in \mathscr{C}_\infty). \tag{2.1}$$

For $T > 0$, let $(C[0, T], \mathscr{W}_T)$ be the space of continuous functions $f : [0, T] \to \mathbb{R}^d$ equipped with the Borel σ-algebra defined relative to the supremum topology.

Here is our main result:

Theorem 2.1 *Suppose $d \geq 2$ and $\mathbb{P}(\omega_b > 0) > p_c(d)$. For $\omega \in \{0 \in \mathscr{C}_\infty\}$, let $(X_n)_{n\geq 0}$ be the random walk with law $P_{\omega,0}$ and let*

$$B_n(t) = \frac{1}{\sqrt{n}}\big(X_{\lfloor tn\rfloor} + (tn - \lfloor tn\rfloor)(X_{\lfloor tn\rfloor+1} - X_{\lfloor tn\rfloor})\big), \qquad t \geq 0. \tag{2.2}$$

Then for all $T > 0$ and for $\mathbb{P}_0$-almost every ω, the law of $(B_n(t) : 0 \leq t \leq T)$ on $(C[0, T], \mathscr{W}_T)$ converges, as $n \to \infty$, weakly to the law of an isotropic Brownian motion $(B_t : 0 \leq t \leq T)$ with a positive and finite diffusion constant.

Using a variant of [6, Lemma 6.4], from here we can extract a corresponding conclusion for the "agile" version of our random walk (cf. [6, Theorem 1.2]) by which we mean the walk that jumps from x to its neighbor y with probability $\omega_{xy}/\pi_\omega(x)$ where $\pi_\omega(x)$ is the sum of ω_{xz} over all of the neighbors z of x. Replacing discrete times by sums of i.i.d. exponential random variables, these invariance principles then extend also to the corresponding continuous-time processes. Finally, Theorem 2.1 of course implies also an annealed invariance principle, which is the above convergence for the walk sampled from the path measure integrated over the environment.

The remainder of this paper is devoted to the proof of this theorem. The main line of attack is similar to the above 1D solution: We define a harmonic coordinate φ_ω—an analogue of (1.5)—and then prove an a.s. invariance principle for $t \mapsto \varphi_\omega(X_{\lfloor nt \rfloor})/\sqrt{n}$ along the argument sketched before. The difficulty comes with showing the sublinearity bound $\varphi_\omega(x) - x = o(x)$. As in Berger and Biskup [6], sublinearity can be proved directly along coordinate directions by soft ergodic-theory arguments. The crux is to extend this to a bound throughout d-dimensional boxes.

Following the $d \geq 3$ proof of [6], the bound along coordinate axes extends to *sublinearity on average*, meaning that the set of sites at which $|\varphi_\omega(x) - x|$ exceeds $\epsilon|x|$ has zero density. The extension of sublinearity on average to pointwise sublinearity is the main novel part of the proof which, unfortunately, still makes non-trivial use of the "heat-kernel technology." A heat-kernel upper bound of the form (1.7) would do but, to minimize the extraneous input, we show that it suffices to have a diffusive bound for the expected displacement of the walk from its starting position. This step still requires detailed control of isoperimetry, volume growth and the comparison between the graph-theoretic and Euclidean distances, but avoids many spurious calculations that are needed for the full-fledged heat-kernel estimates.

Of course, the required isoperimetric inequalities may not be true on $\mathscr{C}_\infty$ because of the presence of weak bonds. As in [7] we circumvent this by observing the random walk on the set of sites that have a connection to infinity by bonds with *uniformly* positive conductances. Specifically we pick $\alpha > 0$ and let $\mathscr{C}_{\infty,\alpha}$ denote the set of sites in $\mathbb{Z}^d$ that are connected to infinity by a path whose edges obey $\omega_b \geq \alpha$. Here we note:

Proposition 2.2 *Let $d \geq 2$ and $p = \mathbb{P}(\omega_b > 0) > p_c(d)$. Then there exists $c(p,d) > 0$ such that if α satisfies*

$$\mathbb{P}(\omega_b \geq \alpha) > p_c(d) \tag{2.3}$$

and

$$\mathbb{P}(0 < \omega_b < \alpha) < c(p,d) \tag{2.4}$$

then $\mathscr{C}_{\infty,\alpha}$ is nonempty and $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ has only finite components a.s. In fact, if $\mathscr{F}(x)$ is the set of sites (possibly empty) in the finite component of $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ containing x, then

$$\mathbb{P}\bigl(x \in \mathscr{C}_\infty \;\&\; \operatorname{diam}\mathscr{F}(x) \geq n\bigr) \leq C\mathrm{e}^{-\eta n}, \qquad n \geq 1, \tag{2.5}$$

for some $C < \infty$ and $\eta > 0$. Here "diam" is the diameter in the ℓ_∞ distance on $\mathbb{Z}^d$.

The restriction of φ_ω to $\mathscr{C}_{\infty,\alpha}$ is still harmonic, but with respect to a walk that can "jump the holes" of $\mathscr{C}_{\infty,\alpha}$. A discrete-time version of this walk was utilized heavily in [7]; for the purposes of this paper it will be more convenient to work with its continuous-time counterpart $Y = (Y_t)_{t\geq 0}$. Explicitly, sample a path of the random walk $X = (X_n)$ from $P_{\omega,0}$ and denote by $T_1, T_2, \dots$ the time intervals between successive visits of X to $\mathscr{C}_{\infty,\alpha}$. These are defined recursively by

$$T_{j+1} = \inf\bigl\{n \geq 1\colon X_{T_1+\dots+T_j+n} \in \mathscr{C}_{\infty,\alpha}\bigr\}, \tag{2.6}$$

with $T_0 = 0$. For each $x, y \in \mathscr{C}_{\infty,\alpha}$, let

$$\hat{\omega}_{xy} = P_{\omega,x}(X_{T_1} = y) \tag{2.7}$$

and define the operator

$$(\mathcal{L}_\omega^{(\alpha)} f)(x) = \sum_{y\in\mathscr{C}_{\infty,\alpha}} \hat{\omega}_{xy}\,\bigl[f(y) - f(x)\bigr]. \tag{2.8}$$

The continuous-time random walk Y is a Markov process with this generator; alternatively take the standard Poisson process $(N_t)_{t\ge 0}$ with jump-rate one and set

$$Y_t = X_{T_1+\dots+T_{N_t}}. \tag{2.9}$$

Note that, while Y may jump "over the holes" of $\mathscr{C}_{\infty,\alpha}$, all of its jumps are finite. The counting measure on $\mathscr{C}_{\infty,\alpha}$ is still invariant for this random walk, $\mathcal{L}_\omega^{(\alpha)}$ is self-adjoint on the corresponding space of square integrable functions and $\mathcal{L}_\omega^{(\alpha)}\varphi_\omega = 0$ on $\mathscr{C}_{\infty,\alpha}$ (see Lemma 5.2).

The skeleton of the proof is condensed into the following statement whose proof, and adaptation to the present situation, is the main novel part of this note:

Theorem 2.3 *Fix α as in (2.3–2.4) and let $\psi_\omega\colon \mathscr{C}_{\infty,\alpha}\to\mathbb{R}^d$ be a function and let $\theta>0$ be a number such that the following holds for a.e. ω:*

(1) *(Harmonicity) $\mathcal{L}_\omega^{(\alpha)}(x+\psi_\omega)=0$ on $\mathscr{C}_{\infty,\alpha}$.*

(2) *(Sublinearity on average) For every $\epsilon>0$,*

$$\lim_{n\to\infty}\frac{1}{n^d}\sum_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}} 1_{\{|\psi_\omega(x)|\ge\epsilon n\}} = 0. \tag{2.10}$$

(3) *(Polynomial growth)*

$$\lim_{n\to\infty}\max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}}\frac{|\psi_\omega(x)|}{n^\theta} = 0. \tag{2.11}$$

Let $Y=(Y_t)$ be the continuous-time random walk with generator $\mathcal{L}_\omega^{(\alpha)}$ and suppose also:

(4) *(Diffusive upper bounds) For a deterministic sequence $b_n = o(n^2)$ and a.e. ω,*

$$\sup_{n\ge1}\max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}}\sup_{t\ge b_n}\frac{E_{\omega,x}|Y_t-x|}{\sqrt{t}}<\infty \tag{2.12}$$

and

$$\sup_{n\ge1}\max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}}\sup_{t\ge b_n} t^{d/2}P_{\omega,x}(Y_t=x)<\infty. \tag{2.13}$$

Then for almost every ω,

$$\lim_{n\to\infty}\max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}}\frac{|\psi_\omega(x)|}{n} = 0. \tag{2.14}$$

This result shows that $\varphi_\omega(x)-x=o(x)$ on $\mathscr{C}_{\infty,\alpha}$ which then extends to $\mathscr{C}_\infty$ by the maximum principle applied to φ_ω on the finite components of $\mathscr{C}_\infty\setminus\mathscr{C}_{\infty,\alpha}$ and using that the component sizes obey a polylogarithmic upper bound. The assumptions (1-3) are known to hold for the corrector of the supercritical bond-percolation cluster and the proof applies, with minor modifications, to the present case as well. The crux is to prove (2.12–2.13) which requires ideas from the "heat-kernel technology." For our purposes it will suffice to take $b_n = n$ in part (4).

