The Linear Theory of S*-Algebras and Their Applications

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Abstract

In this paper, we provide an introduction to the theory of isotopes in infinite dimensional spaces. Although we consider this to be an introduction, most of the results a new, and have never appeared in print. We restrict ourselves to Hilbert spaces and develop the linear theory, providing detailed proofs for all major results. After a few examples, in the first section we consider an isotope as a change in operator multiplication on the space of bounded linear operators over a fixed Hilbert space in the second section. The basic theory is developed leading to the notion of an S*-algebra (in honor of R. M. Santilli), which is a natural generalization of C*-algebras. The basic theory is then used in the third section to develop a complete theory of one-parameter linear iso-semigroups of operators, which extend the theory of one-parameter semigroups of operators, which have played, and still play an important role in applied analysis. In the fourth section we apply our theory of iso-semigroups of operators to unify and simply two different approaches to the important class of Sobolev-Galpern equations. We close with a discussion of the general nonlinear case, where the operators may be nonlinear, singular and/or multi-valued.

This paper is dedicated to Professor R. M. Santilli on the occasion of his seventieth birthday. He has been a mentor, a friend, a supporter and an inspiration to us for over twenty-five years.

Introduction

It is well known that there are many Hamiltonians (Lagrangians) associated with a given set of Hamilton's (Euler-Lagrange) equations. Professor R. M. Santilli (see [1], [2], [3]) was the first to observe that many of these Hamiltonians can be obtained from a fixed one via a change in the definition of the Lie algebra bracket. He called these related algebras Lie-isotopes, by analogy with a similar phenomenon in nuclear physics where the same atom can have a varying number of neutrons. Gill et al [4] have used isotopes as a basic tool in one approach to the construction of a relativistic particle theory. This theory has been quantized by Gill [5], leading to a new approach to the inclusion of geometry in relativistic quantum theory. In this introduction, we provide a number of distinct examples, all showing how natural this concept is. Our purpose is to prepare the way for the first general study of isotopes as a change in the definition of operator composition on the space of closed linear operators on Hilbert space. This work leads to a natural generalization of C*-algebras, which we call S*-algebras.

Example 1

This is a modified version of an example due to Santilli and has almost all the basics for the general case. Let $(\mathrm{so(n)}, +, [\cdot,\cdot])$ be the Lie algebra of real $\mathrm{n} \times \mathrm{n}$ skew-symmetric matrices with the standard product $[A,B] = AB - BA$, where $A, B \in so(n)$. If J is a symmetric invertible real $\mathrm{n} \times \mathrm{n}$ matrix, then define a new product:

$$[A,B]_J = A{\bullet}B - B{\bullet}A = AJB - BJA\,. \tag{1.1}$$

Since $A^t = -A$, $B^t = -B$ and $J^t = J$, it is easy to see that $[A,B] \in (\mathrm{so(n)}, +, [\cdot,\cdot]_{\mathrm{J}})$, so that the algebra is closed under the new product and hence, is a Lie algebra. It is clear that a change in the product at the algebraic level requires a change at other levels. In particular, if I is the identity in the standard case, so that $I \cdot I = I$, then with the new product "$\bullet$", one must find $\hat{I}$ such that $\hat{I}{\bullet}\hat{I} = \hat{I}$. This implies that $\hat{I} = J^{-1}$. To construct the group, use the universal enveloping algebra, so that

$$
\begin{aligned}
g(s) &= \hat{I} + sA + (1/2!)(sA)\bullet(sA) \ + \cdots \\
&= \hat{I}(\exp\{sJA\}) \ = (\exp\{sAJ\})\hat{I},
\end{aligned}
\tag{1.2}
$$

$$
g(s)^t \bullet g(s) = \hat{I}, \quad dg(s)/ds\big|_{s=0} = A. \tag{1.3}
$$

Thus, it follows that $g(s)$ is a one-parameter group for the new product. Denote the groups and their corresponding algebras by:

$$
G_1 = (\ SO(n),\ \cdot),\ \ g_1 = (so(n), +,\ [\cdot,\cdot]), \tag{1.4}
$$

$$
G_2 = (\ SO(n),\ \bullet),\ \ g_2 = (so(n), +,\ [\cdot,\cdot]_J), \tag{1.5}
$$

The properties of G_1 are well known, however, for G_2, we now have two independent ways to define the inner product. For $\mathbf{X} \in \mathbb{R}^n$, we have:

$$
(g(s)\bullet\mathbf{X})^t \bullet (g(s)\bullet\mathbf{X}) = \mathbf{X}^t J\mathbf{X}. \tag{1.6a}
$$

$$
(g(s)\bullet\tilde{\mathbf{X}})_J^t \bullet (g(s)\bullet\tilde{\mathbf{X}})_J = (\tilde{\mathbf{X}}^t J\tilde{\mathbf{X}})\tilde{I}. \tag{1.6b}
$$

In the first case the internal vector operations and the length scale along each coordinate axis can changed. In the second case, in addition to the changes induced by the first case, the definition of scalar multiplication can change, independently of the change in operator multiplication. Thus, $\tilde{I}$ is the unit for the new number system (isounit), which is independent of $\hat{I}$. In order to get a sense of the possibilities, let $n = 3$ and consider the concrete case:

$$
J = \begin{bmatrix} a & a_{12} & a_{13} \\ a_{12} & b & a_{23} \\ a_{13} & a_{23} & c \end{bmatrix}, \ \det[J] \neq 0 \ \ \hat{I} = J^{-1}. \tag{1.7}
$$

In the first case, we have that:

$$\|\mathbf{X}\|_J^2 = [\mathbf{X}^{\mathrm{t}} J\mathbf{X}] = \{ax_1^2 + a_{12}x_1x_2 + a_{23}x_1x_3 \\ +a_{12}x_2x_1 + bx_2^2 + a_{23}x_2x_3 \\ +a_{13}x_1x_3 + a_{23}x_2x_3 + cx_3^2\}. \tag{1.8a}$$

Thus, the length of the vector $\mathbf{X}$ has change relative to the reference value one would normally compute in $\mathbb{R}^3$. It is easy to see that the components of the vector and the angular relations have also changed.

In order to provide a possible physical interpretation of the meaning of this, let us imagine that we have a physical system of interest that is moving in a given environment under the influence of forces and fields that do not affect the properties of the system that interests us. For example, consider the motion of a ball (in air) moving near the earth under the influence of gravity, to be by convention, the standard case so that no isotope is required. If this motion is to be compared to that of the same ball in some medium with properties (not radically) different from those of air, we can account for the difference in the motion of the ball by an isotope of the first kind and, the matrix J would incorporate the difference in the properties of this new medium (relative to air) as it affects the motion of the ball (e.g., index of refraction, viscosity, etc.).

In the second case, we have that: $\mathbf{X} \to \tilde{\mathbf{X}} = [\tilde{x}_1, \quad \tilde{x}_2, \quad \tilde{x}_3]^{\mathrm{t}}$, $\tilde{x}_i = x_i\tilde{I}$ and:

$$[\tilde{\mathbf{X}}^{\mathrm{t}} J\tilde{\mathbf{X}}]\tilde{I} = \{ax_1^2 + a_{12}x_1x_2 + a_{23}x_1x_3 \\ +a_{12}x_2x_1 + bx_2^2 + a_{23}x_2x_3 \\ +a_{13}x_1x_3 + a_{23}x_2x_3 + cx_3^2\}\tilde{I}. \tag{1.8b}$$

The second case comes into play when the fields, forces and/or new media begin to affect the physical properties and relative internal relationship between the material constituents of the ball, in such a way that it might began to exhibit properties and motion that are unrelated to any known motion of a ball moving near the earth. Then an isotope of the second kind would give us an additional degree of freedom in order to provide a faithful representation of the systems behavior.

