On distribution of energy and vorticity for solutions of 2D Navier-Stokes equations with small viscosity

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Abstract

We study distributions of some functionals of space-periodic solutions for the randomly perturbed 2D Navier-Stokes equation, and of their limits when the viscosity goes to zero. The results obtained give explicit information on distribution of the velocity field of space-periodic turbulent 2D flows.

0 Introduction

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We consider the 2D Navier-Stokes equation (NSE) under periodic boundary conditions, perturbed by a random force:

$$\begin{aligned} &v'_\tau - \varepsilon\Delta v + (v\cdot\nabla)v + \nabla\tilde p = \varepsilon^a\,\tilde\eta(\tau,x),\\ &\operatorname{div} v = 0,\quad v = v(\tau,x)\in\mathbb{R}^2,\quad \tilde p = \tilde p(\tau,x),\quad x\in\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2). \end{aligned} \tag{0.1}$$

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Here $0<\varepsilon\ll 1$ and the scaling exponent a is a real number. We assume that $a<\frac32$ since $a\ge\frac32$ corresponds to non-interesting equations with small solutions (see [Kuk06a], Section 10.3). It is also assumed that $\int v\,dx \equiv \int\tilde\eta\,dx\equiv 0$ and that the force $\tilde\eta$ is a divergence-free Gaussian random field, white in time and smooth in x. Under mild non-degeneracy assumption on $\tilde\eta$ (see in Section 1) the Markov process which the equation defines in the function space $\mathcal{H}$,

$$\mathcal{H} = \{u(x)\in L^2(\mathbb{T}^2;\mathbb{R}^2)\mid \operatorname{div} u = 0,\ \int_{\mathbb{T}^2} u\,dx = 0\},$$

has a unique stationary measure. We are interested in asymptotic (as $\varepsilon \to 0$) properties of this measure and of the corresponding stationary solution. The substitution

$$v = \varepsilon^b u\,, \quad \tau = \varepsilon^{-b} t\,, \quad \nu = \varepsilon^{3/2-a}\,,$$

where $b = a - 1/2$, reduces eq. (0.1) to

$$\dot u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \sqrt{\nu}\, \eta(t,x), \quad \operatorname{div} u = 0, \tag{0.2}$$

where $\dot u = u'_t$ and $\eta(t) = \varepsilon^{b/2} \tilde\eta(\varepsilon^{-b} t)$ is a new random field, distributed as $\tilde\eta$ (see [Kuk06a]). Below we study eq. (0.2).

Let μ_ν be the unique stationary measure for (0.2) and $u_\nu(t) \in \mathcal{H}$ be the corresponding stationary solution, i.e., $\mathcal{D} u_\nu(t) \equiv \mu_\nu$ (here and below $\mathcal{D}$ signifies the distribution of a random variable). Comparing to other equations (0.1), the equation (0.2) has the special advantage: when $\nu \to 0$ along a subsequence $\{\nu_j\}$, stationary solution u_{ν_j} converges in distribution to a stationary process $U(t) \in \mathcal{H}$, formed by solutions of the Euler equation

$$\dot u(t,x) + (u \cdot \nabla) u + \nabla p = 0\,, \quad \operatorname{div} u = 0\,. \tag{0.3}$$

Accordingly, $\mu_{\nu_j} \rightharpoonup \mu_0$, where $\mu_0 = \mathcal{D} U(0)$ is an invariant measure for (0.3) (see below Theorem 1.1). The solution U is called the *Eulerian limit*. This is a random process of order one since $\mathbf{E} |\nabla_x U(t,\cdot)|^2_{\mathcal{H}}$ equals to an explicit non-zero constant. The goal of this paper is to study properties of the measure μ_0 since they are responsible for asymptotical properties of solutions for equation (0.1).

The first main difficulty in this study is to rule out the possibility that with a positive probability the energy $E(u_\nu)$ of the process u_ν, equal to $\frac12 \int |u_\nu(t,x)|^2\, dx$, becomes very small with ν (and that the energy of the Eulerian limit vanishes with a positive probability). In Section 2 we show that this is not the case and that

$$\mathbf{P}\{E(u_\nu) < \delta\} \le C \delta^{1/4}, \quad \forall\, \delta > 0, \tag{0.4}$$

for each ν. To prove the estimate we develop further some ideas, exploited in [KP06] in a similar situation. Namely, we construct a new process $\tilde u_\nu \in \mathcal{H}$, coupled to the process u_ν, such that $E(\tilde u_\nu(\tau)) = E(u_\nu(\tau \nu^{-1}))$ and $\tilde u_\nu$ satisfies an Ito equation, independent from ν. Next we use Krylov's result [Kry87] on distribution of Ito integrals to estimate $\mathcal{D} \tilde u_\nu(\tau)$ and recover (0.4).

In Section 3 we use (0.4) to prove that the distribution of energy of the Eulerian limit U has a density against the Lebesgue measure, i.e.

$$\mathcal{D}E(U) = e(x)\,dx, \quad e \in L_1(\mathbb{R}_+).$$

The functionals $\Phi_f(u(\cdot)) = \int f(\operatorname{rot} u(x))\,dx$ are integrals of motion for the Euler equation. An analogy with the averaging theory for finite-dimensional stochastic equations (e.g., see [FW03]) suggests that their distributions behave well when $\nu \to 0$. Accordingly, in Section 4 we study the distributions of vector-valued random variables

$$\Phi_{\mathbf{f}}(u_\nu(t)) = \big(\Phi_{f_1}(u_\nu(t), \dots, \Phi_{f_m}(u_\nu(t)\big) \in \mathbb{R}^m\,,$$

and of $\Phi_{\mathbf{f}}(U(t))$. Assuming that the functions f_j are analytic, linearly independent and satisfy certain restriction on growth, we show that the distribution of $\Phi_{\mathbf{f}}(U(t))$ has a density against the Lebesgue measure:

$$\mathcal{D}\big(\Phi_{\mathbf{f}}U(t)\big) = p_{\mathbf{f}}(x)\,dx'\,, \quad p_{\mathbf{f}} \in L_1(\mathbb{R}^m).$$

To prove this result we show that the measures $\mathcal{D}\Phi_{\mathbf{f}}u_\nu(t)$ are absolutely continuous with respect to the Lebesgue measure, uniformly in ν. The proof crucially uses (0.4) as well as obtained in [Kuk06b] uniform in ν bounds on exponential moments of the random variables $\operatorname{rot}(u_\nu(t,x))$.

Since m is arbitrary, then this result implies that the measure μ_0 is genuinely infinite dimensional in the sense that any compact set of finite Hausdorff dimension has zero μ_0-measure.

Other equations. The results and the methods of this work apply to other PDE of the form

$$\langle\text{Hamiltonian equation}\rangle + \nu\langle\text{dissipation}\rangle = \sqrt{\nu}\,\langle\text{random force}\rangle\,, \qquad (0.5)$$

provided that the corresponding Hamiltonian PDE has at least two 'good' integrals of motion. In particular, they apply to the randomly forced complex Ginzburg-Landau equation

$$\dot u - (\nu + i)\Delta u + i|u|^2 u = \sqrt{\nu}\,\eta(t,x), \quad \dim x \le 4, \qquad (0.6)$$

supplemented with the odd periodic boundary conditions. The corresponding Hamiltonian PDE is the NLS equation, having two 'good' integrals: the

Hamiltonian H and the total number of particles $E = \frac{1}{2}\int |u|^2\,dx$. Eq. (0.6) was considered in [KS04], where it was proved that for stationary in time solutions u_ν of (0.6) an inviscid limit $V(t)$ (as $\nu \to 0$ along a subsequence) exists and possesses properties, similar to those, stated in Theorem 1.1. The methods of this work allow to prove that the random variable $E(u_\nu(t))$ satisfies (0.4) uniformly in $\nu > 0$, that $H(u_\nu(t))$ meets similar estimates and that V is distributed in such a way that $\mathcal{D}\big(H(V(t))\big)$ and $\mathcal{D}\big(E(V(t))\big)$ are absolutely continuous with respect to the Lebesgue measure.

