Exterior flows at low Reynolds numbers: concepts, solutions, and applications

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Abstract

We discuss fluid structure interaction for exterior flows, for Reynolds numbers ranging from about one to several thousand. New applications demand a better quantitative understanding of the details of such flows, and this has stimulated a revival of interest in this topic. Astonishingly, in spite of the apparent simplicity of low Reynolds number flows, their precise prediction turns out to be numerically demanding even with today's computers. On the basis of simple examples we review the progress that has been made over the recent years in the analysis of such problems through a combined use of techniques from asymptotic analysis, symbolic computation, and computational fluid dynamics, and discuss open problems which one should be able to solve with the techniques presented here.

Contents

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1 Introduction

We discuss fluid structure interaction at low Reynolds numbers. Fluids filling up all of space and flowing past spheres, plates, and cylinders of various cross sections are much studied examples of such situations. Examples of more complicated arrangements are the motion of bubbles rising in a liquid close to a wall and the sedimentation of particles that undergo collisions. In all these cases it turns out to be of great practical importance to be able to determine the forces that the fluid exerts on the structure with good precision. The vertical speed of bubbles rising near a wall depends for example on the drag, and the exact distance from the wall at which the bubbles rise requires one to find the position relative to the wall where the transverse force is zero. This is an example of a computation that turns out to be very delicate, since at low Reynolds numbers the transverse forces typically are orders of magnitude smaller than the forces along the flow. Such questions can therefore only be answered with the help of high precision computations, which, if done by brute force, are excessively costly even with today's computers. Luckily there are better ways: in what follows we review the progress that has been made over recent years in the analysis of such problems through a combination of techniques from asymptotic analysis, symbolic computation, and computational fluid dynamics. We focus this review on the theoretical framework of the analysis. The goal is to present the general ideas of the theory in a self-contained way since it is the basis also for ongoing research. The different sections are written at various levels of mathematical rigor. Section 2 contains a pedagogical introduction to the analysis of the large time asymptotics of solutions of parabolic and elliptic partial differential equations as it has been developed over the last ten years. The analysis is based on simple examples but contains detailed proofs and serves as a basis for the other sections. In particular it provides easy access to the results in Section 3. The section also contains an introduction to the function spaces that have been used successfully for the study of many more difficult problems, including the stationary Navier-Stokes equations. In Section 3 we explain how to apply the techniques of Section 2 in order to obtain precise analytic results for the downstream asymptotics of stationary and time periodic solutions of the Navier-Stokes equations. We give a detailed review of existing results and state conjectures which we expect to be proved using the same techniques. Section 4 describes on a much more heuristic level the connection between the results of Section 3, *i.e.*, the downstream asymptotic behavior of a solution for a given problem, and the asymptotic behavior in other directions. The section also contains a proof of the connection between certain invariant quantities at large distance and the forces that act on the structures. In Section 5 we formulate artificial boundary conditions for flows in two and three dimensions. These artificial boundary conditions are based on the results in Sections 3 and Section 4. This section is essentially self-contained and a reader who is mainly interested in implementing the boundary conditions can directly start reading there. In Section 6 we discuss the numerical scheme that has been used for solving the Navier-Stokes equations with the artificial boundary conditions in some explicit cases. Section 7 contains a review of some numerical results. Section 8, finally, contains an extensive bibliography of related work on low Reynolds number flows. Throughout the article we also discuss open problems. Whenever possible we have formulated these problems in terms of concrete conjectures which one should be able to prove with the techniques of Section 2 and Section 3. We hope that these conjectures will stimulate further research in this direction and will serve as a starting point for important further development.

1.1 Applications

In many practical applications Reynolds numbers are extremely large and the corresponding flows turbulent, and this is the reason why such flows are most intensively studied. In contrast to these cases we will concentrate here on the regime of flows that are either stationary or time periodic. Surprisingly, in spite of this apparent simplicity when compared with turbulent flows, there is little reliable quantitative information available for such cases. This is not without reason since, as indicated above, the precise prediction of the forces turns out to be computationally demanding. Indeed, linearized theories (Stokes, Oseen) provide a good quantitative description (forces determined within an error of one percent, say) only for Reynolds numbers less than one (see Batchelor (1967)), whereas approximation schemes based on some version of boundary layer theory work well only for Reynolds numbers in excess of some fifty thousand (see Carmichael (1981)). For the intermediate regime where neither the viscous nor the inertial forces dominate, the full Navier-Stokes equations need to be solved. However, when truncating for numerical purposes an infinite exterior domain to a finite sub-domain one is confronted with the problem of finding appropriate boundary conditions on the outer boundary of this sub-domain in order to mimic the boundary conditions at infinity. See for example Heywood et al. (1992), Sergej et al. (2001), Nazarov and Specovius-Neugebauer (2003). It turns out that for the Reynolds numbers under consideration any naïve such choice modifies the hydrodynamic forces significantly unless excessively large computational domains are used (several hundred times the size of the structures). Based on the asymptotic work reviewed in subsequent chapters we present in Section 5 boundary conditions that are simple to implement and allow a significant reduction of the size of the computational domain. Namely, we provide explicit expressions of vector fields that describe the solutions at large distances. These expressions depend on the forces of interest, and these forces can therefore be determined in a self-consistent way as part of the solution process. When compared with other schemes the size of the computational domains that are needed to determine the forces with a given precision are drastically reduced. This leads in turn to an overall gain in computational efficiency of typically several orders of magnitude. We finally note that the topic which is of particular interest here is the study of wakes since, as we will see, the asymptotic behavior of low Reynolds number flows is entirely encoded in the wakes. See also Jaxquim et al. (2003) and Afanasyev and Yakov (2005). In the remainder of this section we provide a bibliographic survey for two main areas of applications: the problem of "flight at low Reynolds numbers", and various problems related to the micro-physics relevant for "climate modeling". The first case includes besides the study of flight as such, the problem of swimming at low Reynolds numbers and questions related to flow control. The second case regroups the bibliography for questions related to the free fall of small particles like ice crystals in clouds and the sedimentation of particles in the oceans that may in addition undergo collisions.

