Non-uniqueness of Gibbs measures relative to Brownian motion

Volker Betz*
Mathematics Institute, University of Warwick,
Coventry CV4 7AL, United Kingdom
v.m.betz@warwick.ac.uk

Olaf Wittich
Department of Mathematics and Computer Science, TU Eindhoven
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.wittich@tue.nl

July 10, 2007

Abstract

We consider Gibbs measures relative to Brownian motion of Feynman-Kac type, with single site potential V. We show that for a large class of V, including the Coulomb potential, there exist infinitely many infinite volume Gibbs measures.

1 Introduction

Gibbs measures relative to Brownian motion have originally been introduced as a tool to study certain models of rigorous quantum field theory [18, 24]. They have seen growing interest in recent years, both in their relation to quantum theory [14, 3] and in their own right, cf. [17, 5, 6, 13, 21] for a few examples.

In the present paper we are studying the simplest type of Gibbs measure relative to Brownian motion, namely the ones arising from the Feynman-Kac formula. For finite $T > 0$, and $x, y \in \mathbb{R}^d$, we define

$$\mu_T^{x,y}(\mathrm{d}\omega) = \frac{1}{Z_T^{x,y}} \mathrm{e}^{-\int_{-T}^{T} V(\omega(s))\,\mathrm{d}s} \mathcal{W}_{[-T,T]}^{x,y}(\mathrm{d}\omega). \tag{1}$$

Here, $\mathcal{W}_{[-T,T]}^{x,y}$ is conditional Wiener measure [22] on $C([-T,T], \mathbb{R}^d)$, starting in x at time $-T$ and ending in y at time T. $V : \mathbb{R}^d \to \mathbb{R}$ is called the single site potential, and $Z_T^{x,y}$ normalizes μ_T to a probability measure. When we compare (1) to a classical spin system, the point evaluations $\omega(s)$ of Brownian motion take the role of the spins; there is no

*Supported by an EPSRC fellowship EP/D07181X/1

explicit interaction potential between them, but the Brownian motion measure provides an infinitesimal harmonic nearest neighbour coupling. By definition, a Gibbs measure associated to the potential V now is any measure μ on $C(\mathbb{R}, \mathbb{R}^d)$ such that its regular conditional expectations are of the form (1) when the path is fixed outside $[-T, T]$; cf. Definition 2.2.

As in the classical theory of Gibbs measures, the first two questions to answer are existence and uniqueness. Existence is easy in this particular case; a sufficient and well-explored criterion is the existence of a ground state ψ_0 of the associated Schrödinger operator

$$H = -\frac{1}{2}\Delta + V,$$

in which case there is a stationary Gibbs measure μ_{stat}; μ_{stat} is the measure of the stationary diffusion process with drift $\nabla \ln \psi_0$ and unit diffusion matrix.

The question of uniqueness is more subtle. On the one hand, we are dealing with a one-dimensional system, and so one might think that uniqueness automatically holds. This is indeed true when we restrict our attention to the class of stationary measures [19]. It is also true when the associated Schrödinger semigroup e^{-tH} is intrinsically ultracontractive [4]. This is known to be the case for potentials V growing faster than $|x|^a$ but more slowly than $|x|^b$ at infinity, with $2 < a < b < 2a - 2$ [10].

However, in general the fact that the spin space $\mathbb{R}^n$ is not compact gives rise to the possibility of Gibbs measures μ for which $\mathbb{E}_\mu(\omega_t)$ diverges as $|t| \to \infty$. Such measures have been found in a few special cases. The best known example is the harmonic oscillator $V(x) = |x|^2$, which has been studied independently by various authors [20, 9, 1]. Besides this explicitly solvable case, the only other examples we are aware of are due to J. T. Cox [9], who terms them nontrivial entrance laws. He uses the speed measure and scale function method and his results are therefore strictly limited to one space dimension.

The main contribution of the present paper is a new, abstract criterion for the existence of non-stationary Gibbs measures (or, nontrivial entrance laws) for a given potential V, which is not limited to any space dimension and easy to check in many important cases. In essence, all we require is the existence of an eigenfunction $\psi_1(x)$ of H decaying more slowly than the ground state $\psi_0(x)$ at infinity, cf. Definition 3.4. Although the actual proof of non-uniqueness is very short, our criterion is quite powerful, and covers in particular a large class of radial potentials $V(x)$ that grow at most quadratically at infinity. Importantly, this includes potentials that do not grow at all at infinity (i.e. for which the Schrödinger operator has absolutely continuous spectrum), like the Coulomb potential $V(x) = 1/|x|$ in dimension $d = 3$. Thus we are able to improve on the results in [9] even in the one-dimensional case, as in that work the potential was required to grow at infinity faster than $|x|^{2/3}$.

Let us finally remark that Gibbs measures relative to Brownian motion play a role in the study of stochastic partial differential equations; formally, they are just the stationary solutions of the SPDE

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t,u) = -\frac{1}{2}(\nabla V)(X(t,u)) + \frac{1}{2}\Delta_u X(t,u) + \mathcal{W}(t,u), \tag{2}$$

where $\mathcal{W}$ is space-time white noise. In case $u \in \mathbb{R}$ and under somewhat restrictive conditions on V, the connection of (2) with Gibbs measures has been shown by Iwata [15, 16]. In the case when these conditions hold, Iwata proves that the set of Gibbs measures is exactly the set of stationary (in t) measures for (2), but unfortunately his conditions imply that V grows at least quadratically at infinity and thus do not cover the cases that our present results are about. It would be interesting to investigate whether the correspondence between stationary solutions to (2) and Gibbs measures continues to hold when the growth restrictions on the potential V are relaxed.

