ALMOST PERIODIC ORBITS AND STABILITY FOR QUANTUM TIME-DEPENDENT HAMILTONIANS

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Abstract. We study almost periodic orbits of quantum systems and prove that for periodic time-dependent Hamiltonians an orbit is almost periodic if, and only if, it is precompact. In the case of quasiperiodic time-dependence we present an example of a precompact orbit that is not almost periodic. Finally we discuss some simple conditions assuring dynamical stability for nonautonomous quantum system.

1. Introduction

The time evolution of a quantum mechanical system with time-dependent Hamiltonians $H(t)$ is determined by the Schrödinger equation

$$i\frac{d\psi(t)}{dt} = H(t)\psi(t),$$

where $H(t)$ is a family of self-adjoint operators in the Hilbert space $\mathcal{H}$ and $\psi(t) \in \mathcal{H}$ for all $t \in \mathbb{R}$. The initial value problem $\psi(0) = \psi$ has a unique solution

$$\psi(t) \doteq U(t,0)\psi,$$

under suitable conditions on $H(t)$ (see [21, 18, 19, 15]) and the propagators, or time evolution operators $U(t,s)$, form a strongly continuous family of unitary operators acting on $\mathcal{H}$, such that

$$U(t,r)U(r,s) = U(t,s), \qquad \forall r,s,t, \in \mathbb{R}$$

$$U(t,t) = \mathrm{I_d}, \qquad \forall t.$$

$\mathrm{I_d}$ denotes the identity operator. If the Hamiltonian is time-periodic with period T, then $U(t+T, r+T) = U(t,r)$ and the Floquet operator at s is defined by $U_F(s) \doteq U(s+T,s)$; $U_F(0)$ is simply called Floquet operator and denoted by U_F, and $U_F(s)$ is unitarily equivalent to $U_F(r)$, $\forall r,s$. Let

$$\mathcal{O}(\psi) \doteq \{U(t,0)\psi : t \in \mathbb{R}\}$$

be the orbit of a vector $\psi \in \mathcal{H}$.

If $H(t) = H$ is independent of t the time evolution operators are $U(t,s) = e^{-iH(t-s)}$. In this case, it is a well-known fact that if ψ is in the point

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subspace of H then the quantum time evolution of the state ψ, $\psi(t)$, is almost periodic, since it can be expanded in terms of the eigenfunctions φ_n of H, with eigenvalues E_n,

$$\psi(t) = \sum_n c_n e^{-iE_n t} \varphi_n.$$

Reciprocally, if $\psi(t)$ is almost periodic then using the results in [20] (Chapter VI) it holds true that $\mathcal{O}(\psi)$ is precompact and then ψ is in the point subspace of H (see Theorem 3 ahead). In this work, we prove that this fact remains true in the periodic case, that is, ψ is in the point subspace of U_F if, and only if, $\psi(t)$ is almost periodic (see Theorem 5).

In the studies of time-dependent systems it is common to consider the quasienergy operator, i.e., a self-adjoint operator formally given by

$$K = -i\frac{d}{dt} + H(t)$$

acting in some enlarged Hilbert space. The quasienergy operator K was previously defined for periodic Hamiltonians [22, 13] and then generalized for general time dependence in [14]. In the periodic case it was proved that

$$e^{-iKT} \simeq \mathrm{I_d} \otimes U_F,$$

where $\simeq$ means unitary equivalence.

A natural framework for considering general time-dependent perturbations, which includes both periodic and the random potentials as special cases, is to write $H(t)$ in the form

$$H(t) = H(g_t(\theta)) = H_0 + V(g_t(\theta)),$$

where $g_t : \Omega \to \Omega$ is an invertible flow on a compact manifold Ω with a probability ergodic measure μ and H_0 is the Hamiltonian of the isolated system (see [16, 2]). Again, under suitable conditions on V there exists a unitary time evolution operator $U_\theta(t,s)$ and the generalized quasienergy operator is defined [16] on $L^2(\Omega, \mathcal{H}, d\mu)$ by

$$(e^{-i\tilde{K}t} f)_\theta = \mathcal{F}_{-t} U_\theta(t,0) f_\theta = U_\theta(0,-t)\mathcal{F}_{-t} f_\theta,$$

where $\mathcal{F}_{-t} f_\theta = f_{g_{-t}(\theta)}$; we refer to this construction as *Jauslin-Lebowitz formulation.* The operator $\tilde{K}$ acts as

$$(\tilde{K} f)_\theta = i\frac{d}{dt} f_{g_{-t}(\theta)}\Big|_{t=0} + H_\theta f_\theta.$$

In the case of a periodic potential one has $\Omega = S^1 \equiv [0, 2\pi)$, $g_t(\theta) = \theta + \omega t$ and $d\mu = \frac{d\theta}{2\pi}$.

For quasiperiodic potentials with two incommensurate frequencies $\omega_1/\omega_2 \notin \mathbb{Q}$ the manifold Ω is $S^1 \times S^1$, $g_t(\theta_1, \theta_2) = (\theta_1 + \omega_1 t, \theta_2 + \omega_2 t)$ and $d\mu = \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}$. We denote the two periods by $T_j = \frac{2\pi}{\omega_j}$. In this case the generalized Floquet operator acting on $\mathcal{K}_1 \doteq L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi})$ is defined by

$$U_{\mathrm{F}} = \mathcal{T}_{-T_2} u_1, \tag{1}$$

where $u_1(\theta_1) = U_{(\theta_1,0)}(T_2, 0)$ ($\doteq$ monodromy operator) and $(\mathcal{T}_{-T_2}\phi)(\theta_1) = \phi(\theta_1 - \omega_1 T_2)$.

Let $A : \mathrm{dom}\ A \subset \mathcal{H} \to \mathcal{H}$ be an unbounded positive self-adjoint operator with discrete spectrum which we call a *probe operator*. Assuming that if $\psi \in \mathrm{dom}\ A$, then $U(t, 0)\psi \in \mathrm{dom}\ A$ for all $t \geq 0$, a very interesting question is about the behavior of the expectation value of A, that is,

$$E^A_\psi(t) \equiv \langle U(t, 0)\psi, AU(t, 0)\psi\rangle.$$

We say the system is A-dynamically stable if $E^A_\psi(t)$ is a bounded function of time, and A-dynamically unstable otherwise. A particular case is when the Hamiltonian has the form $H(t) = H_0 + V(t)$ and $A = H_0$. In this work we discuss some simple conditions assuring dynamical stability, mainly when either the Floquet or quasienergy operator has purely point spectrum; recall that in the periodic case it is known that continuous spectrum of the Floquet operator implies dynamical instability (see Section 2).

Usually it is not a simple task to get results on dynamical (in)stability in the original Hilbert space H through properties of K or $\tilde{K}$ acting in the corresponding enlarged space. We present some theoretical results about this point in Section 4. An important result in the periodic case was proved in [11], i.e., that the applicability of the KAM method for the quasienergy operator K, which is a technique to find out a unitary operator U such that $UKU^{-1} = D$, where D is pure point, gives a uniform bound at the expectation value of the energy for a class of time-periodic Hamiltonians of the form $H(t) = H_0 + V(t)$ considered in [10].