1.2 Flight at low Reynolds numbers

The wish to construct aircraft that can fly at low Reynolds numbers is not new. Important publications which are concerned with the design of low speed airfoils and which are still of relevance today are due to Werle (1974), Eppler and Somers (1979), Eppler and Somers (1981), Lissaman (1983), Eppler and Somers (1985), and Ladson et al. (1996). When these papers were written, Reynolds numbers of the order of fifty to hundred thousand were considered small, but nowadays the interest focuses on flows with Reynolds number in the range from as low as some hundred to several thousand. Quite recently a new experimental facility has been built with the specific goal of measuring flows for this range of Reynolds numbers. See Hanf (2004). Earlier studies are due to Mueller (1999). See also Suhariyono et al. (2006). The goal of these experimental works and of the corresponding theoretical studies is the engineering of so-called micro-air vehicles (MAV). There is an extensive recent literature on the subject. See in particular Pelletier and Mueller (2000), Abdulrahim and Cocquyt (2000), Mueller (2001), and Mueller and DeLaurier (2003). The size of such micro-air vehicles is close to the size of large insects and, even though the discussion of subsequent sections mainly concerns the case of so-called fixed wing vehicles, it is a natural question to study the so-called flapping wing technology which leads in the regime of Reynolds numbers under consideration to time periodic flows. Such questions are extensively studied in Mueller (2001). Interesting information concerning these questions can also be obtained by studying directly insect flight. See for example Dickinson et al. (1999), Wang (2000) and Michelson and Naqvi (2003a).

A typical application of micro-air vehicles are reconnaissance missions, but they are also a much awaited tool for field studies in climate research, where they will allow cheap in situ measurement of various parameters of the atmosphere at variable height. Such measurements are considered an important input for future improvements of climate models. See the next section for more details. Other applications of low Reynolds number flight include the possibility to fly at very low speed at high altitudes, again either for reconnaissance missions or for the study of the physics of high lying clouds for climate research purposes. See for example the work by Greer et al. (1999). At high altitudes Reynolds numbers are small because of the low density of the atmosphere, but at the same time the Mach number is small for the same reason. Subsonic flight at such altitudes is therefore only possible at Reynolds numbers which are again in the range of interest in this review. There is also an increasing interest in flight in the atmosphere on Mars, which leads again to low Reynolds numbers, see Michelson and Naqvi (2003b). Finally there are many questions which are related to, but go beyond what we will be able to discuss here. To mention just a few further examples there is a whole body of research concerning the design of airfoils with high critical Mach number, see Kropinski (1997), questions related to flow control, see Bewley (2001), and questions related to the swimming of certain marine animals like small mollusc, see for example Childress and Dudley (2004) and Avron et al. (2004).

1.3 Climate modeling

There are several examples of low Reynolds number flows that play a role in climate modeling and weather prediction. Of particular importance is the need to predict the terminal velocity of ice crystals and rain drops falling within clouds and in the high atmosphere, as well as the speed of sedimentation of particles and the motion of small bubbles in the oceans. For the case of ice crystals undergoing free fall in the atmosphere the knowledge of their terminal velocity together with a knowledge of the speed of the upwind allows to predict the size of the ice crystals that populate the clouds. This in turn influences the albedo value of these clouds which is an important parameter in climate modeling. See for example Weinstein (1969), Heymsfield (1972), Baker (1997), and Heymsfield and Iaquinta (1999). A related question is the determination of the terminal speed of raindrops that undergoing free fall, see Gunn and Kinzer (1949), Beard and Pruppacher (1969), Foote and du Toit (1969), Sostarecz and Belmonte (2003) and Necasova (2004). Such questions are the reason why even the flow around a sphere as the simplest example of a falling particle is still a subject of interest today in spite of its long history, see Le Clair et al. (1970), Brown and Lawler (2003). Other cases of low Reynolds number flows that are relevant for climate modeling are the description of bubbly flows, see Esmaeeli and Tryggvason (1999) and Esmaeeli and Tryggvason (2000) as well as the prediction of the speed of sedimentation of particles in the ocean, see Yuan and Li (2006). For dense populations of ice crystals the collision between the crystals also needs to be described, see for example Wang and Ji (2000), Lamura et al. (2001) or Ripoll et al. (2004). Further related questions are discussed in Johnson and Wu (1979) and Vaidya (2006).

2 Large time asymptotics

The goal of this section is to provide a simple and self contained introduction to the so-called renormalization group technique which has been developed over the past ten years as a tool to prove existence and to analyze the long time behavior of solutions of nonlinear parabolic and elliptic partial differential equations. An extensive bibliography of the corresponding body of work can be found in Section 2.10. The same ideas can be applied to some extent to hyperbolic equations, but this case will not be discussed here.

Basically, in all cases, the goal will be to show that the dominant asymptotic behavior of the solution of a given nonlinear problem is given by the solution of an appropriate linear problem. The strength of the method is its robustness with respect to the addition of general nonlinear terms (universality). In some cases the appropriate linear problem will be easy to identify, but in other cases already the identification of the correct linear problem is in itself an important step. We will also give an introduction to the techniques needed for this task.

As we will see, the renormalization group technique comes in several flavors. There are two continuous versions and one discrete version. The discrete version is the most robust one inasmuch the addition of nonlinear terms is concerned. One of the continuous versions can be understood as a special case of the discrete version but involving only one iterative step. It is this version that is most useful for elliptic problems. The second continuous version is more ambitious in its aim. It does not only provide existence and an asymptotic analysis of the solutions but offers a geometric description of the results through the construction of attractive invariant manifolds. This version is very appealing because of the elegance of the mathematical description of the results, but it is technically more involved and less robust in its applications.

