HARDY AND RELLICH INEQUALITIES WITH REMAINDERS

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Abstract. In this paper our primary concern is with the establishment of weighted Hardy inequalities in $L^p(\Omega)$ and Rellich inequalities in $L^2(\Omega)$ depending upon the distance to the boundary of domains $\Omega \subset \mathbb{R}^n$ with a finite diameter $D(\Omega)$. Improved constants are presented in most cases.

1. Introduction

Recently, considerable attention has been given to extensions of the multi-dimensional Hardy inequality of the form

$$\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geq \mu(\Omega) \int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + \lambda(\Omega) \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x}, \quad u \in H_0^1(\Omega), \tag{1.1}$$

where Ω is an open connected subset of $\mathbb{R}^n$ and

$$\delta(\mathbf{x}) := dist(\mathbf{x}, \partial\Omega).$$

It is known that for $\mu(\Omega) = \frac{1}{4}$ there are smooth domains for which $\lambda(\Omega) \leq 0$, and for $\lambda(\Omega) = 0$, there are smooth domains for which $\mu(\Omega) < \frac{1}{4}$ - see M. Marcus, V.J. Mizel, and Y. Pinchover [8] and T. Matskewich and P.E. Sobolevskii [9]. In [2], H. Brezis and M. Marcus showed that for domains of class C^2 inequality (1.1) holds for

$$\mu(\Omega) = \frac{1}{4} \quad \text{and some} \quad \lambda(\Omega) \in (-\infty, \infty)$$

and when Ω is convex

$$\lambda(\Omega) \geq \frac{1}{4D(\Omega)^2} \tag{1.2}$$

in which $D(\Omega)$ is the diameter of Ω.

M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and A. Laptev [6] answered a question posed by H. Brezis and M. Markus in [2] by establishing the improvement to (1.2) that (1.1) holds for a convex domain

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Ω, with

$$\mu(\Omega) = \frac{1}{4}, \quad \lambda(\Omega) \geq \frac{K(n)}{4|\Omega|^{\frac{2}{n}}}, \quad \text{and} \quad K(n) := n\Big[\frac{s_{n-1}}{n}\Big]^{2/n} \tag{1.3}$$

in which $s_{n-1} := |\mathbb{S}^{n-1}|$ and $|\Omega|$ is the volume of Ω.

For a convex domain Ω and $\mu(\Omega) = 1/4$, a lower bound for $\lambda(\Omega)$ in (1.1) in terms of $|\Omega|$ was also obtained by S. Filippas, V. Maz'ya, and A. Tertikas in [5] as a special case of results on L^p Hardy inequalities. They prove that $\lambda(\Omega) \geq 3D_{int}(\Omega)^{-2}$, where $D_{int}(\Omega) = 2\sup_{x\in\Omega}\delta(x)$, the internal diameter of Ω. Since $3D_{int}(\Omega)^{-2} \geq \frac{3}{4n}K(n)/|\Omega|^{2/n}$, their result is an improvement of (1.3) for $n = 2, 3$, but the estimates don't compare for $n > 3$.

In this paper we show that (1.1) holds for (1.3) replaced by

$$\mu(\Omega) = \frac{1}{4} \qquad \text{and} \qquad \lambda(\Omega) \geq \frac{3K(n)}{2|\Omega|^{\frac{2}{n}}}$$

as well as proving weighted versions of the Hardy inequality in $L^p(\Omega)$ for $p > 1$.

In the case $p = 2$, the following are special cases of our results. If Ω is convex and $\sigma \in (0, 1]$, then

$$\int_\Omega \big|\nabla u(\mathbf{x})\big|^2 d\mathbf{x} \geq \frac{2^\sigma n(1-\sigma)^2}{4D(\Omega)^\sigma} \int_\Omega \Big\{\frac{B(n, 2-\sigma)}{\delta(\mathbf{x})^{2-\sigma}} + 3\Big(\frac{s_{n-1}}{n|\Omega|}\Big)^{\frac{2-\sigma}{n}}\Big\}|u(\mathbf{x})|^2 d\mathbf{x} \tag{1.4}$$

for

$$B(n, p) := \frac{\Gamma(\frac{p+1}{2}) \cdot \Gamma(\frac{n}{2})}{\sqrt{\pi} \cdot \Gamma(\frac{n+p}{2})}. \tag{1.5}$$

If $\sigma \in [\frac{2-n}{2}, 0]$ and Ω is convex, then

$$\begin{aligned}\int_\Omega \delta(\mathbf{x})^\sigma \big|\nabla u(\mathbf{x})\big|^2 d\mathbf{x} \quad &\geq \quad \frac{n(1-\sigma)^2}{4} B(n, 2-\sigma) \int_\Omega \delta(\mathbf{x})^{\sigma-2}|u(\mathbf{x})|^2 d\mathbf{x} \\ &+ \quad \frac{C_H(n,\sigma)}{|\Omega|^{\frac{2(1-\sigma)}{n}}} \int_\Omega \delta(\mathbf{x})^{|\sigma|}|u(\mathbf{x})|^2 d\mathbf{x}.\end{aligned}$$

for $C_H(n, \sigma)$ given in (3.4). Similar results for weighted forms of the Hardy inequality in $L^p(\Omega)$ are given in section 4.

Finally, we show that our one-dimensional inequalities in §2 lead to improved constants for the Rellich inequality obtained by G. Barbatis in [1] for $n \geq 4$.

2. One-dimensional inequalities

As is the case in [6], our proofs are based on one-dimensional Hardy-type inequalities coupled with the use of the mean-distance function introduced by Davies to extend to higher dimensions; see [4]. The basic one-dimensional inequality is as follows:

Lemma 1. *Let* $u \in C_0^1(0,2b), \rho(t) := \min\{t, 2b-t\}$ *and let* $f \in C^1[0,b]$ *be monotonic on* $[0,b]$*. Then for* $p>1$

$$\int_0^{2b} |f'(\rho(t))||u(t)|^p dt \le p^p \int_0^{2b} \frac{|f(\rho(t)) - f(b)|^p}{|f'(\rho(t))|^{p-1}} |u'(t)|^p dt. \tag{2.1}$$

Proof. First let $u := v\chi_{(0,b]}$, the restriction to $(0,b]$ of some $v \in C_0^1(0,2b)$. For any constant c

$$\begin{aligned} -\textstyle\int_0^b [f(t)-c]'|u(t)|^p dt = & -[f(t)-c]|u(t)|^p\Big|_0^b \\ & + \textstyle\int_0^b [f(t)-c]\frac{p}{2}[|u(t)|^2]^{\frac{p}{2}-1}[|u(t)|^2]' dt. \end{aligned}$$

By choosing $c = f(b)$, we have that

$$-\textstyle\int_0^b f'(t)|u(t)|^p dt = \; p\int_0^b [f(t)-f(b)]|u(t)|^{p-2}\mathfrak{Re}[\overline{u(t)}u'(t)]dt. \tag{2.2}$$

Similarly, for $u = v\chi_{[b,2b)}$, $v \in C_0^1(0,2b)$, we have

$$\begin{aligned} & -\textstyle\int_b^{2b} f'(2b-s)|u(s)|^p ds \\ & \quad = p\textstyle\int_b^{2b} [f(2b-s)-f(b)]|u(s)|^{p-2}\mathfrak{Re}[\overline{u(s)}u'(s)]ds. \end{aligned}$$

Therefore, since f is monotonic, for any $u \in C_0^1(0,2b)$

$$\begin{aligned} & \textstyle\int_0^{2b} |f'(\rho(t))||u(t)|^p dt \\ & \quad = p\textstyle\int_0^{2b} |f(\rho(t)) - f(b)||u(t)|^{p-2}\mathfrak{Re}[\overline{u(t)}u'(t)]dt \\ & \quad \le p\textstyle\int_0^b |f'(\rho(t))|^{\frac{p-1}{p}}|u(t)|^{p-1}\frac{|f(\rho(t))-f(b)|}{|f'(\rho(t))|^{\frac{p-1}{p}}}|u'(t)|dt \\ & \quad \le p\Big[\textstyle\int_0^b |f'(\rho(t))||u(t)|^p dt\Big]^{\frac{p-1}{p}}\Big[\textstyle\int_0^b \frac{|f(\rho(t))-f(b)|^p}{|f'(\rho(t))|^{p-1}}|u'(t)|^p dt\Big]^{\frac{1}{p}} \end{aligned}$$

on applying Hölder's inequality. Inequality (2.1) now follows. □

The next lemma provides the one-dimensional result needed to improve (1.3), which was proved in [6].