Before turning to other examples, let us see what mathematical advantages occur independent of physics. In the first case, suppose that $a_{ij} = 0,\ i \neq j$ and $a = c = -b = 1$, so that

$$J = \hat{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With this definition of J, it is easy to see that $G_2 = (\ SO(3),\ \bullet) \approx (\ SO(2,1),\ \cdot)$. Since the group $(SO(3),\ \cdot)$ is compact while $(SO(2,1),\ \cdot)$ is noncompact, this is a nontrivial result. It implies that one can study noncompact groups via their isotopic relationship to the corresponding compact group (see Sourlas and Tsaras [6]). In order to understand the geometric and analytic sides of this example, consider the following two Hamiltonians:

$$\mathrm{H}_1 = \frac{1}{2m}\boldsymbol{P}^t\boldsymbol{P} + \frac{k}{2}\boldsymbol{X}^t\boldsymbol{X} = \sum_{i=1}^{3}\left\{\frac{1}{2m}p_i^2 + \frac{k}{2}x_i^2\right\}, \tag{1.9}$$

$$\mathrm{H}_2 = \frac{1}{2m}\boldsymbol{P}^t\bullet\boldsymbol{P} + \frac{k}{2}\boldsymbol{X}^t\bullet\boldsymbol{X} = \sum_{i=1}^{2}\left\{\frac{1}{2m}p_i^2 + \frac{k}{2}x_i^2\right\} - \left\{\frac{1}{2m}p_2^2 + \frac{k}{2}x_2^2\right\}. \tag{1.10}$$

A simple calculation shows that both H_1 and H_2 lead to Newton's equations of motion for a (3-dimensional) harmonic oscillator, $\mathbf{F} = -k\mathbf{X}$. Clearly, H_1 is invariant under $SO(3)$ while H_2 is invariant under $SO(2,1)$. It is easy to see that both H_1 and H_2 are conserved, and are in involution. *This is an example of a bi-Hamiltonian structure for the oscillator.*

Example 2.

Another important example is the Feshbach-Villars representation [8] for the Klein-Gordon equation. In a very important and insightful paper, Feshbach and Villars showed that the Klein-Gordon equation could be transformed into a system of coupled differential equations, which are first order in time. This Schrödinger form made it possible to clearly demonstrate the charge degrees of freedom. The transformation is easy and shows that Feshbach and Villars had actually constructed what we will later define as an S^*- algebra. Writing the Klein-Gordon equation as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \Delta\psi - \frac{mc^2}{\hbar^2}\psi, \tag{1.11}$$

set $\psi = \varphi + \chi$, and $i\hbar\,\partial\Psi/\partial t = (mc^2)(\varphi - \chi), \Psi = (\varphi\,,\chi)^t$. Equation (1.11) becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = \mathrm{H}\Psi = (\boldsymbol{\tau}_3 + \mathrm{i}\boldsymbol{\tau}_2)\frac{\mathbf{p}^2}{2m}\Psi + mc^2\boldsymbol{\tau}_3\Psi. \tag{1.12}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\tau}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\tau}_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \boldsymbol{\tau}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.13}$$

The above (Pauli) matrices satisfy the conditions $\boldsymbol{\tau}_{\mathrm{i}}^2 = 1$, $\boldsymbol{\tau}_{\mathrm{i}}\boldsymbol{\tau}_{\mathrm{j}} - \boldsymbol{\tau}_{\mathrm{j}}\boldsymbol{\tau}_{\mathrm{i}} = 2\boldsymbol{\tau}_{\mathrm{k}}$, (i, j, k cyclic). In order to obtain the isotope form, set $J = \boldsymbol{\tau}_3 \Rightarrow \hat{I} = \boldsymbol{\tau}_3$, and define H_1 as

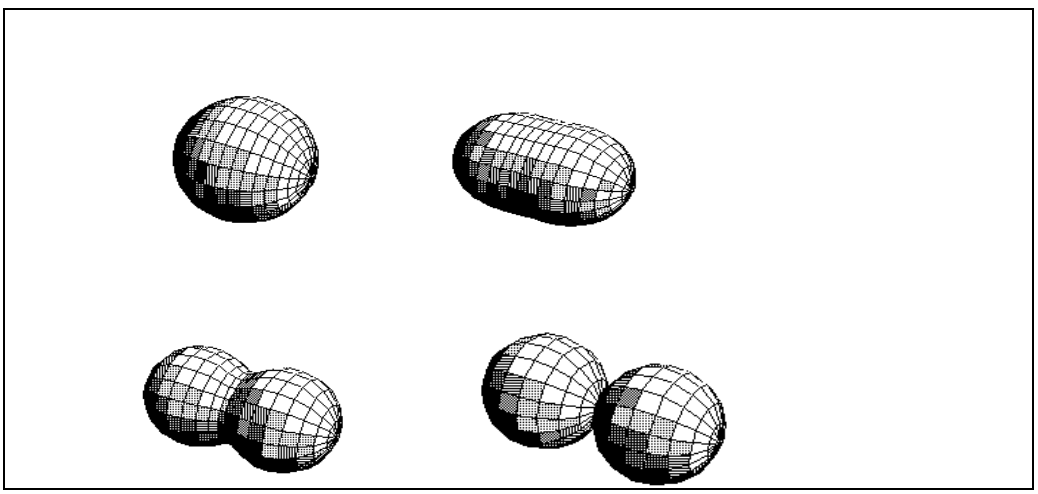
$$\mathrm{H}_1 = (\mathrm{I} - \mathrm{i}\boldsymbol{\tau}_2)\frac{\mathbf{p}^2}{2m} + mc^2\mathrm{I} \;\Rightarrow\; i\hbar\frac{\partial\Psi}{\partial t} = \mathrm{H}_1 \bullet \Psi. \tag{1.14}$$

It is quite remarkable that their approach was so complete. They recognized the need to change the definition of a unitary operator, inner product and expectation value, thus giving one of the first examples of an isotope in quantum theory.

Example 3

The next example is due to Gill and Chachere (see [4], [9]) and offers a different approach to the *problem of non-conservation of species number*. Let $J = \left(a(t)x^2 + b(t)(y^2 + z^2)\right)^{-1/2}$, $\langle \mathbf{r}, \mathbf{r} \rangle_J = J(x^2 + y^2 + z^2)$, where $a(t) = 1 + 3t$ and $b(t) = 1 - t$. If the norm is constrained to $\langle \mathbf{r}, \mathbf{r} \rangle_J = 1$, for all $t \in [0,1]$ then $a(t)x^2 + b(t)(y^2 + z^2) = (x^2 + y^2 + z^2)^2$ when $t = 0$, so that $(x^2 + y^2 + z^2)(x^2 + y^2 + z^2 - 1) = 0$ (see fig.1 for a few snap shots of the change as t varies). At $t = 1$, $((x\text{-}1)^2 + y^2 + z^2 - 1)((x+1)^2 + y^2 + z^2 - 1) = 0$, which gives two unit spheres (touching).

Figure 1



It is shown in [4] that $V = -mc^2\left[1 - \sqrt{a(t)x^2 + b(t)(y^2 + z^2)}\right]$ is the potential energy, which generates the above geometric/topological process while the Hamiltonian is of the harmonic oscillator type (in the proper time formulation of relativistic quantum theory).