The plan of the rest of this paper is a follows: Sect. 3 is devoted to some basic percolation estimates needed in the rest of the paper. In Sect. 4 we define the corrector χ, which is a random function marking the difference between the harmonic coordinate $\varphi_\omega(x)$ and the geometric coordinate x. Then we prove Theorem 4.1. In Sect. 5 we establish the a.s. sublinearity of the corrector as stated in Theorem 2.3 subject to the diffusive bounds (2.12–2.13). Then we assemble all facts into the proof of Theorem 2.1. Finally, in Sect. 6 we adapt some arguments from Barlow [2] to prove (2.12–2.13); first in rather general Propositions 6.1 and 6.2 and then for the case at hand.

3. PERCOLATION ESTIMATES

In this section we provide a proof of Proposition 2.2 and also of a lemma dealing with the maximal distance a random walk Y can travel in a given number of jumps. We will need to work with the "static renormalization" (cf. Grimmett [14, Section 7.4]) whose salient features we will now recall. The underlying ideas go back to the work of Kesten and Zhang [16], Grimmett and Marstrand [15] and Antal and Pisztora [1].

We say that an edge b is occupied if $\omega_b > 0$. Consider the lattice cubes

$$B_L(x) = x + [0, L]^d \cap \mathbb{Z}^d \quad \text{and} \quad \tilde{B}_{3L}(x) = x + [-L, 2L]^d \cap \mathbb{Z}^d \tag{3.1}$$

and note that $\tilde{B}_{3L}(x)$ consists of 3^d copies of $B_L(x)$ that share only sites on their adjacent boundaries. Let $G_L(x)$ be the "good event" which is the set of configurations such that:

(1) Every side of $B_L(Lx)$ is connected to a site on the inner boundary of $\tilde{B}_{3L}(Lx)$ by an occupied path.
(2) Any two paths as in (1) are connected by using only occupied bonds with both endpoints in $\tilde{B}_{3L}(Lx)$.

The sheer existence of infinite cluster implies that (1) occurs with high probability once L is large (see Grimmett [14, Theorem 8.97]) while the situation in (2) occurs with large probability once there is percolation in half space (see Grimmett [14, Lemma 7.89]). It follows that

$$\mathbb{P}\bigl(G_L(x)\bigr) \underset{L\to\infty}{\longrightarrow} 1 \tag{3.2}$$

whenever $\mathbb{P}(\omega_b > 0) > p_c(d)$. A crucial consequence of the above conditions is that, if $G_L(x)$ and $G_L(y)$ occur for neighboring sites $x, y \in \mathbb{Z}^d$, then the largest connected components in $\tilde{B}_{3L}(x)$ and $\tilde{B}_{3L}(y)$—sometimes referred to as *spanning clusters*—are connected. Thus, if $G_L(x)$ occurs for all x along an infinite path on $\mathbb{Z}^d$, the corresponding spanning clusters are subsets of $\mathscr{C}_\infty$.

A minor complication is that the events $\{G_L(x) : x \in \mathbb{Z}^d\}$ are not independent. However, they are 4-dependent in the sense that if (x_i) and (y_j) are such that $|x_i - y_j| > 4$ for each i and j, then the families $\{G_L(x_i)\}$ and $\{G_L(y_j)\}$ are independent. It follows (cf [14, Theorem 7.65]) that the indicators $\{1_{G_L(x)} : x \in \mathbb{Z}^d\}$, regarded as a random process on $\mathbb{Z}^d$, dominate i.i.d. Bernoulli random variables whose density (of ones) tends to one as $L \to \infty$.

Proof of Proposition 2.2. In $d = 2$ the proof is actually very simple because it suffices to choose α such that (2.3) holds. Then $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha} \subset \mathbb{Z}^2 \setminus \mathscr{C}_{\infty,\alpha}$ has only finite (subcritical) components whose diameter has exponential tails (2.5) by, e.g., [14, Theorem 6.10].

To handle general dimensions we will have to invoke the above static renormalization. Let $G_L(x)$ be as above and consider the event $G_{L,\alpha}(x)$ where we in addition require that $\omega_b \notin (0, \alpha)$

for every edge with both endpoints in $B_L(Lx)$. Clearly,

$$\lim_{L\to\infty}\lim_{\alpha\downarrow 0}\mathbb{P}\bigl(G_{L,\alpha}(x)\bigr)=1. \tag{3.3}$$

Using the aforementioned domination by site percolation, and adjusting L and α to have a sufficiently high density of good blocks, we can thus ensure that the set

$$\bigl\{x\in\mathbb{Z}^d\colon G_{L,\alpha}(x)\text{ occurs}\bigr\} \tag{3.4}$$

has a unique infinite component $\mathcal{C}_\infty$, whose complement has only finite components. Moreover, if $\mathcal{G}(0)$ is the finite connected component of $\mathbb{Z}^d\setminus\mathcal{C}_\infty$ containing the origin, then a standard Peierls argument yields

$$\mathbb{P}\bigl(\operatorname{diam}\mathcal{G}(0)\ge n\bigr)\le \mathrm{e}^{-\zeta n} \tag{3.5}$$

for some $\zeta>0$. To prove (2.5), we claim that

$$\mathscr{F}(0)\subset\bigcup_{x\in\mathcal{G}(0)}B_L(Lx). \tag{3.6}$$

Indeed, if $z\in\mathscr{F}(0)$ and x are such that $z\in B_L(Lx)$ then either $B_L(Lx)$ contains a bond with $\omega_b\in(0,\alpha)$, in which case $G_L(x)$ does not occur, or not. If not then z lies in a finite connected component of bonds with $\omega_b\ge\alpha$ whose diameter exceeds L. It suffices to show that any such component lies in a finite connected component of the set in (3.4). This is a standard consequence of properties (1-2) in the definition of $G_L(x)$: If x were adjacent to an infinite path in the set (3.4), then the finite cluster intersecting $B_L(Lx)$ would have to be part of $\mathscr{C}_{\infty,\alpha}$, a contradiction. □

Let $\mathrm{d}(x,y)$ be the "Markov distance" on $V=\mathscr{C}_{\infty,\alpha}$, i.e., the minimal number of jumps the random walk $Y=(Y_t)$ needs to make to get from x to y. Note that $\mathrm{d}(x,y)$ could be quite smaller than the graph-theoretic distance on $\mathscr{C}_{\infty,\alpha}$. To control the volume growth for the Markov graph of the random walk Y—cf. the end of Sect. 6—we will need to know that $\mathrm{d}(x,y)$ is nevertheless comparable with the Euclidean distance $|x-y|$:

Lemma 3.1 *There exists $\varrho>0$ and for each $\gamma>0$ there is $\alpha>0$ obeying (2.3–2.4) such that*

$$\mathbb{P}\bigl(0,x\in\mathscr{C}_{\infty,\alpha}\ \&\ \mathrm{d}(0,x)\le\varrho|x|\bigr)\le\mathrm{e}^{-\gamma|x|},\qquad x\in\mathbb{Z}^d. \tag{3.7}$$

Proof. Suppose α is as in the proof of Proposition 2.2. Let (η_x) be independent Bernoulli that dominate the indicators $1_{A_{L,\alpha}}$ from below and let $\mathcal{C}_\infty$ be the unique infinite component of the set $\{x\in\mathbb{Z}^d\colon\eta_x=1\}$. We may "wire" the "holes" of $\mathcal{C}_\infty$ by putting an edge between every pair of sites on the external boundary of each finite component of $\mathbb{Z}^d\setminus\mathcal{C}_\infty$; we use $\mathrm{d}'(0,x)$ to denote the distance between 0 and x on the induced graph. The processes η and $(1_{G_{L,\alpha}(x)})$ can be coupled so that each connected component of $\mathscr{C}_\infty\setminus\mathscr{C}_{\infty,\alpha}$ with diameter exceeding L is "covered" by a finite component of $\mathbb{Z}^d\setminus\mathcal{C}_\infty$, cf. (3.6). As is easy to check, this implies

$$\mathrm{d}(0,x)\ge\mathrm{d}'(0,x')\quad\text{and}\quad|x'|\ge\frac{1}{L}|x| \tag{3.8}$$

whenever $x\in B_L(Lx')$. It thus suffices to show the above bound for distance $\mathrm{d}'(0,x')$.