If $\dim x = 1$, then the NLS equation is integrable and the inviscid limit V may be analysed further, using the methods, developed in [KP06] to study the damped/driven KdV equation (which is another example of the system (0.5)).

Certainly our methods as well apply to some finite-dimensional systems of the form (0.5). In particular – to Galerkin approximations for the 3D NSE under periodic boundary conditions, perturbed by a random force, similar to (1.2). It is easy to establish for that system analogies of results in Sections 1-3. More interesting example is given by system (0.5), where the Hamiltonian equation is the Euler equation for a rotating solid body [Arn89]. This system can be cautiously regarded as a finite-dimensional model for (0.1); see Appendix.[1]

1 Preliminaries

Using the Leray projector $\Pi : L^2(\mathbb{T}^2;\mathbb{R}^2) \to \mathcal{H}$ we rewrite eq. (0.2) as the equation for $u(t) = u(t,\cdot) \in \mathcal{H}$:

$$\dot{u} + \nu A(u) + B(u) = \sqrt{\nu}\,\eta(t). \tag{1.1}$$

Here $A(u) = -\Pi\Delta u$ and $B(u) = \Pi(u\cdot\nabla)u$. We denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the L_2-norm and scalar product in $\mathcal{H}$. Let $(e_s, s \in \mathbb{Z}^2\setminus\{0\})$ be the standard trigonometric basis of this space:

$$e_s(x) = \frac{\sin(s\cdot x)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix} \quad \text{or} \quad e_s(x) = \frac{\cos(s\cdot x)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix},$$

[1] We are thankful to V. V. Kozlov and members of his seminar in MSU for drawing our attention to this equation.

depending whether $s_1+s_2\delta_{s_1,0}>0$ or $s_1+s_2\delta_{s_1,0}<0$. The force η is assumed to be a Gaussian random field, white in time and smooth in x:

$$\eta=\frac{d}{dt}\zeta(t,x),\quad \zeta=\sum_{s\in\mathbb{Z}^2\setminus\{0\}} b_s\beta_s(t)e_s(x)\,, \tag{1.2}$$

where $\{b_s\}$ is a set of real constants, satisfying

$$b_s=b_{-s}\neq 0\quad \forall\, s,\qquad \sum|s|^2b_s^2<\infty\,,$$

and $\{\beta_s(t)\}$ are standard independent Wiener processes.

This equation is known to have a unique stationary measure μ_ν.[2] This is a probability Borel measure in the space $\mathcal{H}$ which attracts distributions of all solutions for (1.1). Let $u_\nu(t,x)$ be a corresponding stationary solution, i.e.

$$\mathcal{D}u_\nu(t)\equiv\mu_\nu.$$

Apart from being stationary in t, this solution is known to be stationary (=homogeneous) in x.

For any $l\geq 0$ we denote by $\mathcal{H}^l, l\geq 0$, the Sobolev space $\mathcal{H}\cap H^l(\mathbb{T}^2;\mathbb{R}^2)$, given the norm

$$\|u\|_l=\Big(\int\big((-\Delta)^{l/2}u(x)\big)^2\,dx\Big)^{1/2} \tag{1.3}$$

(so $\|u\|_0=\|u\|$). A straightforward application of Ito's formula to $\|u_\nu(t)\|^2$ and $\|u_\nu(t)\|_1^2$ implies that

$$\mathbf{E}\,\|u_\nu(t)\|_1^2\equiv\frac12\,B_0\,,\quad \mathbf{E}\|u_\nu(t)\|_2^2\equiv\frac12\,B_1\,, \tag{1.4}$$

where for $l\in\mathbb{R}$ we denote $B_l=\sum|s|^{2l}b_s^2$ (note that $B_0,B_1<\infty$ by assumption); e.g. see in [Kuk06a].

The theorem below describes what happen to the stationary solutions $u_\nu(t,x)$ as $\nu\to 0$. For the theorem's proof see [Kuk06a].

[2] Due to results of the recent work [HM06], the stationary measure μ_ν is unique if $b_s\neq 0$ for $|s|\leq N$, where N is a ν-independent constant. Theorems 1.1 and 2.1 below remain true under this weaker assumption, but our arguments in Sections 3, 4 use essentially that all coefficients b_s are non-zero.

Theorem 1.1. *Any sequence $\tilde\nu_j \to 0$ contains a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}U(\cdot) \quad in \quad \mathcal{P}\big(C(0,\infty;\mathcal{H}^1)\big)\,. \tag{1.5}$$

The limiting process $U(t) \in \mathcal{H}^1$, $U(t) = U(t,x)$, is stationary in t and in x. Moreover,

1)a) every its trajectory $U(t,x)$ is such that

$$U(\cdot) \in L_{2\,loc}(0,\infty;\mathcal{H}^2), \quad \dot U(\cdot) \in L_{1\,loc}(0\,\infty;\mathcal{H}^1)\,.$$

b) It satisfies the free Euler equation (0.3), so $\mu_0 = \mathcal{D}(U(0))$ is an invariant measure for (0.3),

c) $\|U(t)\|_0$ and $\|U(t)\|_1$ are time-independent quantities. If g is a bounded continuous function, then $\int_{\mathbb{T}^2} g(\mathrm{rot}\,U(t,x))\,dx$ also is a time-independent quantity.

2) For each $t \geq 0$ we have $\mathbf{E}\|U(t)\|_1^2 = \frac12 B_0$, $\mathbf{E}\|U(t)\|_2^2 \leq \frac12 B_1$ and $\mathbf{E}\exp\big(\sigma\|U(t)\|_1^2\big) \leq C$ for some $\sigma > 0, C \geq 1$.

Amplification. If $B_2 < \infty$, then the convergence (1.5) holds in the space $\mathcal{P}(C(0,\infty;\mathcal{H}^\varkappa))$, for any $\varkappa < 2$.

See [Kuk06a], Remark 10.4.

Due to 1b), the measure $\mu_0 = \mathcal{D}U(0)$ is invariant for the Euler equation. By 2) it is supported by the space $\mathcal{H}^2$ and is not a δ-measure at the origin. The process U is called the *Eulerian limit* for the stationary solutions u_ν of (1.1). Note that apriori the process U and the measure μ_0 depend on the sequence ν_j.