Example 4. Iso-Dual

The next example, (due to Santilli [10]) shows that even in the simplest of cases, the use of isotopes offers new physical insights. Let $(\mathbb{R},+,\cdot)$ be the field of real numbers and define a new field by changing the definition of multiplication so that $a\cdot b \to a\bullet b = (a)(\text{-}1)(b)$ so that $(\mathbb{R},+,\bullet)$ is a field with J = -1. Santilli calls this field the Iso-dual numbers. This clearly induces an isomorphism on $\mathbb{R}$ which is equivalent to reversing the direction of the real line so that the new unit becomes -1. Note that to be consistent conceptually, we should replace $\mathbb{R}$ by $\overline{\mathbb{R}} = \{\bar{a} \mid \bar{a} = \text{-}a,\ a\ \in \mathbb{R}\}$, so that $(\overline{\mathbb{R}},+,\bullet)$ is the Iso-dual numbers. Let $\boldsymbol{\gamma}^{\mu}$, $\mu = 0,\cdots,3$, be the Dirac gamma matrices (see Greiner [7]), so that:

$$\boldsymbol{\beta} = \boldsymbol{\gamma}^{0} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},\ \boldsymbol{\gamma}^{\mathrm{i}} = \begin{bmatrix} 0 & \boldsymbol{\tau}_i \\ -\boldsymbol{\tau}_i & 0 \end{bmatrix},\ (i = 1,2,3)\,. \tag{1.15}$$

If we split the above representation by replacing the second row by its isodual, they become:

$$\boldsymbol{\gamma}^{0} = \begin{bmatrix} I & 0 \\ 0 & \bar{I} \end{bmatrix},\ \boldsymbol{\gamma}^{\mathrm{i}} = \begin{bmatrix} 0 & \boldsymbol{\tau}_i \\ \bar{\boldsymbol{\tau}}_i & 0 \end{bmatrix},\ (i = 1,2,3)\,. \tag{1.16}$$

It follows that the Dirac gamma matrices are invariant under the Iso-dual transformation. With a straightforward interpretation, Santilli used the above to show that the Dirac equation is invariant under the Iso-dual transformation if we identify $\mathcal{H}$ as a vector space over $(\mathbb{R},+,\cdot)$ and $\mathcal{H}^{*}$ (the dual Hilbert space), as a vector space over $(\overline{\mathbb{R}},+,\bullet)$. He also provided a consistent formulation of the Stuckelberg-Feynman theory of anti-matter as matter in the iso-dual state over the dual (Hilbert space $\mathcal{H}^{*}$), and proved the equivalence of charge conjugation and iso-duality.

2.0 S^*-algebras

We now introduce a class of algebras, called S^*-algebras, as the natural framework to study isotopes of the first kind, on infinite dimensional spaces. We restrict ourselves to the most transparent case, which is sufficient for a first introduction. A more general theory is in preparation but will not materially affect the results of this paper. Let J be a positive, bounded, invertible operator on a fixed Hilbert space $\mathcal{H}$, with dense range. If $L(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$, and $\mathbf{C}[\mathcal{H}]$ is the set of closed and densely defined linear operators on $\mathcal{H}$, then define $L_J(\mathcal{H})$, $\mathbf{C}_J(\mathcal{H})$ by:

Definition 2.1

$$L_J(\mathcal{H}) = \{A \in \mathbf{C}(\mathcal{H}) \mid \overline{(AJ)},\ (closure) \in L(\mathcal{H})\},$$
$$\mathbf{C}_J(\mathcal{H}) = \{A \in \mathbf{C}(\mathcal{H}) \mid \overline{(AJ)} \in \mathbf{C}(\mathcal{H})\}.$$

Definition 2.2 Let $A, B \in L_J(\mathcal{H})$ and let $f \in \mathcal{H}$, then define $A \bullet f =: \overline{(AJ)} f$ and $(A \bullet B) \bullet f =: \overline{(AJBJ)} f$.

The next result follows from our definition.

Theorem 2.3 If $A, B \in L_J(\mathcal{H})$ and $f, g \in \mathcal{H}$ and a, b, c are scalars, then

$$A \bullet (f + g) = A \bullet f + A \bullet g, \quad A \bullet (af) = aA \bullet f, \tag{2.1}$$
$$(aA + bB) \bullet f = aA \bullet f + bB \bullet f. \tag{2.2}$$

The proof of our next result follows from the definition of $L_J(\mathcal{H})$ and the properties of the identity operator.

Theorem 2.4 $(L_J(\mathcal{H}), +, \bullet)$ is a * algebra of operators such that

1) $L_J(\mathcal{H}) \supseteq L(\mathcal{H})$, for $J \in L(\mathcal{H})$ and

2) $J^{-1} = \hat{I},\ \hat{I} \bullet \hat{I} = \hat{I},\ \hat{I} \bullet A = A \bullet \hat{I} = A,\ \forall\ A \in \mathbf{C}_J(\mathcal{H})$.

Theorem 2.5 If $A \in L_J[\mathcal{H}]$ and $A^{-1} \in L(\mathcal{H})$ then the operators $\hat{A}^{-1} = \hat{I}A^{-1}\hat{I}$ and $\tilde{A} = \hat{I}A\hat{I}$ satisfy:

1) $A^{-1} \in L_J[\mathcal{H}]$,
2) $A \bullet \hat{A}^{-1} = \hat{A}^{-1} \bullet A = \hat{I}$ and
3) $A^{-1} \bullet \tilde{A} = \tilde{A} \bullet A^{-1} = \hat{I}$.

Proof: The proofs of 1 and 2 follow from the definitions and properties of the identity operator. To prove 3, we first note that $(L_J[\mathcal{H}],+,\bullet)$ is a *algebra and then use the uniqueness of the inverse. (It should be noted that, in general, neither $\tilde{A}$ nor $\hat{A}^{-1}$ is in $L_J[\mathcal{H}]$, but both are in $\mathbf{C}_J(\mathcal{H})$).

Definition 2.6 The operators in $L_J[\mathcal{H}]$ are called bounded iso-linear operators.

If Λ is a bounded linear functional on $\mathcal{H}$, the Riesz Representation theorem assures us that $\Lambda f = \Lambda_g f = \langle f | g \rangle$ for a unique element $g \in \mathcal{H}$, where $\langle f | g \rangle$ is the inner product on $\mathcal{H}$ (see Yosida [11], page 90). It follows that, as an element of $L_J[\mathcal{H}]$, Λ induces a unique representation of the form: $\Lambda \bullet f = \Lambda_g \bullet f = \langle f | g \rangle_J = \langle f | J g \rangle$, for a unique element $g \in \mathcal{H}$.

Definition 2.7 $A \in L_J[\mathcal{H}]$ is iso-bounded if $\|A\|_J = \sup\left\{\langle A \bullet f \,|\, A \bullet f \rangle_J\right\}^{1/2} < \infty$, where the sup is over all $f \in \mathcal{H}$ with $\|f\|_J = 1$.

Lemma 2.8 $\|\hat{I}\|_J = 1$ (In general $\hat{I}$ is unbounded as an operator on $\mathcal{H}$ in the standard norm.)

Definition 2.9 An operator U is said to be:
1) Iso-self-adjoint (Hermitian) iff $\langle U \bullet f \,|\, g \rangle_J = \langle f \,|\, U \bullet g \rangle_J$ for all $f, g \in \mathcal{H}$,

2) Iso-unitary iff $U \bullet U^* = U^* \bullet U = \hat{I}$.

Lemma 2.10 An operator U is:
1) Iso-Hermitian if and only if it is Hermitian.
2) Iso-unitary if and only if $U^* = \hat{I} U^{-1} \hat{I}$.

Before going further, we need a recent result on bounded approximation for closed operators (see Gill et al [12]). Let $V[\mathcal{H}]$ denote the set of contraction operators on $\mathcal{H}$ ($A \in \mathbf{V}(\mathcal{H}) \Rightarrow \|Af\| \le \|f\|$). The following result is due to Kaufman [13], and extends earlier work of von Neumann [14].

Theorem 2.11 If $A \in \mathbf{C}(\mathcal{H})$ then $\left(I + A^* A\right)^{-1/2} \in L[\mathcal{H}]$ and the operator $\Gamma(A) = A\left(I + A^* A\right)^{-1/2}$, defines an invertible function from $\mathbf{C}(\mathcal{H})$ onto $V[\mathcal{H}]$, with inverse defined by $\Gamma^{-1}(C) = C\left(\mathrm{I} - C^* C\right)^{-1/2}$ for all $C \in \mathbf{C}[\mathcal{H}]$.

Definition 2.15 An operator T is said to be relatively bounded with respect to A if $D(T) \supseteq D(A)$ and there are constants a, b such that

$$\|Tf\| \le a\|f\| + b\|Af\|, \quad \forall\, f \in D(A).$$

Definition 2.16 An operator $W \in L[\mathcal{H}]$ is called a partial isometry if there is a closed linear subspace $\mathcal{M} \subset \mathcal{H}$, with $\|Wf\| = \|f\|$ on $\mathcal{M}$ and $Wf = 0$ on $\mathcal{M}^{\perp}$.

The following result can be found in Kato [15].