2.1 Introduction

In what follows we consider the Cauchy problem for the one dimensional heat equation

$$\begin{aligned} \dot{u}(x,t) &= u''(x,t) \ , \\ u(x,t_0) &= u_0(x), \qquad u_0 \in L^1(\mathbf{R},dx) \ . \end{aligned} \tag{1}$$

where $x \in \mathbf{R}$, $t \geq t_0$, and where by definition $\dot{u}(x,t) \equiv \partial_t u(x,t)$ and $u'(x,t) \equiv \partial_x u(x,t)$. For the moment we set $t_0 = 0$, but later we will usually choose $t_0 = 1$. The main question that we want to answer is: what can one say about the limit $u(x,t) \rightarrow_{t\rightarrow\infty}$? "as a function of" u_0? The solution u of (1) is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4t}} \ u_0(y) \ dy \ . \tag{2}$$

Note that for $t > 0$ u is smooth as a function of x, even for non-smooth initial conditions $u_0 \in L^1(\mathbf{R},dx)$. For what follows, the regularity of the initial conditions u_0 will only play a minor role, but the behavior of u_0 at infinity will be essential. From (2) we immediately get the following two basic inequalities

$$\|u_t\|_{L^1} \leq \|u_0\|_{L^1} \ , \tag{3}$$

$$\|u_t\|_{L^\infty} \leq \frac{1}{\sqrt{4\pi t}} \|u_0\|_{L^1} \ , \tag{4}$$

where, by definition, $u_t(x) = u(x,t)$. Inequality (4) implies that u_t converges pointwise to zero at least like $t^{-1/2}$, but if we integrate u_t with respect to x over the whole space we find that

$$\int_{\mathbf{R}} u_t(x) \ dx = \int_{\mathbf{R}} u_0(x) \ dx = a_0 \ , \tag{5}$$

i.e., the constant a_0 is an invariant of the time evolution. These two observations motivate to introduce the scaled functions $\tilde{u}$,

$$\tilde{u}(x,t) = \sqrt{t}u(\sqrt{t}x,t) \ . \tag{6}$$

For the function $\tilde{u}_t$, $\tilde{u}_t(x) = \tilde{u}(x,t)$, we still have the invariant quantity a_0,

$$\int_{\mathbf{R}} \tilde{u}_t(x) \ dx = a_0 \ , \tag{7}$$

but now we have instead of (3) and (4) the inequalities

$$\|\tilde{u}_t\|_{L^1} \leq \|u_0\|_{L^1} \ , \tag{8}$$

$$\|\tilde{u}_t\|_{L^\infty} \leq \frac{1}{\sqrt{4\pi}} \|u_0\|_{L^1} \ , \tag{9}$$

i.e., $\tilde{u}_t$ does not anymore converge pointwise to zero like u_t did. Indeed, the following proposition shows that the function $\tilde{u}_t$, has an interesting long time behavior.

Proposition 1 *(universality) Let $u_0 \in L^1(\mathbf{R}, dx)$, let $\tilde{u}_t$ and a_0 be as defined above, and let $\tilde{u}_{\mathrm{as}}$ be defined by*

$$\tilde{u}_{\mathrm{as}}(x) = \frac{a_0}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} \ . \tag{10}$$

Then

$$\lim_{t\to\infty} \|\tilde{u}_t - \tilde{u}_{\mathrm{as}}\|_{L^\infty} = 0 \ . \tag{11}$$

Remark 2 *The fact that $\tilde{u}_{\mathrm{as}}$ depends on u_0 only through the quantity a_0 is what we mean when we say that the limit is "universal". The number a_0 labels the so-called "universality classes", i.e., all initial conditions with the same value of a_0 have the same limit.*

Remark 3 *Since u is smooth (11) implies in particular that for all $x \in \mathbf{R}$, $\tilde{u}_t(x) \to_{t\to\infty} \tilde{u}_{\mathrm{as}}(x)$, i.e., we have pointwise convergence of $\tilde{u}_t$ to the limit $\tilde{u}_{\mathrm{as}}$.*

Remark 4 *Let, for $t > 0$, $x \in \mathbf{R}$,*

$$u_{\mathrm{as}}(x,t) = \frac{1}{\sqrt{t}} \tilde{u}_{\mathrm{as}}(\frac{x}{\sqrt{t}}) = \frac{a_0}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \ .$$

Since, for any fixed $t \geq 1$,

$$\begin{aligned} \sup_{x\in\mathbf{R}} |\tilde{u}_t(x) - \tilde{u}_{\mathrm{as}}(x)| &= \sup_{x\in\mathbf{R}} |\tilde{u}_t(x/\sqrt{t}) - \tilde{u}_{\mathrm{as}}(x/\sqrt{t})| \\ &= \sqrt{t} \sup_{x\in\mathbf{R}} |(u - u_{\mathrm{as}})(x,t)| \ , \end{aligned}$$

we can write instead of (11) equivalently

$$\lim_{t\to\infty} \sqrt{t} \left(\sup_{x\in\mathbf{R}} |(u - u_{\mathrm{as}})(x,t)| \right) = 0 \ . \tag{12}$$

For all of Section 2 we will use the more compact notation in (11). But starting with Section 3, when we discuss the down stream behavior of the Navier-Stokes equation, we will rather use the notation in (12), since the asymptote will have a more complicated structure involving two different scalings.

We now give a proof of Proposition 1. The interest of this proof is that it will carry over to the nonlinear case with very few modifications. The main tool is the Fourier transform (see for example Stein and Weiss (1975) or Titchmarsh (1937)). Let

$$\hat{u}_0(k) = \mathcal{F}(u_0)(k) = \int_{\mathbf{R}} e^{ikx} u_0(x) \ dx \ . \tag{13}$$

By the Riemann-Lebesgue lemma $\hat{u}_0 \in C_\infty(\mathbf{R})$, the space of continuous functions that converge to zero at infinity. For the invariant quantity a_0 we find

$$a_0 = \hat{u}_0(0) \ .$$

From (13) we immediately get the inequality (a special case of the Hausdorff-Young inequality)

$$\|\hat{u}_0\|_{L^\infty} \leq \|u_0\|_{L^1} \ . \tag{14}$$

We will make extensive use of this inequality in what follows. Let

$$\hat{u}(k,t) = \int_{\mathbf{R}} e^{ikx} u(x,t) \ dx \ . \tag{15}$$

In Fourier space the equation (1) becomes

$$\begin{aligned} \frac{d}{dt}\hat{u}(k,t) &= -k^2 \hat{u}(k,t) \ , \\ \hat{u}(k,0) &= \hat{u}_0(k) \ , \end{aligned} \tag{16}$$

which has the solution

$$\hat{u}(k,t) = e^{-k^2 t}\hat{u}_0(k) \ . \tag{17}$$

Note that all nonzero frequencies k are exponentially damped by the evolution (17). Only a rescaled region near $k = 0$ survives at large times, which is the reason for the scaling $\sqrt{t}$ introduced above. Namely, from (15) we find for the Fourier transform $\widehat{\tilde{u}}$ of the scaled function $\tilde{u}$ the expression

$$\begin{aligned} \widehat{\tilde{u}}(k,t) &= \int_{\mathbf{R}} e^{ikx}\tilde{u}(x,t) \ dx = \sqrt{t}\int_{\mathbf{R}} e^{ikx} u(\sqrt{t}x,t) \ dx \\ &= \int_{\mathbf{R}} e^{i\frac{k}{\sqrt{t}}x} u(x,t) \ dx = \hat{u}(\frac{k}{\sqrt{t}},t) \ . \end{aligned} \tag{18}$$

