BOLTZMANN LIMIT AND QUASIFREENESS FOR A HOMOGENOUS FERMI GAS IN A WEAKLY DISORDERED RANDOM MEDIUM

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Abstract. In this note, we discuss some basic aspects of the dynamics of a homogenous Fermi gas in a weak random potential, under simplifying assumptions on the particle pair interactions. We are particularly interested in studying the delocalizing effects due to the Pauli principle. We derive the kinetic hydrodynamic limit, determined by a linear Boltzmann equation, for the momentum distribution function. Moreover, we prove that if the initial state is quasifree, then the time evolved state averaged over the randomness, which is by itself not quasifree, has a quasifree hydrodynamic limit. We show that the momentum distributions determined by the Gibbs states of a free fermion field persist into the diffusive time scale; this includes the limit of zero temperature.

1. Introduction

In this note, we address a problem related to the delocalization conjecture in the theory of random Schrödinger operators. The latter poses the question if in dimensions $d \geq 3$, the Anderson model at weak disorders exhibits an absolutely continuous spectral component corresponding to delocalized generalized eigenstates. Some crucial advances elucidating aspects of this problem have been achieved in [9, 10, 11, 12, 15, 21] through the analysis of macroscopic hydrodynamic limits of the microscopic quantum dynamics; see also [7, 8, 19]. We refer also to [1, 5, 6, 18, 20] for related works.

Here, we investigate some basic aspects of the related question if for electrons in a disordered medium, the inter-particle repulsion due to the Pauli principle enhances the persistence of delocalized states when a weak random potential is added. At least in the simplified setting considered here, we find some affirmative indications. We study the dynamics of a homogenous fermion gas in a weak static random potential where the pair interactions between particles are treated in a simplified mean field approximation. With the single-particle dynamics of the Anderson model at weak disorders in mind, we are especially interested in the interplay between the delocalizing effects due to the Pauli repulsion and the localizing effects due to the weak random potential.

We consider a gas of fermions on the lattice $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^d \cap \mathbb{Z}^d$ in dimension $d \geq 3$ and with periodic boundary conditions, for $L \gg 1$. We denote the dual lattice by $\Lambda_L^* = \Lambda_L / L$, and write $\int dp \equiv \frac{1}{L^d} \sum_{p \in \Lambda_L^*}$ for brevity. Letting $\mathfrak{F} =$

$\bigoplus_{n\geq 0} \bigwedge_1^n \ell^2(\Lambda_L)$ denote the Fock space accounting for scalar fermions on Λ_L, we denote the creation- and annihilation operators by a_p^+, a_p, with $p \in \Lambda_L^*$, satisfying the usual canonical anticommutation relations.

Let $\mathfrak{A}$ denote the C^*-algebra of bounded operators on $\mathfrak{F}$. We let ρ_0 denote a translation invariant, normalized state on $\mathfrak{A}$, which preserves the particle number (i.e., $\rho_0(NA) = \rho_0(AN)$ for all $A \in \mathfrak{A}$, where $N = \sum_x a_x^+ a_x$ is the number operator).

We consider the Hamiltonian

$$H_\omega \,:=\, \int dp\, E(p)\, a_p^+\, a_p \,+\, \eta\, V_\omega \tag{1.1}$$

which generates the dynamics of a free Fermi gas coupled to a random potential

$$V_\omega \,:=\, \sum_{x\in\Lambda_L} \omega_x\, a_x^+\, a_x \tag{1.2}$$

where $\{\omega_x\}_{x\in\Lambda_L}$ are Gaussian i.i.d. random variables, and $0 < \eta \ll 1$ is a small coupling constant accounting for the disorder strength. We assume that the kinetic energy function is given by

$$E(p) \,=\, \sum_{j=1}^d \cos(2\pi p_j)\,, \tag{1.3}$$

i.e., the Fourier multiplication operator determined by the centered nearest neighbor Laplacian $(\Delta f)(x) = \sum_{|y-x|=1} f(y)$ on $\mathbb{Z}^d$.

Due to the translation invariance of the initial state, the dynamics of a significant class of simplified mean field models reduces to the one generated by H_ω, as we point out in Section 3.3. Therefore, we essentially formulate everything for H_ω.

We are interested in the long-time dynamics of the fermion field described by

$$\rho_t(\,A\,) \,:=\, \rho_0(\,e^{itH_\omega}\, A\, e^{-itH_\omega}\,)\,, \tag{1.4}$$

where $A \in \mathfrak{A}$. While the direct pair interactions between the electrons will not be important, due to our model assumptions, the effective interaction between the particles through their coupling to the random potential, and due to the Pauli principle remain significant. We prove the following. In a time scale $t = \frac{T}{\eta^2}$ where $T > 0$ denotes a macroscopic time variable, we find, in the thermodynamic limit, that for all $T > 0$ and for all test functions f, g of Schwartz class $\mathcal{S}(\mathbb{T}^d)$,

$$\Omega_T^{(2)}(f;g) \,:=\, \lim_{\eta\to 0}\lim_{L\to\infty} \mathbb{E}[\,\rho_{T/\eta^2}(\,a^+(f)\,a(g)\,)\,] \,=\, \int_{\mathbb{T}^d} dp\, F_T(p)\,\overline{f(p)}\,g(p)\,, \tag{1.5}$$

where $F_T(p)$ satisfies the linear Boltzmann equation

$$\partial_T F_T(p) \,=\, 2\,\pi \int du\, \delta(\,E(u) - E(p)\,)\,(\,F_T(u) - F_T(p)\,) \tag{1.6}$$

with initial condition $F_0(p) \;=\; \lim_{L\to\infty} \frac{1}{L^d}\rho_0(\,a_p^+\, a_p\,)$. The proof is based on a generalization of methods due to Erdös and Yau in [15], and extended in [7], for the derivation of linear Boltzmann equations from the random Schrödinger dynamics in the weakly disordered 1-particle Anderson model. The same strategy can likewise be applied to prove that if ρ_0 is spatially inhomogenous, the long-time dynamics generated by H_ω is governed by a linear Boltzmann equation with a spatial transport term; however, we will not address this issue here.

A simple but important special case of ρ_0 is given by the Gibbs distribution of the free fermion field for inverse temperature β and chemical potential μ. It is of particular interest because the corresponding momentum occupation density

$$F_0(p) \,=\, \frac{1}{1+e^{\beta(E(p)-\mu)}} \tag{1.7}$$

is an *equilibrium solution* of the linear Boltzmann equation (1.6), for all $\beta > 0$. This is also valid in the zero temperature limit $\beta \to \infty$ where in the weak sense,

$$\frac{1}{1+e^{\beta(E(p)-\mu)}} \,\to\, \chi[E(p) < \mu]\,, \tag{1.8}$$

which is nontrivial if $\mu > 0$. Erdös, Salmhofer and Yau have proved in their landmark work [10, 11, 12] that for a time t beyond the kinetic scale η^{-2}, the effective dynamics of a single electron is *diffusive*; i.e., in this time scale, a wave packet evolves in position space according to the solution of a heat equation. Thus, there arises the question if given an initial condition ρ_0 corresponding to a Gibbs state of the free fermion field, high frequency fermions will eventually "slow down" over a diffusive time scale, i.e., that the probability for small momenta to be occupied increases in time, while the probability for large momenta to be occupied decreases due to the localizing effect of the random potential. On the other hand, the Pauli principle has a stabilizing effect on the Fermi sea, even if it is perturbed by a weak random potential, and imposes a natural restriction on the validity of the picture just described. As we prove here, the momentum density corresponding to the initial free Gibbs state in fact persists for a time extending beyond the onset of diffusive dynamics.

We are interested in the stability of (1.7) because $F_T(p) \neq 0$ is an indication for the presence of electron states which are in some sense delocalized (in position space). In the analysis presented here, the interactions between the particles are omitted or suitably simplified such that they become inessential. The Hamiltonian determining the translation invariant model without the random potential (i.e., $\eta = 0$) but including the full repulsive particle pair interaction is given by

$$\widetilde{H}_\lambda \,:=\, \int dp\, E(p)\, a_p^+\, a_p \,+\, \lambda \sum_{x,y\in\Lambda_L} a_y^+\, a_x^+\, v(x-y)\, a_x\, a_y\,. \tag{1.9}$$

It is widely believed that in a time scale $t = \frac{T}{\lambda^2}$, the momentum density $F_T(p) := \lim_{\lambda\to 0}\lim_{L\to\infty}\frac{1}{L^d}\rho_{T/\lambda^2}(\,a_p^+ a_p\,)$ for the dynamics generated by $\widetilde{H}_\lambda$ satisfies the *Boltzmann-Uhlenbeck-Uehling equation*

$$\begin{aligned}\partial_T F_T(p) \quad &= \quad -4\,\pi \int dp_1\, dp_2\, dq_1\, dq_2\, |\,\widehat{v}(p_1-q_1) - \widehat{v}(p_1-q_2)\,|^2\, \delta(\,p-p_1\,) \\ &\qquad \delta(\,p_1+p_2-q_1-q_2\,)\,\delta(\,E(p_1)+E(p_2)-E(q_1)-E(q_2)\,) \\ &\qquad \Big[\,F_T(p_1)F_T(p_2)\widetilde{F}_T(q_1)\widetilde{F}_T(q_2) \,-\, F_T(q_1)F_T(q_2)\widetilde{F}_T(p_1)\widetilde{F}_T(p_2)\,\Big]\,,\end{aligned}$$

where $\widetilde{F}_T(p) := 1 - F_T(p)$. The derivation of (1.10) from the microscopic quantum dynamics is an extremely challenging open problem; for some work in this direction, see [4, 13, 16, 22]. We note that (1.7) is also an equilibrium solution of (1.10); one easily sees this by observing that $\widetilde{F}_0(p) = e^{\beta(E(p)-\mu)}F_0(p)$. This is connected to the fact that (1.7) is a function of the kinetic energy $E(p)$ which is a *collision invariant* in both (1.6) and (1.10). As a matter of fact, all functions of the form $f(E(p))$ are equilibria of (1.6); on the other hand, the special structure of (1.7), apart from being a function of $E(p)$, is necessary for it to be an equilibrium of (1.10). For a combined Boltzmann limit of the coupled model with $\lambda, \eta > 0$ (which is an open problem), what has been noted above appears to support the conjecture that the kinetic energy $E(p)$ will remain a collision invariant, and that the momentum distribution (1.7) will still persist for a time beyond the diffusive scale.

We note that the question of the stability of the Fermi sea for a gas of interacting fermions is a quintessential problem in mathematical physics which has in recent years received much attention, especially due to the landmark works of Feldman, Knörrer, and Trubowitz summarized in [14].