Since $\|u\|_1^2 \leq \|u\|_0\|u\|_2$ and $\mathbf{E}\,\|u\|_1^2 \leq (\mathbf{E}\|u\|_0^2)^{1/2}(\mathbf{E}\|u\|_1^2)^{1/2}$, then (1.4) implies that

$$\frac12 B_0^2 B_1^{-1} \leq \mathbf{E}\|u_\nu(t)\|_0^2 \leq \frac12 B_1 \tag{1.6}$$

for all ν. That is, the characteristic size of the solution u_ν remains ~ 1 when $\nu \to 0$. Since the characteristic space-scale also is ~ 1, then the Reynolds number of u_ν grows as ν^{-1} when ν decays to zero. Hence, Theorem 1.1 describes a transition to turbulence for space-periodic 2D flows, stationary in time. Recall that eq. (0.2) is the only 2D NSE (0.1), having a limit of order one as $\nu \to 0$ (cf. [Kuk06a], Section 10.3). Thus the various Eulerian limits as in Theorem 1.1 with different coefficients $\{b_s\}$ (corresponding to different spectra of the applied random forces) describe all possible 2D space-periodic stationary turbulent flows.

Our goal is to study further properties of the Eulerian limit.

2 Estimate for energy of solutions

2.1 The result

The energy $E_\nu(t) = \frac{1}{2}\|u_\nu(t)\|_0^2$ of a stationary solution u_ν is a stationary process. It satisfies the relations

$$\frac{1}{4}B_0^2 B_1^{-1} \le \mathbf{E}E_\nu(t) = \frac{1}{4}B_0\,, \quad \mathbf{E}\exp(\sigma E_\nu(t)) \le C\,, \tag{2.1}$$

where $\sigma, C > 0$ are independent from ν (see (1.6) and [Kuk06a], Section 4.3). Let us arrange the numbers $|b_s|$ in the decreasing order: $|b_{s_1}| \ge |b_{s_2}| \ge \dots$.

Theorem 2.1. *There exists a constant $C > 0$, depending only on B_1 and $|b_{s_2}|$, such that*

$$\mathbf{P}\{E_\nu(t) < \delta\} \le C\delta^{1/4}, \tag{2.2}$$

uniformly in $\nu \in (0, 1]$.

Due to the convergence (1.5), the energy $E_0(t) = \frac{1}{2}\|U(t)\|^2$ of the Eulerian limit also satisfies (2.2).

Introducing the fast time $\tau = t\nu^{-1}$ we get for $u(\tau) = u(\tau, x)$ the equation

$$du(\tau) = (-Au - \nu^{-1}B(u))d\tau + \sum_s b_s e_s\, d\beta_s(\tau)\,, \tag{2.3}$$

where $\{\beta_s(\tau) = \sqrt{\nu}\,\beta_s(\nu\tau),\ s \in \mathbb{Z}^2\backslash 0\}$, are new standard independent Wiener processes.

2.2 Beginning of proof

The proof goes in five steps. We start with a geometrical lemma which is used below in the heart of the construction.

Let us denote by S the sphere $\{u \in \mathcal{H} \mid \|u\|_0 = 1\}$. Let $\{e_j, j \ge 1\}$, be the basis $\{e_s, s \in \mathbb{Z}^2 \setminus \{0\}\}$, re-parameterised by the natural numbers in such a way that $e_j = e_{s(j)}$, where $|s(j)| \ge |s(i)|$ if $j \ge i$.

Lemma 2.2. *There exists $\delta > 0$ with the following property. Let v_0, $\tilde v_0$ be any two points in S. Then for $(v, \tilde v) \in S \times S$ such that*

$$\|v - v_0\|_0 < \delta, \quad \|\tilde v - \tilde v_0\|_0 < \delta \tag{2.4}$$

there exists an unitary operator $U_{(v,\tilde v)} = U^{(v_0,\tilde v_0)}_{(v,\tilde v)}$ *of the space* $\mathcal{H}$*, satisfying*

i) U *is an operator-valued Lipschitz function of* v *and* $\tilde v$ *with a Lipschitz constant* ≤ 2*;*

ii) $U_{(v,\tilde v)}(\tilde v) = v$*;*

iii) there exists a unitary vector $\eta = \eta(v, \tilde v)$ *in the plane* span $\{e_1, e_2\}$ *such that the vector* $U_{(v,\tilde v)}(\eta)$ *makes with this plane an angle* $\le \pi/4$*. Accordingly,*

$$\max_{i,j\in\{1,2\}} \left|(U_{(v,\tilde v)} e_i, e_j)\right| \ge c_*, \tag{2.5}$$ 2.5

where $c_* > 0$ *is an absolute constant.*

Proof. Let us start with the following observation:

There exists $\delta > 0$ *such that for any* $v_0 \in S$ *and* $v_1 \in \{v \in S \mid \|v - v_0\|_0 < \delta\}$ *there exists an unitary transformation* W_{v_1,v_0} *of the space* $\mathcal{H}$ *with the following property:* $W_{v_0,v_0} = id$*,* $W_{v_1,v_0}(v_0) = v_1$ *and* W *is a Lipschitz function of* v_1 *and* v_0 *with a Lipschitz constant* ≤ 2*.*

To prove the assertion let us denote by $\mathcal{A}$ the linear space of bounded anti self-adjoint operators in $\mathcal{H}$ (given the operator norm), and consider the map

$$\mathcal{A} \times S \to S, \quad (A, v) \mapsto e^A v\,.$$

Note that the differential of this map in A, evaluated at $A = 0, v = v_0$, is the map $A' \mapsto A'v_0$, which sends $\mathcal{A}$ to the space $T_{v_0}S = \{v \in \mathcal{H} \mid (v, v_0) = 0\}$ and admits a right inverse operator of unit norm. So the assertion with $W = e^A$, where A satisfies the equation $e^A v_0 = v_1$, follows from the implicit function theorem.

To prove the lemma we choose unit vectors $\eta_0, \tilde\eta_0 \in$ span $\{e_1, e_2\}$ such that $(v_0, \eta_0) = 0$ and $(\tilde v_0, \tilde\eta_0) = 0$. Next we choose an unitary transformation U, such that $U(\tilde v_0) = v_0$ and $U(\tilde\eta_0) = \eta_0$. For vectors $v, \tilde v$, satisfying (2.4), denote $U(\tilde v) = \tilde\xi$. Then $\|\tilde\xi - v_0\|_0 < \delta$. Let $W_{v,\tilde\xi}$ be the operator from the assertion above. We set $U_{v,\tilde v} = W_{v,\tilde\xi} \circ U$. This operator obviously satisfies i) and ii). Since $\|U_{v,\tilde v}(\tilde\eta_0) - \eta_0\|_0 \le C\delta$, then choosing $\delta < C^{-1}2^{-1/2}$ we achieve iii) with $\eta = \tilde\eta_0$. $\square$

Remark. Let j_1 and j_2 be any two different natural numbers. The same arguments as above prove existence of an unitary operator U, satisfying i), ii) and such that $\max_{i\in\{1,2\},\, j\in\{j_1,j_2\}} \left|(Ue_i, e_j)\right| \ge c_*\,.$

For any $(v_0, \tilde v_0) \in S \times S$ let $\mathcal{O}_\delta(v_0, \tilde v_0) \subset S \times S$ be the open domain, formed by all pairs $(v, \tilde v)$, satisfying (2.4). Let $\mathcal{O}^1, \mathcal{O}^2, \dots$ be a countable system of domains $\mathcal{O}_{\delta/2}(v_j, \tilde{v}_j) =: \mathcal{O}^j$, $j \geq 1$, which cover $S \times S$. We call $(v_j, \tilde{v}_j)$ the *centre* of the domain $\mathcal{O}^j$.