Lemma 2. *Let* $\sigma \le 1$ *and define* $\mu(t) := 2b - \rho(t)$*. For all* $u \in C_0^1(0,2b)$

$$\int_0^{2b} \rho(t)^\sigma |u'(t)|^2 dt \ge \Big(\frac{1-\sigma}{2}\Big)^2 \int_0^{2b} \rho(t)^{\sigma-2}\Big[1 + k(\sigma)\Big(\frac{2\rho(t)}{\mu(t)}\Big)^{1-\sigma}\Big]^2 |u(t)|^2 dt, \tag{2.3}$$

for

$$k(\sigma) := \begin{cases} [1 - 2^{\frac{1}{\sigma}-1}]^{-\sigma}, & \sigma < 0, \\ 1, & \sigma \in [0,1]. \end{cases}$$

Proof. On setting $f(t) = t^{\sigma-1}$ in (2.1) we get

$$|1-\sigma|^p \int_0^{2b} \rho(t)^{\sigma-2}|u(t)|^p dt \le p^p \int_0^{2b} \rho(t)^{p+\sigma-2}\Big|1 - \Big[\frac{\rho(t)}{b}\Big]^{1-\sigma}\Big|^p |u'(t)|^p dt. \tag{2.4}$$

With $u \in C_0^1(0,2b)$, let $p=2$ and substitute $v(t) = [1-\left(\frac{\rho(t)}{b}\right)^{1-\sigma}]u(t)$ in (2.4). We claim that this gives

$$\int_0^{2b} \rho^\sigma(t)|v'(t)|^2dt \geq \left(\frac{1-\sigma}{2}\right)^2 \int_0^{2b} \rho(t)^{\sigma-2}\Big[1-\Big(\frac{\rho(t)}{b}\Big)^{1-\sigma}\Big]^{-2}|v(t)|^2dt \tag{2.5}$$

for any real number σ. The substitution gives

$$\rho(t)^{\sigma/2}v'(t) = -(1-\sigma)b^{\sigma-1}\rho(t)^{-\sigma/2}\rho'(t)u(t)+\rho(t)^{\sigma/2}\big[1-\big(\frac{\rho(t)}{b}\big)^{1-\sigma}\big]u'(t).$$

Consequently,

$$\begin{aligned}\rho(t)^\sigma|v'(t)|^2 = {} & (1-\sigma)^2b^{2\sigma-2}\rho(t)^{-\sigma}|u(t)|^2 + \rho(t)^\sigma\big[1-\big(\tfrac{\rho(t)}{b}\big)^{1-\sigma}\big]^2|u'(t)|^2 \\ & -(1-\sigma)b^{\sigma-1}\rho'(t)\big[1-\big(\tfrac{\rho(t)}{b}\big)^{1-\sigma}\big][|u|^2]'\end{aligned}$$

which implies that

$$\begin{aligned}\textstyle\int_0^{2b}\rho(t)^\sigma|v'(t)|^2dt = {} & \textstyle\int_0^{2b}\rho(t)^\sigma\big[1-\big(\frac{\rho(t)}{b}\big)^{1-\sigma}\big]^2|u'(t)|^2dt \\ & \textstyle+\int_0^{2b}(1-\sigma)^2b^{2\sigma-2}\rho(t)^{-\sigma}|u(t)|^2dt \\ & \textstyle+(1-\sigma)b^{\sigma-1}\int_0^{2b}\frac{d}{dt}\Big[\rho'(t)\big[1-\big(\frac{\rho(t)}{b}\big)^{1-\sigma}\big]\Big]|u|^2dt \\ = {} & \textstyle\int_0^{2b}\rho(t)^\sigma\big[1-\big(\frac{\rho(t)}{b}\big)^{1-\sigma}\big]^2|u'(t)|^2dt\end{aligned} \tag{2.6}$$

since $\rho'(t)=1$ in $(0,b)$ and -1 in $(b,2b)$. Therefore, (2.5) follows from (2.4).

Since $2b = \mu(t)+\rho(t)$

$$\begin{aligned}\big[1-\big(\tfrac{\rho(t)}{b}\big)^{1-\sigma}\big]^{-2} = {} & \Big[1+\tfrac{\rho(t)^{1-\sigma}}{b^{1-\sigma}-\rho(t)^{1-\sigma}}\Big]^2 \\ = {} & \Big[1+2^{1-\sigma}\big(\tfrac{\rho(t)}{\mu(t)}\big)^{1-\sigma}k_\sigma\big(\tfrac{\rho(t)}{\mu(t)}\big)\Big]^2\end{aligned} \tag{2.7}$$

for

$$k_\sigma(x) := \frac{1}{(1+x)^{1-\sigma}-(2x)^{1-\sigma}}, \qquad x\in[0,1), \qquad \sigma\neq 1.$$

For $\sigma<1$, $k_\sigma(x)>0$ in $(0,1)$, $k_\sigma(0)=1$ and $k_\sigma(x)\to\infty$ as $x\to 1^-$. By examining the derivative of $k_\sigma(x)$

$$k'_\sigma(x) = \frac{-(1-\sigma)((1+x)^{-\sigma}-2^{1-\sigma}x^{-\sigma})}{[(1+x)^{1-\sigma}-(2x)^{1-\sigma}]^2}$$

we see that

$$\lim_{x\to 0^+} k'_\sigma(x) = \begin{cases} -(1-\sigma), & \sigma<0, \\ 1, & \sigma=0, \\ \infty, & 0<\sigma<1. \end{cases}$$

For $\sigma<0$, $k_\sigma(x)$ is minimized at

$$x_\sigma := 1/(2^{1-\frac{1}{\sigma}}-1) < 1.$$

Calculations show that

$$k_\sigma(x_\sigma) = \big[1-2^{\frac{1}{\sigma}-1}\big]^{-\sigma} =: k(\sigma).$$

For $\sigma \in [0,1)$, $k'_\sigma(x)$ is never zero in $(0,1)$ indicating that $k_\sigma(x)$ is minimized at $x = 0$ for $\sigma \in [0,1)$ and $x \in [0,1)$. The inequality (2.3) now follows. □

In order to treat the case in which $p \neq 2$, we make use of the methods of Tidblom [11] and prove a weighted version of Theorem 1.1 in [11].