The next question we address is how strongly the electrons are effectively correlated through their interactions with the random potential. To this end, we assume that ρ_0 is number preserving, homogenous, and *quasifree*. That is, for any tuple of test functions $f_1, \dots, f_r$, $g_1, \dots, g_s$,

$$\rho_0(\, a^+(f_1) \cdots a^+(f_r)\, a(g_1) \cdots a(g_s)\,) \,=\, \delta_{r,s} \det[\, \rho_0(\, a^+(f_j)\, a(g_\ell)\,)\,]_{j,\ell=1}^r \,. \tag{1.10}$$

We consider the dynamics generated by H_ω, and observe that since H_ω is bilinear in a^+, a, the time evolved state ρ_t is almost surely quasifree. However, the state averaged over the randomness is *not* quasifree,

$$\lim_{L\to\infty} \mathbb{E}[\, \rho_t(\, f_1, \dots, f_r\, ;\, g_1, \dots, g_r\,)\,] \,\neq\, \det[\, \lim_{L\to\infty} \mathbb{E}[\, \rho_t(\, f_j\, ;\, g_\ell\,)\,]\,]_{j,\ell=1}^r \,, \tag{1.11}$$

for any $\eta > 0$. This is not surprising because quasifreeness is a nonlinear condition. We prove that in the hydrodynamic limit stated above, the limiting $2r$-correlation functions are quasifree,

$$\begin{aligned} \Omega_T^{(2r)}(\, & f_1, \dots, f_r\, ;\, g_1, \dots, g_r\,) \\ & := \lim_{\eta\to 0} \lim_{L\to\infty} \mathbb{E}[\, \rho_{T/\eta^2}(\, a^+(f_1) \cdots a^+(f_r)\, a(g_1) \cdots a(g_2)\,)\,] \\ & = \det[\, \Omega_T^{(2)}(\, f_j\, ;\, g_\ell\,)\,]_{j,\ell=1}^r \end{aligned} \tag{1.12}$$

for any $r \in \mathbb{N}$. The proof is based on an extension of the proof in [8] for the 1-particle Anderson model at weak disorders that the random Schrödinger evolution converges in *arbitrary higher mean* to a linear Boltzmann evolution. Quasifreeness of the $2r$-point correlation functions is a significant ingredient in some approaches to the problem of quantum charge transport; see for instance [2] and the references therein.

2. Definition of the model

We consider a fermion gas in a finite box $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^d \cap \mathbb{Z}^d$ of side length $L \gg 1$, with periodic boundary conditions, in dimensions $d \geq 3$. We denote its dual lattice by $\Lambda_L^* := \Lambda_L / L \subset \mathbb{T}^d$. For the Fourier transform, we use the convention

$$\widehat{f}(p) := \sum_{x \in \Lambda_L} e^{-2\pi i p \cdot x} f(x) \,, \tag{2.1}$$

where $p \in \Lambda_L^*$, and

$$f(x) = \frac{1}{L^d} \sum_{p \in \Lambda_L^*} e^{2\pi i p \cdot x} \widehat{f}(p) \tag{2.2}$$

for its inverse. For brevity, we will use the notation

$$\int dp \equiv \frac{1}{L^d} \sum_{p \in \Lambda_L^*} \tag{2.3}$$

in the sequel, which recovers its usual meaning in the thermodynamic limit $L \to \infty$.

We denote the fermionic Fock space of scalar electrons by

$$\mathfrak{F} = \bigoplus_{n \geq 0} \mathfrak{F}_n \,, \tag{2.4}$$

where

$$\mathfrak{F}_0 = \mathbb{C} \quad , \quad \mathfrak{F}_n = \bigwedge_1^n \ell^2(\Lambda_L) \;,\; n \geq 1 \,. \tag{2.5}$$

We introduce creation- and annihilation operators a_p^+, a_q, for p, $q \in \Lambda_L^*$, satisfying the canonical anticommutation relations

$$a_p^+ \, a_q + a_q \, a_p^+ = \delta(p-q) := \begin{cases} L^d & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

We first study a Fermi gas coupled to a random potential without direct interactions between the particles. As we will explain below, this also covers the dynamics in some simplified mean field models.

We define the fermionic manybody Hamiltonian

$$H_\omega := T + \eta \, V_\omega \tag{2.7}$$

where

$$T = \int dp \, E(p) \, a_p^+ \, a_p \tag{2.8}$$

is the kinetic energy operator, and

$$V_\omega := \sum_{x \in \Lambda_L} \omega_x \, a_x^+ a_x \tag{2.9}$$

couples the fermions to a static random potential; $\{\omega_x\}_{x \in \Lambda_L}$ is a field of i.i.d. random variables which we assume to be centered, normalized, and Gaussian for simplicity. Thus,

$$\mathbb{E}[\,\omega_x\,] = 0 \;,\; \mathbb{E}[\,\omega_x^2\,] = 1 \tag{2.10}$$

for all $x \in \Lambda_L$. Moreover, we assume that

$$E(p) \, = \, \sum_{j=1}^{d} \cos(2\pi p_j) \, , \tag{2.11}$$

which defines the Fourier multiplier corresponding to the nearest neighbor Laplacian on $\mathbb{Z}^d$.

Let

$$N \, := \, \sum_{x \in \Lambda_L} a_x^+ a_x \tag{2.12}$$

denote the particle number operator. It is clear that

$$[H_\omega, N] \, = \, 0 \tag{2.13}$$

holds.

Let $\mathfrak{A}$ denote the C^*-algebra of bounded operators on $\mathfrak{F}$. We consider the dynamics on $\mathfrak{A}$ given by

$$\alpha_t(A) \, = \, e^{itH_\omega} \, A \, e^{-itH_\omega} \tag{2.14}$$

generated by the random Hamiltonian H_ω.

3. Statement of the main results

We consider a normalized, translation-invariant, deterministic state

$$\rho_0 : \mathfrak{A} \longrightarrow \mathbb{C} \, . \tag{3.1}$$

We define the time-evolved state

$$\rho_t(A) \, := \, \rho_0(\, e^{itH_\omega} \, A \, e^{-itH_\omega} \,) \, , \tag{3.2}$$

with $t \in \mathbb{R}$, and initial condition given by ρ_0. We particularly focus on the dynamics of the averaged two-point functions

$$\mathbb{E}[\, \rho_t(\, a_p^+ a_q \,) \,] \, , \tag{3.3}$$

where $p, q \in \Lambda_L^*$. Clearly,

$$\mathbb{E}[\, \rho_0(\, a_p^+ a_q \,) \,] \, = \, \rho_0(\, a_p^+ a_q \,) \, = \, \delta(p-q) \, \frac{1}{L^d} \, \rho_0(\, a_p^+ a_p \,) \, , \tag{3.4}$$

where

$$\delta(k) \, := \, L^d \delta_k \, , \tag{3.5}$$

and where

$$\delta_k \, = \, \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

denotes the Kronecker delta on the lattice Λ_L^* (mod $\mathbb{T}^d$). We remark that for fermions,

$$0 \, \leq \, \frac{1}{L^d} \, \rho_0(\, a_p^+ a_p \,) \, \leq \, 1 \, , \tag{3.7}$$

since $\|a_p^{(+)}\| = L^{d/2}$ in operator norm, $\forall p \in \Lambda_L^*$.

3.1. **The Boltzmann limit.** We denote the microscopic time, position, and velocity variables by (t,x,p), and the corresponding macroscopic variables by $(T,X,V)=(\eta^2 t,\eta^2 x,v)$. We prove that the momentum distribution $f_t(q)$ converges to a solution of a linear Boltzmann equation in the limit $\eta\to 0$.

Theorem 3.1. *We assume that ρ_0 is translation invariant. Then, the averaged two-point functions are translation invariant,*

$$\mathbb{E}[\rho_t(\,a^+(f)a(g)\,)] \;=\; \int dp\,\overline{f(p)}\,g(p)\,\mathbb{E}[\rho_t(\,a_p^+ a_p\,)]\,, \tag{3.8}$$

(i.e., diagonal in a^+,a) for any $f,g\in\mathcal{S}(\mathbb{T}^d)$ of Schwartz class, and the thermodynamic limit

$$\Omega_T^{(2;\eta)}(f;g) \;:=\; \lim_{L\to\infty}\mathbb{E}[\rho_{T/\eta^2}(\,a^+(f)\,a(g)\,)] \tag{3.9}$$

exists for all $f,g\in\mathcal{S}(\mathbb{T}^d)$, and $T>0$.

For any $T>0$ and all $f,g\in\mathcal{S}(\mathbb{T}^d)$, the limit

$$\Omega_T^{(2)}(f;g) \;:=\; \lim_{\eta\to 0}\Omega_T^{(2;\eta)}(f;g) \tag{3.10}$$

exists, and is the inner product of f,g with respect to a Borel measure $F_T(p)dp$,

$$\Omega_T^{(2)}(f;g) \;=\; \int dp\,F_T(p)\,\overline{f(p)}\,g(p)\,, \tag{3.11}$$

where $F_T(V)$ satisfies the linear Boltzmann equation

$$\partial_T F_T(V) \;=\; 2\,\pi\int_{\mathbb{T}^d} dU\,\delta(\,E(U)-E(V)\,)\,(\,F_T(U)-F_T(V)\,)\,, \tag{3.12}$$

with initial condition

$$F_0(p) \;=\; \lim_{L\to\infty}\frac{1}{L^d}\,\rho_0(\,a^+_{p_{\Lambda_L^*}}\,a_{p_{\Lambda_L^*}}\,) \tag{3.13}$$

for $p\in\mathbb{T}^d$, where $p_{\Lambda_L^*}:=Q_{\frac{1}{2L}}(p)\cap\Lambda_L^*$, and $Q_\delta(p):=p+[-\delta,\delta)^d$.

We note that there exists a unique $p_{\Lambda_L^*}\in\Lambda_L^*$ such that $|p-p_{\Lambda_L^*}|\le\frac{1}{2L}$, for every $p\in\mathbb{T}^d$.

An initial condition of particular interest is the Gibbs state (with inverse temperature β and chemical potential μ) for a non-interacting fermion gas,

$$\rho_0(A) \;=\; \frac{1}{Z_{\beta,\mu}}\,\mathrm{Tr}(\,e^{-\beta(T-\mu N)}A\,) \tag{3.14}$$

where $Z_{\beta,\mu}:=\mathrm{Tr}(\,e^{-\beta(T-\mu N)}\,)$. The corresponding momentum distribution function

$$\lim_{L\to\infty}\frac{1}{L^d}\,\rho_0(\,a_p^+a_p\,) \;=\; \frac{1}{1+e^{\beta(E(p)-\mu)}} \tag{3.15}$$

is a *fixed point* of the linear Boltzmann equation (3.12), for all $\beta>0$, including the zero temperature limit $\beta\to\infty$ where in the weak sense,

$$\frac{1}{1+e^{\beta(E(p)-\mu)}} \;\to\; \chi[E(p)<\mu]\,, \tag{3.16}$$

which is nontrivial if $\mu > 0$. We note that all our results in this paper remain valid in the limit $\beta \to \infty$.