Let $z_0 = 0, z_1, \dots, z_n = x$ be a nearest-neighbor path on $\mathbb{Z}^d$. Let $G(z_i)$ be the unique finite component of $\mathbb{Z}^d \setminus \mathcal{C}_\infty$ that contains z_i—if $z_i \in \mathcal{C}_\infty$, we have $G(z_i) = \emptyset$. Define

$$\ell(z_0, \dots, z_n) := \sum_{i=0}^{n} \operatorname{diam} G(z_i) \Bigl(\prod_{j<i} 1_{\{z_j \notin G(z_i)\}} \Bigr). \tag{3.9}$$

We claim that for each $\lambda > 0$ we can adjust L and α so that

$$\mathbb{E} e^{\lambda \ell(z_0, \dots, z_n)} \le e^n \tag{3.10}$$

for all $n \ge 1$ and all paths as above. To verify this we note that the components contributing to $\ell(z_0, \dots, z_n)$ are distance at least one from one another. So conditioning on all but the last component, and the sites in the ultimate vicinity, we may use the Peierls argument to estimate the conditional expectation of $e^{\lambda \operatorname{diam} G(z_n)}$. (The result is finite because $\operatorname{diam} G(z_n)$ is at most order of the boundary of $G(z_n)$.) Proceeding by induction, (3.10) follows.

As the number of nearest-neighbor paths $(z_0 = 0, \dots, z_n = x)$ is bounded by $(2d)^n$, we can adjust L and α so that

$$\mathbb{P}\Bigl(\exists (z_0 = 0, \dots, z_n = x) \colon \ell(z_0, \dots, z_n) > \frac{n}{2} \Bigr) \le e^{-\gamma n} \tag{3.11}$$

for any given $\gamma > 0$. But if $(z_0 = 0, \dots, z_n = n)$ is the shortest nearest-neighbor interpolation of a path that achieves $d'(0, x)$, then

$$d'(0, x) \ge n - \ell(z_0, \dots, z_n). \tag{3.12}$$

Since, trivially, $|x| \le n$ we deduce $\mathbb{P}(d'(0, x) \le \frac{1}{2}|x|) \le e^{-\gamma |x|}$. □

4. CORRECTOR

The purpose of this section is to define, and prove some properties of, the *corrector* $\chi(\omega, x) = \varphi_\omega(x) - x$. This object could be defined probabilistically by the limit

$$\chi(\omega, x) = \lim_{n \to \infty} \bigl(E_{\omega,x}(X_n) - E_{\omega,0}(X_n) \bigr) - x \tag{4.1}$$

unfortunately, at this moment we seem to have no direct (probabilistic) argument showing that the limit exists. The traditional definition of the corrector involves spectral calculus (Kipnis and Varadhan [17]); we will invoke a projection construction from Mathieu and Piatnitski [21].

Let $\mathbb{P}$ be an i.i.d. law on $(\Omega, \mathscr{F})$ where $\Omega = [0,1]^{\mathbb{B}_d}$ and $\mathscr{F}$ is the natural product σ-algebra. Let $\tau_x \colon \Omega \to \Omega$ denote the shift by x, i.e., $(\tau_z \omega)_{xy} = \omega_{x+z, y+z}$, and note that $\mathbb{P} \circ \tau_x^{-1} = \mathbb{P}$ for all $x \in \mathbb{Z}^d$. Recall that $\mathscr{C}_\infty$ is the infinite connected component of edges with $\omega_b > 0$ and, for $\alpha > 0$, let $\mathscr{C}_{\infty,\alpha}$ denote the set of sites connected to infinity by edges with $\omega_b \ge \alpha$. If $\mathbb{P}(0 \in \mathscr{C}_{\infty,\alpha}) > 0$, let

$$\mathbb{P}_\alpha(-) = \mathbb{P}(- | 0 \in \mathscr{C}_{\infty,\alpha}) \tag{4.2}$$

and let $\mathbb{E}_\alpha$ be the corresponding expectation. Given $\omega \in \Omega$ and sites $x, y \in \mathscr{C}_{\infty,\alpha}(\omega)$, let $d_\omega^{(\alpha)}(x, y)$ denote the graph distance between x and y as measured on $\mathscr{C}_{\infty,\alpha}$. We will also use $\mathcal{L}_\omega$ to denote the generator of the continuous-time version of the walk X, i.e.,

$$(\mathcal{L}_\omega f)(x) = \frac{1}{2d} \sum_{y\colon |y-x|=1} \omega_{xy} \bigl[f(y) - f(x) \bigr]. \tag{4.3}$$

The following theorem summarizes all relevant properties of the corrector:

Theorem 4.1 *Suppose $\mathbb{P}(0 \in \mathscr{C}_\infty) > 0$. There exists a function $\chi\colon \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ such that the following holds $\mathbb{P}_0$-a.s.:*

(1) *(Gradient field) $\chi(0,\omega) = 0$ and, for all $x, y \in \mathscr{C}_{\infty,\alpha}(\omega)$,*

$$\chi(\omega, x) - \chi(\omega, y) = \chi(\tau_y\omega, x-y). \tag{4.4}$$

(2) *(Harmonicity) $\varphi_\omega(x) := x + \chi(\omega, x)$ obeys $\mathcal{L}_\omega\varphi_\omega = 0$.*

(3) *(Square integrability) There is $C = C(\alpha) < \infty$ such that for all $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$,*

$$\mathbb{E}_\alpha\bigl(|\chi(\cdot, y) - \chi(\cdot, x)|^2 \,\omega_{xy}\, 1_{\{x\in\mathscr{C}_\infty\}}\bigr) < C \tag{4.5}$$

Let $\alpha > 0$ be such that $\mathbb{P}(0 \in \mathscr{C}_{\infty,\alpha}) > 0$. Then we also have:

(4) *(Polynomial growth) For every $\theta > d$, a.s.,*

$$\lim_{n\to\infty} \max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}} \frac{|\chi(\omega, x)|}{n^\theta} = 0. \tag{4.6}$$

(5) *(Zero mean under random shifts) Let $Z\colon \Omega \to \mathbb{Z}^d$ be a random variable such that*
   (a) *$Z(\omega) \in \mathscr{C}_{\infty,\alpha}(\omega)$,*
   (b) *$\mathbb{P}_\alpha$ is preserved by $\omega \mapsto \tau_{Z(\omega)}(\omega)$,*
   (c) *$\mathbb{E}_\alpha(\mathrm{d}_\omega^{(\alpha)}(0, Z(\omega))^q) < \infty$ for some $q > 3d$.*

Then $\chi(\cdot, Z(\cdot)) \in L^1(\Omega, \mathscr{F}, \mathbb{P}_\alpha)$ and

$$\mathbb{E}_\alpha\bigl[\chi(\cdot, Z(\cdot))\bigr] = 0. \tag{4.7}$$

As noted before, to construct the corrector we will invoke a projection argument from [21]. Abbreviate $L^2(\Omega) = L^2(\Omega, \mathscr{F}, \mathbb{P}_0)$ and let $B = \{\hat{\mathrm{e}}_1, \dots, \hat{\mathrm{e}}_d\}$ be the set of coordinate vectors. Consider the space $L^2(\Omega \times B)$ of square integrable functions $u\colon \Omega \times B \to \mathbb{R}^d$ equipped with the inner product

$$(u, v) = \mathbb{E}_0\Bigl(\sum_{b\in B} u(\omega, b)\cdot v(\omega, b)\,\omega_b\Bigr). \tag{4.8}$$

We may interpret $u \in L^2(\Omega \times B)$ as a flow by putting $u(\omega, -b) = -u(\tau_{-b}\omega, b)$. Some, but not all, elements of $L^2(\Omega \times B)$ can be obtained as gradients of local functions, where the *gradient* ∇ is the map $L^2(\Omega) \to L^2(\Omega \times B)$ defined by

$$(\nabla\phi)(\omega, b) = \phi\circ\tau_b(\omega) - \phi(\omega). \tag{4.9}$$

Let L^2_∇ denote the closure of the set of gradients of all *local* functions—i.e., those depending only on the portion of ω in a finite subset of $\mathbb{Z}^d$—and note the following orthogonal decomposition $L^2(\Omega \times B) = L^2_\nabla \oplus (L^2_\nabla)^\perp$.