Consider the mapping

$$S \times S \to \mathbb{N}, \quad (v, \tilde{v}) \mapsto n(v, \tilde{v}) = \min\{j \mid (v, \tilde{v}) \in \mathcal{O}^j\}\,. \tag{2.6}$$ map

It is measurable with respect to the Borel sigma-algebras. Finally, for $j = 1, 2, \ldots$ and $(v, \tilde{v}) \in \mathcal{O}^j$ we define the operators

$$U^j_{v,\tilde{v}} = U^{(v_j, \tilde{v}_j)}_{v,\tilde{v}}.$$

2.3 Step 1: equation for $\tilde{u}(t)$

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Till the end of Section 2 for any $u \in \mathcal{H}$ we will denote

$$v = u/\|u\|_0 \;\text{ if } u \neq 0 \text{ and } v = e_1 \text{ if } u = 0. \tag{2.7}$$ denote

Let us fix any $T_0 > 0$. We start to construct a process $\tilde{u}(\tau)$, $0 \leq \tau \leq T_0$, with continuous trajectories, satisfying $\|\tilde{u}(\tau)\|_0 \equiv \|u(\tau)\|_0$. The process will be constructed as a solution of a stochastic equation, in terms of some stopping times $0 = \tau_0 \leq \tau_1 < \tau_2 < \ldots$.

We set $\tau_0 = 0$ and define a random variable $n_0 = n(v(0), v(0)) \in \mathbb{N}$ (see (2.6)). Let us consider the following stochastic equation for $\mathbf{u}(\tau) = (u(\tau), \tilde{u}(\tau)) \in \mathcal{H} \times \mathcal{H}$:

$$du(\tau) = (-Au - \nu^{-1}B(u))d\tau + \sum_s b_s e_s \, d\beta_s(\tau), \tag{2.8}$$ e1

$$d\tilde{u}(\tau) = -U^*_{\mathbf{u}} Au \, d\tau + \sum_s U^*_{\mathbf{u}} b_s e_s \, d\beta_s(\tau). \tag{2.9}$$ e2

Here $U^*_{\mathbf{u}}$ is the adjoint to the unitary operator $U_{\mathbf{u}} = U^{n_0(\omega)}_{v,\tilde{v}}$ (where $v = v(u)$ and $\tilde{v} = \tilde{v}(\tilde{u})$, see (2.7)). Let us fix any $\gamma \in (0, 1]$ and define the stopping times

$$T_\gamma = \inf\{\tau \in [0, T_0] \mid \|u(\tau)\|_0 \wedge \|\tilde{u}(\tau)\|_0 \leq \gamma \;\text{ or }\; \|u(\tau)\|_2 \geq \gamma^{-1}\}\,,$$

$$\tau_1 = \inf\{\tau \in [0, T_0] \mid \mathbf{u}(\tau) \notin \mathcal{O}_\delta(v_{n_0}, \tilde{v}_{n_0})\} \wedge T_\gamma\,.$$

Here and in similar situations below $\inf \emptyset = T_0$, and $(v_{n_0}, \tilde{v}_{n_0})$ is the centre of the domain $\mathcal{O}^{n_0}$.

For $0 \le \tau \le \tau_1$ the operator $U_{\mathbf{u}}$ is a Lipschitz function of $\mathbf{u}$ since $\|u\|_0 \ge \gamma$ and $\|\tilde{u}\|_0 \ge \gamma$. As $\|u(\tau)\|_2 \le \gamma^{-1}$ for $\tau \le T_\gamma$, then it is not hard to see that the system (2.8),(2.9), supplemented with the initial condition

$$\mathbf{u}(0) = (u(0), u(0)) \tag{2.10}$$ 2.8

has a unique strong solution $\mathbf{u}(\tau)$, $0 \le \tau \le \tau_1$, satisfying

$$\mathbf{E} \sup_{0 \le \tau \le \tau_1} \|\tilde{u}(\tau)\|_0^2 \le C(T_0, \nu, \gamma). \tag{2.11}$$ 2.9

Next we set $n_1 = n(v(\tau_1), \tilde{v}(\tau_1))$ and for $\tau \ge \tau_1$ re-define the operator $U_{\mathbf{u}}$ in (2.9) as $U_{v,\tilde{v}}^{n_1(\omega)}$ (as before, $v = v(u(\tau))$ and $\tilde{v} = \tilde{v}(\tilde{u}(\tau))$). We set

$$\tau_2 = \inf\{\tau \in [\tau_1, T_0] \mid \mathbf{u}(\tau) \notin \mathcal{O}_\delta(v_{n_1}, \tilde{v}_{n_1})\} \wedge T_\gamma,$$

where $(v_{n_1}, \tilde{v}_{n_1})$ is the centre of $\mathcal{O}^{n_1}$, and consider the system (2.8), (2.9) for $\tau_1 \le \tau \le \tau_2$ with the initial condition at τ_1, obtained by continuity. The system has a unique strong solution and (2.11) holds with τ_1 replaced by τ_2. Iterating this construction we obtain stopping times $\tau_0 \le \tau_1 \le \tau_2 \le \dots$, the operator $U_{\mathbf{u}}(\tau)$, piecewise constant in τ and discontinuous at points $\tau = \tau_j$, as well as a strong solution $\mathbf{u}(\tau)$ of (2.8)-(2.10), defined for $0 \le \tau < \lim_{j\to\infty} \tau_j \le T_\gamma$, and satisfying (2.11) with τ_1 replaced by any τ_j. Clearly $\tau_j < \tau_{j+1}$, unless $\tau_j = \tau_{j+1} = T_\gamma$.

2.4 Step 2: growth of stopping times τ_j

For any $\tau \ge 0$ let us write $\tilde{u}(\tau \wedge T_\gamma)$ as

$$\tilde{u}(\tau \wedge T_\gamma) = u(0) - \int_0^{\tau \wedge T_\gamma} U^* A(u)\, d\theta + \int_0^{\tau \wedge T_\gamma} \sum_s b_s U^* e_s\, d\beta_s =: (\tilde{u}_1 + \tilde{u}_2)(\tau)\,.$$

Since $\|u\|_2 \le \gamma^{-1}$, then the process $\tilde{u}_1(\tau) \in \mathcal{H}$ is Lipschitz in τ. A straightforward application of the Kolmogorov criterion implies that the process $\tilde{u}_2(\tau) \in \mathcal{H}$ a.s. satisfies the Hölder condition with the exponent $1/3$. So the process $\tilde{u}(\tau \wedge T_\gamma)$ is a.s. Hölder. The process $u(\tau \wedge T_\gamma)$ is Hölder as well, so

$$\|\mathbf{u}\big((\tau_j + \Delta) \wedge T_\gamma; \omega\big) - \mathbf{u}(\tau_j; \omega)\|_0 \le K(\omega) \Delta^{1/3}\,.$$

Since $\|\mathbf{u}(\tau_{j+1}) - \mathbf{u}(\tau_j)\|_0 \ge \frac{\delta}{2}$ unless $\tau_{j+1} = T_\gamma$, then $|\tau_{j+1} - \tau_j| \ge (\delta/2K(\omega))^3$ or $\tau_{j+1} = T_\gamma$. As $\tau_j \le T_\gamma \le T_0$, then

$$\tau_j = T_\gamma \quad \text{for} \quad j \ge j(\gamma; \omega)\,, \tag{2.12}$$ 2.10

where $j(\gamma) < \infty$ a.s.