Theorem 2.17 If $A \in \mathbf{C}[\mathcal{H}]$, then $T = A^* A$ is a positive self-adjoint operator, $A = WT^{1/2}$, where W is a partial isometry. Furthermore, $D(T^{1/2}) = D(A)$ and $\|Af\| = \|T^{1/2} f\|$.

Theorem 2.18 If $A \in \mathbf{C}[\mathcal{H}]$, set $T = A^*A$ and $R(\lambda, T) = (\lambda I - T)^{-1}$, with $\lambda > 0$. If $A_\lambda = \lambda A R(\lambda, T)$, then for $f \in D(A)$, $A_\lambda \in L(\mathcal{H})$ and $\lim_{\lambda \to \infty} A_\lambda f = Af$.

Proof: Let $\|Af\| = \|T^{1/2} f\|$, and recall (Pazy [16]) that T is the generator of a contraction semigroup and $\|R(\lambda, T)\| \le \frac{1}{\lambda}$. Use theorem 2.11 to get that $A_\lambda \in L(\mathcal{H})$. Now $A_\lambda f - A_{\lambda'} f = A\left[R(\lambda, T) - R(\lambda', T)\right] f$. From the resolvent identity, we have

$$\begin{aligned} A_\lambda f - A_{\lambda'} f &= (\lambda - \lambda') A\left[R(\lambda, T) R(\lambda', T)\right] f \\ &= (\lambda - \lambda') W\left[R(\lambda, T) R(\lambda', T)\right] T^{1/2} f \end{aligned}.$$

In the last result, we use the fact that $T^{1/2}$ commutes with $R(\lambda, T)$. Taking norms, and using theorem 2.17, it is easy to see that

$$\|A_\lambda f - A_{\lambda'} f\| \le \left|\frac{1}{\lambda} - \frac{1}{\lambda'}\right| \|T^{1/2} f\|.$$

Thus, the family $\{A_\lambda f \,|\, \lambda > 0\}$ is Cauchy on $D(A)$, so that the limit exists.

Theorem 2.19 Suppose that A generates a contraction semigroup. Then $\lim_{\lambda \to \infty} e^{\tau A_\lambda} f = e^{\tau A} f$.

Proof: Let $f \in D(A)$, then

$$\begin{aligned} \left[\exp\{\tau A_\lambda\} - \exp\{\tau A_{\lambda'}\}\right] f &= \int_0^1 \frac{d}{ds}\left[\exp\{s\tau A_\lambda\} \exp\{(1-s)\tau A_{\lambda'}\}\right] f ds \\ &= \int_0^1 \tau\left\{\exp\{s\tau A_\lambda\} \exp\{(1-s)\tau A_{\lambda'}\}\left[A_\lambda - A_{\lambda'}\right]\right\} f ds \end{aligned}.$$

Taking norms we have

$$\left\| \left[\exp\{\tau A_\lambda\} - \exp\{\tau A_{\lambda'}\} \right] f \right\| \le \tau \left\| \left[A_\lambda - A_{\lambda'} \right] f \right\|.$$

Using theorem 2.18, the limit exists for $f \in D(A)$ which is dense in $\mathcal{H}$. Using the uniform boundedness theorem, the limit exists on $\mathcal{H}$.

Theorem 2. 20 (Fundamental Theorem of Iso-Bounded Linear Operators) For each $A \in \mathbf{C}[\mathcal{H}]$, there is an operator $J \in L[\mathcal{H}]$ such that A is iso-bounded with respect to J (i.e. $A \in L_J[\mathcal{H}]$), and $\|J\| \le 1$. Furthermore, if T is relatively bounded with respect to A, then $T \in L_J[\mathcal{H}]$.

Proof: To prove 1, set $J = \left(I + A^* A\right)^{-1}$ and use theorem 2.18. To prove 2, note that if T is relatively bounded with respect to A then, for all $f \in \mathcal{H}$, we have $\|TJf\| \le a\|Jf\| + b\|AJf\|$. Divide by $\|Jf\|$ to get that $\|TJf\|/\|Jf\| \le a + b\|AJf\|/\|Jf\|$. It follows that TJ is a bounded operator, hence $T \in L_J[\mathcal{H}]$.

Theorem 2. 21 If $\mathcal{A} \subseteq L[\mathcal{H}]$ is a C*-algebra, then so is $\mathcal{A}_J \subseteq L_J[\mathcal{H}]$. We call $\mathcal{A}_J$ the S*-algebra over $\mathcal{H}$ associated with J.

A study of S*-algebras, which parallels the research program on C*-algebras represents an interesting and fruitful (open) research area. As one might have noticed, $L_J[\mathcal{H}]$ can contain certain classes of unbounded operators depending on the properties of J. The study of unbounded operator algebras is more natural for physical applications. (For some of the early work in this area, see Powers [17], [18].). Antoine, Inoue and Trapani [19] provide a recent review of the work on unbounded operator algebras and give a clear discussion of the problems and prospects.

3.0 Iso- Semigroups of Operators

Isotopes have shown up in physics, engineering and mathematics wearing a number of different faces. In this section, we use the theory of Section 2 to develop a general theory of iso-semigroups that will be used in Section 4 to unify these diverse approaches under the natural umbrella of isotopes.

Definition 3.1 A family of linear operators $\{S(t), 0 \le t < \infty\}$ (not necessarily bounded), defined on $\mathcal{H}$, is an iso-semigroup if

1) $S(t+s)\varphi = S(t) \bullet S(s)\varphi$ for $\varphi \in D$, the domain of the $S(t)$ and $0 < t < \infty$.

2) The family is iso-strongly continuous if $\lim_{\tau \to 0} S(t+\tau) \bullet \varphi = S(t) \bullet \varphi$, $\forall \varphi \in D$, $t > 0$.

3) $S(t)$ is a C_0-iso-semigroup if $D = \mathcal{H}$, it is iso-strongly continuous, $S(0) = \hat{I}$ and $\lim_{t \to 0} S(t) \bullet \varphi = \varphi$.

4) The semigroup $S(t)$ is a contraction C_0-iso-semigroup if 3) holds and $\|S(t)\|_J \le 1$.

5) The family $S(t)$ is an C_0-iso -unitary group if $S(t) \bullet S(t)^* = S(t)^* \bullet S(t) = \hat{I}$, and $S(-t) = S(t)^*$.

6) $S(t)$ is uniformly continuous if it is C_0 and $\lim_{t \downarrow 0} \left\| S(t) - \hat{I} \right\|_J = 0$.

7) The operator A defined by: $D(A) = \{ \varphi \in \mathcal{H} \,|\, \lim_{t \downarrow 0} (1/h)[S(h) \bullet \varphi - \varphi] \text{ exists} \}$ and $A \bullet \varphi = \lim_{t \downarrow 0} (1/h)[S(h) \bullet \varphi - \varphi]$, $\varphi \in D(A)$, is called the generator of $S(t)$, and $D(A)$ is the domain of A.

Definition 3.2 The operator A is said to be iso-dissipative if, for $\varphi \in D(A)$, $\mathbf{Re}\langle A{\bullet}\varphi, \varphi\rangle_J \le 0$.

Theorem 3.3 Let $\mathrm{S}(t)$ be a C_0-iso-semigroup of contraction operators on $\mathcal{H}$; then

1) $A{\bullet}\varphi = \lim_{t \to 0} \dfrac{\mathrm{S}(t){\bullet}\varphi - \varphi}{t}$ exists for φ in a dense set and $A \in \mathbf{C}_J(\mathcal{H})$.

2) A is iso-dissipative and $\mathcal{R}(\hat{I} - A) = \mathcal{H}$ (range).

3) $R(\lambda,\ A) = \hat{I}(\lambda\hat{I} - A)^{-1}\hat{I}$ exists for $\lambda > 0$ and $\|R(\lambda,\ A)\|_J \le \dfrac{1}{\lambda}$.

Definition 3.4 If an operator A is densely defined, and satisfies 2), it is called an m-iso-dissipative operator. The next two results follow the proofs in Pazy [16].

Theorem 3.5 Suppose A is an m-iso-dissipative operator. Then A is the generator of a C_0-iso-semigroup $\mathrm{S}(t)$ of contraction operators on $\mathcal{H}$.