The elements of $(L^2_\nabla)^\perp$ can be characterized using the concept of *divergence*, which for $u\colon \Omega\times B\to\mathbb{R}^d$ is the function $\operatorname{div} u\colon \Omega\to\mathbb{R}^d$ defined by

$$\operatorname{div} u(\omega)=\sum_{b\in B}\bigl[\omega_b u(\omega,b)-\omega_{-b}u(\tau_{-b}\omega,b)\bigr]. \tag{4.10}$$

The sums converge in $L^2(\Omega\times B)$. Using the interpretation of u as a flow, $\operatorname{div} u$ is simply the net flow out of the origin. The characterization of $(L^2_\nabla)^\perp$ is now as follows:

Lemma 4.2 *$u\in(L^2_\nabla)^\perp$ if and only if $\operatorname{div} u(\omega)=0$ for $\mathbb{P}_0$-a.e. ω.*

Proof. Let $u\in L^2(\Omega\times B)$ and let $\phi\in L^2(\Omega)$ be a local function. A direct calculation and the fact that $\omega_{-b}=(\tau_{-b}\omega)_b$ yield

$$(u,\nabla\phi)=-\mathbb{E}_0\Bigl(\phi(\omega)\ \operatorname{div} u(\omega)\Bigr). \tag{4.11}$$

If $u\in(L^2_\nabla)^\perp$, then $\operatorname{div} u$ integrates to zero against all local functions. Hence $\operatorname{div} u=0$. □

It is easy to check that every $u\in L^2_\nabla$ is *curl-free* in the sense that for any oriented loop $(x_0,x_1,\dots,x_n)$ on $\mathscr{C}_\infty(\omega)$ with $x_n=x_0$ we have

$$\sum_{j=0}^{n-1}u(\tau_{x_j}\omega,x_{j+1}-x_j)=0. \tag{4.12}$$

On the other hand, every $u\colon\Omega\times B\to\mathbb{R}^d$ which is curl-free can be integrated into a unique function $\phi\colon\Omega\times\mathscr{C}_\infty(\cdot)\to\mathbb{R}^d$ such that

$$\phi(\omega,x)=\sum_{j=0}^{n-1}u(\tau_{x_j}\omega,x_{j+1}-x_j) \tag{4.13}$$

holds for any path $(x_0,\dots,x_n)$ on $\mathscr{C}_\infty(\omega)$ with $x_0=0$ and $x_n=x$. This function will automatically satisfy the *shift-covariance* property

$$\phi(\omega,x)-\phi(\omega,y)=\phi(\tau_y\omega,x-y),\qquad x,y\in\mathscr{C}_\infty(\omega). \tag{4.14}$$

We will denote the space of such functions $\mathcal{H}(\Omega)$. To denote the fact that ϕ is assembled from the shifts of u, we will write

$$u=\operatorname{grad}\phi, \tag{4.15}$$

i.e.,"grad" is a map from $\mathcal{H}(\Omega)$ to functions $\Omega\times B\to\mathbb{R}^d$ assigning $\phi\in\mathcal{H}(\Omega)$ the collection of values $\{\phi(\cdot,b)-\phi(\cdot,0)\colon b\in B\}$.

Lemma 4.3 *Let $\phi\in\mathcal{H}(\Omega)$ be such that $\operatorname{grad}\phi\in(L^2_\nabla)^\perp$. Then ϕ is (discrete) harmonic for the random walk on $\mathscr{C}_\infty$, i.e., for $\mathbb{P}_0$-a.e. ω and all $x\in\mathscr{C}_\infty(\omega)$,*

$$(\mathcal{L}_\omega\phi)(\omega,x)=0. \tag{4.16}$$

Proof. Our definition of divergence is such that "div grad = $2d\,\mathcal{L}_\omega$" holds. Lemma 4.2 implies that $u\in(L^2_\nabla)^\perp$ if and only if $\operatorname{div} u=0$, which is equivalent to $(\mathcal{L}_\omega\phi)(\omega,0)=0$. The translation covariance extends this to all sites in $\mathscr{C}_\infty$. □

Proof of Theorem 4.1(1-3). Consider the function $\phi(\omega, x) = x$ and let $u = \text{grad}\,\phi$. Clearly, $u \in L^2(\Omega \times B)$. Let $G \in L^2_\nabla$ be the orthogonal projection of $-u$ onto L^2_∇ and define $\chi \in \mathcal{H}(\Omega)$ to be the unique function such that

$$G = \text{grad}\,\chi \quad \text{and} \quad \chi(\cdot, 0) = 0. \tag{4.17}$$

This definition immediately implies (4.4), while the definition of the inner product on $L^2(\Omega \times B)$ directly yields (4.5). Since u projects to $-G$ on L^2_∇, we have $u + G \in (L^2_\nabla)^\perp$. But $u + G = \text{grad}\,[x + \chi(\omega, x)]$ and so, by Lemma 4.3, $x \mapsto x + \chi(\omega, x)$ is harmonic with respect to $\mathcal{L}_\omega$. □

For the remaining parts of Theorem 4.1 we will need to work on $\mathscr{C}_{\infty,\alpha}$. However, we do not yet need the full power of Proposition 2.2; it suffices to note that $\mathscr{C}_{\infty,\alpha}$ has the law of a supercritical percolation cluster.

Proof of Theorem 4.1(4). Let $\theta > d$ and abbreviate

$$R_n = \max_{\substack{x \in \mathscr{C}_{\infty,\alpha} \\ |x| \le n}} \bigl|\chi(\omega, x)\bigr|. \tag{4.18}$$

By Theorem 1.1 of Antal and Pisztora [1],

$$\lambda(\omega) := \sup_{x \in \mathscr{C}_{\infty,\alpha}} \frac{\text{d}^{(\alpha)}_\omega(0, x)}{|x|} < \infty, \qquad \mathbb{P}_\alpha\text{-a.s.} \tag{4.19}$$

and so it suffices to show that $R_n/n^\theta \to 0$ on $\{\lambda(\omega) \le \lambda\}$ for every $\lambda < \infty$. But on $\{\lambda(\omega) \le \lambda\}$ every $x \in \mathscr{C}_{\infty,\alpha}$ with $|x| \le n$ can be reached by a path on $\mathscr{C}_{\infty,\alpha}$ that does not leave $[-\lambda n, \lambda n]^d$ and so, on $\{\lambda(\omega) \le \lambda\}$,

$$R_n \le \sum_{\substack{x \in \mathscr{C}_{\infty,\alpha} \\ |x| \le \lambda n}} \sum_{b \in B} \sqrt{\frac{\omega_{x,x+b}}{\alpha}} \bigl|\chi(\omega, x + b) - \chi(\omega, x)\bigr|. \tag{4.20}$$

Invoking the bound (4.5) we then get

$$\| R_n \, 1_{\{\lambda(\omega) \le \lambda\}} \|_2 \le C n^d \tag{4.21}$$

for some constant $C = C(\alpha, \lambda, d) < \infty$. Applying Chebyshev's inequality and summing n over powers of 2 then yields $R_n/n^\theta \to 0$ a.s. on $\{\lambda(\omega) \le \lambda\}$. □

Proof of Theorem 4.1(5). Let Z be a random variable satisfying the properties (a-c). By the fact that $G \in L^2_\nabla$, there exists a sequence $\psi_n \in L^2(\Omega)$ such that

$$\psi_n \circ \tau_x - \psi_n \underset{n \to \infty}{\longrightarrow} \chi(\cdot, x) \qquad \text{in } L^2(\Omega \times B). \tag{4.22}$$

Abbreviate $\chi_n(\omega, x) = \psi_n \circ \tau_x(\omega) - \psi_n(\omega)$ and without loss of generality assume that $\chi_n(\cdot, x) \to \chi(\cdot, x)$ almost surely.

By the fact that Z is $\mathbb{P}_\alpha$-preserving we have $\mathbb{E}_\alpha(\chi_n(\cdot, Z)) = 0$ as soon as we can show that $\chi_n(\cdot, Z) \in L^1(\Omega)$. It thus suffices to prove that

$$\chi_n\bigl(\cdot, Z(\cdot)\bigr) \underset{n \to \infty}{\longrightarrow} \chi\bigl(\cdot, Z(\cdot)\bigr) \qquad \text{in } L^1(\Omega). \tag{4.23}$$

Abbreviate $K(\omega)=\mathrm{d}_\omega^{(\alpha)}(0,Z(\omega))$ and note that, as in part (4),

$$\bigl|\chi_n(\omega,Z(\omega))\bigr|\le\sum_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\|x|\le K(\omega)}}\sum_{b\in B}\sqrt{\frac{\omega_{x,x+b}}{\alpha}}\bigl|\chi_n(\omega,x+b)-\chi_n(\omega,x)\bigr|. \tag{4.24}$$

The quantities $\sqrt{\omega_{x,x+b}}|\chi_n(\omega,x+b)-\chi_n(\omega,x)|\,1_{\{x\in\mathscr{C}_{\infty,\alpha}\}}$ are bounded in L^2, uniformly in x, b and n, and assumption (c) tells us that $K\in L^q$ for some $q>3d$. Ordering the edges in $\mathbb{B}_d$ according to their distance from the origin, Lemma 4.5 of Berger and Biskup [6]—with the choices $p=2$, $s=q/d$ and $N=(2K+1)^d\in L^s(\Omega)$—implies that $\|\chi_n(\cdot,Z(\cdot))\|_r$ are bounded uniformly in n, for some $r>1$. Hence, the family $\{\chi_n(\cdot,Z(\cdot))\}$ is uniformly integrable and (4.23) thus follows by the fact that $\chi_n(\cdot,Z(\cdot))$ converge almost surely. □

5. CONVERGENCE TO BROWNIAN MOTION

Here we will prove Theorem 2.1. We commence by establishing the conclusion of Theorem 2.3 whose proof draws on an idea, suggested to us by Yuval Peres, that sublinearity on average plus heat kernel upper bounds imply pointwise sublinearity. We have reduced the extraneous input from heat-kernel technology to the assumptions (2.12–2.13). These imply heat-kernel upper bounds but generally require significantly less work to prove.