The study of precompacity (and related properties) of orbits of a time-dependent quantum system and their connection with spectral type and stability was carried out, e.g., in [12, 6, 5, 3, 16, 2]. In this work we prove that in the periodic case (including the autonomous case) the orbit $\mathcal{O}(\psi)$ is precompact if, and only if, $\psi(t)$ is an almost periodic function. Moreover, already in the quasiperiodic case we present an example with precompact orbits which are not almost periodic.

This paper is organized as follows. In Section 2 we recall some subspaces of $\mathcal{H}$ that were studied in the literature and the results that connect this subspaces with dynamical (in)stability and spectral properties of the Floquet or quasienergy operators. In Section 3 we present ours results about almost periodic orbits. In Section 4 we discuss some simple conditions assuring dynamical stability; we pay special attention to connection between enlarged spaces and the original quantum Hilbert space. A number of known results are recalled in the text in order to make it as readable as possible.

2. Preliminaries

In this section we present a short account of suitable subspaces and relations among them, in order to put our results in context.

Consider a time-dependent Hamiltonian $H(t)$ acting in a separable Hilbert space $\mathcal{H}$, which may be nonperiodic, and let $U(t,0)$ the corresponding propagators. Denote by $A : \mathrm{dom}\ A \subset \mathcal{H} \to \mathcal{H}$ a probe operator, such that dom A is invariant under time evolution $U(t,0)$. Let $F(A > E)$ be the spectral projection onto the closed space spanned by the eigenvectors of A corresponding to the eigenvalues larger than $E \in \mathbb{R}$. The relevant definitions are as follows [12, 6, 5, 3].

Definition 1. *(i)* $\mathcal{H}_{pc} \doteq \{\xi \in \mathcal{H} : \mathcal{O}(\xi)$ *is precompact in* $\mathcal{H}\}$.
(ii) $\mathcal{H}_{\rm f} \doteq \big\{\xi \in \mathcal{H} : \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \|CU(t,0)\xi\| dt = 0$ *for any compact operator* $C\big\}$.
(iii) $\mathcal{H}_{\rm be} \doteq \{0 \neq \xi \in \mathcal{H} : \lim_{E\to\infty} \sup_{t\in\mathbb{R}} \|F(A > E)U(t,0)\frac{\xi}{\|\xi\|}\| = 0\} \cup \{0\}$.
(iv) $\mathcal{H}_{\rm ue} \doteq \{0 \neq \xi \in \mathcal{H} : \lim_{E\to\infty} \sup_{t\in\mathbb{R}} \|F(A > E)U(t,0)\frac{\xi}{\|\xi\|}\| = 1\} \cup \{0\}$.
(v) $\mathcal{S}^{\rm bd}(A) \doteq \{\xi \in \mathrm{dom}\ A :$ *the function* $t \mapsto E_\xi^A(t)$ *is bounded*$\}$.
(vi) $\mathcal{S}^{\rm un}(A) \doteq \{\xi \in \mathrm{dom}\ A :$ *the function* $t \mapsto E_\xi^A(t)$ *is unbounded*$\}$.

Important compact operators are the projections onto finite subspaces of $\mathcal{H}$, so that the elements of $\mathcal{H}_{\rm f}$ are interpreted as the vectors that under time evolution leave, on average, any finite-dimensional subspace of $\mathcal{H}$.

Some basic properties of the sets that appeared in the above definition are summarized ahead. For proofs we refer the reader to [5, 6, 12, 3].

Theorem 1. *Let* $H(t)$ *be a time-dependent Hamiltonian and* A *as above; then:*
(a) $\mathcal{H}_{\rm f}$ *and* $\mathcal{H}_{\rm pc}$ *are closed subspaces of* $\mathcal{H}$.
(b) $\mathcal{H}_{\rm pc} \perp \mathcal{H}_{\rm f}$.
(c) $\mathcal{H}_{\rm be} = \mathcal{H}_{\rm pc}$ *and* $\mathcal{H}_{\rm f} \subset \mathcal{H}_{\rm ue}$.
(d) If $\xi \in \mathrm{dom}\ A$ *and* $\xi \notin \mathcal{H}_{\rm pc}$ *then* $\xi \in \mathcal{S}^{\rm un}(A)$*, that is,* $\mathcal{S}^{\rm bd}(A) \subset \mathcal{H}_{\rm pc}$*. In particular,* $(\mathrm{dom}\ A \cap \mathcal{H}_{\rm f}) \setminus \{0\} \subset \mathcal{S}^{\rm un}(A)$.

Note that if the Hamiltonian $H(t)$ has the form $H(t) = H_0 + V(t)$ with H_0 an unbounded, positive, self-adjoint operator with discrete spectrum, then Theorem 1(d) holds true for $A = H_0$.

2.1. **Periodic Case.** If $H(t)$ is periodic of period T and $U_F = U(T,0)$ is the corresponding Floquet operator, we denote by $\mathcal{H}_{\rm p}$ the point spectral subspace and by $\mathcal{H}_{\rm c}$ the continuous subspace of the Floquet operator $U_{\rm F}$. Recall the important

Theorem 2 (RAGE). *Let* $C : \mathcal{H} \to \mathcal{H}$ *be a compact operator and* $\xi \in \mathcal{H}_{\rm c}$*, then*

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \|CU(t,0)\xi\| dt = 0.$$

A detailed proof of Theorem 2 can be found in [12]; this result was firstly proved for the autonomous case (see, e.g., [1]). As a consequence of this theorem it follows that if $\xi \in \mathcal{H}_{\mathrm{c}}$ then $\xi \in \mathcal{H}_{\mathrm{f}}$, so by Theorem 1(d) it follows that $\langle U(t,0)\xi, AU(t,0)\xi\rangle$ is unbounded. Thus, as it is well known, the presence of continuous spectrum for the Floquet operator is a signature of quantum instability. In principle, one would expect that a Floquet operator with purely point spectrum would imply quantum stability, however there are examples with purely point spectrum and dynamically unstable; see [9, 17, 7] for examples in the autonomous case and [8] for the time-periodic case.

Using the above theorem and a series of technical lemmas in [6], one gets

Theorem 3. *If the Hamiltonian operator is periodic in time, then*
(a) $\mathcal{H}_{\mathrm{p}} = \mathcal{H}_{\mathrm{be}} = \mathcal{H}_{\mathrm{pc}}$*;*
(b) $\mathcal{H}_{\mathrm{c}} = \mathcal{H}_{\mathrm{ue}} = \mathcal{H}_{\mathrm{f}}$*.*

We observe that Theorem 3 also holds in the autonomous case $H(t) = H$ and with $\mathcal{H}_{\mathrm{p}}$ and $\mathcal{H}_{\mathrm{c}}$ denoting, respectively, the point and continuous subspace of the Hamiltonian H.