2 Basic facts and definitions

Let us first fix the notation. When $I \subset \mathbb{R}$ is a finite union of (bounded or unbounded) intervals, we denote by $C(I, \mathbb{R}^d)$ the space of all continuous functions $I \to \mathbb{R}^d$. The σ-field $\mathcal{F}_I$ on $C(I, \mathbb{R}^d)$ is generated by the point evaluations. The same symbol $\mathcal{F}_I$ denotes the σ-field on $\Omega = C(\mathbb{R}, \mathbb{R}^d)$ generated by the point evaluations at time-points inside I. We write $\mathcal{F}$ instead of $\mathcal{F}_{\mathbb{R}}$, $\mathcal{F}_T$ instead of $\mathcal{F}_{[-T,T]}$, and $\mathcal{T}_T$ instead of $\mathcal{F}_{[-T,T]^c}$ for $T > 0$, where $[-T, T]^c$ denotes the complement of $[-T, T]$. For $s, t \in \mathbb{R}$ with $s < t$ and $x, y \in \mathbb{R}^d$ we denote by $\mathcal{W}^{x,y}_{[s,t]}$ the conditional Wiener measure (non-normalized Brownian bridge) starting in x at time s and ending in y at time t. $\mathcal{W}^{x,y}_{[s,t]}$ is a measure on $C([s, t], \mathbb{R}^d)$. For $T > 0$, we write $\mathcal{W}^{x,y}_T$ instead of $\mathcal{W}^{x,y}_{[-T,T]}$. For $\bar{\omega} \in \Omega$, let $\delta^{\bar{\omega}}_T$ be the Dirac measure on $C([-T, T]^c, \mathbb{R}^d)$ concentrated in $\bar{\omega}$. Note that $\delta^{\bar{\omega}}_T$ does not depend on the part of $\bar{\omega}$ inside $[-T, T]$. Finally, we define

$$\mathcal{W}^{\bar{\omega}}_T := \mathcal{W}^{\bar{\omega}(T_1),\bar{\omega}(T_2)}_T \otimes \delta^{\bar{\omega}}_T. \tag{3}$$

We can (and will) regard $\mathcal{W}^{\bar{\omega}}_T$ as a measure on $C(\mathbb{R}, \mathbb{R}^d)$.

Let us now fix the model we are working with. We begin with the potential V.

Definition 2.1 *A measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is said to be in the* **Kato class** *[22] $\mathcal{K}(\mathbb{R}^d)$, if*

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|\leq 1\}} |V(y)|\, dy < \infty \qquad \textit{in case } d = 1,$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{\{|x-y|\leq r\}} g(x-y)|V(y)|\, dy = 0 \qquad \textit{in case } d \geq 2.$$

Here,

$$g(x) = \begin{cases} -\ln|x| & \textit{if } d = 2 \\ |x|^{2-d} & \textit{if } d \geq 3. \end{cases}$$

V is locally in Kato class, i.e. in $\mathcal{K}_{\text{loc}}(\mathbb{R}^d)$, if $V 1_K \in \mathcal{K}(\mathbb{R}^d)$ for each compact set $K \subset \mathbb{R}^d$. V is **Kato decomposable** *[7] if*

$$V = V^+ - V^- \quad \textit{with} \quad V^- \in \mathcal{K}(\mathbb{R}^d), \quad V^+ \in \mathcal{K}_{\text{loc}}(\mathbb{R}^d),$$

where V^+ is the positive part and V^- is the negative part of V.

Kato-decomposable potentials cover all physically interesting cases known so far. An advantage of assuming Kato-decomposability for the potential V is that a lot is known about the Schrödinger operator $H = -\frac{1}{2}\Delta + V$ and its semigroup e^{-tH} [22]. Most important for our purposes is that for every $t > 0$, e^{-tH} is an integral operator with continuous, bounded kernel K_t, i.e. the map $(t, x, y) \mapsto K_t(x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, bounded if t is bounded away from zero, and the Feynman-Kac formula

$$K_{t-s}(x, y) = \int e^{-\int_s^t V(\omega_r)\, dr}\, d\mathcal{W}_{[s,t]}^{x,y}(\omega) \quad \forall s < t \in \mathbb{R},\ \forall x, y \in \mathbb{R}^d \tag{4}$$

holds. Part of the content of (4) is that the exponential appearing there is actually integrable with respect to the Brownian motion measure. Of course, $K_{t-s}(x, y)$ from equation (4) is just $Z_T^{x,y}$ from equation (1) when $T = |t - s|/2$. This close relationship between the measure μ_T and the associated Schrödinger operator H will be crucial for our arguments.

For Kato-decomposable V and $T > 0$ we now define a probability kernel μ_T from $(\Omega, \mathcal{T}_T)$ to $(\Omega, \mathcal{F})$ by

$$\mu_T(A, \bar{\omega}) := \frac{1}{Z_T(\bar{\omega}_{-T}, \bar{\omega}_T)} \int 1_A(\omega) e^{-\int_{-T}^T V(\omega_s)\, ds}\, d\mathcal{W}_T^{\bar{\omega}}(\omega) \quad (A \in \mathcal{F}, \bar{\omega} \in \Omega). \tag{5}$$

Definition 2.2 *A probability measure μ over Ω is a* **Gibbs measure relative to Brownian motion** *for the potential V if for each $A \in \mathcal{F}$ and $T > 0$,*

$$\mu_T(A, \cdot) = \mu(A|\mathcal{T}_T) \qquad \mu\text{-almost surely}, \tag{6}$$

where $\mu(A|\mathcal{T}_T)$ denotes conditional expectation given $\mathcal{T}_T$.

Equation (6) is the continuum analog to the DLR equations in the lattice context.

Let us now assume that the bottom of the spectrum of H is an eigenvalue. Then necessarily it is of multiplicity one by the Perron-Frobenius theorem, and the corresponding eigenvector ψ_0 can be chosen strictly positive. It is checked directly that if we define a probability measure μ on $(\Omega, \mathcal{F})$ (i.e., a stochastic process) by putting

$$\mu(A) = \int dx\, \psi_0(x) \int dy\, \psi_0(y) \int 1_A(\omega) e^{-\int_{-T}^T V(\omega_s)\, ds}\, d\mathcal{W}_T^{x,y}(\omega) \tag{7}$$

for $A \in \mathcal{F}_T$, we can extend it to a measure on $\mathcal{F}$, which is a Gibbs measure for the potential V. However, we are interested just in the Gibbs measures that are different from μ, and so we need a method for constructing them. Fortunately, a well-known method from the theory of lattice Gibbs measures [12] can be adapted directly to our context, cf. [2].