Invoking the results of Erdös, Salmhofer and Yau proven in [10, 11, 12], the following long-time stability result holds.

Theorem 3.2. *Assume ρ_0 is the Gibbs state for a non-interacting fermion gas such that*

$$F_0(p) \;=\; \frac{1}{1+e^{\beta(E(p)-\mu)}}\,, \tag{3.17}$$

is a fixed point solution of the Boltzmann equation,

$$F_T(p) \;=\; F_0(p) \tag{3.18}$$

for all $T \geq 0$, $p \in \mathbb{T}^d$, and $0 < \beta \leq \infty$. Accordingly,

$$\rho_0(\,a^+(f)\,a(g)\,) \;=\; \int dp\, F_0(p)\,\overline{f(p)}\,g(p)\,. \tag{3.19}$$

Then, the corresponding microscopic state is long-time stable beyond the diffusive time scale $t = O(\eta^{-2})$; that is, for

$$t \;=\; \frac{T}{\eta^{2+\delta}}\,, \tag{3.20}$$

where $0 < \delta < \frac{1}{2000}$, and any $0 < T < \infty$,

$$\lim_{L\to\infty}\Big|\,\mathbb{E}[\,\rho_{T/\eta^{2+\delta}}(\,a^+(f)\,a(g)\,)\,] \;-\; \rho_0(\,a^+(f)\,a(g)\,)\Big| \;=\; o_\eta(1)\,, \tag{3.21}$$

for all $f, g \in \mathcal{S}(\mathbb{T}^d)$.

3.2. **Quasifreeness.** We prove that if in addition to the conditions formulated above, the initial state ρ_0 is *quasifree*, then $\mathbb{E}[\rho_t]$, which is not quasifree for $\eta > 0$, becomes quasifree in the hydrodynamic scaling limit described in Theorem 3.1.

A state ρ_0 is quasifree if for any normal ordered product of creation- and annihilation operators

$$a^+_{p_1}\cdots a^+_{p_r}a_{q_1}\cdots a_{q_s}\,, \tag{3.22}$$

with arbitrary $r, s \in \mathbb{N}$ and $p_i, q_j \in \Lambda^*_L$,

$$\rho_0(\,a^+_{p_1}\cdots a^+_{p_r}a_{q_1}\cdots a_{q_s}\,) \;=\; \delta_{r,s}\,\det\big[\,\rho_0(\,a^+_{p_i}a_{q_j}\,)\,\big]_{1\leq i,j\leq r}\,. \tag{3.23}$$

That is, any higher order correlation function decomposes into the determinant of the matrix of pair correlations. In its most general form, a particle number conserving quasifree state $\rho_0 : \mathfrak{A} \to \mathbb{C}$ can be written as

$$\rho_0(A) \;:=\; \frac{1}{Z_K}\,\mathrm{Tr}(\,e^{-K}A\,) \tag{3.24}$$

for $A \in \mathfrak{A}$, with

$$Z_K \;:=\; \mathrm{Tr}(\,e^{-K}\,)\,, \tag{3.25}$$

and

$$K \;=\; \int dp\,dq\,\kappa(p,q)\,a^+_p\,a_q \tag{3.26}$$

bilinear in a^+, a; for a proof, see [3]. We assume K to be deterministic (with respect to $\{\omega_x\}_x$).

If in addition, translation invariance is imposed, such that

$$[K, T] \, = \, 0 \tag{3.27}$$

then

$$K \, = \, \int dp \, h(p) \, a_p^+ \, a_p \tag{3.28}$$

is bilinear and diagonal in a^+, a.

Since H_ω is bilinear in the creation- and annihilation operators, it is immediately clear that

$$K(t) \, := \, e^{itH_\omega} \, K \, e^{-itH_\omega} \tag{3.29}$$

is also bilinear in a^+, a. Therefore,

$$\rho_t(A) \, = \, \frac{1}{Z_K} \, \mathrm{Tr}(\, e^{-K(t)} A \,) \tag{3.30}$$

is quasifree with probability 1. However, since quasifreeness is a *nonlinear* condition on determinants, almost sure quasifreeness does *not* imply that $\mathbb{E}[\rho_t(\, \cdot \,)]$ is quasifree.

In fact, $\mathbb{E}[\rho_t(\, \cdot \,)]$ is *not* quasifree for any $\eta > 0$.

However, we prove in Theorem 3.3 below that it possesses a hydrodynamic limit (in the sense of Theorem 3.1) which is quasifree.

Theorem 3.3. *Assume that ρ_0 is number conserving and quasifree, and translation invariant. Then, the following holds. For any normal ordered monomial in creation- and annihilation operators,*

$$a^+(f_1) \cdots a^+(f_r) \, a(g_1) \cdots a(g_r) \, , \tag{3.31}$$

with $r, s \in \mathbb{N}$ and Schwartz class test functions $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$, and any $T > 0$, the macroscopic $2r$-point function

$$\begin{aligned} &\Omega_T^{(2r)}(\, f_1, \ldots, f_r \, ; \, g_1, \ldots, g_r \,) \\ &\quad := \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}[\rho_{T/\eta^2}(\, a^+(f_1) \cdots a^+(f_r) \, a(g_1) \cdots a(g_r) \,)] \end{aligned} \tag{3.32}$$

exists and is quasifree,

$$\Omega_T^{(2r)}(\, f_1, \ldots, f_r \, ; \, g_1, \ldots, g_r \,) \, = \, \det\big[\, \Omega_T^{(2)}(\, f_i \, , \, g_j \,) \, \big]_{1 \le i,j \le r} \, . \tag{3.33}$$

The macroscopic 2-point function is the same as in Theorem 3.1,

$$\Omega_T^{(2)}(\, f \, ; \, g \,) \, = \, \int dp \, F_T(p) \, \overline{f(p)} \, g(p) \, , \tag{3.34}$$

and $F_T(p)$ solves the linear Boltzmann equation (3.12) with initial condition (3.13).

We note that the assumption of translation invariance can easily be dropped. However, we do not address inhomogenous Fermi gases in this text.

3.3. **Mean field interactions.** We briefly discuss interactions between electrons in a simplified mean field approximation. The interacting model is described by the Hamiltonian

$$H_{\omega,\lambda} := T + \eta \sum_{x\in\Lambda_L} \omega_x a_x^+ a_x + \lambda \sum_{x,y\in\Lambda_L} a_y^+ \, a_x^+ \, v(x-y) \, a_x \, a_y \,, \tag{3.35}$$

where v extends to a positive translation- and rotation invariant function on $\mathbb{R}^d$.

Since $\{\omega_x\}_{x\in\Lambda_L}$ are i.i.d. random variables, and $H_{\lambda,0}$ is translation invariant, the state

$$\rho_t^{(\lambda,\eta)}(\,\cdot\,) := \rho_0(\, e^{itH_{\omega,\lambda}} \, (\,\cdot\,) \, e^{-itH_{\omega,\lambda}} \,) \tag{3.36}$$

has a translation-invariant $\mathbb{E}$-average. In particular, $\mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\, a_x^+ \, a_x \,)\,]$ does not depend on x; hence,

$$\mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\, a_x^+ \, a_x \,)\,] = \frac{1}{L^d} \sum_{x\in\Lambda_L} \mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\, a_x^+ \, a_x \,)\,] = \frac{1}{L^d} \mathbb{E}[\,\rho_t^{(\lambda,\eta)}(N)\,] \,, \tag{3.37}$$

where N denotes the number operator. Since for every realization of the random potential we have $[H_{\omega,\lambda}, N] = 0$, and since $[K, N] = 0$ in the definition of ρ_0, it follows that

$$\frac{1}{L^d} \rho_t^{(\lambda,\eta)}(\, N \,) = \frac{1}{L^d} \rho_0(\, N \,) \,, \tag{3.38}$$

which is the particle density (we note that $0 < \frac{1}{L^d} \rho_0(\, N \,) \leq 1$ for all L, and especially for $L \to \infty$). In particular, this result is non-random, and thus equals its $\mathbb{E}$-average. Moreover, it is independent of time t, and of the coupling parameters λ and η.

If the electron pair interactions are modeled by the mean field Hamiltonian

$$\overline{H}_{\omega,\lambda} := T + \eta \sum_{x\in\Lambda_L} \omega_x a_x^+ \, a_x + \lambda \sum_{x,y\in\Lambda_L} \mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\, a_y^+ \, a_y \,)\,] \, v(x-y) \, a_x^+ \, a_x \tag{3.39}$$

with $\rho_t^{(\lambda,\eta)}$ as above, the problem reduces to the one formulated for H_ω (this is explained below). However, inclusion of the fermion exchange term would lead to a random nonlinear evolution equation for the mean field; this issue is beyond the scope of the present work.

By translation invariance of $\mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\, a_y^+ a_y \,)\,]$, the mean field interaction is given by

$$\lambda \sum_{x,y\in\Lambda_L} \mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\, a_y^+ \, a_y \,)\,] \, v(x-y) \, a_x^+ \, a_x = \lambda \frac{1}{L^d} \rho_0(N) \Big(\sum_y v(y) \Big) N \,, \tag{3.40}$$

i.e., it is a constant multiple of the number operator N.

Since $[N, \overline{H}_{0,\eta}] = 0$, $[K, N] = 0$ in ρ_0, and $[N, a_p^+ a_q] = 0$, it immediately follows that

$$\mathbb{E}[\,\rho_0(\, e^{it\overline{H}_{\omega,\lambda}} a^+(f) \, a(g) e^{it\overline{H}_{\omega,\lambda}} \,)\,] = \mathbb{E}[\,\rho_0(\, e^{itH_\omega} a^+(f) \, a(g) e^{-itH_\omega} \,)\,] \tag{3.41}$$

for all test functions f, g, where $H_\omega \equiv \overline{H}_{0,\eta}$ is the random Hamiltonian considered in Theorems 3.1 and 3.3, which involves no two-body interactions.

Thus, we obtain the following result about the translation invariant system with interactions between particles modeled in mean-field approximation.

Theorem 3.4. *Assume that ρ_0 is translation invariant. Then, all results stated in Theorems 3.1, 3.2, and 3.3 (i.e., the Boltzmann limit, long-time stability, and quasifreeness) remain valid if H_ω is replaced by the mean field Hamiltonian $\overline{H}_{\omega,\lambda}$.*

We remark that evidently, $\mathbb{E}[\,\rho_t^{(\lambda,\eta)}(\,a_y^+ a_y\,)\,]$ in (3.39) can be replaced by any arbitrary translation invariant average density, and Theorem 3.4 remains valid.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is obtained from an extension of the analysis in [7, 15].