We therefore define in Fourier space the rescaled function $\widetilde{\hat{u}}$ by

$$\widetilde{\hat{u}}(k,t) = \widehat{\tilde{u}}(k,t) = \hat{u}(\frac{k}{\sqrt{t}},t) \ . \tag{19}$$

From (17) we find for $\widetilde{\hat{u}}$,

$$\widetilde{\hat{u}}(k,t) = e^{-k^2}\hat{u}_0(\frac{k}{\sqrt{t}}) \ , \tag{20}$$

and for the Fourier transform $\widehat{\tilde{u}}_{\rm as}$ of $\tilde{u}_{\rm as}$ we have

$$\widehat{\tilde{u}}_{\rm as}(k) = a_0 e^{-k^2} \ , \tag{21}$$

and we set $\widetilde{\hat{u}}_{\rm as} = \widehat{\tilde{u}}_{\rm as}$. Note that $\widetilde{\hat{u}}_{\rm as} \in L^1(\mathbf{R}, dk)$. Now let $t \geq 1$ and let $\widetilde{\hat{u}}_t(k) = \widetilde{\hat{u}}(k,t)$. First, since $\hat{u}_0$ is a continuous function, we find that $\widetilde{\hat{u}}_t$ converges pointwise to $\widetilde{\hat{u}}_{\rm as}(k)$, *i.e.*, for all $k \in \mathbf{R}$,

$$\widetilde{\hat{u}}_t(k) \to_{t\to\infty} \widetilde{\hat{u}}_{\rm as}(k) \ . \tag{22}$$

Second, the family of functions $\widetilde{\hat{u}}_t$ is uniformly bounded by a function which is in $L^1(\mathbf{R}, dk)$. Namely, using (14) we find that

$$|\widetilde{\hat{u}}_t(k)| \leq \|u_0\|_{L^1} \ e^{-k^2} \ . \tag{23}$$

From (22) and (23) it follows from the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} ||\widetilde{\hat{u}}_t - \widetilde{\hat{u}}_{\rm as}||_{L^1} = 0 \ , \tag{24}$$

and finally, using the Hausdorff-Young inequality for the inverse Fourier transform, we see that (24) implies (11). This completes the proof of Proposition 1.

2.2 Other function spaces: a counter example

In later subsections we will be confronted with initial conditions u_0 that are not in $L^1(\mathbf{R}, dx)$. It is therefore instructive to investigate what can be said about the large time behavior in such cases. So assume for a moment that $u_0 \in L^p(\mathbf{R}, dx)$, for some $1 < p \leq 2$, and define for given $x \in \mathbf{R}$ and $t \geq 1$ the function $g_{x,t}$ by

$$g_{x,t}(y) \equiv e^{-\frac{(x-y)^2}{4t}} .$$

Using Hölder's inequality we find from (2) that u_t satisfies the following pointwise bound,

$$\begin{aligned} |u_t(x)| &\leq \frac{1}{\sqrt{4\pi t}} \left\|g_{x,t} u_0\right\|_{L^1} \leq \left\|g_{x,t}\right\|_{L^q} \left\|u_0\right\|_{L^p} \\ &\leq \frac{1}{\sqrt{4\pi t}} \left(\frac{4\pi t}{q}\right)^{1/q} \left\|u_0\right\|_{L^p} \ , \end{aligned} \tag{25}$$

where $1/q + 1/p = 1$, *i.e.* $q = p/(p-1)$. This shows that we should not in general expect to find (11) for initial conditions that are not in $L^1(\mathbf{R}, dx)$. The following counter example (an adaptation of an example given in Collet and Eckmann (1992b)), shows that it is indeed not obvious how to enlarge the function space of initial conditions without losing Proposition 1.

Proposition 5 *There is an initial condition u_0, with $u_0 \in L^p(\mathbf{R}, dx)$ for all $p > 1$, and an increasing sequence of times T_n, $T_n < T_{n+1}$ with $\lim_{n\to\infty} T_n = \infty$, such that*

$$\lim_{n\to\infty} (-1)^n \tilde{u}(0, T_n) = 1 \ .$$

The proof of Proposition 5 is by explicit construction of an appropriate initial condition u_0. The function u_0 is identically zero for $x < 0$. For $x > 0$ it is a sequence of more and more spread apart positive and negative peaks of compact support. The surface below the first peak is equal to one and below subsequent peaks minus or plus two. The positions of the peaks are such that at adequately constructed times the convolution in (2) is essentially given by the sum of the areas of the first n peaks. The details of the construction are given in Appendix 2.8.

2.3 Power counting, asymptotic expansions

We next discuss the question of what can be said about the limit when t goes to infinity in the case when $a_0 = \int_{\mathbf{R}} u_0(x)\ dx = 0$ and, a related question, what can be said about higher order corrections to the asymptotics.