According to the above-quoted results, for periodic systems we have

$$\mathcal{H} = \mathcal{H}_{\mathrm{pc}} \oplus \mathcal{H}_{\mathrm{f}}. \tag{2}$$

In [5] was presented an example for which relation (2) does not hold for nonperiodic time dependence. It was defined the "unusual" subspace $\mathcal{H}_{\mathrm{a}}$ by the relation

$$\mathcal{H} = \mathcal{H}_{\mathrm{pc}} \oplus \mathcal{H}_{\mathrm{f}} \oplus \mathcal{H}_{\mathrm{a}},$$

and constructed a nonperiodic Hamiltonian such that $\mathcal{H} = \mathcal{H}_{\mathrm{a}}$. The example is given by the Floquet operator generated by the kicked Hamiltonian

$$H(t) = p^2 + x\sum_{n=1}^{\infty} \epsilon_n \delta(t-n), \qquad x \in [0, 2\pi),$$

acting on $\mathcal{H} = L^2(\mathbb{T})$ and $\epsilon_n \in \{-1, 0, 1\}$ adequately chosen. This example illustrates some possible unusual properties of nonstationary quantum systems.

2.2. **Quasiperiodic Case.** In this case we have the generalized Floquet operator U_{F} as defined in (1), acting on the enlarged space $\mathcal{K}_1 = L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi})$, and the generalized quasienergy operator $\tilde{K}$ acting in $L^2(S^1 \times S^1, \mathcal{H}, \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi})$. We denote, respectively, by $\mathcal{K}_{1,p}$ and $\mathcal{K}_{1,\mathrm{c}}$ the point and continuous subspace of the generalized Floquet operator U_{F}.

For each fixed t let the unitary operator $U(t) : \mathcal{K}_1 \to \mathcal{K}_1$ be given by $(U(t)\psi)(\theta_1) = U_{(\theta_1,0)}(t,0)\psi(\theta_1)$, that is,

$$U(t) = \int_{S^1}^{\oplus} U_{(\theta_1,0)}(t,0)\frac{d\theta_1}{2\pi},$$

and given $\psi \in \mathcal{K}_1$ let $\tilde{\mathcal{O}}(\psi) = \{U(t)\psi : t \in \mathbb{R}\}$ be the orbit of ψ in the enlarged space $\mathcal{K}_1$.

Let $A : \mathrm{dom}\ A \subset \mathcal{K}_1 \to \mathcal{K}_1$ be a probe operator with $U(t)\mathrm{dom}\ A \subset \mathrm{dom}\ A$ and $F(A > E)$ as before. The relevant definitions are as follows [16, 2, 6]:

Definition 2. *(a)* $\mathcal{K}_{1,\mathrm{f}} \doteq \big\{\psi \in \mathcal{K}_1 : \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \|CU(t)\psi\|_{\mathcal{K}_1} dt = 0$ *for any compact operator* C *in* $\mathcal{K}_1\big\}$.
(b) $\mathcal{K}_{1,\mathrm{pc}} = \{\psi \in \mathcal{K}_1 : \tilde{\mathcal{O}}(\psi)$ *is precompact in* $\mathcal{K}_1\}$.
(c) $\mathcal{K}_{1,\mathrm{be}} \doteq \{0 \neq \psi \in \mathcal{K}_1 : \lim_{E\to\infty} \sup_{t\in\mathbb{R}} \|F(A > E)U(t)\frac{\psi}{\|\psi\|}\| = 0\} \cup \{0\}$.
(d) $\mathcal{K}_{1,\mathrm{ue}} \doteq \{0 \neq \psi \in \mathcal{K}_1 : \lim_{E\to\infty} \sup_{t\in\mathbb{R}} \|F(A > E)U(t)\frac{\psi}{\|\psi\|}\| = 1\} \cup \{0\}$.

In [16] it was proved the analog of the RAGE Theorem for the quasiperiodic case. The proof is an adaptation of the similar statement in the periodic case discussed in [12]. As in the periodic case one has:

Theorem 4. *If the Hamiltonian operator is quasiperiodic in time, then*
(a) $\mathcal{K}_{1,p} = \mathcal{K}_{1,\mathrm{pc}} = \mathcal{K}_{1,\mathrm{be}}$;
(b) $\mathcal{K}_{1,\mathrm{c}} = \mathcal{K}_{1,\mathrm{ue}} = \mathcal{K}_{1,\mathrm{f}}$.

It is worth mentioning that the relation between the energy growth and the characterizations in Definition 2 is not as direct as in the case of periodic and autonomous potentials. The above theorem holds on the enlarged space $\mathcal{K}_1$ so that a generalized operator with continuous spectrum does not ensure unbounded energy growth in the original Hilbert space $\mathcal{H}$, although it does in $\mathcal{K}_1$. See [16, 2] for interesting examples on systems with time-quasiperiodic dependence.

3. Almost Periodic Orbits

Let $\mathcal{B}$ be a Banach space. A continuous function $f : \mathbb{R} \to \mathcal{B}$ is called *almost periodic* if for any number $\epsilon > 0$, one can find a number $l(\epsilon) > 0$ such that any interval of the real line of length $l(\epsilon)$ contains at least one point τ with the property that

$$\|f(t+\tau) - f(t)\| < \epsilon, \qquad \forall t \in \mathbb{R}.$$

For properties of almost periodic functions we refer the reader to [4, 20]. Now we introduce the following subset of $\mathcal{H}$:

$$\mathcal{H}_{\mathrm{ap}} \doteq \{\xi \in \mathcal{H} : \text{the function } \mathbb{R} \ni t \mapsto \xi(t) = U(t,0)\xi \text{ is almost periodic}\}.$$

By abuse of language sometimes we say that the orbit $\mathcal{O}(\xi)$ is almost periodic.

For general time dependence one has

Proposition 1. *$\mathcal{H}_{\rm ap}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{H}_{\rm ap} \subset \mathcal{H}_{\rm pc}$.*

Proof. Clearly $0 \in \mathcal{H}_{\rm ap}$. If $\xi, \psi \in \mathcal{H}_{\rm ap}$ then $\xi(t) = U(t,0)\xi$ and $\psi(t) = U(t,0)\psi$ are almost periodic functions. Since the sum of two almost periodic functions with values in $\mathcal{H}$ is an almost periodic function, it follows that $\xi(t) + \psi(t) = U(t,0)\xi + U(t,0)\psi = U(t,0)(\xi+\psi) = (\xi+\psi)(t)$ is an almost periodic function. So $\xi + \psi \in \mathcal{H}_{\rm ap}$. Now, let $\xi \in \mathcal{H}_{\rm ap}$ and λ a complex number, then $\xi(t) = U(t,0)\xi$ is an almost periodic function. Since $\lambda\xi(t) = \lambda U(t,0)\xi = U(t,0)(\lambda\xi)$ is an almost periodic function, it follows that $\lambda\xi \in \mathcal{H}_{\rm ap}$. So $\mathcal{H}_{\rm ap}$ is a vector subspace of $\mathcal{H}$.

Suppose that $\{\xi_j\} \subset \mathcal{H}_{\rm ap}$ and $\lim_{j\to\infty} \xi_j = \xi$. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|\xi_j - \xi\| < \epsilon$ for all $j \geq N$; thus, there exists N as above such that $j \geq N$ implies that $\forall\, t \in \mathbb{R}$

$$\|\xi(t) - \xi_j(t)\| = \|U(t,0)\xi - U(t,0)\xi_j\| \leq \|\xi - \xi_j\| < \epsilon.$$

So $\xi_j(t) \to \xi(t)$ uniformly in $\mathbb{R}$ in the sense of convergence in the norm. Since each $\xi_j(t)$ is an almost periodic function, it follows that $\xi(t)$ is an almost periodic function (Theorem 6.4 in [4]) and $\xi \in \mathcal{H}_{\rm ap}$, which shows that $\mathcal{H}_{\rm ap}$ is a closed vector subspace of $\mathcal{H}$.