4.1. Duhamel expansion.

We consider the Heisenberg evolution of the creation- and annihilation operators. We define

$$a_p(t) \,:=\, e^{itH_\omega} a_p e^{-itH_\omega}\,, \tag{4.1}$$

and

$$a(f,t) \,:=\, e^{itH_\omega} a(f) e^{-itH_\omega}\,. \tag{4.2}$$

We make the key observation that

$$a(f,t) \,=\, a(f_t) \tag{4.3}$$

where f_t is the solution of the 1-particle random Schrödinger equation

$$i\partial_t f_t \,=\, H_\omega^{(1)} f_t \,:=\, \Delta f_t \,+\, \eta V_\omega^{(1)} f_t \tag{4.4}$$

with initial condition

$$f_0 \,=\, f\,. \tag{4.5}$$

Here, Δ denotes the nearest neighbor Laplacian on Λ_L, and $H_\omega^{(1)} = H_\omega|_{\mathfrak{F}_1}$ is the 1-particle Anderson Hamiltonian at weak disorders studied in [7, 8, 15]. $V_\omega^{(1)} = V_\omega|_{\mathfrak{F}_1}$ is the 1-particle multiplication operator $(V_\omega^{(1)} f)(x) = \omega_x f(x)$.

To prove (4.4), (4.5), we observe that since H_ω is bilinear in a^+, a, it follows that $a(f,t)$ is a linear superposition of annihilation operators. Therefore, there exists a function f_t such that $a(f,t) = a(f_t)$. In particular,

$$\begin{aligned} i\partial_t a(f_t) &= [\,H_\omega\,,\,a(f_t)\,] \\ &= \int dp\, f_t(p)\, E(p)\, a_p \,+\, \eta \int dp \int du\, f_t(p)\, \widehat{\omega}(u-p)\, a_u \\ &= a(\,\Delta f_t\,) \,+\, a(\,\eta V_\omega^{(1)} f_t\,)\,, \end{aligned} \tag{4.6}$$

and moreover, it is clear that $a(f,0) = a(f_0) = a(f)$. This implies (4.4), (4.5).

Thus,

$$\begin{aligned}\rho_t(\,a^+(f)\,a(g)\,) &= \rho_0(\,a^+(f_t)\,a(g_t)\,)\\ &= \int dp\,dq\,\rho_0(\,a_p^+\,a_q)\,\overline{f_t(p)}\,g_t(q)\\ &= \int dp\,J(p)\,\overline{f_t(p)}\,g_t(p) \qquad (4.7)\end{aligned}$$

where

$$\rho_0(\,a_p^+\,a_q) \,=\, \delta(p-q)\,J(p) \qquad (4.8)$$

due to translation invariance, with

$$0 \,\leq\, J(p) \,=\, \frac{1}{L^d}\rho_0(\,a_p^+ a_p\,) \,\leq\, 1\,, \qquad (4.9)$$

see (3.7). In particular, this implies (3.8).

For $N \in \mathbb{N}$, which we determine later, we expand f_t, g_t into the truncated Duhamel series at level N,

$$f_t \,=\, f_t^{(\leq N)} \,+\, f_t^{(>N)}\,, \qquad (4.10)$$

with

$$f_t^{(\leq N)} \,:=\, \sum_{n=0}^{N} f_t^{(n)}\,, \qquad (4.11)$$

and where the Duhamel term of n-th order (in powers of η) is given by

$$\begin{aligned}f_t^{(n)}(p) \;&:=\; (i\eta)^n \int ds_0\cdots ds_n\,\delta(t-\sum_{j=0}^{n} s_j) \qquad (4.12)\\ &\qquad \int dk_0\cdots dk_n\,\delta(p-k_0)\Big(\prod_{j=0}^{n} e^{is_jE(k_j)}\Big)\Big(\prod_{j=1}^{n}\widehat{\omega}(k_j-k_{j-1})\Big)\,f(k_n)\\ &=\; \eta^n\,e^{\epsilon t}\int d\alpha\,e^{it\alpha}\int dk_0\cdots dk_n\,\delta(p-k_0) \qquad (4.13)\\ &\qquad \Big(\prod_{j=0}^{n}\frac{1}{E(k_j)-\alpha-i\epsilon}\Big)\Big(\prod_{j=1}^{n}\widehat{\omega}(k_j-k_{j-1})\Big)\,f(k_n)\,.\end{aligned}$$

The remainder term is given by

$$f_t^{(>N)} \,=\, i\eta\int_0^t ds\,e^{i(t-s)H_\omega}\,V_\omega^{(1)}\,f_t^{(N)}(s)\,. \qquad (4.14)$$

We choose

$$\epsilon \,=\, \frac{1}{t} \qquad (4.15)$$

so that the factor $e^{\epsilon t}$ remains bounded for all t. Accordingly,

$$\rho_t(\,a^+(f)\,a(g)\,) \,=\, \rho_0(\,a^+(f_t)\,a(g_t)\,) \,=\, \sum_{n,\widetilde{n}=0}^{N+1}\rho_t^{(n,\widetilde{n})}(f;g) \qquad (4.16)$$

where

$$\rho_t^{(n,\widetilde{n})}(f;g) \,:=\, \rho_0(\,a^+(f_t^{(n)})\,a(g_t^{(\widetilde{n})})\,) \qquad (4.17)$$

if $n, \widetilde{n} \leq N$, and

$$\rho_t^{(n,N+1)}(f;g) \;:=\; \rho_0(\, a^+(f_t^{(n)})\, a(g_t^{(>N)})\,)\,,$$
$$\rho_t^{(N+1,\widetilde{n})}(f;g) \;:=\; \rho_0(\, a^+(f_t^{(>N)})\, a(g_t^{(\widetilde{n})})\,) \tag{4.18}$$

if $n \leq N$, respectively if $\widetilde{n} \leq N$, and

$$\rho_t^{(N+1,N+1)}(f;g) \;:=\; \rho_0(\, a^+(f_t^{(>N)})\, a(g_t^{(>N)})\,)\,. \tag{4.19}$$

In particular, for $n, \widetilde{n} \leq N$,

$$\begin{aligned}\rho_t^{(n,\widetilde{n})}(f;g) \;=\; \eta^{n+\widetilde{n}}\, e^{2\epsilon t} \int d\alpha\, d\widetilde{\alpha}\, e^{it(\alpha-\widetilde{\alpha})}\\ \int dk_0 \cdots dk_n \int d\widetilde{k}_0 \cdots d\widetilde{k}_{\widetilde{n}}\, \overline{f(k_n)}\, g(\widetilde{k}_{\widetilde{n}})\; J(k_0)\, \delta(k_0 - \widetilde{k}_0)\,)\\ \prod_{j=0}^{n} \frac{1}{E(k_j)-\alpha-i\epsilon} \prod_{\ell=0}^{n} \frac{1}{E(\widetilde{k}_\ell)-\widetilde{\alpha}+i\epsilon}\\ \prod_{j=1}^{n} \widehat{\omega}(k_j - k_{j-1}) \prod_{\ell=1}^{n} \widehat{\omega}(\widetilde{k}_{\ell-1} - \widetilde{k}_\ell)\,.\end{aligned} \tag{4.20}$$

This expression, and the expressions involving n and / or $\widetilde{n} = N+1$, are completely analogous to those appearing in the truncated Duhamel expansion of the Wigner transform in [7, 15].

This permits us to use the methods of [7, 15] to prove Theorem 3.1. We will here only sketch the strategy; for the detailed proof, we refer to [7, 15]. In our subsequent discussion, we will compare the expressions appearing in the given problem to those treated in [7, 15].

To begin with, we introduce a more convenient notation. Clearly, if $n, \widetilde{n} \leq N$, and $n+\widetilde{n}$ is odd, $\mathbb{E}[\rho_t^{(n,\widetilde{n})}(p,q)] = 0$. Thus, we let

$$\bar{n} \;:=\; \frac{n+\widetilde{n}}{2} \;\in\; \mathbb{N}\,, \tag{4.21}$$

and we define $\{u_j\}_{j=0}^{2\bar{n}+1}$ by

$$u_j \;:=\; \begin{cases} k_{n-j} & \text{if } j \leq n \\ \widetilde{k}_{j-n-1} & \text{if } j \geq n+1\,. \end{cases} \tag{4.22}$$

Consequently,

$$\mathbb{E}[\rho_t^{(n,\widetilde{n})}(f;g)] \;=\; \eta^{2\bar{n}}\, e^{2\epsilon t} \sum_{\pi\in\Gamma_{n,\widetilde{n}}} \int d\alpha\, d\widetilde{\alpha}\, e^{it(\alpha-\widetilde{\alpha})} \tag{4.23}$$

$$\begin{aligned}\int du_0 \cdots du_{2\bar{n}+1}\, \overline{f(u_0)}\, g(u_{2\bar{n}+1})\, J(u_n)\, \delta(u_n - u_{n+1})\\ \prod_{j=0}^{n} \frac{1}{E(u_j)-\alpha-i\epsilon} \prod_{\ell=n+1}^{2\bar{n}+1} \frac{1}{E(u_\ell)-\widetilde{\alpha}+i\epsilon}\\ \prod_{j=1}^{n} \widehat{\omega}(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{\omega}(u_j - u_{j-1})\end{aligned} \tag{4.24}$$

in these new variables.

4.2. **Graph expansion.** Next, we take the expectation with respect to the random potential. To this end, we introduce the set of *Feynman graphs* $\Gamma_{n,\widetilde{n}}$, with $n+\widetilde{n} \in 2\mathbb{N}$, as follows.

We consider two horizontal solid lines, which we refer to as *particle lines*, joined by a distinguished vertex which we refer to as the ρ_0-vertex (corresponding to the term $\rho_0(\,a^+_{u_n} a_{u_{n+1}}\,)$. On the line on its left, we introduce n vertices, and on the line on its right, we insert $\widetilde{n}$ vertices. We refer to those vertices as *interaction vertices*, and enumerate them from 1 to $2\bar{n}$ starting from the left. The edges between the interaction vertices are referred to as *propagator lines*. We label them by the momentum variables u_0, ..., $u_{2\bar{n}+1}$, increasingly indexed starting from the left. To the j-th propagator line, we associate the resolvent $\frac{1}{E(u_j)-\alpha-i\epsilon}$ if $0 \leq j \leq n$, and $\frac{1}{E(u_j)-\widetilde{\alpha}+i\epsilon}$ if $n+1 \leq j \leq 2\bar{n}+1$. To the ℓ-th interaction vertex (adjacent to the edges labeled by $u_{\ell-1}$ and u_ℓ), we associate the random potential $\widehat{\omega}(u_\ell - u_{\ell-1})$, where $1 \leq \ell \leq 2\bar{n}+1$.

A *contraction graph* associated to the above pair of particle lines joined by the ρ_0-vertex, and decorated by $n+\widetilde{n}$ interaction vertices, is the graph obtained by pairwise connecting interaction vertices by dashed *contraction lines*. We denote the set of all such contraction graphs by $\Gamma_{n,\widetilde{n}}$; it contains

$$|\Gamma_{n,\widetilde{n}}| \;=\; (2\bar{n}-1)(2\bar{n}-3)\cdots 3\cdot 1 \;=\; \frac{(2\bar{n})!}{\bar{n}!2^{\bar{n}}} \;=\; O(\bar{n}!) \tag{4.25}$$

elements.