Lemma 3. *Let $u \in C_0^1(0,2b)$, $p \in (1,\infty)$, and $\sigma \leq p-1$. Then*

$$\int_0^{2b} \rho(t)^\sigma |u'(t)|^p dt \geq \left[\frac{p-\sigma-1}{p}\right]^p \int_0^{2b} \{\rho(t)^{\sigma-p} + (p-1)b^{\sigma-p}\} |u(t)|^p dt.$$

Proof. We may assume that $\sigma \neq p-1$ since otherwise the conclusion is trivial. According to (2.2) for a monotonic function f and a positive function g,

$$\begin{aligned}\int_0^b |f'(t)||u(t)|^p dt &\leq \int_0^b p|f(t)-f(b)||u(t)|^{p-1}|u'(t)|dt \\ &\leq p\left[\int_0^b g(t)|u'(t)|^p dt\right]^{1/p}\left[\int_0^b \left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{1/(p-1)} |u(t)|^p dt\right]^{1-1/p}.\end{aligned}$$

Consequently,

$$p^p \int_0^b g(t)|u'(t)|^p dt \geq \frac{\left(\int_0^b |f'(t)||u(t)|^p dt\right)^p}{\left(\int_0^b \left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{1/(p-1)} |u(t)|^p dt\right)^{p-1}}.$$

Now, as in [11], using a corollary to Young's inequality, namely

$$A^p/B^{p-1} \geq pA - (p-1)B,$$

with $A = \int_0^b |f'(t)||u(t)|^p dt, B = \int_0^b \left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{1/p-1} |u(t)|^p dt$, it follows that

$$\begin{aligned}&p^p \int_0^b g(t)|u'(t)|^p dt \\ &\qquad \geq \int_0^b \left\{p|f'(t)| - (p-1)\left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{1/(p-1)}\right\} |u(t)|^p dt.\end{aligned}$$

Choose $f(t) = t^{\sigma-p+1}$ and $g(t) = (p-\sigma-1)^{-(p-1)} t^\sigma$. Then

$$\begin{aligned}\left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{1/(p-1)} &= (p-\sigma-1)\left[\frac{\left|t^{\sigma-p+1}-b^{\sigma-p+1}\right|^p}{t^\sigma}\right]^{\frac{1}{p-1}} \\ &= (p-\sigma-1)t^{\sigma-p}\left[\left(1-\left(\frac{t}{b}\right)^{p-\sigma-1}\right)^p\right]^{\frac{1}{p-1}}.\end{aligned}$$

Consequently, for $t \in (0,b)$

$$\begin{aligned} p|f'(t)| &- (p-1)\left(\tfrac{|f(t)-f(b)|^p}{g(t)}\right)^{1/(p-1)} \\ &= (p-\sigma-1)\left\{pt^{\sigma-p} - (p-1)t^{\sigma-p}\left[\left(1-\left(\tfrac{t}{b}\right)^{p-\sigma-1}\right)^p\right]^{\frac{1}{p-1}}\right\} \\ &= (p-\sigma-1)t^{\sigma-p}\left\{1+(p-1)\left(1-\left[1-\left(\tfrac{t}{b}\right)^{p-\sigma-1}\right]^{\frac{p}{p-1}}\right)\right\} \\ &\geq (p-\sigma-1)t^{\sigma-p}\left\{1+(p-1)\left(\tfrac{t}{b}\right)^{p-\sigma-1}\right\} \\ &\geq (p-\sigma-1)\left\{t^{\sigma-p}+(p-1)\left(\tfrac{1}{b^{p-\sigma}}\right)\right\}. \end{aligned}$$

and the inequality follows. In the inequality above we have used the fact that

$$\left[1-\left(\frac{t}{b}\right)^{p-\sigma-1}\right]^{\frac{p}{p-1}} \leq 1-\left(\frac{t}{b}\right)^{p-\sigma-1}.$$

The proof is completed by following the last part of the proof of Lemma 1. □

For a certain range of values taken by σ, $\sigma \in [-c_\sigma, 1)$ with $c_\sigma > 0$, the inequality in $L^2(\Omega)$ given by Lemma 2 gives a better bound than Lemma 3 with $p = 2$. In fact for $\sigma < 1$

$$\rho(t)^{\sigma-2}\left[1+k(\sigma)\Big(\frac{2\rho(t)}{\mu(t)}\Big)^{1-\sigma}\right]^2 = \rho(t)^{\sigma-2} + \frac{2^{2-\sigma}k(\sigma)}{\rho(t)\mu(t)^{1-\sigma}} + \frac{2^{2-2\sigma}k(\sigma)^2}{\rho(t)^\sigma\mu(t)^{2-2\sigma}}$$

with

$$\begin{aligned} \frac{2^{2-\sigma}k(\sigma)}{\rho(t)\mu(t)^{1-\sigma}} &+ \frac{2^{2-2\sigma}k(\sigma)^2}{\rho(t)^\sigma\mu(t)^{2-2\sigma}} \\ &\geq \begin{cases} \frac{5}{2^\sigma}b^{\sigma-2}, & \sigma \in [0,1), \\ \left[2-\sigma+b^\sigma k(\sigma)\rho(t)^{|\sigma|}\right]k(\sigma)b^{\sigma-2}, & \sigma < 0. \end{cases} \end{aligned} \tag{2.8}$$

Since $k(\sigma)$ decreases to 0 for $\sigma < 0$ as $|\sigma| \to \infty$ and $k(-3) \approx 0.22$, then the left-hand side of (2.8) is greater than $b^{\sigma-2}$ for $\sigma \in [-3,1)$.

3. A Hardy inequality in $L^2(\Omega)$

We need the following notation (c.f.[6]). For each $\mathbf{x} \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$,

$$\begin{gathered} \tau_\nu(\mathbf{x}) := \min\{s > 0 : \mathbf{x} + s\nu \notin \Omega\}; \\ D_\nu(\mathbf{x}) := \tau_\nu(\mathbf{x}) + \tau_{-\nu}(\mathbf{x}); \\ \rho_\nu(\mathbf{x}) := \min\{\tau_\nu(\mathbf{x}), \tau_{-\nu}(\mathbf{x})\}; \\ \mu_\nu(\mathbf{x}) := \max\{\tau_\nu(\mathbf{x}), \tau_{-\nu}(\mathbf{x})\} = D_\nu(\mathbf{x}) - \rho_\nu(\mathbf{x}); \\ D(\Omega) := \sup_{\mathbf{x}\in\Omega,\ \nu\in\mathbb{S}^{n-1}} D_\nu(\mathbf{x}); \\ \Omega_\mathbf{x} := \{\mathbf{y} \in \Omega : \mathbf{x} + t(\mathbf{y}-\mathbf{x}) \in \Omega,\ \forall t \in [0,1]\}. \end{gathered}$$

Note that $D(\Omega)$ is the diameter of Ω and $\Omega_\mathbf{x}$ is the part of Ω which can be "seen" from the point $\mathbf{x} \in \Omega$. The volume of $\Omega_\mathbf{x}$ is denoted by $|\Omega_\mathbf{x}|$.

Let $d\omega(\nu)$ denote the normalized measure on $\mathbb{S}^{n-1}$ (so that $1 = \int_{\mathbb{S}^{n-1}} d\omega(\nu)$) and define

$$\rho(\mathbf{x}; s) := \int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^s d\omega(\nu). \tag{3.1}$$

Hence $\rho^{-1/2}(\mathbf{x}; -2) = \rho(\mathbf{x})$ the "mean-distance" function introduced by Davies in [4]. For

$$B(n,p) := \int_{\mathbb{S}^{n-1}} |\cos(\mathbf{e}, \nu)|^p d\omega(\nu) = \frac{\Gamma(\frac{p+1}{2}) \cdot \Gamma(\frac{n}{2})}{\sqrt{\pi} \cdot \Gamma(\frac{n+p}{2})}, \quad \mathbf{e} \in \mathbb{R}^n, \tag{3.2}$$

it is known that

$$\rho(\mathbf{x}; -p) := \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(\mathbf{x})^p} d\omega(\nu) \geq \frac{B(n,p)}{\delta(\mathbf{x})^p} \tag{3.3}$$

for convex domains Ω – see Exercise 5.7 in [4], [3], and [11]. Note that $B(n,2) = n^{-1}$. This fact can be applied to most of the results below when Ω is convex.