We have constructed a process $\mathbf{u}(\tau)$, $\tau \in [0, T_\gamma]$, which satisfies (2.8)-(2.10), where the operator $U_{\mathbf{u}}$ is a piecewise constant function of τ.

2.5 Step 3: $\|\tilde u(\tau)\|_0 \equiv \|u(\tau)\|_0$ for $\tau \le T_\gamma$

For $j = 0, 1, \dots$ we will prove the following assertion:

$$\text{if } \|\tilde u(\tau_j)\|_0 = \| u(\tau_j)\|_0 \text{ a.s., then} \qquad \|\tilde u(\tau)\|_0 = \| u(\tau)\|_0 \text{ for } \tau_j \le \tau \le \tau_{j+1}, \text{ a.s.} \tag{2.13}$$

Since $\tilde u(\tau_0) = u(\tau_0)$, then (2.12) and (2.13) would imply that

$$\|\tilde u(\tau)\|_0 = \| u(\tau)\|_0 \quad \forall\, 0 \le \tau \le T_\gamma\,, \tag{2.14}$$

for any $\gamma > 0$.

To prove (2.13) we consider (following Lemma 7.1 in [KP06]) the quantities $E(\tau) = \frac12 \|u(\tau)\|_0^2$ and $\tilde E(\tau) = \frac12 \|\tilde u(\tau)\|_0^2$. Due to Ito's formula we have

$$dE = (u, -Au)\, d\tau + \frac12 B_0 d\tau + (u, \sum_s b_s e_s\, d\beta_s(\tau))$$

and

$$\begin{aligned} d\tilde E =& (\tilde u, -U^* Au)\, d\tau + \frac12 \sum b_s^2 |U^* e_s|^2 d\tau + (\tilde u, \sum_s b_s (U^* e_s)\, d\beta_s(\tau)) \\ =& \frac{\|\tilde u\|_0}{\|u\|_0}(u, -Au)\, d\tau + \frac12 B_0 d\tau + \frac{\|\tilde u\|_0}{\|u\|_0}(u, \sum_s b_s e_s\, d\beta_s(\tau))\,. \end{aligned}$$

Therefore,

$$\begin{aligned} d(E-\tilde E)^2 =& 2(E-\tilde E)\frac{\|u\|_0 - \|\tilde u\|_0}{\|u\|_0}(u, -Au)\, d\tau \\ & \Big(\frac{\|u\|_0 - \|\tilde u\|_0}{\|u\|_0}\Big)^2 \sum_s b_s^2 (u, e_s)^2 d\tau + \mathcal{M}_\tau\,, \end{aligned}$$

where $\mathcal{M}_\tau$ stands for the corresponding stochastic integral.

For $0 \le \tau \le T_\gamma$ let us denote $J(\tau) = (E - \tilde E)^2\big((\tau \vee \tau_i) \wedge \tau_{i+1}\big)$. Then

$$\begin{aligned} \frac{d}{d\tau}\mathbf{E} J(\tau) =& 2\,\mathbf{E}\Big((E-\tilde E)\frac{\|u\|_0 - \|\tilde u\|_0}{\|u\|_0}(u - Au) I_{\tau_i \le \tau \le \tau_{i+1}}\Big) \\ & + \mathbf{E}\Big(\Big(\frac{\|u\|_0 - \|\tilde u\|_0}{\|u\|_0}\Big)^2 \sum b_s^2 (u, e_s)^2 I_{\tau_i \le \tau \le \tau_{i+1}}\Big)\,. \end{aligned}$$

Since $\|u\|_0 - \|\tilde{u}\|_0 = \frac{2(E-\tilde{E})}{\|u\|_0+\|\tilde{u}\|_0}$ and $|(u,-Au)| \le \gamma^{-2}$, $\|u\|_0, \|\tilde{u}\|_0 \ge \gamma$, then $\frac{d}{d\tau}\mathbf{E}J(\tau) \le C_\gamma \mathbf{E}J(\tau)$. As $J(0)=0$, then $\mathbf{E}J(\tau)\equiv 0$ and (2.13) is established. Accordingly (2.14) also is proved.

2.6 Step 4: limit $\gamma \to 0$

Since $B_2 < \infty$, then $u(\tau)$ satisfies the γ-independent estimate

$$\mathbf{E} \sup_{0\le\tau\le T_0} \|u(\tau)\|_2 \le C(T_0,\nu)$$

(see [Kuk06a], Section 4.3). Accordingly

$$\mathbf{P}\Big\{ \sup_{0\le\tau\le T_0} \|u(\tau)\|_2 \le \gamma^{-1} \Big\} \to 1 \quad \text{as} \quad \gamma \to 0. \tag{2.15} \qquad \boxed{2.20}$$

Let us denote by $\hat{u}(\tau)$ the 4-vector $(u_1(\tau),\dots,u_4(\tau))$, where $u(\tau) = \sum u_j(\tau)e_j$ (we recall that $e_1, e_2, \dots$ are the basis vectors e_s, re-parameterised by natural numbers). Then

$$\hat{u}_j(\tau) = u_j(0) + \int_0^\tau F_j ds + b_j\beta_j(s), \quad j=1,\dots,4\,,$$

where F_j is the j-th component of the drift in (2.3). Since $\hat{u}$ is a stationary process, then $\mathbf{P}\{\hat{u}(0)=0\}=0$ (this follows, say, from Krylov's result, used in the next subsection). Setting $F_j^R = F_j \wedge R$, we denote by $\hat{u}^R(\tau) \in \mathbb{R}^4$ the process

$$\hat{u}_j^R(\tau) = u_j(0) + \int_0^\tau F_j^R ds + b_j\beta_j(s), \quad j=1,\dots,4\,.$$

By the Girsanov theorem, distribution of the process $\hat{u}^R(\tau), 0 \le \tau \le T_0$, is absolutely continuous with respect to the process $(b_1\beta_1,\dots,b_4\beta_4)+\hat{u}(0)$. Therefore

$$\mathbf{P}\{ \min_{0\le\tau\le T_0} |\hat{u}^R(\tau)| = 0\} = 0\,, \tag{2.16} \qquad \boxed{2.21}$$

for any R. Since $\max_{0\le\tau\le T_0} |\hat{u}^R(\tau) - \hat{u}(\tau)| \to 0$ as $R\to\infty$ in probability, then the process $\hat{u}(\tau)$ also satisfies (2.16). Jointly with (2.15) this implies that

$$\mathbf{P}\{T_\gamma = T_0\} \to 1 \quad \text{as} \quad \gamma \to 0\,,$$

and we derive from (2.14) the relation

$$\|\tilde{u}(\tau)\|_0 = \|u(\tau)\|_0 \quad \forall\, 0 \le \tau \le T_0, \;\; \text{a.s.}$$

2.7 Step 5: end of proof

ss2.7

The advantage of the process $\tilde{u}$ compare to u is that it satisfies the ν-independent Ito equation (2.9). Let us consider the first two components of the process:

$$d\tilde{u}_j = -\big(U^*_{u,\tilde{u}}(\tau)A(u)\big)_j\, d\tau + \sum_{l=1}^{\infty} \big(U^*_{u,\tilde{u}}(\tau)\big)_{jl} b_l\, d\beta_l(\tau)\,, \qquad (2.17)$$ 2.23

where $j = 1, 2$. Denoting $a_j(\tau) = \sum_{l=1}^{\infty} \big(U^*_{jl} b_l\big)^2 = \sum_{l=1}^{\infty} \big(U_{lj} b_l\big)^2$ and using (2.5) we find that a.s.