If in a given graph $\pi \in \Gamma_{n,\widetilde{n}}$, the ℓ-th and the ℓ'-th vertex are joined by a contraction line, we write

$$\ell \sim_\pi \ell' \,, \tag{4.26}$$

and we associate the delta distribution

$$\delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})) \;=\; \mathbb{E}[\,\widehat{\omega}(\,u_\ell - u_{\ell-1}\,)\,\widehat{\omega}(\,u_{\ell'} - u_{\ell'-1}\,)\,] \tag{4.27}$$

to this contraction line.

4.3. **Classification of graphs.** For the proof of Theorem 3.1, we classify Feynman graphs as follows; see [7, 15], and Figure 1.

- A subgraph consisting of one propagator line adjacent to a pair of vertices ℓ and $\ell+1$, and a contraction line connecting them, i.e., $\ell \sim_\pi \ell+1$, where both ℓ, $\ell+1$ are either $\leq n$ or $\geq n+1$, is called an *immediate recollision*.
- The graph $\pi \in \Gamma_{n,n}$ (i.e., $n = \widetilde{n} = \bar{n}$) with $\ell \sim_\pi 2n-\ell$ for all $\ell = 1,\ldots,n$, is called a *basic ladder* diagram. The contraction lines are called *rungs* of the ladder. We note that a rung contraction always has the form $\ell \sim_\pi \ell'$ with $\ell \leq n$ and $\ell' \geq n+1$. Moreover, in a basic ladder diagram one always has that if $\ell_1 \sim_\pi \ell_1'$ and $\ell_2 \sim_\pi \ell_2'$ with $\ell_1 < \ell_2$, then $\ell_2' < \ell_1'$.
- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called a *decorated ladder* if any contraction is either an immediate recollision, or a rung contraction $\ell_j \sim_\pi \ell_j'$ with $\ell_j \leq n$ and $\ell_j' \geq n$ for $j = 1,\ldots,k$, and $\ell_1 < \cdots < \ell_k$, $\ell_1' > \cdots > \ell_k'$. Evidently, a basic ladder diagram is the special case of a decorated ladder which contains no immediate recollisions (so that necessarily, $n = \widetilde{n}$).

- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called *crossing* if there is a pair of contractions $\ell \sim_\pi \ell'$, $j \sim_\pi j'$, with $\ell < \ell'$ and $j < j'$, such that $\ell < j$.
- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called *nesting* if there is a subdiagram with $\ell \sim_\pi \ell + 2k$, with $k \geq 1$, and either $\ell \geq n+1$ or $\ell + 2k \leq n$, with $j \sim_\pi j+1$ for $j = \ell+1, \ell+3, \dots, \ell+2k-1$. The latter corresponds to a progression of $k-1$ immediate recollisions.

We note that any diagram that is not a decorated ladder contains at least a crossing or a nesting subdiagram.

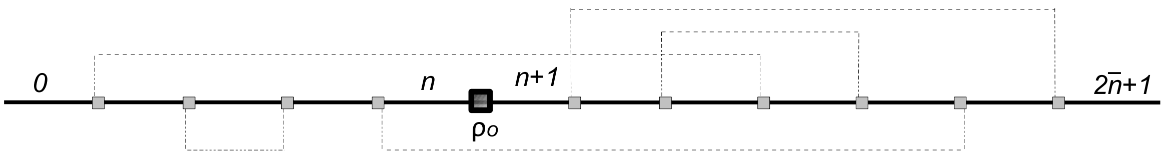

Figure 1. An example of a Feynman graph, $\pi \in \Gamma_{n,\widetilde{n}}$, with $n = 4$, $\widetilde{n} = 6$. The distinguished vertex is the ρ_0-vertex.

4.4. **Feynman amplitudes.** Next, we average (4.20) with respect to the random potential. Accordingly, $\mathbb{E}[\prod \widehat{\omega}(u_\ell - u_{\ell-1})]$ splits into the sum of all possible products of pair correlations, according to Wick's theorem (we recall that $\{\omega_x\}$ are assumed to be i.i.d. Gaussian). This implies that

$$\mathbb{E}[\rho_t^{(n,\widetilde{n})}(f;g)] = \sum_{\pi \in \Gamma_{n,\widetilde{n}}} \mathrm{Amp}_\pi(f;g;\epsilon;\eta) \tag{4.28}$$

with

$$\begin{aligned} \mathrm{Amp}_\pi(f;g;\epsilon;\eta) := \eta^{2\bar{n}} e^{2\epsilon t} \int d\alpha\, d\widetilde{\alpha}\, e^{it(\alpha - \widetilde{\alpha})} \\ \int du_0 \cdots du_{2\bar{n}+1} \overline{f(u_0)}\, g(u_{2\bar{n}+1})\, J(u_n)\, \delta(u_n - u_{n+1})\,) \\ \delta_\pi(\{u_j\}_{j=0}^{2\bar{n}+1}) \prod_{j=0}^{n} \frac{1}{E(u_j) - \alpha - i\epsilon} \prod_{\ell=n+2}^{2\bar{n}} \frac{1}{E(u_\ell) - \widetilde{\alpha} + i\epsilon}\,, \end{aligned} \tag{4.29}$$

and $\epsilon = \frac{1}{t}$. Here,

$$\delta_\pi(\{u_j\}_{j=0}^{2\bar{n}+1}) := \prod_{\ell \sim_\pi \ell'} \delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})) \tag{4.30}$$

is the product of the delta distributions associated to all contraction lines in π. Moreover, we recall that

$$\delta(u_n - u_{n+1})\, J(u_n) = \rho_0(a^+_{u_n} a_{u_{n+1}})\,, \tag{4.31}$$

see (4.8). We note that

$$u_0 - u_{2\bar{n}+1} = 0\,, \tag{4.32}$$

as one easily sees by summing up the arguments of all delta distributions. This holds for any $n, \widetilde{n}$ and again implies (3.8).

We observe that the rôle of (4.31) in (4.29) is analogous to that of the rescaled Schwartz class function J_ϵ in [7, 15], and that the test functions f, g here correspond to the initial state $\widehat{\phi}_0$ in [7, 15].

4.5. **Contribution from crossing and nesting diagrams.** The amplitude of any graph $\pi \in \Gamma_{n,\widetilde{n}}$ that contains either a crossing or a nesting can be estimated by

$$\lim_{L\to\infty} |\mathrm{Amp}_\pi(f;g;\epsilon;\eta)| \;\leq\; \|\, f\,\|_2 \|\, g\,\|_2 \,\|\, J\,\|_\infty \, \epsilon^{1/5} \, (\log \frac{1}{\epsilon})^4 (c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon})^{\bar{n}} \,, \quad (4.33)$$

see [7, 15]. We note that similarly as in [7, 15], the bounds on all error terms will only depend on the L^2-norm of the initial condition, which in [7, 15] is normalized by $\|\widehat{\phi}_0\|_2^2 = 1$.

The existence of the the thermodynamic limit, as $L \to \infty$, is obtained precisely in the same manner as in [7, 8]. Let

$$\Gamma_{n,\widetilde{n}}^{c-n} \;\subset\; \Gamma_{n,\widetilde{n}} \quad (4.34)$$

denote the subset of diagrams of crossing or nesting type. The number of graphs in

$$\Gamma_{2\bar{n}}^{c-n} \;:=\; \bigcup_{n+\widetilde{n}=2\bar{n}} \Gamma_{n,\widetilde{n}}^{c-n} \quad (4.35)$$

is bounded by $2^{\bar{n}}\bar{n}!$.

Thus, the sum of amplitudes associated to all crossing and nesting diagrams is bounded by

$$\sum_{1\leq \bar{n}\leq N} \sum_{\pi\in\Gamma_{2\bar{n}}^{c-n}} \lim_{L\to\infty} |\mathrm{Amp}_\pi(f;g;\epsilon;\eta)| \quad (4.36)$$
$$<\; (N+1)!\, \epsilon^{1/5} \, (\log \frac{1}{\epsilon})^4 (c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon})^{N}$$

noting that evidently, $\|f\|_2, \|g\|_2 < C$ for f, g of Schwartz class, and recalling from (3.7) that

$$\|\, J\,\|_\infty \;\leq\; 1\,, \quad (4.37)$$

which in particular is the case for $J(p) = (1 + e^{\beta(E(p)-\mu)})^{-1}$ associated to a Gibbs state of the free Fermi field, for all $0 \leq \beta \leq \infty$.

4.6. **Remainder term and time partitioning.** If at least one of the indices $n, \widetilde{n}$ equals $N+1$, we first use

$$|\, \mathbb{E}[\, \rho_t^{(N+1,\widetilde{n})}(f;g)\,]\,| \;\leq\; (\mathbb{E}[\, \rho_t^{(\widetilde{n},\widetilde{n})}(g;g)\,])^{1/2} \, (\mathbb{E}[\, \rho_t^{(N+1,N+1)}(f;f)\,])^{1/2} \quad (4.38)$$

by the Schwarz inequality (assuming without any loss of generality that $n = N+1$). If $\widetilde{n} \leq N$, the term $\mathbb{E}[\, \rho_t^{(\widetilde{n},\widetilde{n})}(g;g)\,]$ admits a bound of the form (4.47) below.