The main technical part of Theorem 2.1 is encapsulated into the following lemma:

Lemma 5.1 *Abusing the notation from* (4.18) *slightly, let*

$$R_n=\max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\|x|\le n}}\bigl|\psi_\omega(x)\bigr|. \tag{5.1}$$

Under the conditions (1,2,4) of Theorem 2.1, for each $\epsilon>0$ and $\delta>0$, there exists an a.s. finite random variable $n_0=n_0(\omega,\epsilon,\delta)$ such that

$$R_n\le\epsilon n+\delta R_{3n}.\qquad n\ge n_0. \tag{5.2}$$

Before we prove this, let us see how this and (2.11) imply (2.14).

Proof of Theorem 2.3. Suppose that $R_n/n\not\to0$ and pick c such that $0<c<\limsup_{n\to\infty}R_n/n$. Let θ be is as in (2.11) and choose

$$\epsilon=\frac{c}{2}\quad\text{and}\quad\delta=\frac{1}{3^{\theta+1}}, \tag{5.3}$$

Note that then $c'-\epsilon\ge3^\theta\delta c'$ for all $c'\ge c$. If $R_n\ge cn$—which happens for infinitely many n's—and $n\ge n_0$, then (5.2) implies

$$R_{3n}\ge\frac{c-\epsilon}{\delta}n\ge3^\theta cn \tag{5.4}$$

and, inductively, $R_{3^kn}\ge3^{k\theta}cn$. However, that contradicts (2.11) by which $R_{3^kn}/3^{k\theta}\to0$ as $k\to\infty$ (with n fixed). □

The idea underlying Lemma 5.1 is simple: We run a continuous-time random walk (Y_t) for time $t=o(n^2)$ starting from the maximizer of R_n and apply the harmonicity of $x\mapsto x+\psi_\omega(x)$

to derive an estimate on the expectation of $\psi(Y_t)$. The right-hand side of (5.2) expresses two characteristic situations that may occur at time t: Either $|\psi_\omega(Y_t)| \le \epsilon n$—which, by "sublinearity on average," happens with overwhelming probability—or Y will not yet have left the box $[-3n, 3n]^d$ and so $\psi_\omega(Y_t) \le R_{3n}$. The point is to show that these are the dominating strategies.

Proof of Lemma 5.1. Fix $\epsilon, \delta > 0$ and let $C_1 = C_1(\omega)$ and $C_2 = C_2(\omega)$ denote the suprema in (2.12) and (2.13), respectively. Let z be the site where the maximum R_n is achieved and denote

$$\mathscr{O}_n = \bigl\{x \in \mathscr{C}_{\infty,\alpha} \colon |x| \le n,\ |\chi(\omega, x)| \ge \tfrac{1}{2}\epsilon n\bigr\}. \tag{5.5}$$

Let $Y = (Y_t)$ be a continuous-time random walk on $\mathscr{C}_{\infty,\alpha}$ with expectation for the walk started at z denoted by $E_{\omega,z}$. Define the stopping time

$$S_n = \inf\bigl\{t \ge 0 \colon |Y_t| \ge 2n\bigr\} \tag{5.6}$$

and note that, in light of Proposition 2.2, we have $|Y_{t\wedge S_n} - z| \le 3n$ for all $t > 0$ provided $n \ge n_1(\omega)$ where $n_1(\omega) < \infty$ a.s. The harmonicity of $x \mapsto x + \psi_\omega(x)$ and the Optional Stopping Theorem yield

$$R_n \le E_{\omega,z}\bigl|\psi_\omega(Y_{t\wedge S_n}) + Y_{t\wedge S_n} - z\bigr|. \tag{5.7}$$

Restricting to t satisfying

$$t \ge b_n \quad \text{and} \quad t \ge b_{3n}, \tag{5.8}$$

we will now estimate the expectation separately on the events $\{S_n < t\}$ and $\{S_n \ge t\}$.

On the event $\{S_n < t\}$, the absolute value in the expectation can simply be bounded by $R_{3n}+3n$. To estimate the probability of $S_n < t$ we decompose according to whether $|Y_{2t} - z| \ge \frac{3}{2}n$ or not. For the former, (5.8) and (2.12) imply

$$P_{\omega,z}\bigl(|Y_{2t} - z| \ge \tfrac{3}{2}n\bigr) \le \frac{E_{\omega,z}|Y_{2t} - z|}{\frac{3}{2}n} \le \frac{2}{3}C_1\frac{\sqrt{2t}}{n}. \tag{5.9}$$

For the latter we invoke the inclusion

$$\bigl\{|Y_{2t} - z| \le \tfrac{3}{2}n\bigr\} \cap \{S_n \le t\} \subset \bigl\{|Y_{2t} - Y_{S_n}| \ge \tfrac{1}{2}n\bigr\} \cap \{S_n \le t\} \tag{5.10}$$

and note that $2t - S_n \in [t, 2t]$, (5.8) and (2.12) give us similarly $P_{\omega,x}(|Y_s - x| \ge n/2) \le 2C_1\sqrt{2t}/n$ for the choice $x = Y_{S_n}$ and $s = 2t - S_n$. From the Strong Markov Property we thus conclude that this serves also as a bound for $P_{\omega,z}(S_n < t, Y_{2t} \ge \frac{3}{2}n)$. Combining both parts and using $\frac{8}{3}\sqrt{2} \le 4$ we thus have

$$P_{\omega,z}(S_n < t) \le \frac{4C_1\sqrt{t}}{n}. \tag{5.11}$$

The $S_n < t$ part of the expectation (5.7) is bounded by $R_{3n} + 3n$ times as much.

On the event $\{S_n \ge t\}$, the expectation in (5.7) is bounded by

$$E_{\omega,z}\bigl(|\psi_\omega(Y_t)|\,1_{\{S_n \ge t\}}\bigr) + E_{\omega,z}\bigl|Y_t - z\bigr|. \tag{5.12}$$

The second term on the right-hand side is then less than $C_1\sqrt{t}$ provided $t \ge b_n$. The first term is estimated depending on whether $Y_t \notin \mathscr{O}_{2n}$ or not:

$$E_{\omega,z}\bigl(|\psi_\omega(Y_t)|\,1_{\{S_n \ge t\}}\bigr) \le \frac{1}{2}\epsilon n + R_{3n}P_{\omega,z}(Y_t \in \mathscr{O}_{2n}). \tag{5.13}$$

For the probability of $Y_t \in \mathscr{O}_{2n}$ we get

$$P_{\omega,z}(Y_t \in \mathscr{O}_{2n}) \le \sum_{x\in\mathscr{O}_{2n}} P_{\omega,z}(Y_t = x) \tag{5.14}$$

which, in light of the Cauchy-Schwarz estimate

$$P_{\omega,z}(Y_t = x)^2 \le P_{\omega,z}(Y_t = z)P_{\omega,x}(Y_t = x) \tag{5.15}$$

and the definition of C_2, is further estimated by

$$P_{\omega,z}(Y_t \in \mathscr{O}_{2n}) \le C_2\,\frac{|\mathscr{O}_{2n}|}{t^{d/2}}. \tag{5.16}$$

From the above calculations we conclude that

$$R_n \le (R_{3n} + 3n)\frac{4C_1\sqrt{t}}{n} + C_1\sqrt{t} + \frac{1}{2}\epsilon n + R_{3n}C_2\,\frac{|\mathscr{O}_{2n}|}{t^{d/2}}. \tag{5.17}$$

Since $|\mathscr{O}_{2n}| = o(n^d)$ as $n \to \infty$ by (2.10), we can choose $t = \xi n^2$ with $\xi > 0$ sufficiently small so that (5.8) applies and (5.2) holds for the given ϵ and δ once n is sufficiently large. □

We now proceed to prove convergence of the random walk $X = (X_n)$ to Brownian motion. Most of the ideas are drawn directly from Berger and Biskup [6] so we stay rather brief. We will frequently work on the truncated infinite component $\mathscr{C}_{\infty,\alpha}$ and the corresponding restriction of the random walk; cf (2.6–2.8). We assume throughout that α is such that (2.3–2.4) hold.