2.3.1 The case of compact support

Consider an initial condition of compact support, *i.e.*, $u_0 \in L^1(\mathbf{R}, dx) \cap C_0(\mathbf{R})$. In this case the Fourier transform is an entire analytic function. Namely, let $I = [-L, L]$ be a finite interval containing the support of u_0. Then

$$\hat{u}_0(k) = \int_I e^{ikx} u_0(x)\ dx = \sum_{m\geq 0} a_m\, (ik)^m \ , \tag{26}$$

where

$$a_m = \frac{1}{m!} \int_I x^m u_0(x)\ dx \ . \tag{27}$$

Since we have the bound

$$\left| \int_I x^m u_0(x)\ dx \right| \leq L^m \left\|u_0\right\|_1 \ , \tag{28}$$

we find that the sum in (26) is absolutely convergent for all $k \in \mathbf{C}$. Using this representation we find that

$$\widetilde{u}(k,t) = \sum_{m\geq 0} t^{-m/2} a_m (ik)^m e^{-k^2} \ . \tag{29}$$

In direct space we therefore have

$$\tilde{u}(x,t) = \frac{1}{\sqrt{4\pi}} \sum_{m\geq 0} t^{-m/2} a_m (-1)^m \partial_x^m e^{-\frac{x^2}{4}} \ . \tag{30}$$

Note that

$$(-1)^m \partial_x^m e^{-\frac{x^2}{4}} = e^{-\frac{x^2}{4}} H_m(\frac{x}{2}) \ ,$$

with H_m the m^{th} Hermite Polynomial. We conclude that, for initial conditions with compact support, the term proportional to a_0 (which corresponds to the limit in Proposition 1) is nothing else than the first term of a more general expansion of the solution in inverse powers of $\sqrt{t}$. In particular, if $a_0 = 0$ but $a_1 \neq 0$, then the dominant term in the asymptotics will be the one with amplitude a_1, and the decay of the solution will be proportional to t^{-1} instead of $t^{-1/2}$. For initial conditions of non-compact support there still exists an asymptotic expansion for the solution, but the number of terms in this expansion is limited by the decay of the initial condition at infinity, since in particular the numbers a_m in (27) need to exist. This is the content of the next subsection.

2.3.2 Weighted L^p spaces

To illustrate somewhat further the dependence of the results as a function of the decay of the initial condition we consider now for $n = 0, 1, \ldots$ the weighted spaces $L^1(\mathbf{R}, (1+x^2)^{n/2} dx)$. Similar results can be obtained in weighted L^2 spaces. See Wayne (1997) and the appendix in Gallay and Wayne (2002b).

Proposition 6 *(asymptotic expansion) Let $t \mapsto u_t$ be the solution of the heat equation with initial condition $u_0 \in L^1(\mathbf{R}, (1+x^2)^{n/2} dx)$. Let, for $m = 0, \ldots, n$,*

$$\tilde{u}_{\text{as}}^m(x) = \frac{a_m}{\sqrt{4\pi}} (-1)^m e^{-\frac{x^2}{4}} H_m(\frac{x}{2}) \ ,$$

with

$$a_m = \frac{1}{m!} \int_{\mathbf{R}} x^m u_0(x) \ dx \ ,$$

and let $\tilde{u}_t(x) = \sqrt{t} u_t(\sqrt{t} x)$. Then,

$$\lim_{t\to\infty} t^{n/2} || \tilde{u}_t - \sum_{m=0}^{n} t^{-m/2} \tilde{u}_{\text{as}}^m ||_{L^\infty} = 0 \ . \tag{31}$$

Note that for $n = 0$ the Proposition 6 reduces to Proposition 1.

Since $u_0 \in L^1(\mathbf{R}, (1+x^2)^{n/2} dx)$ the functions $x \mapsto x^m u(x)$ are in $L^1(\mathbf{R}, dx)$ for $m = 1, \ldots, n$, and therefore, by the Riemann-Lebesgue lemma, the Fourier transform $\hat{u}_0$ of u_0 is n times continuously differentiable. For the m^{th} derivative of $\hat{u}_0$ we have

$$\hat{u}_0^{(m)}(k) = \int_{\mathbf{R}} e^{ikx} (ix)^m u_0(x) \ dx \ , \tag{32}$$

and therefore

$$a_m = \frac{1}{m!} \lim_{k\to 0} (-i)^m \hat{u}_0^{(m)}(k) \ .$$

Let $t \geq 1$. Expanding in a Taylor series we get

$$\begin{aligned} R_n(k,t) &= t^{n/2} \left(\hat{u}_0(\frac{k}{\sqrt{t}}) - \sum_{m=0}^{n-1} t^{-m/2} a_m (-i)^m k^m \right) \\ &= t^{n/2} \int_0^{k/\sqrt{t}} dk_1 \int_0^{k_1} dk_2 \ldots \int_0^{k_{n-1}} dk_n \ \hat{u}_0^{(n)}(k_n) \\ &= \int_0^{k} dk_1 \int_0^{k_1} dk_2 \ldots \int_0^{k_{n-1}} dk_n \ \hat{u}_0^{(n)}(\frac{k_n}{\sqrt{t}}) \ , \end{aligned} \tag{33}$$

from which we get that pointwise for $k \in \mathbf{R}$

$$\lim_{t\to\infty} R_n(k,t)e^{-k^2} = \frac{k^n}{n!}\hat{u}_0^{(n)}(0)e^{-k^2} = a_n(-ik)^n e^{-k^2} \ . \tag{34}$$

Furthermore, since by (32)

$$|\hat{u}_0^{(n)}(\frac{k}{\sqrt{t}})| \le \int_{\mathbf{R}} |x|^n |u_0(x)| \ dx = \text{const.} < \infty \ ,$$

we get from (33) that

$$|R_n(k,t)e^{-k^2}| \le \text{const.}|k|^n e^{-k^2} \ , \tag{35}$$

i.e., the function $k \mapsto R_n(k,t)e^{-k^2}$ is bounded uniformly in $t \ge 1$ by a function in $L^1(\mathbf{R}, dk)$. From (34) and (35) we conclude by the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} t^{n/2} ||\hat{u}_0(\frac{k}{\sqrt{t}}) - \sum_{m=0}^{n} t^{-m/2} a_m (-i)^m k^m ||_{L^1} = 0 \ . \tag{36}$$

From (36) the Proposition (31) now follows as in the proof of Proposition 1 by using the Hausdorff-Young inequality for the inverse Fourier transform. This completes the proof of Proposition 6.

2.3.3 Power counting

The above discussion motivates to measure the "size" of various functions in inverse powers of $\sqrt{t}$. This will be important when we discuss the nonlinear case on a formal level. In what follows we mean by power counting a formal reference to the above results. Namely, we will summarize these results by saying that, asymptotically as $t \to \infty$, $u \sim t^{-1/2}$, and similarly that $\dot{u} \sim t^{-3/2}$, $u' \sim t^{-1}$ and $u'' \sim t^{-3/2}$.