Theorem 3.6 If A is densely defined with both A and A^* iso-dissipative, then A is m- iso-dissipative.

Theorem 3.7 If $S(t)$ is a C_0-iso-semigroup of contraction operators on $\mathcal{H}$, then

1) $S(t){\bullet}\varphi$ is a continuous function in t for every $\varphi \in \mathcal{H}$.

2) $\lim_{h \downarrow 0}(1/h)\int_t^{t+h} S(\tau){\bullet}\varphi d\tau = S(t){\bullet}\varphi$ for all φ in $\mathcal{H}$.

3) $\int_0^t S(\tau)\bullet\varphi d\tau \in D(A)$ for all φ in $\mathcal{H}$, and $A\bullet\int_0^t S(\tau)\bullet\varphi d\tau = S(t)\bullet\varphi - \varphi$.

4) If $\varphi \in D(A) \Rightarrow S(t)\bullet\varphi \in D(A)$ and $(d/dt)S(t)\bullet\varphi = A\bullet S(t)\bullet\varphi = S(t)\bullet A\bullet\varphi$.

5) If $\varphi \in D(A) \Rightarrow S(t)\bullet\varphi - S(\tau)\bullet\varphi = \int_\tau^t S(\lambda)\bullet A\bullet\varphi d\lambda$.

Proof: To prove 1), suppose that $t, h \geq 0$, then:

$$\left\|S(t+h)\bullet\varphi - S(t)\bullet\varphi\right\|_J \leq \left\|S(t)\right\|_J \left\|S(h)\bullet\varphi - \varphi\right\|_J \leq \left\|S(h)\bullet\varphi - \varphi\right\|_J .$$

If $t \geq h \geq 0$, then

$$\left\|S(t-h)\bullet\varphi - S(t)\bullet\varphi\right\|_J \leq \left\|S(t-h)\right\|_J \left\|\varphi - S(h)\bullet\varphi\right\|_J \leq \left\|\varphi - S(h)\bullet\varphi\right\|_J .$$

The result now follows from strong continuity.

The proof of 2) follows from 1). To prove 3), let $\varphi \in D(A),\ h > 0$, then:

$$\begin{aligned}(1/h)(S(h)-\hat{I})\bullet\int_0^t S(\tau)\bullet\varphi d\tau &= (1/h)\int_0^t [S(\tau+h)\bullet\varphi - S(\tau)\bullet\varphi]d\tau \\ &= (1/h)\int_t^{t+h} S(\tau)\bullet\varphi d\tau - (1/h)\int_0^h S(\tau)\bullet\varphi d\tau \to S(t)\bullet\varphi - \varphi.\end{aligned}$$

To prove 4), let $\varphi \in D(A)$, then

$$(1/h)(S(h)-\hat{I})\bullet S(t)\bullet\varphi = S(t)\bullet(1/h)(S(h)-\hat{I})\bullet\varphi \to S(t)\bullet A\bullet\varphi .$$

Thus we have that

$$A\bullet S(t)\bullet\varphi = S(t)\bullet A\bullet\varphi \Rightarrow (d^+/dt)S(t)\bullet\varphi = A\bullet S(t)\bullet\varphi = S(t)\bullet A\bullet\varphi .$$

To complete the proof, let $h > 0$. Then,

$$
\begin{aligned}
&\lim_{h\downarrow 0}\{(1/h)[S(t)\bullet\varphi - S(t-h)\bullet\varphi] - S(t)\bullet A\bullet\varphi\} \\
&= \lim_{h\downarrow 0}\{(1/h)[S(t)\bullet\varphi - S(t-h)\bullet\varphi] - S(t-h)\bullet A\bullet\varphi\} \\
&+ \lim_{h\downarrow 0}[S(t-h)\bullet A\bullet\varphi - S(t)\bullet A\bullet\varphi] \\
&= \lim_{h\downarrow 0} S(t-h)\bullet\{(1/h)[S(h)\bullet\varphi - \varphi] - A\bullet\varphi\} \\
&+ \lim_{h\downarrow 0}[S(t-h)\bullet A\bullet\varphi - S(t)\bullet A\bullet\varphi] = 0.
\end{aligned}
$$

To prove 5), integrate $(d/dt)S(t)\bullet\varphi = S(t)\bullet A\bullet\varphi$ from τ to t.

Theorem 3.8 If A is the generator of a C_0-iso-semigroup of contractions $S(t), t \geq 0$, then A is closed and densely defined. Furthermore, every $\lambda > 0$, $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leq 1/\lambda$.

Proof: For $h > 0$, we have by part 3 of Theorem 3.7 that $\varphi_h = (1/h)\int_0^h S(\tau)\bullet\varphi d\tau \in D(A)$ and by part 2, $\lim_{h\downarrow 0}\varphi_h = \varphi$. It follows that $\overline{D(A)} = \mathcal{H}$. To prove that A is closed, let $\varphi_n \to \varphi$ and $A\bullet\varphi_n \to \psi$ as $n \to \infty$. From part 5 of Theorem 3.7, we have that:

$$
\begin{aligned}
&(1/t)[S(t)\bullet\varphi_n - \varphi_n] = (1/t)\int_0^t S(\tau)\bullet A\bullet\varphi_n d\tau \\
&\to (1/t)[S(t)\bullet\varphi - \varphi] = (1/t)\int_0^t S(\tau)\bullet\psi d\tau
\end{aligned}
.
$$

Letting $t \searrow 0$, it follows that $\varphi \in D(A)$ and $A\bullet\varphi = \psi$, so that A is closed. To complete our proof, let $\lambda > 0$ and define $R(\lambda)$ by:

$$
R(\lambda)\bullet\varphi = \int_0^\infty e^{-\lambda t} S(t)\bullet\varphi dt .
$$

The above integral is well defined as an improper Riemann integral (since $S(t)\bullet\varphi$ is continuous and uniformly bounded). It follows that $R(\lambda)$ is a bounded linear operator on $\mathcal{H}$ and

$$\left\|R(\lambda)\bullet\varphi\right\|_J = \left\|\int_0^\infty e^{-\lambda t}S(t)\bullet\varphi dt\right\|_J \le \left\|S(t)\bullet\varphi\right\|_J \int_0^\infty e^{-\lambda t}dt \le (1/\lambda)\left\|\varphi\right\|_J .$$

Now note that

$$\begin{aligned}(1/h)(S(h)-I)\bullet R(\lambda)\bullet\varphi &= (1/h)\int_0^\infty e^{-\lambda t}\,[S(t+h)\bullet\varphi - S(t)\bullet\varphi]d\tau \\ &= (1/h)(e^{-\lambda h}-1)\int_0^\infty e^{-\lambda t}S(\tau)\bullet\varphi d\tau - (1/h)e^{-\lambda h}\int_0^h e^{-\lambda t}S(t)\bullet\varphi dt \qquad (3.1)\\ &\xrightarrow{h\searrow 0} \lambda R(\lambda)\bullet\varphi - \varphi \Rightarrow A\bullet R(\lambda)\bullet\varphi = \lambda R(\lambda)\bullet\varphi - \hat{I}\bullet\varphi.\end{aligned}$$

Thus, $R(\lambda)\bullet\varphi\in D(A)$ for all $\varphi\in\mathcal{H}$ and $(\lambda\hat{I}-A)\bullet R(\lambda)=\hat{I}$. On the other hand, for $\varphi\in D(A)$, we have

$$R(\lambda)\bullet A\bullet\varphi = \int_0^\infty e^{-\lambda t}S(t)\bullet A\bullet\varphi dt = A\bullet\int_0^\infty e^{-\lambda t}S(t)\bullet\varphi dt = A\bullet R(\lambda)\bullet\varphi,$$

so that $R(\lambda)\bullet(\lambda\hat{I}-A)=\hat{I} \Rightarrow R(\lambda)=\hat{I}(\lambda\hat{I}-A)^{-1}\hat{I} := R(\lambda,A)$.