Lemma 5.2 *Let χ be the corrector on $\mathscr{C}_\infty$. Then $\varphi_\omega(x) = x + \chi(\omega, x)$ is harmonic for the random walk observed only on $\mathscr{C}_{\infty,\alpha}$, i.e.,*

$$\mathcal{L}_\omega^{(\alpha)}\varphi_\omega(x) = 0, \qquad \forall x \in \mathscr{C}_{\infty,\alpha}. \tag{5.18}$$

Proof. We have

$$(\mathcal{L}_\omega^{(\alpha)}\varphi_\omega)(x) = E_{\omega,x}\bigl(\varphi_\omega(X_{T_1})\bigr) - \varphi_\omega(x) \tag{5.19}$$

But X_n is confined to a finite component of $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ for $n \in [0, T_1]$, and so $\varphi_\omega(X_n)$ is bounded. Since $(\varphi_\omega(X_n))$ is a martingale and T_1 is an a.s. finite stopping time, the Optional Stopping Theorem tells us $E_{\omega,x}\varphi_\omega(X_{T_1}) = \varphi_\omega(x)$. □

Next we recall the proof of sublinearity of the corrector along coordinate directions:

Lemma 5.3 *For $\omega \in \Omega$, let $(x_n(\omega))_{n\in\mathbb{Z}}$ mark the intersections of $\mathscr{C}_{\infty,\alpha}$ and one of the coordinate axis so that $x_0(\omega) = 0$. Then*

$$\lim_{n\to\infty} \frac{\chi(\omega, x_n(\omega))}{n} = 0, \qquad \mathbb{P}_\alpha\text{-a.s.} \tag{5.20}$$

Proof. Let τ_x be the "shift by x" on Ω and let $\sigma(\omega) = \tau_{x_1(\omega)}(\omega)$ denote the "induced" shift. Standard arguments (cf. [6, Theorem 3.2]) prove that σ is $\mathbb{P}_\alpha$ preserving and ergodic. Moreover,

$$\mathbb{E}_\alpha\bigl(\mathrm{d}_\omega^{(\alpha)}(0, x_1(\omega))^p\bigr) < \infty, \qquad p < \infty, \tag{5.21}$$

by [6, Lemma 4.3] (based on Antal and Pisztora [1]). Theorem 4.1(5) tells us that $\Psi(\omega) := \chi(\omega, x_1(\omega))$ obeys

$$\Psi \in L^1(\mathbb{P}_\alpha) \quad \text{and} \quad \mathbb{E}_\alpha\Psi(\omega) = 0. \tag{5.22}$$

But the gradient property of χ implies

$$\frac{\chi(\omega, x_n(\omega))}{n} = \frac{1}{n}\sum_{k=0}^{n-1} \Psi \circ \sigma^k(\omega) \tag{5.23}$$

and so the left-hand side tends to zero a.s. by the Pointwise Ergodic Theorem. □

We will also need sublinearity of the corrector on average:

Lemma 5.4 *For each $\epsilon > 0$ and $\mathbb{P}_\alpha$-a.e. ω:*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}} 1_{\{|\chi(\omega,x)|\ge\epsilon n\}} = 0. \tag{5.24}$$

Proof. This follows from Lemma 5.3 exactly as [6, Theorem 5.4]. □

Finally, we will assert the validity of the bounds on the return probability and expected displacement of the walk from Theorem 2.3:

Lemma 5.5 *Let (Y_t) denote the continuous-time random walk on $\mathscr{C}_{\infty,\alpha}$. Then the diffusive bounds (2.12–2.13) hold for $\mathbb{P}_\alpha$-a.e. ω.*

We will prove this lemma at the very end of Sect. 6.

Proof of Theorem 2.1. Let α be such that (2.3–2.4) hold and let χ denote the corrector on $\mathscr{C}_\infty$ as constructed in Theorem 4.1. The crux of the proof is to show that χ grows sublinearly with x, i.e., $\chi(\omega, x) = o(|x|)$ a.s.

As in the Introduction, let $\varphi_\omega(x) = x + \chi(\omega, x)$. By Lemmas 5.2 and 5.4, Theorem 4.1(4) and Lemma 5.5, the corrector satisfies the conditions of Theorem 2.3. It follows that χ is sublinear on $\mathscr{C}_{\infty,\alpha}$ as stated in (2.14). However, by (2.4) the largest component of $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ in a box $[-2n, 2n]$ is less than $C \log n$ in diameter, for some random but finite $C = C(\omega)$. Invoking the harmonicity of φ_ω on $\mathscr{C}_\infty$, the Optional Stopping Theorem gives

$$\max_{\substack{x\in\mathscr{C}_\infty\\ |x|\le n}} \bigl|\chi(\omega, x)\bigr| \le \max_{\substack{x\in\mathscr{C}_{\infty,\alpha}\\ |x|\le n}} \bigl|\chi(\omega, x)\bigr| + 2C(\omega)\log(2n), \tag{5.25}$$

whereby we deduce that χ is sublinear on $\mathscr{C}_\infty$ as well.

Having proved the sublinearity of χ on $\mathscr{C}_\infty$, we proceed as in the $d = 2$ proof of [6]. Abbreviate $M_n = \varphi_\omega(X_n)$. Fix $\hat{\mathrm{v}} \in \mathbb{R}^d$ and define

$$f_K(\omega) = E_{\omega,0}\bigl((\hat{\mathrm{v}} \cdot M_1)^2\, 1_{\{|a\cdot M_1|\ge K\}}\bigr). \tag{5.26}$$

By Theorem 4.1(3), $f_K \in L^1(\Omega, \mathscr{F}, \mathbb{P}_0)$ for all K. Since the Markov chain on environments, $n \mapsto \tau_{X_n}(\omega)$, is ergodic (cf. [6, Section 3]), we thus have

$$\frac{1}{n}\sum_{k=0}^{n-1} f_K \circ \tau_{X_k}(\omega) \underset{n\to\infty}{\longrightarrow} \mathbb{E}_0 f_K, \tag{5.27}$$

for $\mathbb{P}_0$-a.e. ω and $P_{\omega,0}$-a.e. path $X = (X_k)$ of the random walk. Using this for $K = 0$ and $K = \epsilon\sqrt{n}$ along with the monotonicity of $K \mapsto f_K$ verifies the conditions of the Lindeberg-Feller

Martingale Functional CLT ([11, Theorem 7.7.3]). Thereby we conclude that the random continuous function

$$t \mapsto \frac{1}{\sqrt{n}}\bigl(\hat{\rm v}\cdot M_{\lfloor nt\rfloor} + (nt-\lfloor nt\rfloor)\,\hat{\rm v}\cdot(M_{\lfloor nt\rfloor+1}-M_{\lfloor nt\rfloor})\bigr) \tag{5.28}$$

converges weakly to Brownian motion with mean zero and covariance

$$\mathbb{E}_0\, f_0 = \mathbb{E}_0 E_{\omega,0}\bigl((\hat{\rm v}\cdot M_1)^2\bigr). \tag{5.29}$$

This can be written as $\hat{\rm v}\cdot D\hat{\rm v}$ where D is the matrix with coefficients

$$D_{i,j} = \mathbb{E}_0 E_{\omega,0}\bigl((\hat{\rm e}_i\cdot M_1)(\hat{\rm e}_j\cdot M_1)\bigr). \tag{5.30}$$

Invoking the Cramér-Wold device ([11, Theorem 2.9.5]) and the fact that continuity of a stochastic process in $\mathbb{R}^d$ is implied by the continuity of its d one-dimensional projections we get that the linear interpolation of $t\mapsto M_{\lfloor nt\rfloor}/\sqrt{n}$ scales to d-dimensional Brownian motion with covariance matrix D. The sublinearity of the corrector then ensures, as in [6, (6.11–6.13)], that

$$X_n - M_n = \chi(\omega, X_n) = o(|X_n|) = o(|M_n|) = o(\sqrt{n}), \tag{5.31}$$

and so the same conclusion applies to $t\mapsto B_n(t)$ in (2.2).

The reflection symmetry of $\mathbb{P}_0$ forces D to be diagonal; the rotation symmetry then ensures that $D=\sigma^2\,\mathbb{1}$ where of $\sigma^2 = ({}^1\!/_d)\mathbb{E}_0 E_{\omega,0}|M_1|^2$. To see that the limiting process is not degenerate to zero we note that if $\sigma=0$ then $\chi(\cdot,x)=-x$ a.s. for all $x\in\mathbb{Z}^d$. But that is impossible since, as we proved above, $x\mapsto\chi(\cdot,x)$ is sublinear a.s. □

6. Heat kernel and expected distance

Here we will derive the bounds (2.12–2.13) and thus establish Lemma 5.5. Most of the derivation will be done for a general countable-state Markov chain; we will specialize to random walk among i.i.d. conductances at the very end of this section. The general ideas underlying these derivations are fairly standard and exist, in some form, in the literature. A novel aspect is the way we control the non-uniformity of volume-growth caused by local irregularities of the underlying graph; cf (6.4) and Lemma 6.3(1). A well informed reader may wish to read only the statements of Propositions 6.1 and 6.2 and then pass directly to the proof of Lemma 5.5.