For a Hardy inequality in $L^2(\Omega)$ with weights we will need to define

$$C_H(n,\sigma) := n\Big(\frac{s_{n-1}}{n}\Big)^{\frac{2(1-\sigma)}{n}} k(\sigma)[2^{|\sigma|} + 2^{2|\sigma|-1}k(\sigma)](1-\sigma)^2 \tag{3.4}$$

for $\sigma \in [\frac{2-n}{2}, 0]$ and $n \geq 2$ where as given in Lemma 2

$$k(\sigma) := \begin{cases} \big[1 - 2^{\frac{1}{\sigma}-1}\big]^{-\sigma}, & \sigma < 0, \\ 1, & \sigma \in [0,1]. \end{cases}$$

Note that $C_H(n,0) = \frac{3}{2}K(n)$ for $K(n)$ defined in (1.3).

Theorem 1. *If $\frac{2-n}{2} \leq \sigma \leq 0$, then for any $u \in C_0^1(\Omega)$*

$$\begin{aligned} \int_\Omega \delta(\mathbf{x})^\sigma \big|\nabla u(\mathbf{x})\big|^2 d\mathbf{x} \;&\geq\; \frac{n(1-\sigma)^2}{4} \int_\Omega \rho(\mathbf{x}; \sigma-2)|u(\mathbf{x})|^2 d\mathbf{x} \\ &+\; C_H(n,\sigma) \int_\Omega \frac{\delta(\mathbf{x})^{|\sigma|}}{|\Omega_{\mathbf{x}}|^{\frac{2(1-\sigma)}{n}}} |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned} \tag{3.5}$$

If $0 < \sigma \leq 1$, then

$$\begin{aligned} &\int_\Omega \big|\nabla u(\mathbf{x})\big|^2 d\mathbf{x} \geq \\ &\frac{2^\sigma n(1-\sigma)^2}{4D(\Omega)^\sigma} \int_\Omega \Big\{\rho(\mathbf{x}; \sigma-2) + 3\big(\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|}\big)^{\frac{2-\sigma}{n}}\Big\} |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned} \tag{3.6}$$

If Ω is convex, then for any $u \in C_0^1(\Omega)$

$$\begin{aligned} \int_\Omega \delta(\mathbf{x})^\sigma \big|\nabla u(\mathbf{x})\big|^2 d\mathbf{x} \;&\geq\; \frac{n(1-\sigma)^2}{4} B(n, 2-\sigma) \int_\Omega \delta(\mathbf{x})^{\sigma-2} |u(\mathbf{x})|^2 d\mathbf{x} \\ &+\; \frac{C_H(n,\sigma)}{|\Omega|^{\frac{2(1-\sigma)}{n}}} \int_\Omega \delta(\mathbf{x})^{|\sigma|} |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

when $\sigma \in [\frac{2-n}{2}, 0]$ *and*

$$\int_\Omega \big|\nabla u(\mathbf{x})\big|^2 d\mathbf{x} \geq \frac{2^\sigma n(1-\sigma)^2}{4D(\Omega)^\sigma} \int_\Omega \Big\{B(n, 2-\sigma)\delta(\mathbf{x})^{\sigma-2} + 3\big(\tfrac{s_{n-1}}{n|\Omega|}\big)^{\frac{2-\sigma}{n}}\Big\}|u(\mathbf{x})|^2 d\mathbf{x}.$$

when $\sigma \in (0, 1]$.

Proof. Let $\partial_\nu u$, $\nu \in \mathbb{S}^{n-1}$, denote the derivative of u in the direction of ν, i.e., $\partial_\nu u = \nu \cdot (\nabla u)$. It follows from Lemma 2 that for $\sigma \in (-\infty, 1]$

$$\begin{aligned} &\int_\Omega \rho_\nu^\sigma(\mathbf{x})\big|\partial_\nu u\big|^2 d\mathbf{x} \\ &\geq \big(\tfrac{1-\sigma}{2}\big)^2 \int_\Omega \rho_\nu(\mathbf{x})^{\sigma-2}\big(1 + k(\sigma)\big[\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\big]^{(1-\sigma)}\big)^2 |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned} \tag{3.7}$$

Expanding the integrand in (3.7), we have

$$\begin{aligned} &\rho_\nu(\mathbf{x})^{\sigma-2}\big(1 + k(\sigma)\big[\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\big]^{(1-\sigma)}\big)^2 \\ = \quad &\rho_\nu(\mathbf{x})^{\sigma-2} + 2^{2-\sigma}\frac{k(\sigma)\rho_\nu(\mathbf{x})^{-\sigma}}{(\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x}))^{1-\sigma}} + 2^{2(1-\sigma)}k(\sigma)^2\frac{\rho_\nu(\mathbf{x})^{-\sigma}}{\mu_\nu(\mathbf{x})^{2(1-\sigma)}}. \end{aligned} \tag{3.8}$$

If $\sigma \leq 0$

$$\begin{aligned} &\rho_\nu(\mathbf{x})^{\sigma-2}\big[1 + k(\sigma)\big(\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\big)^{(1-\sigma)}\big]^2 \\ &\geq \rho_\nu(\mathbf{x})^{\sigma-2} + 2^{2-\sigma}\frac{k(\sigma)\delta(\mathbf{x})^{|\sigma|}}{(\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x}))^{1-\sigma}} + 2^{2(1-\sigma)}k(\sigma)^2\frac{\delta(\mathbf{x})^{|\sigma|}}{\tau_\nu(\mathbf{x})^{2(1-\sigma)}+\tau_{-\nu}(\mathbf{x})^{2(1-\sigma)}} \end{aligned} \tag{3.9}$$

since $\rho_\nu(\mathbf{x})^{-\sigma} \geq \delta(\mathbf{x})^{|\sigma|}$ in this case. As in [6], we note that since

$$\begin{aligned} \int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x}))^{1-\sigma}d\omega(\nu) \leq \ &\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x}))^{2(1-\sigma)}d\omega(\nu) \\ \leq \ &\big[\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x}))^n d\omega(\nu)\big]^{\frac{2(1-\sigma)}{n}} \\ = \ &\big[\tfrac{n}{s_{n-1}}|\Omega_\mathbf{x}|\big]^{\frac{2(1-\sigma)}{n}} \end{aligned}$$

for $\sigma \geq \frac{2-n}{2}$, then

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \frac{1}{(\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x}))^{1-\sigma}}d\omega(\nu) \geq \ &\big[\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x}))^{1-\sigma}d\omega(\nu)\big]^{-1} \\ \geq \ &\big[\tfrac{n}{s_{n-1}}|\Omega_\mathbf{x}|\big]^{-\frac{2(1-\sigma)}{n}}. \end{aligned}$$

For the third term in inequality (3.9)

$$\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x})^{2(1-\sigma)} + \tau_{-\nu}(\mathbf{x})^{2(1-\sigma)})d\omega(\nu) = \ 2\int_{\mathbb{S}^{n-1}} \tau_\nu(\mathbf{x})^{2(1-\sigma)}d\omega(\nu)$$

implying that for $\sigma \geq \frac{2-n}{2}$

$$\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x})^{2(1-\sigma)} + \tau_{-\nu}(\mathbf{x})^{2(1-\sigma)})^{-1}d\omega(\nu) \geq \ \tfrac{1}{2}\big[\tfrac{n}{s_{n-1}}|\Omega_\mathbf{x}|\big]^{-\frac{2(1-\sigma)}{n}}.$$