Definition 2.3 *A family μ_T is locally uniformly dominated if for each $S < \infty$ there exists a probability measure ν on $\mathcal{F}_{[-S,S]}$ such that*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall A \in \mathcal{F}_{[-S,S]} : \nu_S(A) < \delta \Rightarrow \limsup_{T\to\infty} \mu_T(A) < \varepsilon.$$

Lemma 2.4 *[12, 2] Let (T_n) be a monotone sequence with $T_n \to \infty$ as $n \to \infty$. If $(\mu_{T_n})_{n>0}$ is uniformly locally dominated, then it has a cluster point as $n \to \infty$, in the local strong topology (setwise convergence on sets $A \in \mathcal{F}_{[-S,S]}$ for each S). This cluster point is itself a probability measure, and an infinite volume Gibbs measure for the potential V.*

In order to prove non-uniqueness, we will now look for a sequence of measures μ_T that has a cluster point which is different from the stationary process.

3 A non-uniqueness criterion

We will work with the standing assumption that the bottom of the spectrum of $H = -\frac{1}{2}\Delta + V$ is an eigenvalue. We add a constant to V such that this eigenvalue is 0, and write ψ_0 for the eigenvector, which we refer to as the ground state.

Usually when one looks for a sequence of finite volume measures converging to a Gibbs measure, one fixes the boundary conditions, possibly dependent on the volume, and sends the volume to infinity. In our case this would amount to studying the measures

$$\mu_T^{\bar{\omega}}(\mathrm{d}\omega) = \frac{1}{Z_T} \mathrm{e}^{-\int_{-T}^{T} V(\omega_s)\,\mathrm{d}s} \mathcal{W}_T^{\bar{\omega}}(\mathrm{d}\omega), \tag{8}$$

for a fixed function $\bar{\omega} : \mathbb{R} \to \mathbb{R}^d$. If the family $\mu_T^{\bar{\omega}}$ has a cluster point as $T \to \infty$, then by the Markov property it is immediate that this cluster point satisfies the DLR equations and is thus an infinite volume Gibbs measure. (8) is indeed the right expression to look at if we are interested in showing uniqueness: in [4] it was shown that if for a set Ω^* of boundary functions $\bar{\omega}$ the measures $\mu_T^{\bar{\omega}}$ converge to the same measure μ, and if Ω^* has full μ measure, then μ is the only Gibbs measure supported on Ω^*.

We are however interested in non-uniqueness rather than uniqueness, and for this purpose it is just as good to fix only the left hand side boundary condition, and leave the other one free. This translates to considering

$$\mu_T^{x(-T),\psi_0}(\mathrm{d}\omega) = \frac{1}{Z_T} \mathrm{e}^{-\int_{-T}^{T} V(\omega_s)\,\mathrm{d}s} \psi_0(\omega_T)\, \mathcal{W}_{-T}^{x(-T)}(\mathrm{d}\omega), \tag{9}$$

where now $\mathcal{W}_{-T}^{x(-T)}$ denotes Brownian motion started in $x(-T)$ at time $-T$. Again, it is immediate from the Markov property that any cluster point as $T \to \infty$ will be a Gibbs measure. The great advantage of the form (9) is that now the normalisation is explicit: from $H\psi_0 = 0$ we conclude

$$Z_T = \int \mathrm{e}^{-\int_{-T}^{T} V(\omega_s)\,\mathrm{d}s}\, \psi_0(\omega_T)\, \mathrm{d}\mathcal{W}_T^{x(-T)}(\omega) = \mathrm{e}^{-2TH} \psi_0(x(-T)) = \psi_0(x(-T)).$$

For the same reason, the image measures of $\mu_T^{x(-T),\psi_0}$ under point evaluations are calculated easily:

$$\begin{aligned} \mu_T^{x(-T),\psi_0}(\omega_0 \in A) &= \frac{1}{\psi(x(-T))} \int \mathrm{e}^{-\int_{-T}^{0} V(\omega_s)\,\mathrm{d}s} \psi_0(\omega_0) 1_A(\omega_0)\, \mathrm{d}\mathcal{W}_{-T}^{x(-T)}(\omega) = \\ &= \frac{1}{\psi_0(x_{-T})} \mathrm{e}^{-TH} (\psi_0 1_A)(x_{-T}). \end{aligned} \tag{10}$$

We should note that $\mu_T^{x(-T),\psi_0}$ is just the solution of the SDE

$$\mathrm{d}X_t = \nabla \ln(\psi_0)(X_t)\,\mathrm{d}t + \mathrm{d}B_t. \tag{11}$$

with starting point $x(t)$ and starting time $-T$. Thus our subsequent study of the measures $\mu_T^{x(T),\psi_0}$ corresponds exactly to the entrance laws studied by Cox [9].

We will now show that knowledge of the quantities appearing in (10) suffices to show local uniform domination. We fix $T \mapsto x(T)$ and write $\mu_T = \mu_T^{x(-T),\psi}$ for simplicity.

Proposition 3.1 *Assume that for each $S > 0$, and for each $\varepsilon > 0$, there exists $M > 0$ such that*

$$\limsup_{T\to\infty} P_{\mu_T}(|\omega_{-S}| > M) < \varepsilon. \tag{12}$$

Then (μ_T) is locally uniformly dominated.