Theorem 3.9 Suppose that A is closed, densely defined and, for every $\lambda>0$, $\lambda\in\rho(A)$ with $\left\|R(\lambda, A)\right\| = \left\|\hat{I}(\lambda\hat{I}-A)^{-1}\hat{I}\right\| \le 1/\lambda$. Then:

1) $\lim_{\lambda\nearrow\infty}\lambda R(\lambda,A)\bullet\varphi=\varphi$ for all $\varphi\in\mathcal{H}$.

2) $A_\lambda = \lambda A\bullet R(\lambda,A) \Rightarrow A_\lambda = \lambda^2 R(\lambda,A) - \lambda\hat{I}$ and, for all $\varphi\in D(A)$ $A_\lambda\varphi \to A\varphi$, as $\lambda\to\infty$.

3) A_λ, known as the Yosida approximator for A, is a (uniformly bounded) generator for a C$_0$-iso-semigroup of contractions and $\forall \lambda, \mu > 0$, $\varphi \in \mathcal{H}$, $\left\| [(e^{tA_\lambda J} - e^{tA_\mu J})\hat{I}] \bullet \varphi \right\|_J \le t \left\| [A_\lambda - A_\mu] \bullet \varphi \right\|_J$.

4) A is the generator of a C$_0$-iso-semigroup of contractions $S(t), t \ge 0$.

5) $R(\lambda, A) - R(\mu, A) = (\lambda - \mu) R(\lambda, A) \bullet R(\mu, A)$ (resolvent equation).

Proof: To prove 1), recall from the last part of equation (3.1) that $A \bullet R(\lambda) \bullet \varphi = \lambda R(\lambda) \bullet \varphi - \varphi$, so that

$$\left\| \lambda R(\lambda, A) \bullet \varphi - \varphi \right\|_J = \left\| A \bullet R(\lambda, A) \bullet \varphi \right\|_J = \left\| R(\lambda, A) \bullet A \bullet \varphi \right\|_J \le (1/\lambda) \left\| A \bullet \varphi \right\|_J .$$

To prove 2), use $A \bullet R(\lambda, A) = \lambda R(\lambda, A) - \hat{I}$ to get that $\lambda A \bullet R(\lambda) = \lambda^2 R(\lambda) - \lambda \hat{I}$. If $\varphi \in D(A)$ then, from $\lambda A \bullet R(\lambda, A) \bullet \varphi = \lambda R(\lambda, A) \bullet A \bullet \varphi$ and 1, we get that $\lim_{\lambda \nearrow \infty} A_\lambda \bullet \varphi = A \bullet \varphi$.

To prove 3, recall from the last part of Theorem 2.2 that $R(\lambda, A) \bullet \varphi \in D(A)$ for all $\varphi \in \mathcal{H}$, so that A_λ is a iso-bounded. It generates a iso-semigroup since $e^{tA_\lambda J} \hat{I} := \left[\sum_{n=0}^{\infty} (tA_\lambda \bullet)^n / n! \right] \hat{I} = \hat{I} e^{JtA_\lambda}$ converges uniformly for all t and $\left\| e^{tA_\lambda J} \hat{I} \right\|_J \le e^{t \| A_\lambda \|_J} = e^{t \| \lambda^2 R(\lambda) - \lambda \hat{I} \|_J} = e^{-\lambda t} e^{t\lambda \| \lambda R(\lambda) \|} \le 1$, so it is a contraction (see Example 1). It is clearly strongly continuous. To complete the proof of 3, observe that

$$\begin{aligned} \left\| [(e^{tA_\lambda J} - e^{tA_\mu J})\hat{I}] \bullet \varphi \right\|_J &= \left\| \int_0^1 \frac{d}{ds} [(e^{tsA_\lambda J} \bullet e^{t(1-s)A_\mu J})\hat{I}] \bullet \varphi ds \right\|_J \\ &\le t \int_0^1 ds \left\| [(e^{tsA_\lambda J} \bullet e^{t(1-s)A_\mu J})\hat{I}] \bullet [A_\lambda - A_\mu] \bullet \varphi \right\|_J \le t \left\| [A_\lambda - A_\mu] \bullet \varphi \right\|_J . \end{aligned} \tag{3.2}$$

To prove 4, note from equation (3.2) that, for all $\varphi \in D(A)$, $[e^{tA_\lambda J}\hat{I}]\bullet\varphi$ converges as $\lambda \to \infty$. Furthermore, the convergence is uniform on bounded intervals and, since $\left\|e^{tA_\lambda J}\hat{I}\right\|_J \leq 1$ and $D(A)$ is dense,

$$\lim_{t\to\infty}[e^{tA_\lambda J}\hat{I}]\bullet\varphi = S(t)\bullet\varphi,\ \forall\varphi \in \mathcal{H}. \tag{3.3}$$

It is clear that $S(t)$ has the semigroup property, $S(0) = \hat{I}$, $\|S(t)\|_J \leq 1$ and the convergence is uniform on bounded intervals. Since $S(t)\bullet\varphi$ is the uniform limit of continuous functions in t, it is a continuous function of t. It follows that $S(t)$ is a C_0-iso-semigroup of contractions on $\mathcal{H}$. Now, if $\varphi \in D(A)$, it follows from equation (3.3), part 3 of Theorem 2.1 and the fact that $[e^{tA_\lambda J}\hat{I}]\bullet A_\lambda \bullet\varphi \to S(t)\bullet A\bullet\varphi$ uniformly on bounded intervals that:

$$\begin{aligned} S(t)\bullet\varphi - \varphi &= \lim_{\lambda\to\infty}[(e^{tA_\lambda J}\hat{I})\bullet\varphi - \varphi] \\ &= \lim_{\lambda\to\infty}\int_0^t [e^{tA_\lambda J}\hat{I}]\bullet A_\lambda\bullet\varphi d\tau = \int_0^t S(\tau)\bullet A\bullet\varphi d\tau \end{aligned}. \tag{3.4}$$

Dividing (3.4) by t and letting $t \downarrow 0$, we see that A is the generator of $S(t)$, which is easily seen to be unique.

To prove 5, note that

$$\begin{aligned} R(\lambda, A) &= R(\lambda, A)\bullet(\mu\hat{I} - A)\bullet R(\mu, A) \\ &= R(\lambda,A)\bullet\{(\mu-\lambda)\hat{I} + (\lambda\hat{I} - A)\}\bullet R(\mu, A) \\ &= (\lambda-\mu)R(\lambda, A)\bullet R(\mu, A) + R(\mu, A). \end{aligned}$$

Combining Theorems 2.2 and 2.3, we get the iso-version of the famous Hille-Yosida Theorem (see Hille and Phillips [20] or Pazy [16]):

Theorem 3.10 (*Hille-Yosida*) The linear operator A is the generator of a C_0-iso-semigroup of contractions $S(t), t \geq 0$, if and only if A is closed, densely defined and every $\lambda > 0 \Rightarrow \lambda \in \rho(A)$, with $\|R(\lambda, A)\|_J \leq 1/\lambda$.

We now develop the Lumer-Phillips approach, which has a number of advantages in applications. Recall that a linear operator A is said to be iso-dissipative if $\operatorname{Re}\langle A \bullet \varphi, \varphi \rangle_J \le 0 \quad \forall \varphi \in \mathrm{D}(A)$. It is said to be m-iso-dissipative if it is iso-dissipative, closed, densely defined and $\mathcal{R}(\lambda \hat{I} - A) = \mathcal{H}$.

Theorem 3.11 Let A be a closed densely defined linear operator on $\mathcal{H}$.

1) If A is iso-dissipative, then:

$$\lambda \|\varphi\|_J \le \left\| [\lambda \hat{I} - A] \bullet \varphi \right\|_J \quad \forall \varphi \in D(A),\ \lambda > 0. \tag{3.5}$$

2) The operator A generates a C_0-iso-semigroup of contractions on $\mathcal{H}$, $\{S(t) \mid 0 \le t < \infty\}$, if and only if A is m-iso-dissipative.

3) If A is closed and densely defined with both A and A' (on $\mathcal{H}'$, the dual of $\mathcal{H}$) iso-dissipative, then A is m-iso-dissipative.