Since the set of values of an almost periodic function with values in $\mathcal{H}$ is precompact in $\mathcal{H}$ (Theorem 6.5 in [4]), it follows that $\mathcal{H}_{\rm ap} \subset \mathcal{H}_{\rm pc}$. $\square$

3.1. **Periodic Systems.** If the Hamiltonian time dependence is periodic (or autonomous) more can be said.

Proposition 2. *If the Hamiltonian operator is periodic in time and $\xi \in \mathcal{H}_{\rm p}$ is an eigenvector of $U_{\rm F}$, that is, $U_{\rm F}\xi = e^{-i\alpha}\xi$, $\alpha \in \mathbb{R}$, then $\xi \in \mathcal{H}_{\rm ap} \subset \mathcal{H}_{\rm pc}$.*

Proof. Since $U(t,0)$ is strongly continuous the map $t \mapsto \xi(t)$ is continuous.

Any $t \in \mathbb{R}$ can be written in the form $t = nT + s$, with $n \in \mathbb{Z}$ and $0 \leq s < T$. We have $U_{\rm F}\xi = e^{-i\alpha}\xi$ and $U_{\rm F}^{-1}\xi = e^{i\alpha}\xi$. Since for $t \geq 0$ $(n \geq 0)$

$$\begin{aligned} U(t,0)\xi &= U(s+nT,nT)U(nT,(n-1)T)\ldots U(T,0)\xi \\ &= U(s,0)\underbrace{U(T,0)\ldots U(T,0)}_{n \text{ factors}}\xi = U(s,0)e^{-in\alpha}\xi, \end{aligned}$$

and for $t < 0$ $(n < 0)$

$$\begin{aligned} U(t,0)\xi &= U(s+nT,nT)U(nT,(n+1)T)\ldots U(-T,0)\xi \\ &= U(s,0)\underbrace{U(T,0)^{-1}\ldots U(T,0)^{-1}}_{n \text{ factors}}\xi = U(s,0)e^{-in\alpha}\xi, \end{aligned}$$

it follows that

$$U(t,0)\xi = U(s,0)e^{-in\alpha}\xi,$$

for $t = nT + s \in \mathbb{R}$, $n \in \mathbb{Z}$ and $0 \le s < T$. So for each $t = nT + s \in \mathbb{R}$

$$\begin{aligned} \xi(t+T) &= U(t+T,0)\xi = U(s,0)e^{-i(n+1)\alpha}\xi \\ &= e^{-i\alpha}U(s,0)e^{-in\alpha}\xi = e^{-i\alpha}\xi U(t,0)\xi = e^{-i\alpha}\xi(t), \end{aligned}$$

so $t \to \xi(t)$ is an almost periodic function and the result is proved. □

Summing up, we conclude:

Theorem 5. *If the Hamiltonian operator is periodic in time, then*
(a) $\mathcal{H}_{\mathrm{p}} = \mathcal{H}_{\mathrm{be}} = \mathcal{H}_{\mathrm{pc}} = \mathcal{H}_{\mathrm{ap}}$*;*
(b) $\mathcal{H}_{\mathrm{c}} = \mathcal{H}_{\mathrm{ue}} = \mathcal{H}_{\mathrm{f}}$*.*

Proof. It is enough to prove that $\mathcal{H}_{\mathrm{pc}} = \mathcal{H}_{\mathrm{ap}}$. The inclusion $\mathcal{H}_{\mathrm{ap}} \subset \mathcal{H}_{\mathrm{pc}}$ was proved in Proposition 1. On the other hand, it is a consequence of Propositions 1 and 2 that $\mathcal{H}_{\mathrm{p}} \subset \mathcal{H}_{\mathrm{ap}}$. Since $\mathcal{H}_{\mathrm{p}} = \mathcal{H}_{\mathrm{pc}}$, it follows that $\mathcal{H}_{\mathrm{pc}} \subset \mathcal{H}_{\mathrm{ap}}$. □

Theorem 5 holds also for autonomous Hamiltonians.

3.2. **Quasiperiodic Systems.** In the above theorem we proved that for time-periodic Hamiltonians an orbit $\mathcal{O}(\xi)$ is precompact if, and only if, $t \mapsto \xi(t)$ is almost periodic. Now we construct an example showing that already in the case of time-quasiperiodic Hamiltonians there are precompact orbits that are not almost periodic.

Example Given the matrix

$$u_1(\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix},$$

it is known (see Lemma 5.1 in [2]) that there exists a quasiperiodic Hamiltonian $H_\theta(t)$, $\theta = (\theta_1, \theta_2)$, acting on $\mathcal{H} = \mathbb{C}^2$, of the form

$$H_\theta(t) = h_0(t)\mathrm{I_d} + \sum_{j=1}^{3} h_j(t)\sigma_j, \tag{3}$$

where σ_j are the Pauli matrices, and $h_j(t)$ are real quasiperiodic functions, i.e., $h_j(t) = \bar{h}_j(\omega_1 t + \theta_1, \omega_2 t + \theta_2)$, where $\bar{h}_j(\theta_1, \theta_2)$ are continuous and 2π-periodic in the two arguments $\theta_1,\ \theta_2 \in S^1$, and $\omega_1,\ \omega_2$ are positive real numbers so that $u_1(\theta_1) = U_{(\theta_1,0)}(T_2, 0)$ is the corresponding monodromy operator. Moreover, the corresponding generalized Floquet operator $U_F = \mathcal{T}_{-T_2}u_1$ has absolutely continuous spectrum for any irrational $\alpha \doteq \frac{\omega_1}{\omega_2}$.

By the construction in the proof of Lemma 5.1 in [2], it is found that for $k \in \mathbb{Z}$, $k > 0$,

$$\begin{aligned}
U_{(\theta_1,0)}(kT_2,0) &= u_1(\theta_1+(k-1)2\pi\alpha)\dots u_1(\theta_1+2\pi\alpha)u_1(\theta_1) \\
&= \begin{pmatrix} e^{i(\theta_1+(k-1)2\pi\alpha)} & 0 \\ 0 & e^{-i(\theta_1+(k-1)2\pi\alpha)} \end{pmatrix} \cdots \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \\
&= \begin{pmatrix} e^{i(\theta_1+(k-1)2\pi\alpha)} \dots e^{i\theta_1} & 0 \\ 0 & e^{-i(\theta_1+(k-1)2\pi\alpha)} \dots e^{-i\theta_1} \end{pmatrix} \\
&= \begin{pmatrix} e^{i(k\theta_1+(1+2+\dots(k-1))2\pi\alpha)} & 0 \\ 0 & e^{-i(k\theta_1+(1+2+\dots(k-1))2\pi\alpha)} \end{pmatrix} \\
&= \begin{pmatrix} e^{ik(\theta_1+(k-1)\pi\alpha)} & 0 \\ 0 & e^{-ik(\theta_1+(k-1)\pi\alpha)} \end{pmatrix};
\end{aligned}$$

for $k < 0$ the same expression is found. Therefore, for all $k \in \mathbb{Z}$

$$U_{(\theta_1,0)}(kT_2,0) = \begin{pmatrix} e^{ik(\theta_1+(k-1)\pi\alpha)} & 0 \\ 0 & e^{-ik(\theta_1+(k-1)\pi\alpha)} \end{pmatrix}.$$