Proof: Pick $\varepsilon > 0$ and choose M according to (12). We claim that the measure

$$\nu(\mathrm{d}\omega) = \frac{1}{\lambda(\{|x|, M\})}\mu_T(\mathrm{d}\omega|\omega_{-S} = x)1_{|x|<M}\mathrm{d}x$$

dominates μ_T uniformly on $\mathcal{F}_{[-S,S]}$. ν is just the solution of (11) started with uniform distribution inside $\{|x| < M\}$. On the other hand, the measure $1_{\{|\omega(-S)|<M\}}\mu_T(\mathrm{d}\omega)$ on $\mathcal{F}_{[-S,S]}$ is just the solution of the same SDE, but started with the distribution $\mu_T(\omega_{-S} \in \mathrm{d}x) \cap \{|x| < M\}$. This distribution has a Lebesgue density given by the integral kernel $K_{T-S}(x_{-T}, y)$ of the Schrödinger semigroup, cf. (4). By the results of [22], this integral kernel is uniformly bounded in $x_{-T}, y \in \mathbb{R}^d$ and $T - S > 1$. Thus $\mu_T(\omega_{-S} \in \mathrm{d}x)$ is absolutely continuous with respect to Lebsegue measure, and the same is true for the solutions of (11) with respective starting distributions. So for $\delta > 0$ there exists $A \in \mathcal{F}_{[-S,S]}$ such that $\nu(A) < \delta$ implies

$$\limsup_{T\to\infty} \mu_T^{x_{-T},\psi}(A \cap \{|\omega(-S) < M|\}) < \varepsilon.$$

Thus $\limsup_{T\to\infty} \mu_T^{x_{-T},\psi}(A) < 2\varepsilon$. □

In many cases, one can even get rid of the "for each S" in the last proposition, and consider only $S = 0$. A sufficient condition is that the diffusion process does not bring paths back from infinity in finite time (see condition (14) below).

Proposition 3.2 *Suppose we can prove that for all $\varepsilon > 0$ there is $R > 0$ such that*

$$\limsup_{T\to\infty} \mu_T(|\omega_0| \geq R) < \varepsilon. \tag{13}$$

Assume further that for all $S > 0$ we find $\alpha > 0$ such that

$$\forall R > 0\, \exists M > 0 : \mu_T\big(|\omega_0| \leq R \,\big|\, |\omega_{-S}| \geq M\big) \leq 1 - \alpha. \tag{14}$$

Then for every $\varepsilon > 0$ and every $S > 0$ we find $R > 0$ such that

$$\limsup_{T\to\infty} \mu_T(|\omega_{-S}| \geq R) < \varepsilon. \tag{15}$$

Proof: Assume that there is $\delta > 0$ such that for all $M > 0$ we have

$$\limsup_{T\to\infty} \mu_T(|\omega_{-s}| \geq M) > \delta.$$

For each $R > 0$ and each $M > 0$, we have

$$\mu_T(|\omega_0| \geq R) \geq \mu_T\big(|\omega_{-S}| \geq M, |\omega_0| \geq R\big) = \mu_T\big(|\omega_0| \geq R \,\big|\, |\omega_{-S}| \geq M\big)\mu_T(|\omega_{-S}| \geq M),$$

and so

$$\limsup_{T\to\infty} \mu_T(|\omega_0| \geq R) \geq \delta\mu_T\big(|x_0| \geq R \,\big|\, |x_{-s}| \geq M\big).$$

Taking M so large that (14) holds, we find $\limsup_{T\to\infty} \mu_T(|\omega_0| \geq R) \geq \delta\alpha$ independently of $R > 0$. Thus (13) cannot hold. □

Condition (14) is necessary, since there are potentials V that bring the path back from infinity in finite time, e.g. those for which e^{-tH} is intrinsically ultracontractive. We refer to Remark 3.6 for a discussion. However, if the potential grows less than quadratically at infinity, (14) holds:

Lemma 3.3 *Assume that there exists a compact set $K \subset \mathbb{R}^d$ and $\alpha < 1$ such that $V(x) \leq |x|^{2\alpha}$ on $\mathbb{R}^d \setminus K$. Then (14) holds.*

Proof: Let $S > 0$ and $R > 0$. It will be enough to show that

$$\mu_T\big(|\omega_0| \leq R \,\big|\, |\omega_{-S}| = y) \to 0$$

as $|y| \to \infty$. Writing $A = 1_{\{|x|\leq R\}}$, we get

$$\begin{aligned}\mu_T\big(|\omega_0| \leq R \,\big|\, |\omega_{-S}| = y\big)^2 &= \left(\frac{1}{\psi_0(y)} \int \mathrm{e}^{-\int_0^S V(\omega_s)\,\mathrm{d}s} \psi_0(\omega_S) 1_A(\omega_S)\, \mathrm{d}\mathcal{W}^y(\omega)\right)^2 \\ &\leq \frac{1}{\psi_0(y)} \int \mathrm{e}^{-2\int_0^S V(\omega_s)\,\mathrm{d}s} \psi_0^2(\omega_S)\, \mathrm{d}\mathcal{W}^y(\omega)\, \mathcal{W}^y(\omega_S \in A)\end{aligned}$$

by the Cauchy-Schwarz inequality. Since V is Kato-decomposable, we have

$$\sup_{y\in\mathbb{R}^d} \int \mathrm{e}^{-2\int_0^t V(\omega_s)\,\mathrm{d}s} \psi_0^2(\omega_t)\, \mathrm{d}\mathcal{W}^y(\omega) < \infty.$$

Under our assumption on V, Carmona's lower bound on ψ_0 [8] reads $\psi_0(y) \geq D\exp(-\delta|y|^{\alpha+1})$ for some $D, \delta > 0$; also, $\mathcal{W}^y(\omega_S \in A) \leq C\exp(-\gamma(\mathrm{dist}(y, A))^2)$ for some $C, \gamma > 0$ since ω_t is Gaussian distributed. Thus there is $y_0 \in \mathbb{R}^n$ with

$$\mu_T\big(|\omega_0| \leq R \,\big|\, |\omega_s| = y\big) \leq C\exp(\delta|y|^{\alpha+1} - \gamma|y - y_0|^2) \to 0$$

as $y \to \infty$, as desired. □

We now state our criterion for non-uniqueness; we need the following

Definition 3.4 *Fix a function $f : \mathbb{R}^d \to \mathbb{R}$ such that $f(x) > 0$ if $|x| > R$ for some $R > 0$. We say that a function g* **dominates f at infinity** *if*

$$\liminf_{|x|\to\infty} \frac{g(x)}{f(x)} = +\infty.$$

We make the following assumption on the potential V:

(A) We assume that $H = -\frac{1}{2}\Delta + V$ has at least one eigenfunction ψ_1 other than its ground state ψ_0, and that ψ_1 dominates ψ_0 at infinity.