Consequently, for $\frac{2-n}{2} \leq \sigma \leq 0$ we have that

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^{\sigma-2}\big[1 + k(\sigma)\big(\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\big)^{(1-\sigma)}\big]^2 d\omega(\nu) \\ &\geq \rho(\mathbf{x}; \sigma-2) + C_H(n,\sigma)\delta(\mathbf{x})^{|\sigma|}/\big[n|\Omega_\mathbf{x}|^{\frac{2(1-\sigma)}{n}}\big]. \end{aligned} \tag{3.10}$$

Upon combining this fact with (3.7) we have

$$\begin{aligned}&\Big(\tfrac{1-\sigma}{2}\Big)^2\int_\Omega\big\{\rho(\mathbf{x};\sigma-2)+\tfrac{C_H(n,\sigma)\delta(\mathbf{x})^{|\sigma|}}{n|\Omega_{\mathbf{x}}|^{2(1-\sigma)/n}}\big\}|u(\mathbf{x})|^2d\mathbf{x}\\&\le\int_\Omega\int_{\mathbb{S}^{n-1}}\rho_\nu(\mathbf{x})^\sigma\big|\partial_\nu u(\mathbf{x})\big|^2d\omega(\nu)d\mathbf{x}\\&=\int_\Omega\delta(\mathbf{x})^\sigma\int_{\mathbb{S}^{n-1}}|\cos(\nu,\nabla u(\mathbf{x}))|^2d\omega(\nu)\big|\nabla u(\mathbf{x})\big|^2d\mathbf{x}\end{aligned}\tag{3.11}$$

for $\sigma\le 0$. Since

$$\int_{\mathbb{S}^{n-1}}|\cos(\nu,\alpha)|^2d\omega(\nu)=\frac{1}{n}\tag{3.12}$$

for any fixed $\alpha\in\mathbb{S}^{n-1}$ (see Tidblom [11], p.2270), inequality (3.5) follows.

For $0<\sigma\le 1$, we consider first the third term on the right-hand side of (3.8). We have

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}\rho_\nu(\mathbf{x})^\sigma\mu_\nu(\mathbf{x})^{2(1-\sigma)}d\omega(\nu)\\&\quad\le\int_{\mathbb{S}^{n-1}}2^{-\sigma}(\tau_\nu(\mathbf{x})+\tau_{-\nu}(\mathbf{x}))^\sigma(\tau_\nu(\mathbf{x})+\tau_{-\nu}(\mathbf{x}))^{2(1-\sigma)}d\omega(\nu)\\&\quad=2^{-\sigma}\|\tau_\nu(\mathbf{x})+\tau_{-\nu}(\mathbf{x})\|^{2-\sigma}_{L^{2-\sigma}(\mathbb{S}^{n-1})}\\&\quad\le 2^{-\sigma}\big[\|\tau_\nu(\mathbf{x})\|_{L^{2-\sigma}(\mathbb{S}^{n-1})}+\|\tau_{-\nu}(\mathbf{x})\|_{L^{2-\sigma}(\mathbb{S}^{n-1})}\big]^{2-\sigma}\\&\quad=2^{2(1-\sigma)}\int_{\mathbb{S}^{n-1}}\tau_\nu(\mathbf{x})^{2-\sigma}d\omega(\nu)\\&\quad\le 2^{2(1-\sigma)}\big[\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x}))^nd\omega(\nu)\big]^{\frac{2-\sigma}{n}}\\&\quad=2^{2(1-\sigma)}\big[\tfrac{n}{s_{n-1}}|\Omega_{\mathbf{x}}|\big]^{\frac{2-\sigma}{n}}\end{aligned}$$

for $n\ge 2$ by the Minkowski and Hölder inequalities. Therefore, the term

$$\int_{\mathbb{S}^{n-1}}\frac{\rho_\nu(\mathbf{x})^{-\sigma}d\omega(\nu)}{\mu_\nu(\mathbf{x})^{2(1-\sigma)}}\ \ge\ \ 2^{2(\sigma-1)}\big(\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|}\big)^{\frac{2-\sigma}{n}}.$$

Similarly, in the second term of (3.8)

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}\rho_\nu(\mathbf{x})\mu_\nu(\mathbf{x})^{1-\sigma}d\omega(\nu)\\&\quad\le\tfrac12\int_{\mathbb{S}^{n-1}}(\tau_\nu(\mathbf{x})+\tau_{-\nu}(\mathbf{x}))(\tau_\nu(\mathbf{x})+\tau_{-\nu}(\mathbf{x}))^{1-\sigma}d\omega(\nu)\\&\quad\le 2^{1-\sigma}\big[\tfrac{n}{s_{n-1}}|\Omega_{\mathbf{x}}|\big]^{\frac{2-\sigma}{n}}\end{aligned}$$

as before implying that

$$\int_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu(\mathbf{x})\mu_\nu(\mathbf{x})^{1-\sigma}}\ \ge\ \ 2^{\sigma-1}\big(\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|}\big)^{\frac{2-\sigma}{n}}.$$

For $0<\sigma<1$ we now have that

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}\rho_\nu(\mathbf{x})^{\sigma-2}\big[1+k(\sigma)\big(\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\big)^{(1-\sigma)}\big]^2d\omega(\nu)\\&\ge\rho(\mathbf{x};\sigma-2)+3\big(\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|}\big)^{\frac{2-\sigma}{n}}\end{aligned}$$

since $k(\sigma)=1$ in this case. Consequently,

$$\begin{aligned}&\int_\Omega\int_{\mathbb{S}^{n-1}}\rho_\nu(\mathbf{x})^\sigma|\cos(\nu,\nabla u(\mathbf{x}))|^2d\omega(\nu)\big|\nabla u(\mathbf{x})\big|^2d\mathbf{x}\\&\ge\Big(\tfrac{1-\sigma}{2}\Big)^2\int_\Omega\big[\rho(\mathbf{x};\sigma-2)+3\big(\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|}\big)^{\frac{2-\sigma}{n}}\big]|u(\mathbf{x})|^2d\mathbf{x}.\end{aligned}$$

According to (3.12) it follows that

$$\int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^\sigma |\cos(\nu, \nabla u(\mathbf{x}))|^2 d\omega(\nu) \leq \frac{D(\Omega)^\sigma}{2^\sigma n}.$$

Therefore, (3.6) holds.

The inequalities in the statement of the theorem for the case of a convex domain Ω follow from (3.3) and the fact that $|\Omega_{\mathbf{x}}| = |\Omega|$ for all $\mathbf{x} \in \Omega$. □

4. An $L^p(\Omega)$ inequality

With the guidance of Tidblom's analysis for the Hardy inequality in [11], L^p versions of the weighted Hardy theorem in the last section can be proved by similar techniques. When $\sigma = 0$, the next theorem reduces to Theorem 2.1 of [11].