Moreover, for $\theta_1 \in S^1$, $0 \le t \le T_2$, define

$$v(t;\theta_1) = \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)} & 0 \\ 0 & e^{-i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)} \end{pmatrix},$$

which is differentiable with respect to t and satisfies

$$v(0;\theta_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I_d}, \qquad v(T_2;\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} = u_1(\theta_1).$$

So for $t \in \mathbb{R}$, $t = kT_2 + \delta_t$, $0 \le \delta_t \le T_2$, one has

$$U_{(\theta_1,0)}(t,0) = v(\delta_t;\theta_1+k2\pi\alpha)U_{(\theta_1,0)}(kT_2,0).$$

Therefore, for $\xi \in \mathcal{H} = \mathbb{C}^2$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, we have

$$U_{(\theta_1,0)}(t,0)\xi = \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)}\xi_1 \\ e^{-i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)}\xi_2 \end{pmatrix}.$$

Since the map, for $0 \neq a \in \mathbb{R}$, $t \mapsto \sin at^2$ is not almost periodic, because it is not uniformly continuous, we conclude that the map $t \mapsto e^{iat^2}$ is not almost periodic. Thus,

$$t \mapsto g(t) = e^{i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)} = e^{i\frac{t}{T_2}\theta_1}e^{it^2\frac{\omega_1\omega_2}{4\pi}}e^{-it\frac{\omega_1}{2}}$$

is not almost periodic, because on the contrary the map

$$e^{-i\frac{t}{T_2}\theta_1}g(t)e^{it\frac{\omega_1}{2}} = e^{it^2\frac{\omega_1\omega_2}{4\pi}}$$

would be almost periodic.

Therefore, if $\xi \neq 0$ then the map $t \mapsto U_{(\theta_1,0)}(t,0)\xi$ is not almost periodic for all $\theta_1 \in S^1$. Hence we have got an example of precompact orbit (a closed and bounded set on $\mathbb{C}^2$ is compact) which is not almost periodic. This finishes the example.

The above example can be extend to the infinite dimensional Hilbert space $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}^2$ of the elements $\xi = (\xi_n)_{n\in\mathbb{N}}$ with $\xi_n \in \mathbb{C}^2$ and $\sum_n |\xi_n|^2 < \infty$. Denote

$$\tilde{u_1}(\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix};$$

we know that there exists a quasiperiodic $\tilde{H}_\theta(t)$ such that $\tilde{u_1}(\theta_1)$ is the corresponding monodromy operator. Moreover, $\sigma(\tilde{U_\mathrm{F}})$ is absolutely continuous for all irrational α. Let

$$u_1(\theta_1) = \begin{pmatrix} \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} & & & \\ & \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} & & \\ & & \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} & \\ & & & \ddots \end{pmatrix}$$

or, writing in the another way, $u_1(\theta_1) = \bigoplus \tilde{u_1}(\theta_1)$. For $\xi \in \mathcal{H}$ one has $u_1(\theta_1)\xi = \bigoplus \tilde{u_1}(\theta_1)\xi_n$. The Floquet operator corresponding to $u_1(\theta_1)$, $U_\mathrm{F} = \mathcal{T}_{-T_2}u_1 : L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi}) \to L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi})$ has absolutely continuous spectrum for all irrational α.

If $H_\theta(t) = \bigoplus_{n\in\mathbb{N}} \tilde{H}_\theta(t)$ then the propagator of $H_\theta(t)$ is $U_\theta(t,0) = \bigoplus \tilde{U}_\theta(t,0)$. Thus, $H_\theta(t)$ has $u_1(\theta_1)$ as the corresponding monodromy operator, and given $0 \neq \xi \in \mathcal{H}$ and $\theta = (\theta_1, 0) \in S^1 \times S^1$ one has

$$\begin{aligned} U_{(\theta_1,0)}(t,0)\xi &= \bigoplus_n \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2}-1)\pi\alpha)} & 0 \\ 0 & e^{-i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2}-1)\pi\alpha)} \end{pmatrix} \xi_n \\ &= \bigoplus_n \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2}-1)\pi\alpha)}\xi_n^1 \\ e^{-i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2}-1)\pi\alpha)}\xi_n^2 \end{pmatrix}. \end{aligned}$$

So $t \mapsto U_{(\theta_1,0)}(t,0)\xi$ is not almost periodic. If ξ satisfies $\xi = \oplus\xi_n$ with $\xi_n \neq 0$ if, and only if, $n = l$, then

$$U_{(\theta_1,0)}(t,0)\xi = \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2}-1)\pi\alpha)}\xi_l^1 \\ e^{-i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2}-1)\pi\alpha)}\xi_l^2 \end{pmatrix},$$

and the orbit is precompact since it lives in a finite dimension subspace. In the same way, if ξ is of the form $\xi = \oplus\xi_n$ with $\xi_n \neq 0$ only for finitely many indices n, we have an example of a theoretical quantum model with precompact orbits which are not almost periodic.

3.3. **Quasienergy Operator and Almost Periodic Orbits.** Let $H(t)$ be a general time-dependent Hamiltonian in a Hilbert space $\mathcal{H}$ such that the propagator $U(t,s)$ is well defined. In this case we have defined the quasienergy operator $K = -i\frac{d}{dt} + H(t)$ acting in the extended Hilbert space $\mathcal{K} = L^2(\mathrm{I\!R}, \mathcal{H}, dt)$. It is known [13, 14] that the quasienergy operator and the propagator are connected by the relation

$$(e^{-iK\sigma}f)(t) = U(t,t-\sigma)f(t-\sigma). \tag{4}$$

Let $\mathcal{K}_{\mathrm{p}}(K)$ and $\mathcal{K}_{\mathrm{c}}(K)$ denote, respectively, the point and continuous subspaces of K. We get the following result:

Proposition 3. *Let $\xi \in \mathcal{H}$ be such that $1 \otimes \xi \in \mathcal{K}_{\mathrm{p}}(K)$. Then:*

i) *The map $t \mapsto U(t,0)^{-1}\xi$ is almost periodic.*
ii) *If the eigenvectors of K have the form $\psi_m = 1 \otimes \xi_m$, with $\xi_m \in \mathcal{H}$, then $\xi \in \mathcal{H}_{\mathrm{ap}}$.*

Proof. If $1 \otimes \xi \in \mathcal{K}_{\mathrm{p}}(K)$ then $1 \otimes \xi = \sum_m c_m \psi_m$, with $K\psi_m = \lambda_m \psi_m$. So

$$e^{iK\sigma}(1 \otimes \xi) = \sum_m c_m e^{i\lambda_m \sigma}\psi_m,$$

therefore by (4) for each $t \in \mathrm{I\!R}$,

$$U(t,t+\sigma)\xi = (e^{iK\sigma}(1 \otimes \xi))(t) = \sum_m c_m e^{-i\lambda_m\sigma}\psi_m(t)$$

and we conclude that, for each fixed t, the map $\sigma \mapsto U(t,t+\sigma)\xi$ is almost periodic. In particular taking $t=0$ we obtain that $\sigma \mapsto U(0,\sigma)\xi$ is almost periodic and i) is proved.