Let V be a countable set and let $(a_{xy})_{x,y\in V}$ denote the collection of positive numbers with the following properties: For all $x,y\in V$,

$$a_{xy}=a_{yx} \quad\text{and}\quad \pi(x) := \sum_{y\in V} a_{xy} < \infty. \tag{6.1}$$

Consider a continuous time Markov chain (Y_t) on V with the generator

$$(\mathcal{L}f)(x) = \frac{1}{\pi(x)}\sum_{y\in V} a_{xy}\bigl[f(y)-f(x)\bigr]. \tag{6.2}$$

We use P^x to denote the law of the chain started from x, and E^x to denote the corresponding expectation. Consider a graph $G=(V,E)$ where E is the set of all pairs (x,y) such that $a_{xy}>0$. Let $\mathrm{d}(x,y)$ denote the distance between x and y as measured on G.

For each $x \in V$, let $B_n(x) = \{y \in V : \mathrm{d}(x,y) \le n\}$. If $\Lambda \subset V$, we use $Q(\Lambda, \Lambda^{\mathrm{c}})$ to denote the sum

$$Q(\Lambda, \Lambda^{\mathrm{c}}) = \sum_{x\in\Lambda}\sum_{y\in\Lambda^{\mathrm{c}}} a_{xy}. \tag{6.3}$$

Suppose that there are constants $d > 0$ and $\nu \in (0, {}^1\!/_2)$ such that, for some $a > 0$,

$$C_{\mathrm{vol}}(x,a) := \sup_{0<s\le a}\Bigl[\, s^d \sum_{y\in V}\pi(y)\mathrm{e}^{-s\mathrm{d}(x,y)}\Bigr] < \infty \tag{6.4}$$

and

$$C_{\mathrm{iso}}(x) := \inf_{n\ge1}\inf\biggl\{\frac{Q(\Lambda,\Lambda^{\mathrm{c}})}{\pi(\Lambda)^{\frac{d-1}{d}}} \colon \Lambda \subset B_{2n}(x),\ \pi(\Lambda) \ge n^{\nu}\biggr\} > 0. \tag{6.5}$$

Let $V(\epsilon) \subset V$ denote the set of all $x \in V$ that are connected to infinity by a self-avoiding path $(x_0 = x, x_1, \dots)$ with $a_{x_i,x_{i+1}} \ge \epsilon$ for all $i \ge 0$. Suppose that

$$a_\star := \inf\bigl\{\epsilon > 0 \colon V(\epsilon) = V\bigr\} > 0. \tag{6.6}$$

(Note that this does not require a_{xy} be bounded away from zero.)

The first observation is that the heat-kernel, defined by

$$q_t(x,y) := \frac{P^x(Y_t = y)}{\pi(y)}, \tag{6.7}$$

can be bounded in terms of the isoperimetry constant $C_{\mathrm{iso}}(x)$. Bounds of this form are well known and have been derived by, e.g., Coulhon, Grigor'yan and Pittet [8] for heat-kernel on manifolds, and by Lovász and Kannan [18], Morris and Peres [19] and Goel, Montenegro and Tetali [13] in the context of countable-state Markov chains. We will use the formulation for infinite graphs developed in Morris and Peres [19].

Proposition 6.1 *There exists a constant $c_1 \in (1,\infty)$ depending only on d and $a_\star$ such that for $t(x) := c_1[\log(C_{\mathrm{iso}}(x) \vee c_1)]^{\frac{1}{1-2\nu}}$ we have*

$$\sup_{z\in B_t(x)}\sup_{y\in V} q_t(z,y) \le c_1 \frac{C_{\mathrm{iso}}(x)^{-d}}{t^{d/2}}, \qquad t \ge t(x). \tag{6.8}$$

Proof. We will first derive the corresponding bound for the discrete-time version of (Y_t). Let $\mathsf{P}(x,y) = a_{xy}/\pi(x)$ and define $\hat{\mathsf{P}} = \frac12(1+\mathsf{P})$. Let $\hat q_n(x,y) = \hat{\mathsf{P}}^n(x,y)/\pi(y)$. We claim that, for some absolute constant c_1 and any $z \in B_n(x)$,

$$\hat q_n(z,y) \le c_1 \frac{C_{\mathrm{iso}}(x)^{-d}}{n^{d/2}}, \qquad n \ge t(x). \tag{6.9}$$

To this end, let us define

$$\phi(r) = \inf\biggl\{\frac{Q(\Lambda,\Lambda^{\mathrm{c}})}{\pi(\Lambda)} \colon \pi(\Lambda) \le r,\ \Lambda \subset B_{2n}(x)\biggr\}. \tag{6.10}$$

Theorem 2 of Morris and Peres [19] then implies that once

$$n \ge 1 + \int_{4(\pi(z)\wedge\pi(y))}^{4/\epsilon} \frac{4\mathrm{d}r}{r\phi(r)^2} \tag{6.11}$$

we have $\hat{q}_n(z,y) \le \epsilon$. Here we noted that, by time n the Markov chain started at $z \in B_n(x)$ will not leave $B_{2n}(x)$ and so the restriction to $\Lambda \subset B_{2n}(x)$ is redundant up to this time. (We can modify the chain by "attaching" a random walk on a binary tree to each site outside $B_{2n}(x)$; this keeps the conductances inside $B_{2n}(x)$ intact and makes $\Lambda \subset B_{2n}(x)$ superfluous up to time n.)

Now (6.5–6.6) give us

$$\phi(r) \ge \frac{1}{2}\bigl(C_{\rm iso}(x)r^{-1/d} \wedge a_\star n^{-\nu}\bigr) \tag{6.12}$$

where the extra half arises due the consideration of time-delayed chain $\hat{\mathsf{P}} = \frac{1}{2}(1+\mathsf{P})$. The two regimes cross over at $r_n := (C_{\rm iso}(x)/a_\star)^d n^{d\nu}$; the integral is thus bounded by

$$\int_{4(\pi(z)\wedge\pi(y))}^{4/\epsilon} \frac{4{\rm d}r}{r\phi(r)^2} \le 4\frac{n^{2\nu}}{a_\star^2}\log\Bigl(\frac{r_n}{4a_\star}\Bigr) + 2dC_{\rm iso}(x)^{-2}\Bigl(\frac{4}{\epsilon}\Bigr)^{2/d}. \tag{6.13}$$

The first term splits into a harmless factor of order $n^{2\nu}\log n = o(n)$ and a term proportional to $n^{2\nu}\log C_{\rm iso}(x)$ which is $O(n)$ by $n \ge t(x)$. To make the second term order n we choose $\epsilon = c[C_{\rm iso}(x)^2 n]^{-d/2}$ for some constant c. Adjusting c appropriately, (6.9) follows.

To extend the bound (6.9) to continuous time, we note that $\mathcal{L} = 2(\hat{\mathsf{P}}-1)$. Thus if N_t is Poisson with parameter $2t$, then

$$q_t(z,y) = E\hat{q}_{N_t}(z,y). \tag{6.14}$$

But $P(N_t \le \frac{3}{2}t \text{ or } N_t \ge 3t)$ is exponentially small in t, which is much less than (6.8) for $t \ge c_1\log C_{\rm iso}(x)$ with c_1 sufficiently large. As $q_t \le (a_\star)^{-1}$, the $N_t \not\in (\frac{3}{2}t, 3t)$ portion of the expectation in (6.14) is negligible. For $N_t \in (t,3t)$ the uniform bound (6.9) implies (6.8). □

Our next item of business is a diffusive bound on the expected (graph-theoretical) distance traveled by the walk Y_t by time t. As was noted by Bass [4] and Nash [23], this can be derived from the above uniform bound on the heat-kernel assuming regularity of the volume growth. Our proof is an adaptation of an argument of Barlow [2].

Proposition 6.2 *There exist constants $c_2 = c_2(d)$, $c_3 = c_3(d)$ and $c_4 = c_4(d)$ such that the following holds: Let $x \in V$ and suppose $A > 0$ and $t(x) > 1$ are numbers for which*

$$\sup_{y\in V} q_t(x,y) \le \frac{A}{t^{d/2}}, \qquad t \ge t(x), \tag{6.15}$$

holds and let $T(x) = \frac{1}{d}(Aa_\star)^{-4/d} \vee [t(x)\log t(x)]$. Then

$$E^x{\rm d}(x,Y_t) \le A'(x,t)\sqrt{t}, \qquad t \ge T(x), \tag{6.16}$$

with $A'(x,t) = c_2 + c_3\log A + c_4 C_{\rm vol}(x,t^{-1/2})$.