Theorem 2. *Let $u \in C_0^1(\Omega)$ and $p \in (1, \infty)$. If $\sigma \leq 0$, then for $B(n,p)$ defined in (3.2)*

$$\begin{aligned} &\int_\Omega \delta(\mathbf{x})^\sigma |\nabla u(\mathbf{x})|^p d\mathbf{x} \geq \\ &\quad \tfrac{[|p-\sigma-1|/p]^p}{B(n,p)} \int_\Omega \Big\{ \rho(\mathbf{x}; \sigma - p) + (p-1) \Big[\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|} \Big]^{\frac{p-\sigma}{n}} \Big\} |u(\mathbf{x})|^p d\mathbf{x} \end{aligned} \tag{4.1}$$

and if $\sigma \in [0, p-1]$, then

$$\begin{aligned} &\int_\Omega |\nabla u(\mathbf{x})|^p d\mathbf{x} \geq \\ &\quad \tfrac{2^\sigma [|p-\sigma-1|/p]^p}{B(n,p)\ D(\Omega)^\sigma} \int_\Omega \Big\{ \rho(\mathbf{x}; \sigma - p) + (p-1) \Big[\tfrac{s_{n-1}}{n|\Omega_{\mathbf{x}}|} \Big]^{\frac{p-\sigma}{n}} \Big\} |u(\mathbf{x})|^p d\mathbf{x}. \end{aligned} \tag{4.2}$$

If Ω is convex, $\rho(\mathbf{x}, \sigma - p)$ can be replaced in (4.1) and (4.2) by the term $B(n, p-\sigma)/\delta(\mathbf{x})^{p-\sigma}$ (in view of (3.3)) and $|\Omega_x|$ by $|\Omega|$.

Proof. From Lemma 3 we have that for $\sigma \leq p-1$, any $\nu \in \mathbb{S}^{n-1}$, and $u \in C_0^1(\Omega)$

$$\begin{aligned} &\int_\Omega \rho_\nu(\mathbf{x})^\sigma |\partial_\nu u(\mathbf{x})|^p d\mathbf{x} \geq \\ &\quad \Big[\tfrac{|p-\sigma-1|}{p}\Big]^p \int_\Omega \Big\{ \rho_\nu(\mathbf{x})^{\sigma-p} + \tfrac{(p-1)2^{p-\sigma}}{D_\nu(\mathbf{x})^{p-\sigma}} \Big\} |u(\mathbf{x})|^p d\mathbf{x}. \end{aligned} \tag{4.3}$$

If $\sigma \leq 0$ we bound $\rho_\nu(\mathbf{x})^\sigma$ for any $\nu \in \mathbb{S}^{n-1}$ by $\delta(\mathbf{x})^\sigma$ in the first integral above. If $\sigma > 0$, we bound it by $D(\Omega)^\sigma/2^\sigma$. As in [11] we may use the fact that

$$\int_{\mathbb{S}^{n-1}} |\partial_\nu u(\mathbf{x})|^p d\omega(\nu) = B(n,p) |\nabla u(\mathbf{x})|^p. \tag{4.4}$$

After bounding $\rho_\nu(\mathbf{x})^\sigma$ as described above, integrate in (4.3) over $\mathbb{S}^{n-1}$ with respect to $d\omega(\nu)$. In order to evaluate the integral of $(2/D_\nu(\mathbf{x}))^{p-\sigma}$, we proceed as in [11]. Since $\sigma \leq p-1$, then $f(t) = t^{\sigma-p}$ is convex for $t > 0$ and we have that

$$\int_{\mathbb{S}^{n-1}} \Big(\frac{2}{D_\nu(\mathbf{x})}\Big)^{p-\sigma} d\omega(\nu) \geq \Big(\int_{\mathbb{S}^{n-1}} \frac{D_\nu(\mathbf{x})}{2} d\omega(\nu) \Big)^{\sigma-p} \geq \Big(\frac{n|\Omega_{\mathbf{x}}|}{s_{n-1}} \Big)^{-\frac{p-\sigma}{n}} \tag{4.5}$$

by Jensen's inequality and Lemma 2.1 of [11]. The conclusion follows. □

5. Rellich's inequality

The methods described above with Proposition 1 below can be used to prove a weighted Rellich inequality which, for $n \geq 4$ and without weights, improves the constant given in a Rellich inequality proved recently by Barbatis ([1], Theorem 1.2). A comparison is made below. The methods used by Barbatis depends upon the identity (5.2) first proved by M.P. Owen ([10], see the proof of Theorem 2.3). In order to incorporate weights, our proof requires the point-wise identity (5.1) which does not follow from the proof of Owen.

Proposition 1. *Let Ω be a domain in $\mathbb{R}^n$. Then, for all $u \in C^2(\mathbb{R}^n)$*

$$\int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(\mathbf{x})|^2 d\omega(\nu) = \frac{1}{n(n+2)}\Big[|\Delta u(\mathbf{x})|^2 + 2\sum_{i,j=1}^n \Big|\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j}\Big|^2\Big], \quad (5.1)$$

and for all $u \in C_0^2(\Omega)$

$$\int_\Omega \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(\mathbf{x})|^2 d\omega(\nu) d\mathbf{x} = \frac{3}{n(n+2)} \int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x}. \quad (5.2)$$

Proof. For $\nu = (\nu_1, \dots, \nu_n)$ we have

$$\begin{aligned} \partial_\nu^2 u = (\nu \cdot \nabla)^2 u = & \textstyle\sum_{\ell,m=1}^n \nu_\ell \nu_m u_{\ell m} \\ = & \textstyle\sum_{\ell=1}^n \nu_\ell^2 u_{\ell\ell} + 2 \sum_{1\leq \ell < m \leq n} \nu_\ell \nu_m u_{\ell m} \end{aligned}$$

in which $u_{pq}(\mathbf{x}) := \frac{\partial^2 u(\mathbf{x})}{\partial x_p \partial x_q}$. Consequently,

$$\begin{aligned} \textstyle\int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u|^2 d\omega(\nu) = & \textstyle\sum_{\ell,m=1}^n u_{\ell\ell}\overline{u_{mm}} \int_{\mathbb{S}^{n-1}} (\nu_\ell)^2 (\nu_m)^2 d\omega(\nu) \\ & +4\textstyle\sum_{m=1}^n \sum_{1\leq p<q\leq n} \Re\mathfrak{e}(u_{mm}\overline{u_{pq}}) \int_{\mathbb{S}^{n-1}} (\nu_m)^2 \nu_p \nu_q d\omega(\nu) \\ & +4 \textstyle\sum_{1\leq j<k\leq n} \sum_{1\leq p<q\leq n} \Re\mathfrak{e}(u_{pq}\overline{u_{jk}}) \int_{\mathbb{S}^{n-1}} \nu_p\nu_q\nu_j\nu_k d\omega(\nu). \end{aligned} \quad (5.3)$$

Let $\theta_j \in [0, \pi]$ for $j = 1, \dots, n-2$, and $\theta_{n-1} \in [0, 2\pi]$. Using the convention that $\Pi_{j=q}^p = 1$ for $p < q$ and $\theta_n = 0$, we have

$$\nu_j = \Pi_{k=1}^{j-1} \sin\theta_k \cos\theta_j, \qquad j = 1, \dots, n,$$

$$d\omega(\nu) := \frac{(n-2)!!}{\gamma_n} \Pi_{k=1}^{n-2} (\sin\theta_k)^{n-1-k} d\theta_k d\theta_{n-1}, \quad (5.4)$$

for $n!! := n \cdot (n-2) \cdot (n-4) \cdots 1$ and

$$\gamma_n = \begin{cases} 2(2\pi)^{(n-1)/2} & \text{for } n \text{ odd,} \\ (2\pi)^{n/2} & \text{for } n \text{ even.} \end{cases}$$

Calculations show that

$$\int_{\mathbb{S}^{n-1}} (\nu_m)^2 \nu_p \nu_q d\omega(\nu) = 0, \qquad m = 1, \dots, n, \quad 1 \leq p < q \leq n$$

implying that the second term on the right-hand side of (5.3) vanishes.