Now, if the eigenvectors of K have the form $\psi_m = 1 \otimes \xi_m$, then

$$\begin{aligned} \xi(t) &= U(t,0)\xi = (e^{-iKt}(1 \otimes \xi))(t) \\ &= \sum_m c_m e^{-i\lambda_m t}\psi_m(t) \\ &= \sum_m c_m e^{-i\lambda_m t}\xi_m. \end{aligned}$$

If the sum is finite the map $t \mapsto \xi(t)$ is almost periodic since it is a trigonometric polynomial. If the sum is infinite then $\sum_{m=1}^k c_m e^{-i\lambda_m t}\xi_m \to \sum_{m=1}^\infty c_m e^{-i\lambda_m t}\xi_m$ uniformly as $k \to \infty$ and so the map $t \mapsto \xi(t)$ is almost periodic, that is, $\xi \in \mathcal{H}_{\mathrm{ap}}$, which is ii). □

4. Bounded Energy

In this section we consider time-dependent Hamiltonians $H(t) = H_0 + V(t)$ for which H_0 is a probe operator.

If $\psi_0 \in \text{dom } H_0$ and $\psi(t) = U(t,0)\psi_0$ is the solution of the Schrödinger equation, under which conditions

$$E^0_{\psi_0}(t) = \langle \psi(t), H_0\psi(t) \rangle$$

is a bounded function on t? Also, when

$$E_{\psi_0}(t) = \langle \psi(t), H(t)\psi(t) \rangle$$

is a bounded function? Next we present a set of simple general conditions related to the boundedness of such energy functions.

4.1. **General Systems.**

Proposition 4. *If $V(t)$ is an uniformly bounded family of operators, that is, $\sup_t \|V(t)\| < \infty$, then $E^0_{\psi_0}(t)$ is bounded if, and only if, $E_{\psi_0}(t)$ is bounded.*

Proof. It is sufficient to note that

$$E_{\psi_0}(t) = \langle \psi(t), H(t)\psi(t) \rangle = E^0_{\psi_0}(t) + \langle \psi(t), V(t)\psi(t) \rangle$$

and

$$\sup_t |\langle \psi(t), V(t)\psi(t) \rangle| \le \sup_t \|\psi(t)\|^2 \|V(t)\| = \sup_t \|\psi_0\|^2 \|V(t)\| < \infty.$$

□

Proposition 5. *If $\psi(t) \in C^1(\mathbb{R}; \mathcal{H})$ is almost periodic and $\psi'(t)$ is uniformly continuous, then $E_{\psi_0}(t)$ is bounded.*

Proof. For each $n \in \mathbb{N}^*$ define

$$f_n(t) = n\left[\psi\left(t + \frac{1}{n}\right) - \psi(t)\right] = n \int_t^{t+\frac{1}{n}} \psi'(s)ds.$$

Since ψ is almost periodic it follows that f_n is almost periodic for each n. As $\psi'(t)$ is uniformly continuous, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|s - t| < \delta$ implies $\|\psi'(t) - \psi'(s)\| < \epsilon$. Given $\epsilon > 0$ let $N(\epsilon)$ the smallest integer larger or equal to $\frac{1}{\delta}$; then for all $n > N(\epsilon)$ and $t \in \mathbb{R}$

$$\begin{aligned} \|f_n(t) - \psi'(t)\| &= \left\| n \int_t^{t+\frac{1}{n}} (\psi'(s) - \psi'(t))ds \right\| \\ &\le n \int_t^{t+\frac{1}{n}} \|\psi'(s) - \psi'(t)\| \, ds < \epsilon. \end{aligned}$$

So $f_n \to \psi'$ uniformly and therefore $\psi'(t)$ is almost periodic. Hence $i\psi'(t)$ and $\psi(t)$ are bounded maps. Since

$$E_{\psi_0}(t) = \langle \psi(t), H(t)\psi(t) \rangle = \langle \psi(t), i\frac{d\psi}{dt}(t) \rangle$$

the result follows. □

Note that the boundedness of energy follows if $t \mapsto \psi(t)$ and $t \mapsto \psi'(t)$ are bounded maps. Though well known, it is worth mentioning Proposition 6 in this set of conditions.

Proposition 6. *If $t \mapsto V(t)$ is strongly C^1 and $\psi'(t) \in \mathrm{dom}\ H(t)$ for all t, then:*

(a) *The map $t \mapsto E_\psi(t)$ is differentiable and*
$$\frac{d}{dt}E_\psi(t) = \langle \psi(t), V'(t)\psi(t)\rangle.$$
(b) $|E_\psi(t) - E_\psi(0)| \le t \times \sup_s \|V'(s)\|$.
(c) *If there are $C > 0, a > 1$ so that $\|V'(t)\| \le \frac{C}{(1+|t|)^a}$, then $E_\psi(t)$ and $E^0_\psi(t)$ are bounded functions.*

Proof. (a) $E_\psi(t) = \langle \psi(t), (H_0 + V(t))\psi(t)\rangle$ and so
$$\begin{aligned}\frac{d}{dt}E_\psi(t) &= \langle \psi'(t), H(t)\psi(t)\rangle + \langle \psi(t), H(t)\psi'(t)\rangle + \langle \psi(t), V'(t)\psi(t)\rangle \\ &= \langle \psi'(t), i\psi'(t)\rangle + \langle i\psi'(t), \psi'(t)\rangle + \langle \psi(t), V'(t)\psi(t)\rangle \\ &= \langle \psi(t), V'(t)\psi(t)\rangle.\end{aligned}$$
(b) Since
$$E_\psi(t) - E_\psi(0) = \int_0^t \frac{d}{ds}E_\psi(s)ds = \int_0^t \langle \psi(s), V'(s)\psi(s)\rangle ds$$
the result follows.
(c) Similar to (b). □

A possibility for the proposition above is $V(t) = B_1 \sin t + \frac{B_2}{(1+|t|)^2}$ with $B_1,\ B_2 \in B(\mathcal{H})$ and self-adjoint. From this we see that certainly the choices of ψ depend on B_1, B_2, since $B_1\psi$ and $B_2\psi$ must be kept in suitable domains so that $E_\psi(t)$ is meaningful.

4.2. **Purely Point Systems.** The next result is restricted to periodic time dependence and Floquet operators with nonempty point spectrum (see [11]).