2.3.4 Higher order terms revisited: formal and asymptotic expansions

Another way to proceed in order to prove the existence of the leading and subleading order terms is to set $u(x,t) = \frac{1}{\sqrt{t}} f(\frac{x}{\sqrt{t}}) + u_1(x,t)$ (we choose $t_0 = 1$ here) and to plug this expression into the equation. In a first step one sets $u_1 \equiv 0$ and gets an ordinary differential equation for f,

$$f''(z) + \frac{1}{2} z f'(z) + \frac{1}{2} f(z) = 0 \ ,$$

which has in particular the solution $f(z) = \tilde{u}_{\text{as}}(z)$, with $\tilde{u}_{\text{as}}$ as defined in (10). With this function f one gets for u_1 again the heat equation but with the initial condition $u_1(x,0) = u_0(x) - \tilde{u}_{\text{as}}(x)$. By definition of $\tilde{u}_{\text{as}}$ one has that $\int_{\mathbf{R}} u_1(x,0) = 0$. Provided the initial condition u_0 decays sufficiently rapidly, higher order terms can be computed in a similar way. Finally, the remainder is estimated by solving the resulting equation in the appropriate function space. This again leads to the construction of asymptotic expansions. This procedure will be the method of choice in the case of nonlinear problems.

2.4 Function spaces

Based on the discussions above we introduce now function spaces directly for the Fourier transforms. These spaces have proved to be well adapted for studying the large time behavior of parabolic and elliptic problems. Related function spaces are used for the case of the Navier-Stokes equations. We set $t_0 = 1$ from now on.

Definition 7 *Let $\alpha \ge 0$. Then, we define $\mathcal{A}_\alpha$ to be the Banach space of continuous, complex valued functions $\hat{f}\colon \mathbf{R} \to \mathbf{C}$, for which the norm $||\hat{f}||_\alpha$ defined by*

$$||\hat{f}||_\alpha = \sup_{k\in\mathbf{R}} (1 + |k|^\alpha)|\hat{f}(k)| \tag{37}$$

is finite.

Remark 8 *For $\alpha > 1$ a function $\hat{f} \in \mathcal{A}_\alpha$ is in $L^q(\mathbf{R}, dk)$ for all $q \geq 1$, and its inverse Fourier transform $f = \mathcal{F}^{-1}(\hat{f})$ is therefore in particular in $C_\infty \cap L^p(\mathbf{R}, dx)$ for $2 \leq p \leq \infty$.*

For functions $\hat{f}$ of $k \in \mathbf{R}$ and $t \geq 1$ we write either $\hat{f}(k,t)$ or $\hat{f}_t(k)$.

Definition 9 *Let α, $\beta \geq 0$ and let $\hat{f}$ be a continuous map from $[1, \infty)$ to $\mathcal{A}_\alpha$. Let*

$$\widetilde{\hat{f}}_t(k) = \hat{f}(\frac{k}{t^{1/2}}, t) \ . \tag{38}$$

Then, we define $\mathcal{B}_{\alpha,\beta}$ to be the Banach space of all such maps for which the norm $||\hat{f}||_{\alpha,\beta}$ defined by

$$||\hat{f}||_{\alpha,\beta} = \sup_{t \geq 1} t^\beta ||\widetilde{\hat{f}}_t||_\alpha \tag{39}$$

is finite.

Note that for all $\hat{f} \in \mathcal{B}_{\alpha,\beta}$ we have for all $k \in \mathbf{R}$ and $t \geq 1$ the bound

$$|\hat{f}(k,t)| \leq \frac{||\hat{f}||_{\alpha,\beta}}{t^\beta} \mu_\alpha(k,t) \ , \tag{40}$$

where

$$\mu_\alpha(k,t) = \frac{1}{1 + (|k|\sqrt{t})^\alpha} \ . \tag{41}$$

Similarly, we find that if a continuous function f satisfies for some constant c the bound

$$|\hat{f}(k,t)| \leq \frac{c}{t^\beta} \mu_\alpha(k,t) \ , \tag{42}$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$, then $\hat{f} \in \mathcal{B}_{\alpha,\beta}$, and $||\hat{f}||_{\alpha,\beta} \leq c$.

2.5 The renormalization group

Let again $t_0 = 1$. Let $\tau > 0$ (typically $\tau \gg 0$) and let $\mathcal{R}_\tau$ be the map that associates to the initial condition u_0 at $t = t_0 = 1$ the rescaled solution $\tilde{u}_t$, at $t = e^\tau$. Explicitly we have in Fourier space

$$\hat{u}(k,t) = e^{-k^2(t-1)} \hat{u}_0(k) \ ,$$

and therefore

$$\widetilde{\hat{u}}(k,t) = e^{-k^2(1-1/t)} \hat{u}_0(\frac{k}{\sqrt{t}}) \ ,$$

so that

$$\mathcal{R}_\tau(\hat{u}_0)(k) = e^{-k^2(1-e^{-\tau})} \hat{u}_0(k e^{-\tau/2}) \ .$$

Note that it follows from the above that $\mathcal{R}_\tau$ is well defined as a map from $\mathcal{A}_\alpha$ to $\mathcal{A}_\alpha$, for all $\alpha \geq 0$. Now let $\hat{u}_1 = \mathcal{R}_{\tau_1}(\hat{u}_0)$ and $\hat{u}_2 = \mathcal{R}_{\tau_2}(\hat{u}_1)$. For the composition $\mathcal{R}_{\tau_2}\mathcal{R}_{\tau_1}$ of the two maps we have

$$\begin{aligned}
(\mathcal{R}_{\tau_2}\mathcal{R}_{\tau_1})(\hat{u}_0)(k) &= \mathcal{R}_{\tau_2}(\hat{u}_1)(k) \\
&= e^{-k^2(1-e^{-\tau_2})} \hat{u}_1(k e^{-\tau_2/2}) \\
&= e^{-k^2(1-e^{-\tau_2})} \left(e^{-k_1^2(1-e^{-\tau_1})} \hat{u}_0(k_1 e^{-\tau_1/2}) \right)\Big|_{k_1 = k e^{-\tau_2/2}} \\
&= e^{-k^2(1-e^{-\tau_2})} e^{-k^2(1-e^{-\tau_1})e^{-\tau_2}} \hat{u}_0(k e^{-\tau_2/2} e^{-\tau_1/2}) \\
&= e^{-k^2(1-e^{-(\tau_1+\tau_2)})} \hat{u}_0(k e^{-(\tau_1+\tau_2)}) \\
&= \mathcal{R}_{\tau_1+\tau_2}(\hat{u}_0)(k) \ .
\end{aligned}$$

This means that $\mathcal{R}_\tau$ has a semi-group structure. This is the so-called Renormalization group which has been enormously successful as a frame of mind, as a way of structuring and organizing the proofs. Note that by construction $\mathcal{R}_\tau(u_{\rm as}) = u_{\rm as}$, *i.e.*, the asymptotic behavior discussed above is recovered here in terms of a fixed point of $\mathcal{R}_\tau$. This is the so-called trivial (Gaussian) fixed point. The rescaled solution $\tilde{u}$ is a (forward) orbit of the Renormalization group, and the above results can be interpreted as saying that the fixed-point $u_{\rm as}$ is attractive.