$$C \geq a_1(\tau) + a_2(\tau) \geq c > 0 \quad \forall\, \tau\,, \qquad (2.18)$$ 2.24

where $C = 2\sqrt{B_0}$ and c depends only on $|b_1| \wedge |b_2|$. Due to (1.4) for each $\tau \geq 0$ we have $\mathbf{E}|U^* A(u(\tau))|_j \leq \sqrt{B_1/2}$. This bound and the first estimate in (2.18) imply that Lemma 5.1 from [Kry87] applies to the Ito equation (2.17) uniformly in ν if we choose the lemma's parameters as follows:

$$d = 1, \;\; \gamma = 1, \;\; A_s = s, \;\; r_s = 1, \;\; c_s = 1, \;\; y_t = t, \;\; \varphi_t = t. \qquad (2.19)$$ kry

Taking in the lemma for $f(t, x)$ the characteristic function of the segment $[-\delta, \delta]$, we get

$$\mathbf{E} \int_0^{\gamma_R} e^{-t} a_j(\tau)^{1/2} I_{\{|\tilde{u}_j(\tau)| \leq \delta\}}\, d\tau \leq C\sqrt{\delta}\,, \quad j = 1, 2,$$

where $\gamma_R \leq 1$ is the first exit time ≤ 1 of the process $\tilde{u}_j$ from the segment $[-R, R]$. Sending R to ∞ we get that

$$\mathbf{E} \int_0^{1} a_j(\tau)^{1/2} I_{\{|\tilde{u}_j(\tau)| \leq \delta\}}\, d\tau \leq C_1\sqrt{\delta}\,, \quad j = 1, 2\,, \qquad (2.20)$$ 2.25

uniformly in ν.

For c as in (2.18) let us consider the event $Q_1^\tau = \{a_1(\tau) \geq \frac{1}{2}c\}$ and denote by Q_2^τ its complement. Then

$$a_1(\tau) \geq \frac{1}{2}\, c \text{ on } Q_1^\tau \text{ and } a_2(\tau) \geq \frac{1}{2}\, c \text{ on } Q_2^\tau. \qquad (2.21)$$ 2.26

Let us set

$$Q^\tau = \{|\tilde{u}_1(\tau)| + |\tilde{u}_2(\tau)| \leq \delta\}.$$

Then

$$\mathbf{P}(Q^\tau) = \mathbf{E}(I_{Q^\tau} I_{Q_1^\tau} + I_{Q^\tau} I_{Q_2^\tau}) \le \mathbf{E}(I_{\{|\tilde u_1(\tau)|\le\delta\}} I_{Q_1^\tau} + I_{\{|\tilde u_2(\tau)|\le\delta\}} I_{Q_2^\tau}).$$

By (2.21) the r.h.s. is bounded by

$$\sqrt{\frac{2}{c}}\,\mathbf{E}\big(I_{\{|\tilde u_1(\tau)|\le\delta\}}\sqrt{a_1} + I_{\{|\tilde u_2(\tau)|\le\delta\}}\sqrt{a_2}\big).$$

Jointly with (2.20) the obtained inequality shows that

$$\int_0^1 \mathbf{P}(Q^\tau)\,d\tau \le C_2\sqrt{\delta}.$$

Since

$$\mathbf{P}\{\|u(\tau)\|_0 \le \frac{\delta}{2}\} = \mathbf{P}\{\|\tilde u(\tau)\|_0 \le \frac{\delta}{2}\} \le \mathbf{P}(Q^\tau)\,,$$

where the l.h.s. is independent from τ, then

$$\mathbf{P}\{\|u(\tau)\|_0 \le \frac{\delta}{2}\} \le C_2\sqrt{\delta}$$

for any $\delta > 0$. This relation implies (2.2).

The constant C in (2.2), as well as all other constants in this section, depend only on B_1 and $|b_1| \wedge |b_2|$. Using the Remark in Section 2.2 we may replace $|b_1| \wedge |b_2|$ by $|b_{j_1}| \wedge |b_{j_2}|$, where j_1 and j_2 correspond to s_1 and s_2. This completes the theorem's proof.

3 Distribution of energy

Again, let $u_\nu(\tau)$ be a stationary solution of (1.1), written in the form (2.3), let $E_\nu(\tau)$ be its energy and $E_0(\tau) = \frac{1}{2}\|U(\tau)\|_0^2$ be the energy of the Eulerian limit.

Theorem 3.1. *For any $R > 0$ let $Q \subset [-R, R]$ be a Borel set. Then*

$$\mathbf{P}\{E_\nu(\tau) \in Q\} \le p_R(|Q|) \tag{3.1}$$

uniformly in $\nu \in (0, 1]$, where $p_R(t) \to 0$ as $t \to 0$

In particular, the measures $\mathcal{D}(E_\nu(\tau))$ are absolutely continuous with respect to the Lebesgue measure. Since $\mathcal{D}(E_{\nu_j}) \rightharpoonup \mathcal{D}(E_0(\tau))$, then $E_0(\tau)$ satisfies (3.1) for any open set $Q \subset [-R, R]$. Accordingly, $\mathbf{P}\{E_0(\tau) \in Q\} = 0$ if $|Q| = 0$ since the Lebesgue measure is regular. We got

Corollary 3.2. *The measure $\mathcal{D}(E_0(\tau))$ is absolutely continuous with respect to the Lebesgue measure.*

Proof of the theorem. For any $\delta > 0$ let us consider the set

$$\mathcal{O} = \mathcal{O}(\delta) = \{u \in \mathcal{H}^2 \mid \|u\|_2 \le \delta^{-\frac{1}{4}},\ \|u\|_0 \ge \delta\}$$

Writing $u = u_\nu$ as $u = \sum u_s e_s$, we set $u^I = \sum_{|s|\le N} u_s e_s$ and $u^{II} = u - u^I$. For any $u \in \mathcal{O}$ we have $\|u^{II}\|_0^2 \le N^{-4}\|u^{II}\|_2^2 \le \delta^{-\frac{1}{2}} N^{-2}$. So $\|u^I\|_0^2 \ge \delta^2 - \delta^{-\frac{1}{2}} N^{-4}$. Choosing $N = N(\delta) = \left[2^{1/4}\delta^{-5/8}\right]$ we achieve

$$\|u^I\|_0^2 \ge \frac{1}{2}\,\delta^2 \quad \forall\, u \in \mathcal{O}.$$

The stationary process $E(u_\nu(\tau))$ satisfies the Ito equation

$$dE = \big(-\|u(\tau)\|_1^2 + \frac{1}{2} B_0\big)\, d\tau + \sum b_s u_s(\tau)\, d\beta_s(\tau)$$

(see in Section (2.5)). The diffusion coefficient $a(\tau)$ satisfies

$$a(\tau) = \sum b_s^2 |u_s(\tau)|^2 \ge \underline{b}_N^2 \|u^I(\tau)\|_0^2,$$

where $\underline{b}_N = \min_{|s|\le N} |b_s| > 0$. So,

$$a(\tau) \ge \frac{1}{2}\,\underline{b}_N^2 \delta^2 \quad \text{if} \quad u(\tau) \in \mathcal{O}. \tag{3.2}$$

Besides,

$$\mathbf{E}|a(\tau)| \le \frac{\max_s b_s^2}{2}\, B_0\,, \quad \mathbf{E}\Big| -\|u(\tau)\|_1^2 + \frac{1}{2}\, B_0\Big| \le B_0\,.$$