A similar consideration for the third term on the right-hand side of (5.3) shows that

$$\int_{\mathbb{S}^{n-1}} \nu_p \nu_q \nu_j \nu_k d\omega(\nu) \neq 0, \qquad 1 \leq p < q \leq n, \quad 1 \leq j < k \leq n,$$

only if $j = p$ and $k = q$. Therefore, (5.3) reduces to

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(\mathbf{x})|^2 d\omega(\nu) = \; & \sum_{\ell,m=1}^n u_{\ell\ell}\overline{u_{mm}} \int_{\mathbb{S}^{n-1}} (\nu_\ell)^2 (\nu_m)^2 d\omega(\nu) \\ & +4 \sum_{1\leq p<q\leq n} |u_{pq}|^2 \int_{\mathbb{S}^{n-1}} (\nu_p)^2 (\nu_q)^2 d\omega(\nu). \end{aligned} \tag{5.5}$$

However, further calculations show that

$$\int_{\mathbb{S}^{n-1}} \nu_p^2 \nu_q^2 d\omega(\nu) = \begin{cases} \frac{1}{n(n+2)} & 1 \leq p < q \leq n, \\ \frac{3}{n(n+2)} & p = q = 1, \dots, n \end{cases} \tag{5.6}$$

implying that

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u|^2 d\omega(\nu) = \; & \frac{3}{n(n+2)} \sum_{m=1}^n |u_{mm}|^2 \\ & + \frac{1}{n(n+2)} \sum_{1\leq p<q\leq n} [4|u_{pq}|^2 + 2\mathfrak{Re}(u_{pp}\overline{u_{qq}})] \\ = \; & \frac{1}{n(n+2)} \left[|\Delta u(\mathbf{x})|^2 + 2\sum_{i,j=1}^n \left| \frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} \right|^2 \right] \end{aligned} \tag{5.7}$$

which is (5.1). Equality (5.2) now follows since

$$\sum_{i,j=1}^n \int_\Omega \left| \frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} \right|^2 d\mathbf{x} = \int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x}.$$

□

Define

$$d(\mathbf{x};\sigma) := \begin{cases} \delta(\mathbf{x})^\sigma, & \sigma < 0, \\ \left(\frac{D(\Omega)}{2}\right)^\sigma, & \sigma \in [0,1]; \end{cases}$$

$$\beta(n,\sigma) := \frac{(1-\sigma)^2(3-\sigma)^2 n(n+2)}{16};$$

and

$$C_R(n,\sigma) := 2^{4-\sigma} k(\sigma-2) \left[\frac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \left(1 + 2^{2-\sigma} k(\sigma-2)\right) \tag{5.8}$$

for $\sigma \leq 1$ and $k(\sigma)$ defined in Lemma 2.

Theorem 3. *For $\sigma \leq 1$ and $u \in C_0^2(\Omega)$,*

$$\begin{aligned} & \int_\Omega d(\mathbf{x};\sigma) \left[|\Delta u(\mathbf{x})|^2 + 2\sum_{i,j=1}^n \left| \frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} \right|^2 \right] d\mathbf{x} \\ & \quad \geq \beta(n,\sigma) \Big\{ \int_\Omega \rho(\mathbf{x};\sigma-4) |u(\mathbf{x})|^2 d\mathbf{x} \\ & \qquad + 2^{4-\sigma} k(\sigma-2) \left[\frac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4-\sigma}{n}}} d\mathbf{x} \Big\} \end{aligned} \tag{5.9}$$

holds when $n \geq 4-\sigma$ and

$$\begin{aligned}\int_\Omega d(\mathbf{x};\sigma)\Big[|\Delta u(\mathbf{x})|^2 + 2\textstyle\sum_{i,j=1}^n \left|\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j}\right|^2\Big] d\mathbf{x} \\ \geq \beta(n,\sigma)\Big\{ \int_\Omega \rho(\mathbf{x};\sigma-4)|u(\mathbf{x})|^2 d\mathbf{x} \\ +2^{4-\sigma}k(\sigma-2)\left[\tfrac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4-\sigma}{n}}} d\mathbf{x} \\ +2^{2(3-\sigma)}k(\sigma-2)^2\left[\tfrac{s_{n-1}}{n}\right]^{\frac{4+t-\sigma}{n}} \int_\Omega \frac{\delta(\mathbf{x})^t|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4+t-\sigma}{n}}} d\mathbf{x}\Big\}\end{aligned} \tag{5.10}$$

holds when $n \geq 4+t-\sigma$ and $t \geq 2-\sigma$.

Proof. For $\sigma \leq 1$, it follows that

$$\begin{aligned}\int_0^{2b} \rho(t)^\sigma |u''(t)|^2 dt &\geq \int_0^{2b} \rho(t)^\sigma \big[1 - (\frac{\rho(t)}{\mu(t)})^{1-\sigma}\big]^2 |u''(t)|^2 dt \\ &\geq \Big(\frac{1-\sigma}{2}\Big)^2 \int_0^{2b} \rho(t)^{\sigma-2}|u'(t)|^2 dt\end{aligned}$$

by (2.4). Therefore, for $\sigma \leq 1$ and $u \in C_0^2(0,2b)$,

$$\begin{aligned}\int_0^{2b} \rho(t)^\sigma |u''(t)|^2 dt \\ \geq \big(\tfrac{(1-\sigma)(3-\sigma)}{4}\big)^2 \int_0^{2b} \rho(t)^{\sigma-4}\big[1 + k(\sigma-2)\big(\tfrac{2\rho(t)}{\mu(t)}\big)^{3-\sigma}\big]^2 |u(t)|^2 dt\end{aligned} \tag{5.11}$$

by (2.3).

From (5.11) we have for $u \in C_0^2(\Omega)$

$$\begin{aligned}\int_\Omega \rho_\nu(\mathbf{x})^\sigma |\partial_\nu^2 u(\mathbf{x})|^2 d\mathbf{x} \\ \geq \Big(\tfrac{(1-\sigma)(3-\sigma)}{4}\Big)^2 \int_\Omega \rho_\nu(\mathbf{x})^{\sigma-4}\Big\{1 + k(\sigma-2)\Big(\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\Big)^{3-\sigma}\Big\}^2 |u(\mathbf{x})|^2 d\mathbf{x}\end{aligned} \tag{5.12}$$

for $\sigma \leq 1$. As in (3.8) we write

$$\begin{aligned}\rho_\nu(\mathbf{x})^{\sigma-4}\Big\{1 + k(\sigma-2)\Big(\tfrac{2\rho_\nu(\mathbf{x})}{\mu_\nu(\mathbf{x})}\Big)^{3-\sigma}\Big\}^2 \\ = \rho_\nu(\mathbf{x})^{\sigma-4} + 2^{4-\sigma}k(\sigma-2)\tfrac{\rho_\nu^{-1}(\mathbf{x})}{\mu_\nu(\mathbf{x})^{3-\sigma}} + 2^{2(3-\sigma)}k(\sigma-2)^2\tfrac{\rho_\nu^{-\sigma+2}(\mathbf{x})}{\mu_\nu(\mathbf{x})^{2(3-\sigma)}}.\end{aligned} \tag{5.13}$$

Since $\rho_\nu(\mathbf{x})\mu_\nu(\mathbf{x}) = \tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x})$, in the second term on the right-hand side of (5.13) we may write

$$\frac{\rho_\nu^{-1}(\mathbf{x})}{\mu_\nu(\mathbf{x})^{3-\sigma}} = \frac{1}{[\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x})]\mu_\nu(\mathbf{x})^{2-\sigma}} =: I(\nu;\mathbf{x}).$$