Note that due to the continuity of ψ_0 and ψ_1 and the positivity of ψ_0 it follows that ψ_1 can be chosen strictly positive outside of some compact set. We will comment more on condition (A) after stating and proving our main theorem.

Theorem 3.5 *Let $H = -\frac{1}{2}\Delta + V$ be a Schrödinger operator with ground state ψ_0. Assume (14) and (A). Then for the potential V there exist infinitely many Gibbs measures relative to Brownian motion.*

Proof: Put $f = \psi_1/\psi_0$. Then by our assumptions $f(y) > 0$ for $|y| > R$ and there exists a sequence $(y_n) \subset \mathbb{R}^d$ such that $f(y_n) \to \infty$ as $n \to \infty$. We can choose (y_n) such that $f(y_n)$ is monotone. By (10),

$$\begin{aligned} \mu_T^{x(-T),\psi_0}(f(\omega_0)) &= \frac{1}{\psi_0(x(-T))} \mathrm{e}^{-TH}(f\psi_0)(x(-T)) = \frac{\mathrm{e}^{-TH}(\psi_1)(x(-T))}{\psi_0(x(-T))} \\ &= \frac{\mathrm{e}^{-T(E_1-E_0)}\psi_1(x(-T))}{\psi_0(x(-T))} = \mathrm{e}^{-T(E_1-E_0)} f(x(-T)). \end{aligned}$$

Here, $E_1 > E_0$ is the eigenvalue corresponding to the eigenfunction ψ_1. Now we insert our sequence y_n from above and define $T_n = (\ln(f(y_n)) + \alpha)/(E_1 - E_0)$ for some (arbitrary) $\alpha \in \mathbb{R}$. Then we find

$$\mu_{T_n}^{y_n,\psi_0}(f(\omega_0)) = e^{-\alpha} > 0.$$

This shows two things. Firstly, the standard Chebyshev type argument gives

$$\mu_{T_n}^{y_n,\psi_0}(|\omega_0| > R) \le \frac{\mu_{T_n}^{y_n,\psi_0}(f(\omega_0) 1_{|\omega_0|>R})}{\inf\{f(x) : |x| > R\}} \le \frac{\mu_{T_n}^{y_n,\psi_0}(f(\omega_0))}{\inf\{f(x) : |x| > R\}} = \frac{e^{-\alpha}}{\inf\{f(x) : |x| > R\}}.$$

Thus goes to zero as $R \to \infty$, showing existence of a cluster point μ_α of the sequence by Lemma 2.4. μ_α is a Gibbs measure, and $\mu_\alpha(f(\omega_0)) = e^{-\alpha}$ by local convergence of the $\mu_{T_n}^{y_n,\psi_0}$ to μ_α. So obviously, $\mu_\alpha \neq \mu_\beta$ for $\alpha \neq \beta$. Moreover, for the stationary Gibbs measure μ_0 we have $\mu_0(f) = \int \psi_0(x)\psi_1(x)\,\mathrm{d}x = 0$ by orthogonality of the eigenfunctions. Thus $\mu_\alpha \neq \mu_0$. □

3.6 Remark: Together with the results in [4], a fairly complete picture of non-uniqueness emerges: as shown there, potentials giving rise to intrinsically ultracontractive semigroups

e^{-tH} (henceforth called IUC potentials) imply a unique Gibbs measure. A sufficient condition for a potential V to be IUC that its positive part V^+ is bounded from above and below by certain functions growing faster than quadratically. More precisely, if there exist constants $C_1, C_3 > 0$, $C_2, C_4 \in \mathbb{R}$ and a, b with $2 < a < b < 2a - 2$ such that

$$C_1|x|^a + C_2 \leq V^+(x) \leq C_3|x|^b + C_4, \tag{16}$$

then V is an IUC potential [10]. On the other hand, we will show in the next section that a large class of radial potentials that grow quadratically or more slowly at infinity in a regular way give rise to non-uniqueness of the Gibbs measure. So as a rule of thumb, up to quadratic growth at infinity implies non-uniqueness, while faster growth implies uniqueness. In this generality this rule is probably not true and there should exist pathological counter-examples, but to construct them one would need quite delicate methods of proving uniqueness, which are unavailable to date. We add a few more comments about IUC potentials and non-uniqueness in general.

a) By definition the semigroup e^{-tH} is IUC if its kernel fulfils

$$K_t(x, y) \leq C\psi_0(x)\psi_0(y)$$

for all $x, y \in \mathbb{R}^d$. Condition (A) must rule out IUC potentials, and indeed this can be shown: Theorem 4.2.4 of [23] says that IUC implies

$$|\psi_j(x)| \leq c_j\psi_0(x) \tag{17}$$

for each eigenfunction ψ_j. It even provides an (abstract) value for c_j. It would be interesting to know whether (17) in turn implies intrinsic ultracontractivity; since (17) is very closely related to condition (A) not being fulfilled, this would suggest that IUC potentials are basically the only potentials with a unique Gibbs measure, at least in the absence of absolutely continuous spectrum. Now

$$K_t(x, y) = \sum_{j=0}^{\infty} e^{-t(E_j - E_0)}\psi_j(x)\psi_j(y),$$

where E_j are the eigenvalues to the eigenfunctions ψ_j, and therefore (17) implies intrinsic ultracontractivity if the growth of the c_j can be controlled. This is not straigthforward, and we are not aware of any investigation of the matter.

b) Closely related to the above is a conjecture due to Martin Hairer, who suggested that the Gibbs measure is unique if and only if the associated diffusion process brings the path back from infinity in finite time, i.e. it (14) fails. This conjecture is natural and intuitively appealing, but we were unable to prove either direction; the "if" direction should be easier.

c) In the light of Lemma 3.3, Hairer's conjecture would imply that non-uniqueness holds for all potentials growing quadratically or slower at infinity, provided H has a ground state. However, there is one example that might disprove this conjecture: It is well known that for a compactly supported square well potential, the strength of the well can be adjusted such that H has exactly one eigenvalue below the bottom of the absolutely continuous spectrum. At the very least this would mean that our method of proving non-uniqueness breaks down there. Moreover, it might well be that for this potential the only possible outcomes of an infinite volume limit are convergence to the stationary process and divergence. We did not investigate this further, but it would be an interesting example.