Proposition 7. *Let V be periodic with period T. If the subset $\{\xi_1, \dots, \xi_n\}$ of eigenvectors of U_F is in $\mathrm{dom}\ H_0$ and $t \mapsto \xi_j(t)$ are C^1 maps, then for $\psi = \sum_{j=1}^n a_j\xi_j$, where $a_j \in \mathbb{C}$, $j = 1, \cdots, n$, the map $E_\psi(t)$ is bounded. If, moreover, $V(t)$ are bounded operators and $\sup_t \|V(t)\| < \infty$, then $E^0_\psi(t)$ is also bounded.*

Proof. Suppose $U_\mathrm{F}\xi_j = e^{i\lambda_j}\xi_j$ with $\lambda_j \in \mathbb{R}$, $1 \le j \le n$. We have
$$E_{\xi_j,\xi_k}(t) \doteq \langle \xi_j(t), H(t)\xi_k(t)\rangle = \langle \xi_j(t), i\frac{d}{dt}\xi_k(t)\rangle$$

and so $t \mapsto E_{\xi_j,\xi_k}(t)$ is continuous. Now

$$\begin{aligned} E_{\xi_j,\xi_k}(t+T) &= \langle U(t+T,0)\xi_j, H(t+T)U(t+T,0)\xi_k\rangle \\ &= \langle U(t+T,T)U_{\mathrm{F}}\xi_j, H(t)U(t+T,T)U_{\mathrm{F}}\xi_k\rangle \\ &= e^{-i\lambda_j}e^{i\lambda_k}\langle U(t,0)\xi_j, H(t)U(t,0)\xi_k\rangle \\ &= e^{i(\lambda_k-\lambda_j)}E_{\xi_j,\xi_k}(t) \end{aligned}$$

and then $t \mapsto E_{\xi_j,\xi_k}(t)$ is an almost periodic function. Since for $\psi = \sum_{j=1}^n a_j\xi_j$ we have $E_\psi(t) = \sum_{j,k=1}^n \bar{a}_j a_k E_{\xi_j,\xi_k}(t)$ it follows that $E_\psi(t)$ is almost periodic and so bounded. The second statement follows by Proposition 4. □

According to Proposition 7, in order to get dynamical stability in the periodic case we need conditions assuring the eigenvectors of U_{F} are in dom H_0 and $t \mapsto \xi_j(t)$ to be C^1 functions. We present some sufficient conditions in terms of the quasienergy operator K.

Lemma 1. *Let $\xi \in \mathcal{H}$ be such that $H(t)U(t,s)\xi$ is well defined. Then the map $t \mapsto H(t)U(t,s)\xi$ is a C^r function if, and only if, $t \mapsto e^{i\lambda(t-s)}U(t,s)\xi$ is a C^{r+1} function for fixed $\lambda,\ s \in \mathbb{R}$.*

Proof. Note that

$$\frac{d}{dt}(e^{i\lambda(t-s)}U(t,s)\xi) = i\lambda e^{i\lambda(t-s)}U(t,s)\xi - ie^{i\lambda(t-s)}H(t)U(t,s)\xi.$$

Thus, if $t \mapsto H(t)U(t,s)\xi$ is C^r then $t \mapsto e^{i\lambda(t-s)}U(t,s)\xi$ is C^{r+1} and reciprocally if $t \mapsto e^{i\lambda(t-s)}U(t,s)\xi$ is C^{r+1} then $t \mapsto H(t)U(t,s)\xi$ is C^r. □

Corollary 1. *If $f^{(\lambda)}$ is an eigenvector of K, $Kf^{(\lambda)} = \lambda f^{(\lambda)}$, then the map $t \mapsto f^{(\lambda)}(t)$ is C^r if, and only if, there exists $s \in \mathbb{R}$ so that $t \mapsto H(t)U(t,s)f^{(\lambda)}(s)$ is C^{r-1}.*

Proof. If $Kf^{(\lambda)} = \lambda f^{(\lambda)}$, then by relation (4),

$$e^{-i\lambda\sigma}f^{(\lambda)}(t) = U(t,t-\sigma)f^{(\lambda)}(t-\sigma);$$

so $f^{(\lambda)}(t) = e^{i\lambda\sigma}U(t,t-\sigma)f^{(\lambda)}(t-\sigma)$ for all $\sigma \in \mathbb{R}$. Denoting $t-\sigma = s$ it follows that $f^{(\lambda)}(t) = e^{i\lambda(t-s)}U(t,s)f^{(\lambda)}(s)$ and the result follows by Lemma 1. □

By using relation (4) one can easily show

Lemma 2. *For periodic systems with period T, one has:*

(a) *If $Kf = \lambda f$ then $U_{\mathrm{F}}(s)f(s) = e^{-i\lambda T}f(s)$, $\forall\ s \in \mathbb{R}$.*
(b) *If $U_{\mathrm{F}}(s)\xi_s = e^{-i\lambda T}\xi_s$, $\xi_s \in \mathcal{H}$, $\forall\ s$, then*

$$f_\xi(t) \doteq e^{i\lambda(t-s)}U(t,s)\xi_s \in \text{dom } K$$

and $Kf_\xi = \lambda f_\xi$.

Corollary 2. *(a) If* $H(t+T) = H(t)$*, and* $\xi^{(\lambda)}$ *is an eigenvector of* $U_{\rm F}(s)$*,* $U_{\rm F}(s)\xi^{(\lambda)} = e^{-i\lambda T}\xi^{(\lambda)}$*, then* $\xi^{(\lambda)} \in \mathrm{dom}\ H(s)$ *if, and only if, there exists an eigenvector* $f_{\xi^{(\lambda)}}$ *of* K*,* $Kf_{\xi^{(\lambda)}} = \lambda f_{\xi^{(\lambda)}}$*, with* $t \mapsto f_{\xi^{(\lambda)}}(t)$ *continuous and differentiable.*
(b) In particular, $U_{\rm F}(s)$ *has a basis of eigenvectors in* $\mathrm{dom}\ H(s)$ *if, and only if,* K *has a basis of eigenvectors* $\{f_j\}$ *such that* $t \mapsto f_j(t)$ *is continuous and differentiable for each* j*.*

Proof. (a) Suppose that $\xi^{(\lambda)} \in \mathrm{dom}\ H(s)$. By Lemma 2 $f_{\xi^\lambda}(t) = e^{i\lambda(t-s)}U(t,s)\xi^\lambda \in \mathrm{dom}\ K$ and $Kf_{\xi^\lambda} = \lambda f_{\xi^\lambda}$. Since $\xi^{(\lambda)} \in \mathrm{dom}\ H(s)$ it follows that $U(t,s)\xi^\lambda \in \mathrm{dom}\ H(t)$ and $i\partial_t U(t,s)\xi^\lambda = H(t)U(t,s)\xi^\lambda$. Thus, $t \mapsto f_{\xi^{(\lambda)}}(t)$ is continuous and differentiable.

Reciprocally, it there exists an eigenvector $f_{\xi^{(\lambda)}}$ of K with $t \mapsto f_{\xi^{(\lambda)}}(t)$ continuous and differentiable, then $f_{\xi^\lambda}(t) = e^{i\lambda(t-s)}U(t,s)\xi^\lambda$ and $Kf_{\xi^{(\lambda)}} = \lambda f_{\xi^{(\lambda)}}$ implies $-i\partial_t f_{\xi^\lambda}(t) + H(t)f_{\xi^\lambda}(t) = \lambda f_{\xi^\lambda}(t)$; therefore, $\xi^\lambda \in \mathrm{dom}\ H(s)$.