Thus

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} I(\nu;\mathbf{x})d\omega(\nu) = & \int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{\sigma-3}(\mathbf{x})\tau_{-\nu}(\mathbf{x})^{-1}d\omega(\nu)\\ & + \int_{\tau_\nu(\mathbf{x})\leq\tau_{-\nu}(\mathbf{x})} \tau_{-\nu}(\mathbf{x})^{\sigma-3}(\mathbf{x})\tau_\nu(\mathbf{x})^{-1}d\omega(\nu)\\ \geq & \int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{\sigma-4}(\mathbf{x})d\omega(\nu)\\ & + \int_{\tau_\nu(x)\leq\tau_{-\nu}(\mathbf{x})} \tau_{-\nu}(\mathbf{x})^{\sigma-4}(\mathbf{x})d\omega(\nu)\end{aligned}$$

and

$$\begin{aligned}\left\{\int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{\sigma-4}d\omega(\nu)\right\}^{-1} \leq & \int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{-\sigma+4}d\omega(\nu)\\ \leq & \left\{\int_{\mathbb{S}^{n-1}} \tau_\nu^n(\mathbf{x})d\omega(\nu)\right\}^{(4-\sigma)/n}\\ = & \left(\frac{n}{s_{n-1}}|\Omega_{\mathbf{x}}|\right)^{(4-\sigma)/n}\end{aligned}$$

for $n\geq 4-\sigma$. Therefore for the second term on the right-hand side of (5.13), for $\sigma\leq 1$ and $n\geq 4-\sigma$, it follows that

$$\int_\Omega\int_{\mathbb{S}^{n-1}} \frac{\rho_\nu^{-1}(\mathbf{x})d\omega(\nu)}{\mu_\nu(\mathbf{x})^{3-\sigma}}|u(\mathbf{x})|^2d\mathbf{x} \geq \left(\frac{s_{n-1}}{n}\right)^{\frac{4-\sigma}{n}}\int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4-\sigma}{n}}}d\mathbf{x}. \tag{5.14}$$

For any $t\in(-\infty,\infty)$, we may write the third term in (5.13) as

$$\frac{\rho_\nu^{-\sigma+2}(\mathbf{x})}{\mu_\nu(\mathbf{x})^{2(3-\sigma)}} = \rho_\nu(\mathbf{x})^t(\tau_\nu(\mathbf{x})\tau_{-\nu}(\mathbf{x}))^{2-\sigma-t}\mu(\mathbf{x})^{-8+3\sigma+t} =: \rho_\nu(\mathbf{x})^tJ(\nu,\mathbf{x}).$$

If $t\geq 2-\sigma$

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} J(\nu;\mathbf{x})d\omega(\nu) \geq & \int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{-4+\sigma-t}d\omega(\nu)\\ & + \int_{\tau_\nu(\mathbf{x})\leq\tau_{-\nu}(\mathbf{x})} \tau_{-\nu}(\mathbf{x})^{-4+\sigma-t}d\omega(\nu).\end{aligned}$$

As before

$$\begin{aligned}\left\{\int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{-4+\sigma-t}d\omega(\nu)\right\}^{-1} \leq & \int_{\tau_\nu(\mathbf{x})\geq\tau_{-\nu}(\mathbf{x})} \tau_\nu(\mathbf{x})^{4-\sigma+t}d\omega(\nu)\\ \leq & \left\{\int_{\mathbb{S}^{n-1}} \tau_\nu^n(\mathbf{x})d\omega(\nu)\right\}^{(4-\sigma+t)/n}\\ = & \left(\frac{n}{s_{n-1}}|\Omega_{\mathbf{x}}|\right)^{(4-\sigma+t)/n}\end{aligned}$$

if $n\geq 4-\sigma+t$. Associated with the third term on the right-hand side of (5.13), we have for $\sigma\leq 1$, $t\geq 2-\sigma>0$, and $n\geq 4-\sigma+t$

$$\int_\Omega\int_{\mathbb{S}^{n-1}} \frac{\rho_\nu^{-\sigma+2}(\mathbf{x})d\omega(\nu)}{\mu_\nu(\mathbf{x})^{2(3-\sigma)}}|u(\mathbf{x})|^2d\mathbf{x} \geq \left(\frac{s_{n-1}}{n}\right)^{\frac{4+t-\sigma}{n}}\int_\Omega \frac{\delta(\mathbf{x})^t|u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4+t-\sigma}{n}}}d\mathbf{x}. \tag{5.15}$$

From (5.12) – (5.15) we obtain

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^\sigma |\partial_\nu^2 u(\mathbf{x})|^2 d\omega(\nu) d\mathbf{x} \geq \tfrac{(1-\sigma)^2(3-\sigma)^2}{16} \Big\{ \int_\Omega \rho(\mathbf{x};\sigma-4)|u(\mathbf{x})|^2 d\mathbf{x}$$
$$+2^{4-\sigma} k(\sigma-2) \left[\tfrac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \tfrac{|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4-\sigma}{n}}} d\mathbf{x}$$
$$+2^{2(3-\sigma)} k(\sigma-2)^2 \left[\tfrac{s_{n-1}}{n}\right]^{\frac{4+t-\sigma}{n}} \int_\Omega \tfrac{\delta(\mathbf{x})^t |u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4+t-\sigma}{n}}} d\mathbf{x} \Big\}$$

provided $\sigma \leq 1$, $t \geq 2-\sigma$, and $n \geq 4+t-\sigma$.

Note, that we may simply choose zero as a lower bound for the third term on the right-hand side of (5.13) and conclude that

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^\sigma \quad |\partial_\nu^2 u(\mathbf{x})|^2 d\omega(\nu) d\mathbf{x} \geq \tfrac{(1-\sigma)^2(3-\sigma)^2}{16} \Big\{ \int_\Omega \rho(\mathbf{x};\sigma-4)|u(\mathbf{x})|^2 d\mathbf{x}$$
$$+2^{4-\sigma} k(\sigma-2) \left[\tfrac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \tfrac{|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4-\sigma}{n}}} d\mathbf{x} \Big\}$$

for $\sigma \leq 1$ and $n \geq 4-\sigma$.

Now, it follows from Proposition 1 that

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^\sigma |\partial_\nu^2 u(\mathbf{x})|^2 d\omega(\nu) d\mathbf{x}$$
$$\leq \tfrac{1}{n(n+2)} \int_\Omega d(\mathbf{x};\sigma) \left[|\Delta u(\mathbf{x})|^2 + 2\sum_{i,j=1}^n \left| \tfrac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} \right|^2 \right] d\mathbf{x}.$$

Thus, (5.9) and (5.10) are proved. □

It follows from Theorem 1.2 of Barbatis [1] that for a convex bounded domain Ω and all $u \in C_0^\infty(\Omega)$

$$\int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x} \geq \frac{9}{16} \int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^4} d\mathbf{x} + \frac{11}{48} n(n+2) \left[\frac{s_{n-1}}{n|\Omega|}\right]^{4/n} \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x}. \tag{5.16}$$

As in Theorem 2, for a convex domain $\Omega \subset \mathbb{R}^n$, we may replace $\rho(\mathbf{x},\sigma-4)$ in Theorem 3 by $B(n,4-\sigma)/\delta(\mathbf{x})^{4-\sigma}$ and $|\Omega_x|$ by $|\Omega|$ to conclude from (5.9) that for $n \geq 4$

$$\int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x} \geq \frac{9}{16} \int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^4} d\mathbf{x} + c_4 n(n+2) \left[\frac{s_{n-1}}{n|\Omega|}\right]^{4/n} \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x} \tag{5.17}$$

for all $u \in C_0^\infty(\Omega)$ in which $c_4 = 3k(-2) \approx 1.25$. Therefore (5.17) improves the bound given by (5.16) for all $n \geq 4$.

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