d) In Theorem 3.5 we showed existence of infinitely many Gibbs measures for certain potentials. Given the general fact [12] that the set of all Gibbs measures for a given potential forms a simplex, it is clear that already two different extremal Gibbs measures will produce infinitely many Gibbs measures via convex combinations. What would be much more interesting is whether there are infinitely many *extremal* Gibbs measures. This would follow from the μ_α begin mutually singular for different α. By the same calculation as in the proof of Theorem 3.5, we find that

$$\mathbb{E}_{\mu_\alpha}(f(\omega_{-t})) = \mathrm{e}^{-\alpha}\mathrm{e}^{t(E_1-E_0)},$$

which suggests that the the measures μ_α are supported on sets of paths with different limiting behaviour as $t \to -\infty$. However, for a rigorous law of large numbers type argument we do not have sufficient control over the fluctuations of the paths under μ_α as $t \to -\infty$, and so we leave the question whether there are infinitely many extremal Gibbs measures as an open problem.

4 Examples

Let us now consider some examples where we can apply Theorem 3.5. We begin with two cases where infinitely many eigenfunctions of the corresponding Schrödinger operator are known, and thus our information is fairly complete.

Example 1. (Harmonic Oscillator) As already mentioned in the introduction, the harmonic oscillator has been dealt with before, and is the prime example for non-uniqeness of Gibbs measures relative to Brownian motion. However, while existing proofs rely on the full knowledge of the transition semigroup (Mehler's formula), for our criterion we only need information about the first few eigenfunctions. We restrict to dimension $d = 1$ for simplicity. The Hamiltonian is then given by

$$H_V := -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 - \frac{1}{2},$$

and its eigenfunctions are

$$\psi_n = h_n(x)\psi_0(x) \qquad \text{with} \qquad \psi_0(x) = \pi^{-1/4}\, e^{-\frac{x^2}{2}},$$

and where $h_n(x)$ is the Hermite polynomial of degree n. The corresponding eigenvalues are $E_n = n, n = 0, 1, 2, \ldots$. Explicitly, we have

$$h_0(x) = 1, \quad h_1(x) = \sqrt{2}x, \quad h_2(x) = 2x^2 - 1.$$

Thus obviously the ground state is dominated at infinity by ψ_2, but a straightforward application of Theorem 3.5 fails since the assumption of 3.3 is not fulfilled. However, in this simple case we can proceed by direct calculations of second moments: we may write $x^2 = 1/2\, h_2(x) + 1/2\, h_0(x)$, and thus choosing $x_{-T} = e^T$ we get

$$\begin{aligned} \mu_T(\omega(-t)^2) &= \frac{1}{\psi_0(x)} e^{-(T-t)H}(\psi_0(x)x^2)\Big|_{x_T=e^T} = \\ &= 1/2\, h_0(x_T) + 1/2\, e^{-2T} h_2(x_T) = 1/2 + 1 + O(e^{-2T}). \end{aligned}$$

Hence, the sequence of measures is tight. On the other hand, $x = h_1(x)/\sqrt{2}$ and

$$\mu_T(x) = \frac{1}{\psi_0(x)} e^{-TH}(x\psi_0(x))\Big|_{x_T=e^T} = \frac{1}{\sqrt{2}} e^{-T} h_1(x_T) = \frac{1}{\sqrt{2}};$$

but if the Gibbs measure would be the stationary measure ν, we would have $\nu(f) = \langle \psi_0, \psi_0\, f\rangle$ and hence $\nu(x) = 0$.

Our second example is the Coulomb potential in three dimensions, which has not been known before to give rise to nontrivial Gibbs measures. Due to the presence of absolutely continuous spectrum there is little hope of obtaining the full transition semigroup as was possible in the case of the harmonic oscilator. But there are infinitely many explicitly known eigenfunctions, and thus our method is applicable.

Example 2. (Coulomb Potential) The Coulomb potential in dimension $n = 3$ (with normalized zero point energy) is given by the Hamiltonian

$$H_V := -\frac{1}{2}\Delta - \frac{1}{|x|} + \frac{1}{2}.$$

It has the ground state $\psi_0(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{|x|}{2}}$ and corresponding eigenvalue $E_0 = 0$. As is well known, the radially symmetric eigenfunctions (corresponding to angular momentum $l = m = 0$) are given (with $r = |x|$) by

$$\psi_{n-1}(r) = c_n\, e^{-r/(n+1)}\, L_n^1(2r/n), \quad (n \geq 1),$$

with corresponding eigenvalues

$$E_n = \frac{1}{2}\left(1 - \frac{1}{(n+1)^2}\right) = \frac{1}{2}\frac{n(n+2)}{(n+1)^2}.$$

The functions $L_n^1(r) = l_n r^n + O(r^{n-1})$ are the *Legendre polynomials*. We can notice already here that ψ_0 is dominated by ψ_2 at infinity, and applying Theorem 3.5 proves non-uniqueness of Gibbs measures. However, since we have such detailed information about all the eigenfunctions, we can extract some more information.
We consider now the functions $f_n = \psi_n/\psi_0$, and put $x(-T) = \gamma(T)$, where $\gamma(T)$ remains

to be determined for now. Then

$$\begin{aligned}
\mu_T(f_n(\omega_0)) &= \frac{1}{\psi_0(x)}\, e^{-TH}(\psi_0 f_n)\big|_{x_T=\gamma(T)} = e^{-\frac{T}{2}\frac{n(n+2)}{(n+1)^2}} f_n(\gamma(T)) \\
&= c_n\sqrt{\pi}\, e^{-\frac{T}{2}\frac{n(n+2)}{(n+1)^2}} e^{\gamma(T)} e^{-\frac{\gamma(T)}{n+1}}\, L_n^1(2\gamma(T)/n) \\
&= c_n l_n \sqrt{\pi}\, e^{n\left(-\frac{T(n+2)}{2(n+1)^2}+\frac{\gamma(T)}{n+1}+\log(2\gamma(T)/n)\right)}(1+O((\gamma(T))^{-1})).
\end{aligned}$$