To bound $\mathbb{E}[\, \rho_t^{(N+1,N+1)}(f;f)\,]$, corresponding to the remainder term in the Duhamel expansion, we use the time partitioning method of [15]; see also [7]. To this end, we further expand the remainder term into, say, $3N$ additional Duhamel terms, and to subdivide the time integration interval $[0,t]$ into $\kappa \in \mathbb{N}$ equal segments

$$[0,t] \,=\, \bigcup_{j=1}^{\kappa}[\tau_{j-1},\tau_j] \;\;,\;\; \tau_j = \frac{jt}{\kappa}\,, \tag{4.39}$$

whereby one obtains

$$f_t^{(>N)} \,=\, f_t^{(N,4N)} \,+\, f_t^{(>4N)}\,, \tag{4.40}$$

where

$$f_t^{(N,4N)} \,:=\, \sum_{j=1}^{\kappa}\sum_{n=N+1}^{4N-1} e^{i(t-\tau_j)H_\omega^{(1)}} f_{\tau_j}^{(n,N,\tau_{j-1})}\,, \tag{4.41}$$

with

$$f_s^{(n,N,\tau)}(p) \,:=\, (i\eta)^{n-N}\int_{\mathbb{R}^{n-N+1}} ds_0\cdots ds_{n-N}\,\delta(\sum_{j=0}^{n-N}s_j-(s-\tau)) \tag{4.42}$$

$$\int du_0\cdots du_{n-N}\,\delta(p-u_0)\prod_{j=0}^{n-N}e^{is_jE(u_j)}\prod_{\ell=1}^{n-N}\widehat{\omega}(u_j-u_{j-1})\,f(u_{n-N})\,,$$

and

$$\widetilde{f}_s^{(n,N,\tau_{j-1})} \,:=\, i\,\eta\,V_\omega^{(1)}f_s^{(n,N,\tau_{j-1})}\,. \tag{4.43}$$

Moreover,

$$f_t^{(>4N)} \,=\, \sum_{j=1}^{\kappa}e^{i(t-\tau_j)H_\omega}\int_{\tau_{j-1}}^{\tau_j}ds\,e^{i(\tau_j-s)H_\omega^{(1)}}\,f_s^{(N,4N,\tau_{j-1})}\,.$$

By the Schwarz inequality,

$$\rho_t^{(N+1,N+1)}(f;f) \,\le\, R_1(f,t) \,+\, R_2(f,t) \tag{4.44}$$

where

$$R_1(f,t) \,:=\, (3N)^2\,\kappa^2 \sup_{\substack{N<n\le 4N\\ 1\le j\le\kappa}} \rho_0(\,a^+(f_{\tau_j}^{(n,N,\tau_{j-1})})\,a(f_{\tau_j}^{(n,N,\tau_{j-1})})\,) \tag{4.45}$$

and

$$R_2(f,t) \,:=\, t^2\sup_{1\le j\le\kappa}\;\sup_{s\in[\tau_{j-1},\tau_j]}\rho_0(\,a^+(\widetilde{f}_s^{(N,4N,\tau_{j-1})})\,a(\widetilde{f}_s^{(N,4N,\tau_{j-1})})\,)\,. \tag{4.46}$$

By separating terms due to decorated ladders from those due to crossing and nesting diagrams, one finds

$$\begin{aligned}\lim_{L\to\infty}&\mathbb{E}[\,\rho_0(\,a^+(f_{\tau_j}^{(n,N,\tau_{j-1})})\,a(f_{\tau_j}^{(n,N,\tau_{j-1})})\,)\,]\\ &=\,\mathbb{E}[\int dp\,J(p)\,|\,f_{\tau_j}^{(n,N,\tau_{j-1})}(p)\,|^2\,]\\ &\le\,\|\,J\,\|_\infty\,\mathbb{E}[\,\|\,f_{\tau_j}^{(n,N,\tau_{j-1})}\,\|_2^2\,]\\ &\le\,\|\,f\,\|_2^2\,\|\,J\,\|_\infty\Big[\,\frac{(c\,\epsilon^{-1}\,\lambda^2)}{(N!)^{1/2}}\,+\,\epsilon^{1/5}\,(\log\frac{1}{\epsilon})^4(c\lambda^2\epsilon^{-1}\log\frac{1}{\epsilon})^{8N}\,\Big]\,.\end{aligned} \tag{4.47}$$

Observing that for a time integral on the interval $[\tau_{j-1},\tau_j]$ of length $\frac{t}{\kappa}$, the parameter $\epsilon = t^{-1}$ can be replaced by $\kappa\epsilon = (\frac{t}{\kappa})^{-1}$, one gets

$$\begin{aligned}\lim_{L\to\infty} \mathbb{E}[\, \rho_0(\, a^+(\widetilde{f}_s^{(N,4N,\tau_{j-1})})\, a(\widetilde{f}_s^{(N,4N,\tau_{j-1})})\,)\,] \\ \leq \; \| \, J \, \|_\infty \, \mathbb{E}[\, \| \, \widetilde{f}_s^{(N,4N,\tau_{j-1})} \, \|_2^2 \,] \\ \leq \; \| \, f \, \|_2^2 \, \| \, J \, \|_\infty \Big[\, \frac{((4N)!)}{\kappa^{2N}} \, (\log \frac{1}{\epsilon})^4 (c\lambda^2\epsilon^{-1} \log \frac{1}{\epsilon})^{8N} \, \Big] \, . \end{aligned} \tag{4.48}$$

The crucial gain of a factor κ^{-2N} expresses that the probability for the occurrence of $O(N)$ collisions in a short time interval of length $\frac{t}{\kappa}$ is small.

One obtains that if at least one of the indices n, $\widetilde{n}$ equals $N+1$,

$$\begin{aligned}\lim_{L\to\infty} |\mathbb{E}[\rho^{(n,\widetilde{n})}(f;f)]| \; \leq \; \| \, f \, \|_2^2 \, \| \, J \, \|_\infty \Big[\, \frac{N^2 \, \kappa^2 \, (c \, \epsilon^{-1} \, \lambda^2)}{(N!)^{1/2}} \\ + \Big(\, N^2 \, \kappa^2 \, \epsilon^{1/5} \, + \, \epsilon^{-2} \, \kappa^{-N} \, \Big) \, ((4N)!) \, (\log \frac{1}{\epsilon})^4 (c\lambda^2 \epsilon^{-1} \log \frac{1}{\epsilon})^{8N} \, \Big] \, , \end{aligned} \tag{4.49}$$

where κ remains to be chosen. The first term on the right hand side of (4.49) bounds the contribution from all basic ladder diagrams contained in the Duhamel expanded remainder term. For a detailed discussion, we refer to [7, 8, 15].

4.7. **Choosing the constants.** We recall from (4.37) that $\| \, J \, \|_\infty \leq 1$. Moreover, $\| \, f \, \|_2$, $\| \, g \, \|_2 < C$ for all test functions f, $g \in \mathcal{S}(\mathbb{T}^d)$. As in [7, 8, 15], we choose

$$\begin{aligned} t \; = \; \frac{1}{\epsilon} \; &= \; \frac{T}{\eta^2} \\ N \; &= \; \frac{\log \frac{1}{\epsilon}}{10 \log \log \frac{1}{\epsilon}} \\ \kappa \; &= \; (\log \frac{1}{\epsilon})^{15} \, . \end{aligned} \tag{4.50}$$

Then,

$$\begin{aligned} \epsilon^{-1/11} \; < \; N! \; &< \; \epsilon^{-1/10} \\ \kappa^N \; &> \; \epsilon^{-3/2} \end{aligned} \tag{4.51}$$

and consequently,

$$(4.36) \, , \, (4.48) \; < \; \eta^{1/15} \tag{4.52}$$

and

$$(4.49) \; < \; \eta^{1/4} \tag{4.53}$$

for η sufficiently small. It follows that the sum of all crossing, nesting, and remainder terms is bounded by $\eta^{1/20}$.

4.8. **Resummation of decorated ladder diagrams.** Let $\Gamma_{n,\widetilde{n}}^{(lad)} \subset \Gamma_{n,\widetilde{n}}$ denote the subset of all decorated ladders based on $n+\widetilde{n}$ vertices. Then, for $T>0$, let

$$\Omega_T^{(2;\eta)}(f;g) \; := \; \sum_{\bar{n}=0}^{N(\epsilon(T,\eta))} \; \sum_{n+\widetilde{n}=2\bar{n}} \; \sum_{\pi \subset \Gamma_{n,\widetilde{n}}^{(lad)}} \lim_{L\to\infty} \mathrm{Amp}_\pi(f;g;\epsilon(T,\eta);\eta) \tag{4.54}$$

with $\epsilon(T,\eta) = \frac{\eta^2}{T}$. In the hydrodynamic limit $\eta \to 0$ with $t = \frac{1}{\epsilon} = T/\eta^2$, one obtains

$$\Omega_T^{(2)}(f;g) := \lim_{\eta\to 0} \Omega_T^{(2;\eta)}(f;g) = \int dp\, F_T(p)\, \overline{f(p)}\, g(p)\,, \tag{4.55}$$

where

$$\begin{aligned} F_T(p) \ &:= \ \lim_{\eta\to 0} F_T^{(\eta)}(p) \\ &= \ e^{-2\pi T\int du\,(E(u)-E(p))} \sum_{\bar n=0}^{\infty} \int_{\mathbb{R}_+^{\bar n+1}} dS_0 \cdots dS_{\bar n}\, \delta(T - \sum_{j=0}^{\bar n} S_j) \\ &\quad \int du_0 \,\cdots\, du_n\, \delta(p-u_0) \Big(\prod_{j=1}^{\bar n} 2\pi\delta(E(u_j)-E(u_{j-1})) \Big) F_0(u_n)\,, \end{aligned} \tag{4.56}$$

with initial condition

$$F_0(u) = \lim_{L\to\infty} J(u_{\Lambda_L^*}) = \lim_{L\to\infty} \frac{1}{L^d}\, \rho_0(\, a^+_{u_{\Lambda_L^*}}\, a_{u_{\Lambda_L^*}}\,) \tag{4.57}$$

(for the definition of $u_{\Lambda_L^*}$, see Theorem 3.1). It can be straightforwardly verified that (4.56) is a solution of the Cauchy problem for the linear Boltzmann equation (3.12), as asserted in Theorem 3.1.

5. Proof of Theorem 3.2

The assertion of Theorem 3.2 follows from a straightforward application of the results in the landmark paper [10, 11, 12] of Erdös, Salmhofer, and Yau. In [10, 11, 12], a diffusive scaling limit is performed for the 1-particle Anderson model on $\mathbb{Z}^3$ with weak disorders where the scaling between the microscopic and macroscopic time and position variables is given by

$$t = \eta^{2+2\delta} T \quad , \quad x = \eta^{2+\delta} X\,, \tag{5.1}$$

for a constant $0 < \delta < \frac{1}{2000}$. This scaling implies that first, a kinetic scaling limit is performed with a scaling factor η^2, whereupon one arrives at the Boltzmann evolution given in Theorem 3.3. Subsequently, one carries out a diffusive scaling where time scales with $\eta^{2\delta}$ while position scales with η^{δ}. The heat equation thereby obtained corresponds to the diffusive limit of the Boltzmann equation in Theorem 3.1. The main task carried out in [10, 11, 12] is to prove that even up to a time of order $O(\eta^{-2-2\delta})$, all quantum fluctuations (determined by crossing and nesting diagrams) around the semiclassical dynamics (determined by decorated ladder diagrams) are bounded by $O(\eta^{\delta'})$ for some $\delta' > 0$. This is accomplished by a subtle classification of Feynman graphs argument, combined with novel estimates related to restriction results in Harmonic Analysis, for surfaces with vanishing Gauss curvature on embedded curves, [10].

In our case, the sum over decorated ladders provides an equilibrium solution of the macroscopic dynamics, hence the estimate (3.21) asserted in Theorem 3.2 is a bound on the sum of crossing and nesting terms, and on the convergence rate of the sum of decorated ladder terms.