Much of the proof boils down to the derivation of rather inconspicuous but deep relations (discovered by Nash [23]) between the following quantities:

$$M(x,t) := E^x{\rm d}(x,Y_t) = \sum_y \pi(y)q_t(x,y){\rm d}(x,y) \tag{6.17}$$

and

$$Q(x,t) := -E^x\log q_t(x,Y_t) = -\sum_y \pi(y)q_t(x,y)\log q_t(x,y). \tag{6.18}$$

Note that $q_t(x,\cdot) \le (a_\star)^{-1}$ implies $Q(x,t) \ge \log a_\star$.

Lemma 6.3 *There exists a constant c_5 such that for all $t \ge 0$ and all $x \in V$,*

(1) $M(x,t)^d \ge \exp\{-1 - C_{\mathrm{vol}}(x, M(x,t)^{-1}) + Q(x,t)\}$
(2) $M'(x,t)^2 \le Q'(x,t)$.

Proof. (1) The proof follows that of [2, Lemma 3.3] except for the use of the quantity $C_{\mathrm{vol}}(x)$. Pick two numbers $a > 0$ and $b \in \mathbb{R}$ and note that the bound $u \log u + \lambda u \ge -\mathrm{e}^{-\lambda-1}$ implies

$$-Q(x,t) + aM(x,t) + b \ge -\sum_y \pi(y)\mathrm{e}^{-b-1-a\mathrm{d}(x,y)} \tag{6.19}$$

Using the definition of $C_{\mathrm{vol}}(x,a)$ and bounding $\mathrm{e}^{-1} \le 1$ we get

$$-Q(x,t) + aM(x,t) + b \ge -C_{\mathrm{vol}}(x,a)\,\mathrm{e}^{-b}a^{-d} \tag{6.20}$$

Now set $\mathrm{e}^{-b} = a^d$ to make the right-hand side a constant times $C_{\mathrm{vol}}(x)$. Setting $a = M(x,t)^{-1}$ then yields the result.

(2) This is identical to the proof of Lemma 3.3 in Barlow [2]. □

These bounds imply the desired diffusive estimate on $M(x,t)$:

Proof of Proposition 6.2. Suppose without loss of generality that $M(x,t) \ge \sqrt{t}$, because otherwise there is nothing to prove. We follow the proof of [2, Proposition 3.4]. The key input is provided by the inequalities in Lemma 6.3. Define the function

$$L(t) = \frac{1}{d}\Big(Q(x,t) + \log A - \frac{d}{2}\log t\Big) \tag{6.21}$$

and note that $L(t) \ge 0$ for $t \ge t(x)$. Let $t_0 = (Aa_\star)^{-2/d} \vee \sup\{t \ge 0\colon L(t) \le 0\}$. We claim that $M(x,t_0) \le \sqrt{dT(x)}$. Indeed, when $t_0 = (Aa_\star)^{-2/d}$ then this follows by

$$M(x,t_0) \le t_0 = (Aa_\star)^{-2/d} \le \sqrt{dT(x)} \tag{6.22}$$

due to our choice of $T(x)$. On the other hand, when $t_0 > (Aa_\star)^{-2/d}$ we use Lemma 6.3(2), the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality to derive

$$M(x,t_0) \le \sqrt{t_0}\big[Q(x,t_0) - Q(x,0)\big]^{1/2}. \tag{6.23}$$

Since $Q(x,0) \ge \log a_\star$ and $L(t_0) = 0$ by continuity, we have

$$M(x,t_0) \le \sqrt{t_0}\,\Big(\frac{d}{2}\log t_0 - \log A - \log a_\star\Big)^{1/2} \le \sqrt{dt_0\log t_0} \tag{6.24}$$

where we used that $t_0 \ge (Aa_\star)^{-d/2}$ implies $\log A + \log a_\star \ge -\frac{d}{2}\log t_0$. As $t_0 \le t(x)$ and $t(x) > 1$, this is again less than $\sqrt{dT(x)}$.

For $t \ge t_0$ we have $L(t) \ge 0$. Lemma 6.3(2) yields

$$\begin{aligned} M(x,t) - M(x,t_0) &\le \sqrt{d}\int_{t_0}^t \Big(\frac{1}{2s} + L'(s)\Big)^{1/2}\mathrm{d}s \\ &\le \sqrt{d}\int_{t_0}^t \Big(\frac{1}{\sqrt{2s}} + L'(s)\sqrt{s/2}\Big)\mathrm{d}s \le \sqrt{2dt} + L(t)\sqrt{dt}, \end{aligned} \tag{6.25}$$

where we used integration by parts and the positivity of L to derive the last inequality. Now put this together with $M(x,t_0) \le \sqrt{dt}$ and apply Lemma 6.3(1), noting that $C_{\rm vol}(z, M(x,t)^{-1}) \le C_{\rm vol}(z,t^{-1/2})$ by the assumption $M(x,t) \ge \sqrt{t}$. Dividing out an overall factor $\sqrt{t}$, we thus get

$$A^{-1/d} e^{-1/d - c_5 C_{\rm vol}(x,t^{-1/2}) + L(t)} \le 3\sqrt{d} + \sqrt{d}\, L(t). \tag{6.26}$$

This implies that $L(t) \le \tilde c_3 \log A + \tilde c_4 C_{\rm vol}(x, t^{-1/2})$ for some constants $\tilde c_3$ and $\tilde c_4$ depending only on d. Plugging this in (6.25), we get the desired claim. □

We are now finally ready to complete the proof of our main theorem:

Proof of Lemma 5.5. We will apply the above estimates to obtain the proof of the bounds (2.12–2.13). We use the following specific choices

$$V = \mathscr{C}_{\infty,\alpha}, \quad a_{xy} = \hat\omega_{xy}, \quad \pi(x) = 2d, \quad \text{and} \quad b_n = n. \tag{6.27}$$

As $a_\star \ge \alpha$, all required assumptions are satisfied.

To prove (2.13), we note that by Lemma 3.3 of Berger, Biskup, Hoffman and Kozma [7] (using the isoperimetric inequality on the supercritical bond-percolation cluster, cf. Benjamini and Mossel [5] and Rau [24, Proposition 1.2]) we have $C_{\rm iso}(0) > 0$ a.s. Hence, Proposition 6.1 ensures that, for all $z \in \mathscr{C}_{\infty,\alpha}$ with $|z| \le n$,

$$t^{d/2} P_{\omega,z}(Y_t = z) \le 2dc_1 C_{\rm iso}(0)^{-d} \tag{6.28}$$

provided t exceeds some t_1 depending on $C_{\rm iso}(0)$. From here (2.13) immediately follows.

To prove (2.12), we have to show that, a.s.,

$$\sup_{n\ge1} \sup_{t>n} \max_{\substack{z\in\mathscr{C}_{\infty,\alpha}\\ |z|\le n}} C_{\rm vol}(z, t^{-1/2}) < \infty. \tag{6.29}$$

To this end we note that Lemma 3.1 implies that there is a.s. finite $C = C(\omega)$ such that for all $z, y \in \mathscr{C}_{\infty,\alpha}$ with $|z| \le n$ and $|z-y| \ge C\log n$,

$$\mathrm{d}(z,y) \ge \varrho |z-y|. \tag{6.30}$$

It follows that, once $1/a > C\log n$, for every $z \in \mathscr{C}_{\infty,\alpha}$ with $|z| \le n$ we have

$$\sum_{y\in\mathscr{C}_{\infty,\alpha}} e^{-a\mathrm{d}(z,y)} \le c_6 a^d + \sum_{\substack{y\in\mathscr{C}_{\infty,\alpha}\\ |y-z|\ge 1/a}} e^{-a\varrho|z-y|} \le c_7 a^d, \tag{6.31}$$

where c_6 and c_7 are constants depending on d and ϱ. Setting $a = t^{-1/2}$, (6.29) follows.

Once we have the uniform bound (6.29), as well as the uniform bound (6.15) from Proposition 6.1, Proposition 6.2 yields the a.s. inequality

$$\sup_{n\ge1} \sup_{t>n} \max_{\substack{z\in\mathscr{C}_{\infty,\alpha}\\ |z|\le n}} \frac{E_{\omega,z}\mathrm{d}(z,Y_t)}{\sqrt{t}} < \infty. \tag{6.32}$$

To convert $\mathrm{d}(z,Y_t)$ into $|z-Y_t|$ in the expectation, we invoke (6.30) one more time. □

ACKNOWLEDGMENTS

The research of M.B. was supported by the NSF grant DMS-0505356. The authors wish to thank N. Berger, C. Hoffman, G. Kozma and G. Pete for discussions on this problem.

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