In the so-called discrete version of the Renormalization group one first constructs a metric space (typically a ball in a Banach space) which, for some large but finite τ, is contracted into itself by $\mathcal{R}_\tau$, which implies the existence of a solution for some large but finite time. Once the existence of the solution is known, enough additional information on this solution is then obtained so that the procedure can be iterated using the semi-group properties, *i.e.*, on analyzes $\mathcal{R}_{n\tau} = \mathcal{R}_\tau^n$ ($n-$fold composition) as $n \in N$ goes to infinity. This is the original technique introduced (for the nonlinear case) by Bricmont and Kupiainen (1994a), Bricmont et al. (1994), Bricmont and Kupiainen (1994b), and Bricmont and Kupiainen (1995), based in part on earlier work by Collet and Eckmann (1992b), Collet and Eckmann (1992a), Collet et al. (1992), Eckmann and Gallay (1993), Eckmann and Wayne (1994), Eckmann and Wayne (1994). See also Pao (1993) and Bona et al. (1994). For higher order asymptotics see for example Bona et al. (1995).

In the simple continuous version of the RG one proves for a (small) set of initial conditions existence and bounds on $\mathcal{R}_\tau$ that are uniform in $\tau \geq 0$. The (attractive) fixed-points of $\mathcal{R}_\tau$ can then be obtained simply by taking the limit $\tau \to \infty$. This is the procedure that we follow below. It has been introduced in Gallay (1994), based on Gallay (1993) and is closest to the standard functional analytic techniques.

The second continuous version of the RG is constructed differently. Instead of studying $\mathcal{R}_\tau$, one sets $u(x,t) = v(\sqrt{t}x, \log(t))$ and analyzes then the semigroup obtained by solving the evolution equation for v rather than the one obtained by solving the equation for u. This leads to a connection with the idea of invariant manifolds and the theory of finite dimensional dynamical systems. See Wayne (1997) for an introduction and Gallay and Wayne (2002b) for an important application.

2.6 Technical lemmas

This section contains the main technical lemmas used in subsequent sections. The propositions are specific to the heat equation but the methods of proof and the basic ideas are independent of the particular case under consideration. It is therefore instructive to give the details of these proofs for this simple case. But, in order to proceed with the general discussion, we have relegated these details to Appendix 2.9 at the end of this section. Here, we only state the results.

The first proposition shows that the function spaces $\mathcal{B}_{\alpha\beta}$ are well adapted for the description of the time evolution generated by the heat equation.

Lemma 10 *Let α, β, $\gamma \geq 0$ with $\alpha + 2\beta \geq \gamma$. Then*

$$e^{-k^2(t-1)} \left(\frac{t-1}{t}\right)^\beta |k|^\gamma \mu_\alpha(k,1) \leq \frac{\text{const.}}{t^{\min\{\beta,\gamma/2\}}} \mu_{\alpha+2\beta-\gamma}(k,t) \ ,$$

uniformly in $t \geq 1$, $k \in \mathbf{R}$, and with μ_α as defined in (41).

The second proposition shows that the scaling built into the function spaces $\mathcal{B}_{\alpha\beta}$ naturally permits to extract the optimal time decay of nonlinear terms. Note that local nonlinear terms in direct space (products of u, u', and u'', say) become convolution products in Fourier space.

Lemma 11 *Let α, $\alpha' > 1$. Then we have*

$$\int_{\mathbf{R}} \mu_\alpha(k-k',t)\mu_{\alpha'}(k',t)dk' \leq \frac{\text{const.}}{\sqrt{t}} \mu_{\min\{\alpha,\alpha'\}}(k,t) \ , \tag{43}$$

uniformly in $t \geq 1$ and $k \in \mathbf{R}$.

The third proposition exhibits properties of the semigroup generated by the heat equation. In particular we have a maximal regularity result in $\mathcal{B}_{\alpha\beta}$ (see (47)), which again shows that the spaces $\mathcal{B}_{\alpha\beta}$ provide a natural functional setting. See for example the book by Lunardi (2003) for a discussion of optimal regularity in parabolic problems.

Lemma 12 *Let $\beta > 1$. For $\alpha \geq 0$, we have the bound,*

$$\int_1^t e^{-k^2(t-s)} \frac{1}{s^\beta} \mu_\alpha(k,s)\ ds \quad \leq \quad \text{const. } \mu_\alpha(k,t)\ , \tag{44}$$

$$\int_1^t \left(e^{-k^2(t-s)} - e^{-k^2(t-1)} \right) \frac{1}{s^\beta} \mu_\alpha(k,s)\ ds \quad \leq \quad \frac{\text{const.}}{t^{\beta-1}}\ \mu_\alpha(k,t)\ , \tag{45}$$

for $\alpha \geq 1$ we have the bound,

$$\int_1^t e^{-k^2(t-s)} \frac{1}{s^\beta} \mu_{\alpha-1}(k,s)\ ds \leq \text{const. } \mu_\alpha(k,t)\ , \tag{46}$$

and for $\alpha \geq 2$ we have the bound (maximal regularity),

$$\int_1^t e^{-k^2(t-s)} \frac{1}{s^\beta} \mu_{\alpha-2}(k,s)\ ds \leq \text{const. } \mu_\alpha(k,t)\ , \tag{47}$$

uniformly in $t \geq 1$ and $k \in \mathbf{R}$.