Thus, we obtain a finite answer if we can show that there is a choice for $\gamma(T)$ with $\gamma(T)\to\infty$ and at the same time

$$\lim_{T\to\infty}\left[-\frac{T(n+2)}{2(n+1)^2}+\frac{\gamma(T)}{n+1}+\log(2\gamma(T)/n)\right]=0$$

(or to some other constant). For that, we consider the equation

$$AX = B\Gamma + \log(C\Gamma) = B\Gamma + \log\Gamma + \log C \tag{18}$$

with given $A, B, C > 0$. The function $F(\Gamma) = B\Gamma + \log\Gamma + \log C$ is strictly monotone, continuous for $\Gamma > 0$ and increasing with $\lim_{\Gamma\to\infty} F(\Gamma) = \infty$. Hence, we find for all $X > 0$ one and only one $\Gamma(X)$ such that (18) holds. In particular, we can choose $\gamma(T)$ in a way such that even

$$-\frac{T(n+2)}{2(n+1)^2}+\frac{\gamma(T)}{n+1}+\log(2\gamma(T)/n)=0$$

for all $T > 0$ and at the same time $\lim_{T\to\infty}\gamma(T)=\infty$. Thus we have for this choice that

$$\mu_T(f_n) = c_n l_n\sqrt{\pi}\,(1+O((\gamma(T))^{-1})) > 0.$$

On the other hand, if the Gibbs measure ν would be the stationary one, we would have

$$\nu(f_n) = \langle\psi_0, \psi_0\, f_n\rangle_{L^2(\mathbb{R}^3,dx)} = \langle\psi_0,\psi_n\rangle_{L^2(\mathbb{R}^3,dx)} = 0.$$

Thus in order to keep $f_n(\omega_0)$ integrable in the limit, we need to take $\gamma_n(T) = \frac{n+2}{2(n+1)}T + o(T)$. In other words, the allowed speed of pulling depends on order of "moments" we want to be finite in the limiting measure.

Note that the way how a higher eigenfunction dominates the ground state is different in both cases. This is due to the sign of the exponent and we will now see this pattern again in the general case.

For a general radial potential in $\mathbb{R}^3$, we may separate radial and angular variables for the eigenfunctions $\psi_E(r,\Theta) = r\,u_E(r)Y_E(\Theta)$ and the corresponding radial equation is given by

$$u_E'' = 2\left(V(r) - E - \frac{l(l+1)}{r^2}\right)u_E$$

where $l = 0, 1, 2, \ldots$ denotes the angular momentum. It will turn out that we need only consider only $l = 0$, i.e. radially symmetric eigenfunctions. Indeed, in the following example we will restrict the discussion to the case $V_a(x) := \mathrm{sgn}(a)\,|x|^a$ where $-2 < a \le 2$ and

$a \neq 0$ in $\mathbb{R}^3$, but will later remark on how this can be generalized.

Example 3. $\mathbf{sgn(a)\, r^a, -2 < a < 2, a \neq 0.}$ The radial equation for $l = 0$ is given by

$$u_E'' = 2\,(\mathrm{sgn}(a) r^a - E)\, u_E.$$

By imposing the additional boundary condition $u_E(0) = 0$ we achieve that that the corresponding problem is self-adjoint on $L^2(\mathbb{R}^3)$. Self-adjointness then implies the existence of infinitely many bound states with energies $E > 0$ if $a > 0$ and $E < 0$ if $a < 0$. We investigate the asymptotic of the eigenfunctions using their WKB-approximations. By [11], Ch. 2, 6.1 (4), p. 50, the eigenfunctions u_E can be written

$$u_E(r) = (2(V_a(r) - E))^{-1/4} \exp\left(- \int_{r_E}^{r} \sqrt{2(V_a(s) - E)} ds \right) (1 + \varepsilon(r)) \tag{19}$$

for all V_a considered above (cf. [11], Ch. 2, 6.1.3, p. 52) where $\varepsilon(x)$ tends to zero as r tends to infinity.
That implies in the case $a > 0$: Let $E_1 > E_0$ be two eigenvalues with corresponding eigenfunctions u_1 and u_0. Then

$$\begin{aligned} u_1/u_0 &= \left[\frac{V_a - E_0}{V_a - E_1} \right]^{1/4} \exp\left(- \int_{r_E}^{r} (\sqrt{2(V_a - E_1)} - \sqrt{2(V_a - E_0)}) ds \right) (1 + \varepsilon(r)) \\ &= \exp\left(\int_{r_E}^{r} \frac{E_1 - E_0}{\sqrt{2V_a}} (1 + O(V_a^{-3/2})) ds \right) (1 + \varepsilon(x)) \end{aligned}$$

But $\int_{r_E}^{\infty} \frac{du}{r^{a/2}} = \infty$ if and only if $a \leq 2$. Hence in this case, u_1 dominates u_0, and the same is true for the corresponding eigenstates of the Schrödinger operator in $\mathbb{R}^3$. Note that the bound $a \leq 2$ fits neatly with the fact that for $a > 2$, the corresponding potentials give rise to intrinisically ultracontractive semigroups, and thus the Gibbs measure is known to be unique.
In the case $a < 0$, we have infinitely many negative eigenvalues and $0 > E_1 > E_0$ implies $|E_0| > |E_1|$. The WKB-approximation in this case reads

$$\begin{aligned} u_1/u_0 &= \left[\frac{|E_0| - r^a}{|E_1| - r^a} \right]^{1/4} \exp\left(\int_{r_E}^{r} (\sqrt{|E_0|} - \sqrt{|E_1|})(1 + O(s^a)) ds \right) (1 + \varepsilon(r)) \\ &= \left| \frac{E_0}{E_1} \right|^{1/4} \exp\left(\int_{r_E}^{r} (\sqrt{|E_0|} - \sqrt{|E_1|})(1 + O(s^a)) ds \right) (1 + \varepsilon(r)). \end{aligned}$$

Thus, u_1 dominates u_0 also in this case.