6. Proof of Theorem 3.3

Because both K (in the definition of ρ_0) and the random Hamiltonian H_ω are bilinear in a^+, a (of the form $\int du_1\, du_2\, k(u_1,u_2)\, a^+_{u_1}\, a_{u_2}$), the same is true for

$$K(t) \,:=\, e^{itH_\omega}\, K\, e^{-itH_\omega}\,, \tag{6.1}$$

with probability 1. Therefore,

$$\rho_t(\,\cdot\,) \,=\, \frac{1}{Z_K}\,\mathrm{Tr}(\, e^{-K(t)}\,(\,\cdot\,)\,) \tag{6.2}$$

is quasifree with probability 1 (see, for instance, [3]). Thus, for $r, s \in \mathbb{N}$,

$$\rho_t(\, a^+(f_1)\cdots a^+(f_r)\, a(g_1)\cdots a(g_s)\,) \,=\, \delta_{r,s}\, \det\Big[\, \rho_t(\, a^+(f_j) a(g_\ell)\,)\,\Big]_{j,\ell=1}^{r}\,, \tag{6.3}$$

where $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$ belong to the Schwartz class. In particular, we can set $r = s$.

We expand the determinant into

$$\begin{aligned} &\det\Big[\, \rho_t(\, a^+(f_j)\, a(g_\ell)\,)\,\Big]_{j,\ell=1}^{r} \\ &\quad = \sum_{s\in S_r} (-1)^{\mathrm{sign}(s)} \prod_{j=1}^{r} \rho_t(\, a^+(f_j)\, a(g_{q_{s(j)}})\,)\,, \end{aligned} \tag{6.4}$$

where S_r is the symmetric group of degree r. We claim that for $T > 0$ and $t = \frac{T}{\eta^2}$, and any choice of $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$,

$$\lim_{L\to\infty} \Big| \mathbb{E}\Big[\, \prod_{j=1}^{r} \rho_{T/\eta^2}(\, a^+(f_j) a(g_{q_{s(j)}})\,)\,\Big] \,-\, \prod_{j=1}^{r} \mathbb{E}[\, \rho_{T/\eta^2}(\, a^+(f_j)\, a(g_{q_{s(j)}})\,)\,] \,\Big| \,<\, \eta^\delta\,, \tag{6.5}$$

for a constant $\delta > 0$ independent of r, $s \in S_r$, η, and T, and for $\eta > 0$ sufficiently small. This immediately implies that, for every fixed $r < \infty$,

$$\begin{aligned} \lim_{L\to\infty} \Big| \mathbb{E}\Big[\, &\rho_{T/\eta^2}(\, a^+(f_1)\cdots a^+(f_r)\, a(g_1)\cdots a(g_r)\,)\,\Big] \\ &- \det\Big[\, \mathbb{E}[\, \rho_{T/\eta^2}(\, a^+(f_j) a(g_\ell)\,)\,]\,\Big]_{j,\ell=1}^{r}\, \Big| \,<\, r!\, \eta^\delta \end{aligned} \tag{6.6}$$

converges to zero as $\eta \to 0$.

This implies that

$$\begin{aligned} &\Omega_T^{(2r)}(\, f_1,\dots,f_r\,;\, g_1,\dots,g_r\,) \\ &\quad := \lim_{\eta\to 0} \lim_{L\to\infty} \mathbb{E}[\rho_{T/\eta^2}(\, a^+(f_1)\cdots a^+(f_r)\, a(g_1)\cdots a(g_r)\,)] \end{aligned} \tag{6.7}$$

is quasifree, i.e.,

$$\Omega_T^{(2r)}(\, f_1,\dots,f_r\,;\, g_1,\dots,g_r\,) \,=\, \det\big[\, \Omega_T^{(2)}(\, f_i\,;\, g_j\,)\,\big]_{1\le i,j\le r}\,, \tag{6.8}$$

where

$$\Omega_T^{(2)}(\, f\,;\, g\,) \,=\, \int dp\, F_T(p)\, \overline{f(p)}\, g(p)\,. \tag{6.9}$$

The function $F_T(p)$ solves the linear Boltzmann equation with initial condition $F_0(p)$, as given in Theorem 3.1.

6.1. **Proof of (6.5).** The inequality (6.5) follows from a straightforward application of the main results in [8] where we refer for details. In this section, we shall only outline the strategy. The expectation

$$\lim_{L\to\infty} \mathbb{E}\Big[\prod_{j=1}^{r} \rho_t(\, a^+(f_j)\, a(g_{s(j)})\,)\Big] \tag{6.10}$$

can be represented by a graph expansion as follows. We expand each of the factors

$$\rho_t(\, a^+(f)a(g)\,) \;=\; \sum_{n,\widetilde{n}=1}^{N+1} \int dp\, J(p)\, \overline{f_t^{(n)}(p)}\, g_t^{(\widetilde{n})}(p) \tag{6.11}$$

separately into a truncated Duhamel series of level N, using the same definitions as in (4.16). For the remainder term (where at least one of the indices $n, \widetilde{n}$ equals $N+1$), we subdivide the time integration interval $[0,t]$ into κ pieces of length $\frac{t}{\kappa}$.

For the expectation (6.10), we introduce the following extension of the classes of Feynman graphs discussed for the proof of Theorem 3.1, see also Figure 2. For $r > 1$, we consider r particle lines parallel to one another, each containing a distinguished ρ_0-vertex separating it into a left and a right part. Enumerating them from 1 to r, the j-th particle line contains n_j interaction vertices on the left of the ρ_0-vertex, and $\widetilde{n}_j$ interaction vertices on its right. We note that for $r > 1$, only $\sum_{j=1}^{r}(n_j + \widetilde{n}_j)$ has to be an even number, but not every individual

$$\widehat{n}_j \;:=\; n_j \,+\, \widetilde{n}_j\,. \tag{6.12}$$

On the j-th interaction line, we label the propagator lines by momentum variables $u_0^{(j)}, \dots, u_{\widehat{n}_j+1}^{(j)}$, with indices increasing from the left.

A *contraction graph* of degree $\{(n_j, \widetilde{n}_j)\}_{j=1}^{r}$ is obtained by connecting pairs of interaction vertices by contraction lines. We denote the set of contraction graphs of degree $\{(n_j, \widetilde{n}_j)\}_{j=1}^{r}$ by $\Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^{r}}$. If the ℓ-th vertex on the j-th particle line is connected by a contraction line to the ℓ'-th vertex on the j'-th particle line, we write

$$(j;\ell) \;\sim_\pi\; (j';\ell')\,. \tag{6.13}$$

To a graph $\pi \in \Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^{r}}$, we associate the *Feynman amplitude*

$$\begin{aligned} \mathrm{Amp}_\pi(\{f_j, g_{s(j)}\};\eta;T) \;:=\; & \eta^{2\sum_{1\le j\le r}(n_j+\widetilde{n}_j)}\, e^{2r\epsilon t} \prod_{j=1}^{r} \int d\alpha_j\, d\widetilde{\alpha}_j\, e^{it(\alpha_j - \widetilde{\alpha}_j)} \\ & \int du_0^{(j)} \cdots du_{\widehat{n}_j+1}^{(j)}\, \overline{f_j(u_0^{(j)})}\, g_{s(j)}(u_{\widehat{n}_j+1}^{(j)})\, J(u_{n_j}^{(j)})\, \delta(\, u_{n_j}^{(j)} - u_{n_j+1}^{(j)}\,) \\ & \delta_\pi(\, \{u_j^{(j)}\}_{j=0}^{\widehat{n}_j+1}\,) \prod_{\ell=0}^{n_j} \frac{1}{E(u_\ell^{(j)}) - \alpha_j - i\epsilon} \prod_{\ell'=n_j+2}^{\widehat{n}_j} \frac{1}{E(u_{\ell'}^{(j)}) - \widetilde{\alpha}_j + i\epsilon}\,, \end{aligned} \tag{6.14}$$

where

$$\epsilon \;=\; \frac{1}{t} \;=\; \frac{\eta^2}{T} \tag{6.15}$$

for $T > 0$. The delta distribution

$$\delta_\pi(\,\{u_j^{(j)}\}_{j=0}^{\widehat{n}_j+1}\,) \;=\; \prod_{(j;\ell)\,\sim_\pi\,(j';\ell')} \delta(\,u_\ell^{(j)} - u_{\ell-1}^{(j)} - (u_{\ell'}^{(j')} - u_{\ell'-1}^{(j')})\,) \tag{6.16}$$

is the product of delta distributions associated to all contraction lines in π.

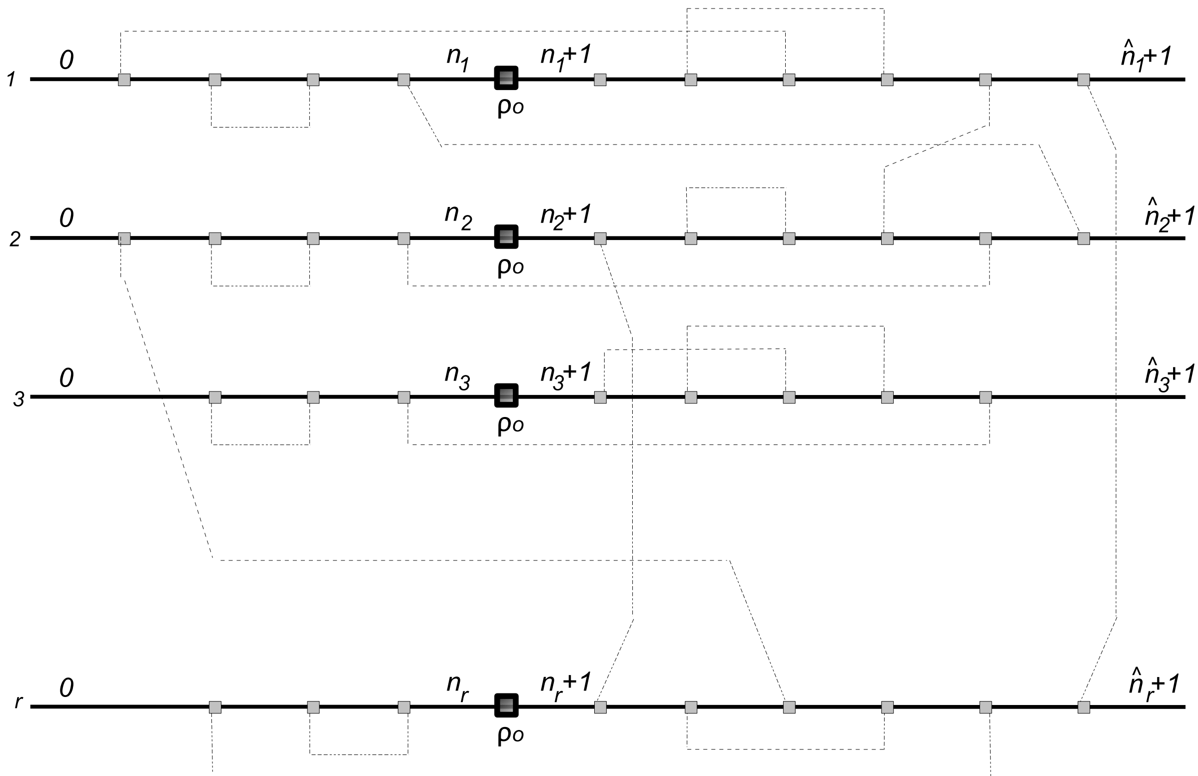

Figure 2. Order r Feynman graph. The particle line indexed by $j = 3$ is disconnected.