Proof: To prove 1, let A is iso-dissipative, $\lambda > 0$ and $\varphi \in D(A)$, then $\operatorname{Re}\langle A \bullet \varphi, \varphi \rangle_J \le 0$. Then:

$$\begin{aligned} \lambda \|\varphi\|_J^2 = \langle \lambda\varphi, \varphi \rangle_J &= \operatorname{Re}\langle \lambda\varphi - A\bullet\varphi, \varphi \rangle_J + \operatorname{Re}\langle A\bullet\varphi, \varphi \rangle_J \le \operatorname{Re}\langle \lambda\varphi - A\bullet\varphi, \varphi \rangle_J \\ &\le \left| \langle \lambda\varphi - A\bullet\varphi, \varphi \rangle_J \right| \le \left\| [\lambda\hat{I} - A]\bullet\varphi \right\|_J \|\varphi\|_J = \left\| [\lambda\hat{I} - A]\bullet\varphi \right\|_J \|\varphi\|_J . \end{aligned}$$

To prove 2, suppose A is m-iso-dissipative, so that $\mathcal{R}(\lambda\hat{I} - A) = \mathcal{H},\ \lambda > 0$. It follows from equation (3.5) that $R(\lambda, A)$ is an iso-bounded linear operator on $\mathcal{H}$ and that, for each $\lambda > 0$, $\|R(\lambda, A)\|_J \le 1/\lambda$. Thus, by the Hille-Yosida theorem, A is the generator of a C_0-iso-semigroup of contractions on $\mathcal{H}$.
On the other hand, if A is the generator of a C_0-iso-semigroup of contractions $\{S(t) \mid 0 \le t < \infty\}$ on $\mathcal{H}$, then $\rho(A) \supset (0, \infty)$, so that $\mathcal{R}(\lambda\hat{I} - A) = \mathcal{H},\ \lambda > 0$.

Furthermore, if $\varphi \in D(A)$, then: $\left|\left\langle S(t)\bullet\varphi, \varphi\right\rangle_J\right| \le \|\varphi\|_J^2$, so that

$$(1/h)\operatorname{Re}\left\langle S(h)\bullet\varphi - \varphi, \varphi\right\rangle_J = (1/h)\left[\operatorname{Re}\left\langle S(h)\bullet\varphi, \varphi\right\rangle_J - \|\varphi\|_J^2\right] \le 0 .$$

Letting $h \downarrow 0$, we get that $\lim_{h\downarrow 0}(1/h)\operatorname{Re}\left\langle S(h)\bullet\varphi - \varphi,\varphi\right\rangle_J = \operatorname{Re}\left\langle A\bullet\varphi,\varphi\right\rangle_J \le 0$, so that A is m-iso-dissipative.

To prove 3, we need only show that for any $\lambda > 0$, $\mathcal{R}(\lambda\hat{I} - A) = \mathcal{H}$. Since A is iso-dissipative and closed, it follows that $\mathcal{R}(\lambda\hat{I} - A)$ is a closed subspace of $\mathcal{H}$. Let $\lambda_0 > 0$ be such that there exists at least one ψ in $\mathcal{H}$ with $\left\langle \lambda_0\varphi - A\bullet\varphi, \psi\right\rangle_J = 0$ for all $\varphi \in D(A)$. This implies that:

$$\begin{aligned} 0 &= \left\langle \lambda_0\varphi - A\bullet\varphi, \psi\right\rangle_J = \left\langle \varphi, \lambda_0\psi\right\rangle_J - \left\langle A\bullet\varphi, \psi\right\rangle_J = \left\langle \varphi, \lambda_0\psi\right\rangle_J - \left\langle \varphi, A'\bullet\psi\right\rangle_J \\ &= \left\langle \varphi, \lambda_0\psi - A'\bullet\psi\right\rangle_J,\ \forall\varphi \in D(A) \Rightarrow \lambda_0\psi - A'\bullet\psi = 0. \end{aligned}$$

Since A' is iso-dissipative, by equation (3.5), we have that $0 = \left\|[\lambda\hat{I} - A']\bullet\psi\right\|_J \ge \lambda\|\psi\|_J$, so that $\psi = 0$, contradicting our assumption that $\psi \ne 0$. Thus, it follows that for any $\lambda > 0$, we must have $\mathcal{R}(\lambda\hat{I} - A) = \mathcal{H}$.

Example

The following example is an extension of one used by deLaubenfels [21] to motivate the development of the theory of C-semigroups. Let $\mathcal{H} = H_0^0(\mathbb{R}^n)$, the Hilbert space of functions mapping $\mathbb{R}^n$ to itself, which vanish at ∞, along all approaches. Consider the Cauchy problem:

$$\frac{d}{dt}\mathbf{u}(\mathbf{x},t) = a\left|\mathbf{x}\right|\mathbf{u}(\mathbf{x},t),\ \ \mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}),$$

where $a = \prod_{i=1}^{n} sign(x_i)$. Let $S(t)\mathbf{f}(\mathbf{x}) = \mathrm{e}^{ta|\mathbf{x}|}\mathbf{f}(\mathbf{x})$, where $\mathbf{x} = [x_1, \cdots, x_n]^t$. It is easy to see that $S(t)$ is a semigroup on $\mathcal{H}$ with generator A such that $A\mathbf{f}(\mathbf{x}) = a|\mathbf{x}|\mathbf{f}(\mathbf{x})$. It follows, that $u(\mathbf{x},t) = S(t)\mathbf{f}(\mathbf{x})$ solves the above initial-value problem. If we compute the resolvent, we get that:

$$R(\lambda, A)\mathbf{f}(\mathbf{x}) = \int_0^\infty e^{-\lambda t} \exp\{-t|\mathbf{x}|\}\mathbf{f}(\mathbf{x})dt = \frac{1}{\lambda - a|\mathbf{x}|}\mathbf{f}(\mathbf{x}).$$

It is clear that the spectrum of A is the real line, so that $R(\lambda, A)$ is an unbounded operator for all real λ. However, it can be checked that the bounded linear operator $A_\lambda = a\lambda|\mathbf{x}|/[\lambda + |\mathbf{x}|]$ converges strongly to A as $\lambda \to \infty$, and $\lim_{\lambda \to 0} S_\lambda(t)\mathbf{f}(\mathbf{x}) = S(t)\mathbf{f}(\mathbf{x})$. We do not prove this since it is a special case of the next theorem.

For any closed densely defined linear operator A on $\mathcal{H}$, let $T = -\left[A^*A\right]^{1/2}$, $\bar{T} = -\left[A^*A\right]^{1/2}$. It is easy to see that $T\,(\bar{T})$ is m-iso-dissipative, and thus, generates a C$_0$-contraction iso-semigroup. We can now write A as $A = W\,T$, where W is a partial isometry. Define $A_\lambda = \lambda A\bullet R(\lambda, T)$, so that $A_\lambda = \lambda^2 W\bullet R(\lambda, T) - \lambda W$ and, although in general $A\bullet R(\lambda, T) \neq R(\lambda, T)\bullet A$, we have that $\lambda A\bullet R(\lambda, T) = \lambda R(\lambda, \bar{T})\bullet A$, which is sufficient for the following theorem, which is a generalization of Theorem 2.18.

Theorem 3.12 If A is a closed densely defined linear operator on $\mathcal{H}$, then:

1. The operator A_λ is bounded and $\lim_{\lambda \to \infty} A_\lambda \bullet \varphi = A\bullet\varphi \;\forall \varphi \in D(A)$,
2. For $\lambda, t > 0$, $\exp\left[tA_\lambda J\right]\hat{I} = S_\lambda(t)$ is a iso-contraction semigroup and
3. If A generates an iso-semigroup $S(t) = \exp\left[tAJ\right]\hat{I}$ on $D \supseteq D(A)$ for $t > 0$, then $\lim_{\lambda \to \infty} \left\| S_\lambda(t)\bullet\varphi - S(t)\bullet\varphi \right\|_J = 0 \;\; \forall \varphi \in D$.