Let $Q \subset [-R,\, R]$ be a Borel set and f be its indicator function. Applying the Krylov lemma with the same choices of parameters as in (2.19), passing to the limit as $R \to \infty$ as in Section 2.7 and taking into account that $E(\tau)$ is a stationary process, we get that

$$\mathbf{E}\big(a(\tau)^{1/2} f(E(\tau)\big) \le C|Q|^{1/2}, \tag{3.3}$$

uniformly in $\nu > 0$. Due to (1.4) and (2.2),

$$\mathbf{P}\{u(\tau) \notin \mathcal{O}\} \le \frac{1}{2} B_1 \sqrt{\delta} + C\sqrt{\delta}.$$

Jointly with (3.2) and (3.3) this estimate implies that

$$\mathbf{P}(E_\nu(\tau) \in Q) = \mathbf{E} f(E(\tau)) \le C(|Q|^{1/2} \underline{b}_N^{-1} \delta^{-1}) + C_1 \sqrt{\delta} \quad \forall\, 0 < \delta \le 1,$$

where $N = N(\delta)$. Now (3.1) follows.

4 Distributions of functionals of vorticity

In his section we assume that $B_6 < \infty$. The vorticity $\zeta = \operatorname{rot} u(t, x)$ of a solution u for (1.1), written in the fast time $\tau = \nu t$, satisfies the equation

$$\zeta'_\tau - \Delta\zeta + \nu^{-1}(u \cdot \nabla)\zeta = \xi(\tau, x). \tag{4.1}$$

Here $\xi = \frac{d}{dt} \sum_{s \in \mathbb{Z}^2 \setminus \{0\}} \beta_s(\tau) \varphi_s(x)$ and

$$\varphi_s = \frac{|s|}{\sqrt{2}\pi} \cos s \cdot x, \quad \varphi_{-s} = -\frac{|s|}{\sqrt{2}\pi} \sin s \cdot x,$$

for any s such that $s_1 + s_2 \delta_{s_1,0} > 0$. We will study eq. (4.1) in Sobolev spaces

$$H^l = \{\zeta \in H^l(\mathbb{T}^2) \mid \int \zeta\, dx = 0\}, \quad l \ge 0,$$

given the norms $\|\cdot\|_l$, defined as in (1.3).

Let us fix $m \in \mathbb{N}$ and choose any m analytic functions $f_1(\zeta), \dots, f_m(\zeta)$, linear independent modulo constant functions.[3] We assume that the functions $f_j(\zeta), \dots, f_j'''(\zeta)$ have at most a polynomial growth as $|\zeta| \to \infty$ and that

$$f_j''(\zeta) \ge -C \quad \forall\, j, \quad \forall \zeta \tag{4.2}$$

(for example, each $f_j(\zeta)$ is a trigonometric polynomial, or a polynomial of an even degree with a positive leading coefficient). Consider the map

$$F : H^l \to \mathbb{R}^m, \ \zeta \mapsto (F_1(\zeta), \dots, F_m(\zeta)),$$
$$F_j = \int_{\mathbb{T}^2} f_j(\zeta(x))\, dx,$$

[3] I.e., $C_1 f_1(\zeta) + \dots + C_m f_m(\zeta) \ne \text{const}$, unless $C_1 = \dots = C_m = 0$.

where $0 < l < 1$. Since for any $P < \infty$ we have $H^l \subset L_P(\mathbb{T}^2)$ if l is sufficiently close to 1, then choosing a suitable $l = l(F)$ we achieve that the map F is C^2-smooth. Let us fix this l. We have

$$dF(\zeta)(\xi) = \Big(\int f_1'(\zeta(x))\xi(x)\,dx, \dots, \int f_m'(\zeta(x))\xi(x)\,dx \Big).$$

12 **Lemma 4.1.** *If $\zeta \not\equiv 0$, then the rank of $dF(\zeta)$ is m.*

Proof. Assume that the rank is $< m$. Then there exists number $C_1, \dots, C_m$, not all equal to zero, such that

$$\int (C_1 f_1'(\zeta) + \cdots + C_m f_m'(\zeta))\xi\,dx = 0 \quad \forall \xi \in H^l. \tag{4.3}$$ 4.2

Denote $P(\zeta) = C_1 f_1'(\zeta) + \cdots + C_m f_m'(\zeta)$. This is a non-constant analytic function. Due to (4.3), $P(\zeta(x)) = \text{const}$. Denote this constant C_*. Then the connected set $\zeta(\mathbb{T}^2)$ lies in the discrete set $P^{-1}(C_*)$. So $\zeta(\mathbb{T}^2)$ is a point, i.e. $\zeta(x) \equiv \text{const}$. Since $\int \zeta\,dx = 0$, then $\zeta(x) \equiv 0$. □

Now let $\zeta(t) = \operatorname{rot} u_\nu(t)$, where u_ν is a stationary solution of (1.1). Applying Ito's formula to the process $F(\zeta(\tau)) \in \mathbb{R}^m$ and using that F_j is an integral of motion for the Euler equation, we get that

$$dF_j(\tau) = \Big(\int f_j'(\zeta(\tau, x))\Delta\zeta(\tau, x)\,dx + \frac{1}{2}\sum_s b_s^2 \int f''_j(\zeta(\tau, x))\varphi_s^2(x)\,dx) \Big) d\tau$$
$$+ \sum_s b_s \Big(\int f'_j(\zeta(\tau, x))\varphi_s(x)\,dx \Big)\,d\beta_s(\tau).$$

Since $b_s \equiv b_{-s}$ and $\varphi_s^2 + \varphi_{-s}^2 \equiv |s|^2/2\pi^2$, then

$$\begin{aligned} dF_j(\tau) &= \Big(\int f''_j(\zeta)(-|\nabla_x\zeta|^2 + \frac{1}{4\pi}B_1)\,dx \Big) d\tau \\ &\quad + \sum_s b_s \Big(\int f'_j(\zeta(\tau, x))\varphi_s(x)\,dx \Big)\,d\beta_s(\tau) \\ &:= H_j(\zeta(\tau))\,d\tau + \sum_s h_{js}(\zeta(\tau))\,d\beta_s(\tau)\,. \end{aligned}$$

Ito's formula applies since under our assumptions all moments of the random variables $\zeta(\tau, x)$ and $|\nabla_x\zeta(\tau, x)|$ are finite (see [Kuk06a], Section 4.3). Using that $F_j(\tau)$ is a stationary process, we get from the last relation that $\mathbf{E}H_j = 0$, i.e.

$$\mathbf{E}\int {f''}_j(\zeta(\tau,x))|\nabla_x\zeta(\tau,x)|^2\,dx = \frac{B_1}{4\pi}\,\mathbf{E}\int {f''}_j(\zeta(\tau,x))\,dx. \qquad (4.4)$$

Since $B_6<\infty$ then all moments of random variables $|\zeta(\tau,x)|$ are bounded uniformly in $\nu\in(0,1]$, see [Kuk06b] and (10.11) in [Kuk06a]. Jointly with (4.3), (4.4) and the equality

$$\mathbf{E}\int|\nabla_x\zeta(\tau,x)|^2dx = \mathbf{E}\|u_\nu(\tau)\|_2^2 = \frac{1}{2}\,B_1$$

this implies that

$$\mathbf{E}|H_j(\zeta(\tau))|\le C_j<\infty \qquad (4.5)$$

uniformly in ν (and for all τ).