Remark. A sufficient condition for the validity of the WKB-approximation formula (19) in an interval $I = [a, \infty)$ is that $V \in C^2(I, \mathbb{R})$, $Q(x) := V(x) - E > 0$ for all $x \in I$ and

$$\frac{1}{8} \frac{Q''}{Q^{3/2}} - \frac{5}{32} \frac{(Q')^2}{Q^{5/2}} \in L^1(I).$$

Thus, the technique just described is not restricted to the potentials considered above.

Although our final example is technically not covered by any of the above examples, it is by now rather straightforward. Its main interest lies in the differences it shows between the "stochastic differential equation" and the Gibbs measures point of view on diffusion processes, as discussed below.

Example 4. Square well. Let $U(x) = -U_0 \, \mathbf{1}_{[-L,L]}(x), \; L, U_0 > 0$ be a finite square potential well, centered at the origin. If U_0 os sufficiently large, the Schrödinger operator has finitely many eigenvalues $E < 0$, and the asymptotics of the associated eigenfunctions is for $x \to \pm\infty$ also given by the WKB-approximation

$$\psi_E(x) = c_E \, e^{\sqrt{2(U_0 - E)}\,|x|}.$$

Hence, the higher eigenstates dominate the ground state as in the preceding examples, and the speed to pull $x(-T)$ to infinity is linear as it was (to leading order) in the Coulomb example. The remarkable point about the present example can be appreciated by looking back to Section 3 where we argued that $\mu_T^{x(-T),\psi_0}$ is just given by the stationary solution of the SDE (11). However, if we would just naively start the SDE (11) at $x(-T) = \alpha T$ for some $\alpha > 0$, then the vast majority of paths would not even reach the support of the potential U by the time $t = 0$ and just perform Brownian motion. In particular, any numerical algorithm relying on purely local information would have to calculate a huge number of paths before even noticing the existence of the potential. On the other hand, the weighting of paths in (9) is a global condition, and so it is clear from the outset that paths that spend long time near the origin will be favoured.

Acknowledgements: We would like to thank Brian Davies for useful remarks on intrinsic ultracontractivity, Vassili Gelfreich for valuable comments on the WKB asymptotics of eigenfunctions, and Martin Hairer and Jochen Voss for stimulating discussions.

References

[1] G. Benfatto, E. Presutti, M. Pulvirenti: DLR Measures for one-dimensional harmonic systems. Z. Wahrscheinlichkeitstheorie verw. Gebiete **41**, 305-312 (1978).

[2] Betz, V.: *Gibbs measures relative to Brownian motion and Nelson's model*, PhD thesis, TU München, 2002

[3] V. Betz, F. Hiroshima, J. Lőrinczi, R. Minlos, H. Spohn: Ground state properties of the Nelson Hamiltonian - A Gibbs measure-based approach. Rev. Math. Phys. **14** No 2, 173-198 (2002)

[4] V. Betz, J. Lőrinczi: Uniqueness of Gibbs measures relative to Brownian motion. Ann. I. H. Poincaré - PR **39**, 877-889 (2003).

[5] Betz, V., Lőrinczi, J. and Spohn, H.: Gibbs measures on Brownian paths: theory and applications. In *Interacting stochastic systems*, Springer, Berlin 2005, 75-102.

[6] Betz, V. and Spohn, H.: A central limit theorem for Gibbs measures relative to Brownian motion. *Probability Theory and Related Fields* **131**, 459-478 (2005)

[7] K. Broderix, D. Hundertmark, H. Leschke: Continuity properties of Schrödinger semigroups with magnetic fields, *Rev. Math. Phys.* **12**, 181-255 (2000).

[8] R. Carmona: Pointwise bounds for Schrödinger eigenstates, *Commun. Math. Phys.* **62**, 97-106 (1978).

[9] J. T. Cox: On one-dimensional diffusions with time parameter set $(-\infty, \infty)$, *Ann. Prob.* **5**, 807-813 (1977).

[10] E. B. Davies, B Simon: Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59**, 335-395 (1984).

[11] M. V. Fedoryuk: Asymptotic Analysis. Springer, Berlin 1993

[12] H.-O. Georgii: Gibbs Measures and Phase Transitions. Berlin, New York: de Gruyter, 1988.

[13] M. Gubinelli: Gibbs measures for self-interacting Wiener paths, to appear in Markov Proc. Rel. Fields (2006)

[14] F. Hiroshima, Functional integral representation of a model in quantum electrodynamics, *Rev. Math. Phys.* **9** (1997), 489–530.

[15] K. Iwata: Reversible measures of a $P(\phi)_1$ time evolution, in Probabilistic Methods in Mathematical Physics, Proc. Taniguchi Symp., Katata-Kyoto, 1985, Academic Press, pp. 195-209.

[16] K. Iwata: An infinite-dimensional stochastic differential equation with state space $C(R)$. Probab. Theory Related Fields **74**, 141–159 (1987)

[17] J. Lőrinczi, R. Minlos: Gibbs measures for Brownian paths under the effect of an external and a small pair potential. J. Stat. Phys. **105**, 607-649 (2001).

[18] E. Nelson: Schrödinger particles interacting with a quantized scalar field, Proceedings of a conference on analysis in function space, Ed. W. T. Martin, I. Segal, MIT Press, Cambridge 1964, p. 87.

[19] G. Royer: Unicité de certaines mesures quasi-invariantes sur $\mathcal{C}(\mathbb{R})$. *Ann. Scient. Ecole Normale Sup., Serie 4,t. 8*, 319-338 (1975).

[20] G. Royer, M. Yor: *Représentation intégrale de certaines mesures quasi-invariantes sur* $\mathcal{C}(\mathbb{R})$ *; mesures extrémales et propriété de Markov*, Ann. Inst. Fourier (Grenoble) **26-2**, 7-24 (1976).

[21] B. Roynette, P. Vallois, M. Yor, Some penalisations of the Wiener measure. *Jpn. J. Math.* **1** (2006), 263–290.

[22] B. Simon: Schrödinger semigroups. *Bull. AMS* **7**, 447-526 (1982).

[23] E. B. Davies: Heat Kernels and Spectral Theory. Cambridge University Press (1998).

[24] H. Spohn: Effective mass of the polaron: a functional integral approach. *Ann. Phys.* **175** No. 2, 278-318 (1987).