(b) It is a directly consequence of (a). □

4.3. **Jauslin-Lebowitz Formulation.** We want to study an analogue of the expectation value of probe operators $A : \mathrm{dom}\ A \subset \mathcal{H} \to \mathcal{H}$ on the formulation presented by Jauslin and Lebowitz [16, 2] briefly recalled in the Introduction. If the generalized quasienergy operator $\tilde{K}$ has pure point spectrum, there exists an orthonormal basis $B \doteq \{f_n\}_{n=1}^{\infty}$ of $\tilde{\mathcal{K}}$ with $\tilde{K}f_n = \lambda_n f_n$. By Theorem 4.2 in [16], if $f = 1 \otimes \xi$ is in the point subspace of $\tilde{K}$ the function $t \mapsto U_\theta(t,0)\xi$ is almost periodic a.e. θ with respect to the ergodic measure μ on the compact manifold Ω (see Section 1).

Denote

$$B_{n,m}(A) \doteq \int_\Omega \langle f_n(\theta), Af_m(\theta)\rangle_{\mathcal{H}}\, d\mu(\theta) = \langle f_n, (1\otimes A)f_m\rangle_{\tilde{\mathcal{K}}}\,.$$

If $f \in \tilde{\mathcal{K}}$ then $f = \sum_n a_n f_n$, with $\sum_n |a_n|^2 = \|f\|^2_{\tilde{\mathcal{K}}}$. For each time t, consider the average over Ω of the expectation value of A, that is,

$$\begin{aligned} A_f(t) &\doteq \int_\Omega \langle U_\theta(t,0)f(\theta), AU_\theta(t,0)f(\theta)\rangle_{\mathcal{H}}\, d\mu(\theta) \\ &= \int_\Omega \langle(\mathcal{F}_t e^{-i\tilde{K}t}f)(\theta), A(\mathcal{F}_t e^{-i\tilde{K}t}f)(\theta)\rangle_{\mathcal{H}} d\mu(\theta) \end{aligned}$$

$$\begin{aligned}
&= \left\langle \mathcal{F}_t e^{-i\tilde{K}t} f, (1 \otimes A)\mathcal{F}_t e^{-i\tilde{K}t} f \right\rangle_{\tilde{\mathcal{K}}} \\
&= \left\langle e^{-i\tilde{K}t} f, (1 \otimes A) e^{-i\tilde{K}t} f \right\rangle_{\tilde{\mathcal{K}}} \\
&= \sum_{n,m} \overline{a_n} a_m e^{-it(\lambda_m - \lambda_n)} \langle f_n, (1 \otimes A) f_m \rangle_{\tilde{\mathcal{K}}} \\
&= \sum_{n,m} \overline{a_n} a_m e^{-it(\lambda_m - \lambda_n)} B_{n,m}(A).
\end{aligned}$$

Note that if this sum is absolutely convergent then $A_f(t)$ is a bounded and almost periodic function of t, and

$$t \mapsto \langle U_\theta(t,0) f(\theta), A U_\theta(t,0) f(\theta) \rangle_{\mathcal{H}}$$

is bounded a.e. θ. We conclude

Proposition 8. *If $f = \sum_{j=1}^m a_j f_j$, where f_j are eigenvectors of $\tilde{K}$ and $f_j(\theta) \in \mathrm{dom}\ A$, for all θ, then $t \mapsto A_f(t)$ is a bounded and almost periodic function. Moreover,*

$$t \mapsto \langle U_\theta(t,0) f(\theta), A U_\theta(t,0) f(\theta) \rangle_{\mathcal{H}}$$

is bounded for almost every θ.

More generally we obtain the following result:

Theorem 6. *Suppose that Ω is a compact manifold, $g_t : \Omega \to \Omega$ a C^1 flow with $\sup_{t,\theta} \|\partial_t g_t(\theta)\| < \infty$, and $\tilde{K} f^{(\lambda)} = \lambda f^{(\lambda)}$ with $\theta \mapsto f^{(\lambda)}(\theta)$ a C^1 map. Then for μ almost every θ one has $U_\theta(t,0) f^{(\lambda)}(\theta) \in \mathrm{dom}\ H_\theta(t)$ and*

$$\left\langle U_\theta(t,0) f^{(\lambda)}(\theta), H_\theta(t) U_\theta(t,0) f^{(\lambda)}(\theta) \right\rangle$$

is a bounded function of t. Moreover, if $H_\theta(t) = H_0 + V(g_t(\theta))$ with $V(g_t(\theta))$ bounded and $\sup_{t,\theta} \|V(g_t(\theta))\| < \infty$, then the energy expectation

$$\left\langle U_\theta(t,0) f^{(\lambda)}(\theta), H_0 U_\theta(t,0) f^{(\lambda)}(\theta) \right\rangle$$

is also bounded.

Proof. Since $\tilde{K} f^{(\lambda)} = \lambda f^{(\lambda)}$ then $f^{(\lambda)}(\theta) \in \mathrm{dom}\ H_\theta(0)$ a.e. θ and therefore $U_\theta(t,0) f^{(\lambda)}(\theta) \in \mathrm{dom}\ H_\theta(t)$ a.e. θ. On the other hand

$$U_\theta(t,0) f^{(\lambda)}(\theta) = \mathcal{F}_t e^{-i\tilde{K}t} f^{(\lambda)}(\theta) = \mathcal{F}_t e^{-i\lambda t} f^{(\lambda)}(\theta) = e^{-i\lambda t} f^{(\lambda)}(g_t(\theta))$$

and from the differentiability hypothesis it follows that

$$i \frac{\partial}{\partial t} U_\theta(t,0) f^{(\lambda)}(\theta) = \lambda e^{-i\lambda t} f^{(\lambda)}(g_t(\theta)) + i e^{-i\lambda t} \frac{d}{d\theta} f^{(\lambda)}(g_t(\theta)) \frac{d}{dt} g_t(\theta),$$

which implies that

$$i \frac{\partial}{\partial t} U_\theta(t,0) f^{(\lambda)}(\theta) = H_\theta(t) U_\theta(t,0) f^{(\lambda)}(\theta)$$

is bounded and the first part of the result is proved. The second one follows as in Proposition 4. □

Corollary 3. *Suppose the hypotheses of the above theorem hold and that for each eigenvector $f^{(\lambda_n)} \in \tilde{\mathcal{K}}$ the function $\theta \mapsto f^{(\lambda_n)}(\theta)$ is C^1. Then for μ almost every θ and for all vectors $\xi \in \mathcal{H}$ of the form*

$$\xi = a_1 f^{(\lambda_1)}(\theta) + \ldots + a_k f^{(\lambda_k)}(\theta),$$

the expectation value of the energy

$$\langle U_\theta(t,0)\xi, H_\theta(t) U_\theta(t,0)\xi\rangle$$

is a bounded function.

In case $\xi = \sum_{n=1}^{\infty} a_n f^{(\lambda_m)}(\theta)$ with $\sum |a_n|^2 < \infty$, a sufficient condition for $U_\theta(t,0)\xi \in \operatorname{dom} H_\theta(t)$ and bounded energy is

$$\sum_{j=1}^{\infty} |a_j| \left(|\lambda_j| + \sup_\theta \|\partial_\theta f^{\lambda_j}(\theta)\| \right) < \infty,$$

since this implies that

$$t \mapsto U_\theta(t,0)\xi = \sum_{j=1}^{\infty} a_j e^{-i\lambda_j t} f^{\lambda_j}(g_t(\theta))$$

is a C^1 function and $i\partial_t U_\theta(t,0)$ is bounded.

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