6.1.1. *Completely disconnected graphs.* The subclass

$$\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} \subset \Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} \tag{6.17}$$

of *completely disconnected* graphs of degree $\{(n_j, \widetilde{n}_j)\}_{j=1}^r$ consists of those graphs in which contraction lines only connect interaction vertices on the same particle line.

It is clear that

$$\lim_{L\to\infty}\sum_{\substack{0\le n_j,\widetilde{n}_j\le N\\ j=1,\dots,r}}\sum_{\pi\in\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}}\mathrm{Amp}_\pi(\{f_j,g_{s(j)}\};\eta;T) \tag{6.18}$$

$$= \lim_{L\to\infty}\prod_{j=1}^r\sum_{n_j,\widetilde{n}_j=1}^N\mathbb{E}[\int dp\,J(p)\,\overline{f_{j,T/\eta^2}(p)}\,g_{s(j),T/\eta^2}(p)]$$

$$= \lim_{L\to\infty}\prod_{j=1}^r\big(\mathbb{E}[\rho_{T/\eta^2}(a^+(f_j)a(g_{s(j)}))] + O(\eta^\delta)\big), \tag{6.19}$$

according to our proof of Theorem 3.1. The term of order $O(\eta^\delta)$ accounts for the remainder term associated to the j-th particle line (i.e., the terms involving $\mathbb{E}[\rho^{(n_j,\widetilde{n}_j)}_{T/\eta^2}(p,q)]$ where at least one of the indices $n_j,\widetilde{n}_j$ equals N). Thus, for any fixed $r\in\mathbb{N}$, we obtain

$$\lim_{\eta\to0}\lim_{L\to\infty}\sum_{\substack{0\le n_j,\widetilde{n}_j\le N\\ j=1,\dots,r}}\sum_{\pi\in\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}}\mathrm{Amp}_\pi(\{f_j,g_{s(j)}\};\eta;T)$$

$$= \prod_{j=1}^r\Omega_T^{(2)}(f_j\,;\,g_{s(j)}). \tag{6.20}$$

That is, the sum over completely disconnected graphs yields the corresponding product of averaged 2-point functions in the hydrodynamic limit.

6.1.2. *Non-disconnected graphs.* We refer to the complement of the set of completely disconnected graphs in $\Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}$,

$$\Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} := \Gamma_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}\setminus\Gamma^{disc}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}, \tag{6.21}$$

as the set of *non-disconnected graphs*. It remains to prove that the sum over non-disconnected graphs, combined with the remainder terms, can be bounded by $O(\eta^\delta)$, for L sufficiently large.

The condition required in [8] for the estimate analogous to (6.5) to hold is that for the initial condition ϕ_0 (corresponding to the test functions f_j, g_ℓ in our case) of the random Schrödinger evolution studied in [8], a "concentration of singularity condition" is satisfied (that is, singularities in momentum space are not too much "spread out" in the limit $\eta\to0$). It states that in frequency space $\mathbb{T}^d$,

$$\widehat{\phi}_0 = \widehat{\phi}_0^{(reg)} + \widehat{\phi}_0^{(sing)}, \tag{6.22}$$

where

$$\|\,\widehat{\phi}_0^{(reg)}\,\|_\infty < c \tag{6.23}$$

and

$$\|\,|\widehat{\phi}_0^{(sing)}|*|\widehat{\phi}_0^{(sing)}|\,\|_2 < c'\,\eta^{3/2} \tag{6.24}$$

are satisfied uniformly in L, as $L\to\infty$.

In the present case, we have to require that f_j, g_ℓ satisfy the concentration of singularity condition. This is, however, evidently fulfilled since f_j, g_ℓ are η-independent Schwartz class functions (in contrast, the initial states considered in [8] are of WKB type, and scale non-trivially with η.)

It is proven in [8] that the amplitude of every non-disconnected graph with $n_j, \widetilde{n}_j \leq N$ for $j = 1, \dots, r$, is bounded by

$$\sup_{\pi \in \Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}} \big| \, \mathrm{Amp}_\pi(\{f_j, g_{s(j)}\}; \eta; T) \, \big| \tag{6.25}$$

$$< \, \epsilon^{1/5} \, (c \, \eta^2 \, \epsilon^{-1} \, \log \frac{1}{\epsilon})^{\frac{r}{2} \sum_{j=1}^r \widehat{n}_j} (\log \frac{1}{\epsilon})^{4r} \, , \tag{6.26}$$

where we recall that $\epsilon = \frac{1}{t} = \frac{\eta^2}{T}$ for $T > 0$. This key estimate is a factor $\epsilon^{1/5}$ smaller than the bound on the sum of disconnected graphs; this improvement is obtained from exploiting the existence of at least one contraction line that connects two different particle lines; see [8].

The number of non-disconnected graphs is bounded by

$$\big| \, \Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r} \, \big| \, \leq \, (\sum_{j=1}^r \widehat{n}_j)! \, \leq \, (2rN)! \tag{6.27}$$

where $\widehat{n}_j = n_j + \widetilde{n}_j$. Therefore, the sum over all non-disconnected graphs with $0 \leq n_j, \widetilde{n}_j \leq N$ is bounded by

$$\sum_{1 \leq j \leq r} \; \sum_{0 \leq n_j, \widetilde{n}_j \leq N} \; \sum_{\pi \in \Gamma^{n-d}_{\{(n_j,\widetilde{n}_j)\}_{j=1}^r}} \big| \, \mathrm{Amp}_\pi(\{f_j, g_{s(j)}\}; \eta; T) \, \big| \tag{6.28}$$

$$\leq \, ((2rN)!)^2 \epsilon^{1/5} \, (c \, \eta^2 \, \epsilon^{-1} \, \log \frac{1}{\epsilon})^{rN} (\log \frac{1}{\epsilon})^{4r} \tag{6.29}$$

since $\#\{(n_j, \widetilde{n}_j)\}_{j=1}^r \, | \, \sum_j \widehat{n}_j = m \, \} \leq m!$.

6.1.3. *Duhamel remainder term.* In case at least one of the indices n_j or $\widetilde{n}_j$ equals $N+1$, the following argument can be applied. Clearly, from a Hölder estimate of the form $\|h_1 \cdots h_s\|_1 \leq \|h_1\|_s \cdots \|h_s\|_s$ with respect to $\mathbb{E}$, we have

$$\Big| \mathbb{E}\Big[\prod_{j=1}^r \rho_t^{(n_j,\widetilde{n}_j)}(f;g) \Big] \Big| \, \leq \, \prod_{j=1}^r \mathbb{E}[\, | \, \rho_t^{(n_j,\widetilde{n}_j)}(f;g) \, |^{2r} \,]^{\frac{1}{2r}} \, . \tag{6.30}$$

Here, we have used an exponent $2r$ instead of r because then, even for r odd, an absolute value of the form $|z|^{2r}$ can be replaced by a product of the form $\overline{z}^r z^r$, where $z \in \mathbb{C}$.

We make a choice of constants

$$\begin{aligned} t \, = \, \frac{1}{\epsilon} \; &= \; \frac{T}{\eta^2} \\ N \; &= \; \frac{\log \frac{1}{\epsilon}}{10r \log \log \frac{1}{\epsilon}} \\ \kappa \; &= \; (\log \frac{1}{\epsilon})^{15r} \, , \end{aligned} \tag{6.31}$$

similarly as in Section 4.7 of the proof of Theorem 3.1.

If n_j or $\widetilde{n}_j$ equals $N+1$, we can use the bounds (4.52) and (4.53).

If both n, $\widetilde{n} \leq N$, we use the a priori bound

$$\sum_{n+\widetilde{n}=2\bar{n}} \sum_{\pi\in\Gamma_{n,\widetilde{n}}} \lim_{L\to\infty} \mathbb{E}[\,|\,\rho_t^{(n,\widetilde{n})}(f;g)\,|^{2r}\,]^{\frac{1}{2r}} \tag{6.32}$$

$$< \Big[\sum_{\ell=0}^{2r} \binom{2r}{\ell} \Big(\frac{(c\eta^2\epsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}} \Big)^{\ell} \epsilon^{1/5}\,((2r-\ell)\bar{n})!\,\Big((\log\frac{1}{\epsilon})^4 (c\eta^2\epsilon^{-1}\log\frac{1}{\epsilon})^{\bar{n}} \Big)^{2r-\ell} \Big]^{\frac{1}{2r}}$$

$$< \frac{(c\eta^2\epsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}} + \eta^{\frac{1}{10}} \tag{6.33}$$

The factor $\frac{(c\eta^2\epsilon^{-1})^{\ell\bar{n}}}{(\bar{n}!)^{\ell}}$ in $[\cdots]$ accounts for ℓ basic ladders on ℓ copies of $\Gamma^{disc}_{n,\widetilde{n}}$, while the remaining factor accounts for all other (not necessarily non-disconnected) contractions on the remaining $2r-\ell$ particle lines; for details, see [7, 8, 15].

Let us without any loss of generality assume that $n_1 = N+1$. Then, keeping n_1 fixed and summing over the remaining indices $\widetilde{n}_1$ and n_j, $\widetilde{n}_j$, with $j = 2, \dots, r$, we find

$$\sum_{\substack{0\leq n_2,n_j,\widetilde{n}_j\leq N+1\\ j=2,\dots,r}} \prod_{j=1}^{r} \mathbb{E}[\,|\,\rho_t^{(n_j,\widetilde{n}_j)}(f;g)\,|^{2r}\,]^{\frac{1}{2r}}$$

$$< \eta^{\frac{1}{15}} \Big[\sum_{\bar{n}=0}^{N} \frac{(c\eta^2\epsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}} + \eta^{\frac{1}{10}} \Big]^{2r-1} \tag{6.34}$$

where the factor $\eta^{\frac{1}{15}}$ accounts for the remainder term indexed by $n_1 = N+1$. We conclude that the sum over all terms (6.30) which contain at least one n_j or $\widetilde{n}_j$ equalling $N+1$ (i.e., which contain at least one Duhamel remainder term) can be bounded by

$$C^r\,\eta^{\frac{1}{15}} \tag{6.35}$$

for a constant C independent of η and r.

Combined with

$$(6.28) < \eta^{\frac{1}{20}}, \tag{6.36}$$

which one easily verifies, this completes the proof of Theorem 3.4. For more details addressing the arguments outlined here, we refer to [8].

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