Proof: To prove 1., first note that $A_\lambda = \lambda^2 W \bullet R(\lambda,T) - \lambda W$, $\|W\|_J = 1$, so that $\|A_\lambda\|_J \le 2\lambda$, and then use the fact that $A_\lambda \bullet \varphi = \lambda R(\lambda,\bar{T}) \bullet A \bullet \varphi$, with $\lim_{\lambda\to\infty} \lambda R(\lambda,\bar{T}) \bullet \varphi = \varphi$.

To prove 2, use $A_\lambda = \lambda^2 W \bullet R(\lambda,T) - \lambda W$, $\|\lambda R(\lambda,T)\|_J \le 1$, and $\|W\|_J = 1$ to get that

$$\|S_\lambda(t)\|_J \le \exp\left[-t\lambda \|W\|_J\right] \exp\left[t\lambda \|W\|_J \|\lambda R(\lambda,T)\|_J\right] \le 1.$$

To prove 3., let $t > 0$ and $\varphi \in D(A)$. Then

$$\begin{aligned}\|S_\lambda(t)\bullet\varphi - S(t)\bullet\varphi\|_J &= \left\|\int_0^t \frac{d}{ds}\left[S_\lambda(t-s)\bullet S(s)\right]\bullet\varphi ds\right\|_J \\ &\le \int_0^t \left\|\left[S_\lambda(t-s)\bullet\left(A - A_\lambda\right)\bullet S(s)\bullet\varphi\right]\right\|_J \le \int_0^t \left\|\left[\left(A - A_\lambda\right)\bullet S(s)\bullet\varphi\right]\right\|_J ds.\end{aligned}$$

Use $\left\|\left[A\bullet S_\lambda(s)\bullet\varphi\right]\right\|_J = \left\|\left[\lambda R(\lambda,\bar{T})\bullet A\bullet S(s)\bullet\varphi\right]\right\|_J \le \left\|\left[A\bullet S(s)\bullet\varphi\right]\right\|_J$ to get $\left\|\left[\left(A - A_\lambda\right)\bullet S(s)\bullet\varphi\right]\right\|_J \le 2\left\|\left[A\bullet S(s)\bullet\varphi\right]\right\|_J$. Now, since $\left\|\left[A\bullet S(s)\bullet\varphi\right]\right\|_J$ is continuous, by the bounded convergence theorem we have

$$\lim_{\lambda\to\infty} \|S(s)\bullet\varphi - S_\lambda(s)\bullet\varphi\|_J \le \int_0^t \lim_{\lambda\to\infty} \left\|\left[\left(A - A_\lambda\right)\bullet S(s)\bullet\varphi\right]\right\|_J ds = 0.$$

4.0 Application (C-semigroups and B-Evolutions)

The theory of C-semigroups was developed by deLaubenfels [21], and applied to a number of interesting problems. In particular, he shows that C-semigroups are general enough to include the theory of integrated semigroups. Unaware of the work of deLaubenfels, Sauer [22] developed the theory of B-evolutions (see also [23]). As an application of the theory in Section 3, we show that the theory of C-semigroups and B-evolutions are contained in the theory of S*-algebras.

Definition 4.1 A one parameter family $S(t),\ 0 \le t < \infty$ of bounded linear operators on $\mathcal{H}$ is a strongly continuous C-regularized semigroup if
i) $S(0) = C$, where C is a bounded and injective linear operator.
ii) $S(t)S(s) = CS(t+s)$, $t,s \ge 0$ (semigroup property).
iii) $\lim_{t \downarrow 0} S(t)f = Cf$, $\forall f \in \mathcal{H}$.

Definition 4.2 A one parameter family $S(t),\ 0 \le t < \infty$ of bounded linear operators on $\mathcal{H}$ is a B-evolution semigroup if B is a closed, densely defined linear operator with a bounded inverse and
i) $S(0) = B^{-1}$,
ii) $S(t)BS(s) = S(t+s)$, $t,s \ge 0$ (semigroup property),
iii) $\lim_{t \downarrow 0} S(t)f = B^{-1}f,\ \forall f \in \mathcal{H}$.

The work of Sauer was motivated by the desire to develop a theory for the class of Sobolev-Galpern type equations. These equations are of the form

$$\left(Bu(x,t)\right)_t = Au(x,t) + f(t),\ u(x,0) = u_0(x)\,, \tag{4.1}$$

where A and B are linear elliptic operators on $\mathcal{H}$ (see Sauer [22], and references therein). These problems received intense investigation in this country starting around 1970 (see Showalter [24], [25], Lagnese [26] and Brill [27]). It should be noted that in general $\left(Bu\right)_t \ne Bu_t$ so that B cannot be factored out in the obvious way.

Theorem 4.3 If the operator B has a bounded inverse, and A is relatively bounded with respect to B , then $A \in L_J[\mathcal{H}]$ with $J = B^{-1}$. If $f \in D(A)$ and $v = Bu$, then the solution to the initial value problem

$$v_t(x,t) = A \bullet v(x,t) + f(t),\ v(x,0) = v_0(x) = \left(Bu\right)(x,t)\,|_{t=0}\,, \tag{4.2}$$

solves (4.1). If the solution iso-semigroup is $S(t)$, then

$$v(x,t) = S(t)\bullet v_0(x) + \int_0^t S(t-s)\bullet f(s)ds, \;\; u(x,t) = B^{-1}v(x,t). \tag{4.3}$$

Note that in this case, A is an iso-bounded linear operator so that $S(t)$ is a uniformly continuous iso-semigroup. In the general (linear) case, A need not be relatively bounded with respect to B, but the theorem still holds.

Discussion (Nonlinear Case)

From the beginning, it was clear that Sobolev-Galpern type equations take on additional importance when A or B (or both) are nonlinear operators (see Showalter [28], [29]). Many of the important cases can be reduced to the case where A is linear and $J = B^{-1}$ is nonlinear. This case includes the porous medium (nonlinear diffusion) equation. The Crandall-Liggett theory of nonlinear semigroups has been used quite successfully to attack problems of this type (see Crandall [30] and Konishi [31], [32]). Work by Bandle, Nanbu and Stakgold [33] has focused on the issue of extinction in finite time for such problems. It should be noted that the nonlinear case was studied earlier by Strauss [34], [35], see also Brezis [36].

From the above discussion, it is clear that we need to develop a "restricted" theory of nonlinear operator algebras as a part of a different approach to problems of this type. The theory of nonlinear operator algebras has not received much attention for obvious reasons. In order to make the approach clear, some definitions are required.

Definition 4.4 A set of operators, $(N[\mathcal{H}],+,\cdot)$ is called a *nonlinear operator algebra* if

1. The pair $(N[\mathcal{H}],+)$ is a vector space,
2. The pair $(N[\mathcal{H}],\cdot)$ is a non-Abelian semigroup with unit I, and

3. $\left(A+B\right)C = AC + BC,\ (ab)A = a(bA),\ A,\ B,\ C \in N[\mathcal{H}],\ a,b \in \mathbb{C}$.

There is no standard terminology; Martin [37] calls $\left(N[\mathcal{H}],+,\cdot\right)$ a near ring while Masani [38] calls it a pseudolinear algebra.

Definition 4.5 If J is a positive nonlinear operator, $L_J[\mathcal{H}]$ is called an NS*-algebra.

The following result shows why $L_J[\mathcal{H}]$ is called a *-algebra.

Theorem 4.6 If $L_J[\mathcal{H}]$ is an NS*-algebra, then for $f, g \in L_J[\mathcal{H}]$, we have:

$$\left\langle A \bullet f,\ g \right\rangle_J = \left\langle A^* \bullet g,\ f \right\rangle_J^c . \tag{4.4}$$

There is much work to be done but it should be clear that this approach provides closer contact between abstract analysis and applications.

Degenerate Parabolic Equations

The recent book on degenerate parabolic equations by DiBenedetto [39] approaches the problem

$$\partial u / \partial t = A \bullet (u) = AJ(u) \tag{4.5}$$

from an analysis point of view with interest in the continuity and growth properties of the solutions (J is nonlinear). More important from our point of view is the issue of the degenerate or singular nature of the problem. This work and the recent book by Favini and Yagi [40], which considers problems of the above type where AJ is a multi-valued operator, makes it clear that the most general class of isotopes is required for problems of this type.

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