Let us consider the diffusion matrix $a(\zeta(\tau))$, $\;a_{jl}(\zeta)=\sum_s h_{js}(\zeta)h_{ls}(\zeta)$, and denote $D(\zeta)=|\det a_{jl}(\zeta)|$. Clearly

$$\mathbf{E}\,tr(a_{jl})(\zeta(\tau))\le C, \qquad (4.6)$$

uniformly in ν. Noting that $h_{js}(\zeta)=b_s(dF(\zeta))_{js}$, we obtain from Lemma 4.1

Lemma 4.2. *The function D is continuous on H^l and $D>0$ outside the origin.*

Now we regard (4.1) as an equation in H^1 and set

$$\mathcal{O}_\delta=\{\zeta\in H^1 \mid \|\zeta\|_1\le\delta^{-1},\ \|\zeta\|_l\ge\delta\}\,.$$

Since $H^1\Subset H^l$, then $D\ge c(\delta)>0$ everywhere in $\mathcal{O}_\delta$.

Estimates (4.5), (4.6) allow to apply Krylov's lemma with $p=d=m$ to the stationary process $F(\zeta_\nu(\tau))\in\mathbb{R}^m$, uniformly in ν. Choosing there for f the characteristic function of a Borel set $Q\subset\{|z|\le R\}$, we find that

$$\mathbf{P}\{F(\zeta_\nu(\tau))\in Q\}\le\mathbf{P}\{\zeta_\nu(\tau)\notin\mathcal{O}_\delta\}+c(\delta)^{-1/(m+1)}C_R|Q|^{1/(m+1)} \qquad (4.7)$$

(cf. the arguments in Section 3). Since $\|\zeta\|_1=\|u\|_2$ and $\|\zeta\|_l\ge\|\zeta\|_0\ge\|u\|_0$ for $\zeta=\mathrm{rot}\,u$, then due to (1.4) and (2.2) the first term in the r.h.s. of (4.7) goes to zero with δ uniformly in ν, and we get that

$$\mathbf{P}\{F(\zeta_\nu(\tau))\in Q\}\le p_R(|Q|)\,,\quad p_R(t)\to0\ \text{ as }\ t\to0\,, \qquad (4.8)$$

uniformly in ν. Evoking Amplification to Theorem 1.1 we derive from (4.8) that the vorticity ζ_0 of the Eulerian limit U satisfies (4.8), if Q is an open subset of B_R. We have got

Theorem 4.3. *If $B_6 < \infty$, then the distribution of the stationary solution for the 2D NSE, written in terms of vorticity (4.1), satisfies (4.8) uniformly in ν. The vorticity ζ_0 of the Eulerian limit U is distributed in such a way that the law of $F(\zeta_0(\tau))$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^m$.*

Corollary 4.4. *Let $X \Subset \mathcal{H} \cap C^1(\mathbb{T}^2; \mathbb{R}^2)$ be a compact set of finite Hausdorff dimension. Then $\mu_0(X) = 0$.*

Proof. Denote the Hausdorff dimension of X by d and choose any $m > d$. Then $(F \circ \mathrm{rot})(X)$ is a subset of $\mathbb{R}^m$ of positive codimension. So its measure with respect to $\mathcal{D}(f(\zeta_0(t))$ equals zero. Since $\mathcal{D}(f(\zeta_0(t)) = (F \circ \mathrm{rot}) \circ \mu_0$, then $\mu_0(X) = 0$.

□

5 Appendix: rotation of solid body

The Euler equation for a freely rotating solid body, written in terms of its momentum $M \in \mathbb{R}^3$, is

$$\dot{M} + [M, A^{-1}M] = 0, \tag{5.1}$$

where A is the operator of inertia and $[\cdot,\cdot]$ is the vector product. The corresponding damped/driven equation (0.5) is

$$\dot{M} + [M, A^{-1}M] + \nu M = \sqrt{\nu}\,\eta(t)\,, \tag{5.2}$$

where the random force is $\eta(t) = \frac{d}{dt}\sum_{j=1}^{3} b_j\beta_j(t)e_j$ with non-zero b_j's, and $\{e_1, e_2, e_3\}$ is the eigenbasis of the operator A. Eq. (5.2) has a unique stationary measure μ_ν. Let $M_\nu(t)$ be a corresponding stationary solution. An inviscid limit, similar to that in Theorem 1.1, holds:

$$\mathcal{D}M_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}M_0(\cdot) \quad \text{as} \;\; \nu_j \to 0\,, \tag{5.3}$$

where $M_0(t) \in \mathbb{R}^3$ is a stationary process, formed by solutions of (5.1). The Euler equation has two quadratic integrals of motion: $H_1(M) = \frac{1}{2}|M|^2$ and $H_2(M) = \frac{1}{2}(A^{-1}M, M)$. Distributions of the random variables $H_1(M_\nu(t))$ and $H_2(M_\nu(t))$, $0 \le \nu \le 1$, satisfy direct analogies of the assertions in Sections 2, 3.

To analyse further the processes M_ν with $\nu \ll 1$ and the inviscid limit M_0, we note that a.e. level set of the vector-integral $H = (H_1, H_2)$ is formed by two periodic trajectories of (5.1) (see [Arn89]). Denote them $S^{\pm}_{(H_1,H_2)}$. It is easy to see that the conditional probabilities for $M_\nu(t)$ to belong to $S^{+}_{(H_1,H_2)}$ or to $S^{-}_{(H_1,H_2)}$ are equal. Since the dynamics, defined by (5.1) on each set $S^{\pm}_{(H_1,H_2)}$ obviously is ergodic with respect to a corresponding measure $\nu^{\pm}_{(H_1,H_2)}$,[4] then the methods of [FW98, FW03, KP06] apply to the process $H(M_{\nu_j}(\tau)) \in \mathbb{R}^2$, $\tau = \nu_j t$, and allow to prove that a limiting process $H_0(\tau)$ exists and satisfies a SDE, obtained from the equation for $H(M(\tau))$ by the usual stochastic averaging with respect to the ergodic measures $\nu^{\pm}_{(H_1,H_2)}$ on the curves $S^{\pm}_{(H_1,H_2)}$. It is very plausible that the averaged equation has a unique stationary measure θ. If so, then

$$\mathcal{D}(H(M_0)) = \theta$$

and

$$\mathcal{D}(M_0) = \sum_{\alpha \in \{+,-\}} \int_{\mathbb{R}^2} \pi_\alpha \nu^{\alpha}_{(H_1,H_2)}\, \theta(dH_1\, dH_2),$$

where $\pi_+ = \pi_- = 1/2$. Cf. Theorem 6.6 in [KP06]. In particular, the convergence (5.3) holds as $\nu \to 0$ (i.e., the limit does not depend on a sequence $\nu_j \to 0$).

The representation above for the measure $\mathcal{D}(M_0)$ is called its *disintegration* with respect to the map $H : \mathbb{R}^3 \to \mathbb{R}^2$, and may be obtained independently from the arguments above (see references in [Kuk07]). The role of the arguments is to represent the measure θ in terms of the averaged equation. The measure $\mu_0 = \mathcal{D}U(0)$, corresponding to the Eulerian limit U (Theorem 1.1) also admits a similar disintegration, see [Kuk07]. In that work we conjecture an averaging procedure to find the measures, involved in the disintegration of μ_0.

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