2.7 Nonlinear problems

We now discuss the simplest non-linear cases. Consider the equation

$$\begin{aligned} \dot{u}(x,t) &= u''(x,t) - u(x,t)^p, \qquad p = 1,2,3\ldots \\ u(x,1) &= u_0(x), \qquad u_0 = \mathcal{F}^{-1}(\hat{u}_0), \text{ with } \hat{u}_0 \in \mathcal{A}_\alpha,\ \alpha > 1\ . \end{aligned} \tag{48}$$

Here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, *i.e.*,

$$\mathcal{F}^{-1}(\hat{u}_0)(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ikx} \hat{u}_0(k)\ dk\ .$$

The strength of the renormalization group method is that it makes no direct reference to the exact form of the nonlinearity. The same technique that we use now first for studying the nonlinearity u^p can then also be used to study a general nonlinearity of the form $F(u, u', u'')$, with F an arbitrary (nonlinear) function that is jointly analytic in its arguments near the origin.

On a formal level, if we assume that the linear heat equation is the relevant problem for the description of the large time asymptotics of the solution of equation (48), *i.e.*, that the nonlinear term becomes negligible at large times t, then we have the following power counting: $u \sim t^{-1/2}$, $\dot{u}, u'' \sim t^{-3/2}$, and $u^p \sim (t^{-1/2})^p = t^{-p/2}$. We therefore find that u^p is indeed negligible when compared with $\dot{u}$ and u'', provided $p > 3$, and we get the following formal classification:

i) $p > 3$, the nonlinearity is "irrelevant" (the trivial fixed point is stable and the linear heat equation is the relevant linear problem at large times).

ii) $p = 3$, the nonlinearity is "marginal" (the trivial fixed point is marginally stable and, depending on the nonlinearity, the linear heat equation is or is not the relevant linear problem).

iii) $p = 1$, 2, the nonlinearity is "relevant" (the trivial fixed-point is unstable, the linear heat equation is not the relevant linear problem).

In what follows we will mainly discuss the point *i)* of this formal classification, since this is the relevant case for the analysis of the Navier-Stokes equation. For the readers interested in the analysis of the cases *ii)* and *iii)* we refer to Bricmont et al. (1994), Bricmont et al. (1996), Bricmont and Kupiainen (1996b), and Uecker (2006).

2.7.1 The case of irrelevant perturbations, I

For the equation (48) we have the following analog of Proposition 1.

Proposition 13 *(universality) Let $p > 3$ and $t \geq 1$ and let $t \mapsto \hat{u}_t$ be the solution of equation (48) with initial condition $\hat{u}_0 \in \mathcal{A}_\alpha$ with $\alpha > 1$. Let $u(x,t) = \mathcal{F}^{-1}(\hat{u}_t)(x)$, and let*

$$\tilde{u}_t(x) = \sqrt{t}u(\sqrt{t}x, t) \ .$$

Define furthermore the function u_{as} by

$$\tilde{u}_{\mathrm{as},p}(x) = \frac{a_{\mathrm{as},p}}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} \ ,$$

where

$$a_{\mathrm{as},p} = \hat{u}_0(0) + \int_1^\infty \hat{u}_s^{*p}(0) \ ds \ . \tag{49}$$

Then,

$$\lim_{t\to\infty} ||\tilde{u}_t - \tilde{u}_{\mathrm{as},p}||_{L^\infty} = 0 \ . \tag{50}$$

Remark 14 *In contrast to Proposition 1, where because of the existence of an invariant quantity the amplitude of the limit could be computed directly from the initial condition, the amplitude (49) involves the solution. There are still universality classes of initial conditions (two initial conditions are equivalent if they have the same limit), but in contrast to the linear case studied above we have to solve the equation in order to know if two initial conditions belong to the same class or not.*

In order to prove Proposition 13 we proceed in two steps. First we prove the existence of a solution, then we analyze its long time behavior. So let $p > 3$ and $t \geq 1$ and let $\hat{u}_0 \in \mathcal{A}_\alpha$ with $\alpha > 1$. We construct solutions of the equation (48) by solving in Fourier space the integral equation

$$\hat{u}(k,t) = e^{-k^2(t-1)}\hat{u}_0(k) + \int_1^t e^{-k^2(t-s)} \ \hat{q}(k,s) \ ds \ , \tag{51}$$

with

$$\hat{q}(k,t) = \hat{u}_t^{*p}(k) \ , \tag{52}$$

with $*$ the convolution product with respect to the variable k. If $\hat{u}$ solves (51), (52), then the inverse Fourier transform,

$$u_t(x) = \mathcal{F}^{-1}(\hat{u}_t)(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-ikx}\hat{u}_t(k) \ dk \ ,$$

solves (48). To prove the existence of a solution to (51), (52) for a given initial conditions $\hat{u}_0$ we will apply the contraction mapping principle to the map $\mathcal{N} = \mathcal{M}\mathcal{L}$, where $\mathcal{L}\colon \hat{q} \mapsto_{(51)} \hat{u}$, is the map that associates to $\hat{q}$ the function $\hat{u}$ using (51), and where $\mathcal{M}\colon \hat{u} \mapsto_{(52)} \hat{q}$, is the map that associates to $\hat{u}$ the function $\hat{q}$ using (52). Note that for $p > 3$ we have that $\beta = (p-1)/2 > 1$.

Proposition 15 *Let $0 < \varepsilon_0$, $\alpha > 1$ and let $\beta = (p-1)/2$. Let $\hat{u}_0 \in \mathcal{A}_\alpha$ with $\|\hat{u}_0\|_\alpha = \varepsilon_0$ and let $\mathcal{U}_{\alpha,\beta}(\varepsilon_0) = \{\hat{q} \in \mathcal{B}_{\alpha,\beta} \mid \|\hat{q}\|_{\alpha,\beta} < \varepsilon_0\}$. Then,*

i) $\mathcal{L}$ is well defined as a map from $\mathcal{B}_{\alpha,\beta}$ to $\mathcal{B}_{\alpha,0}$.

ii) $\mathcal{M}$ is well defined as a map from $\mathcal{B}_{\alpha,0}$ to $\mathcal{B}_{\alpha,\beta}$.

iii) $\mathcal{N}$ is well defined as a map from $\mathcal{B}_{\alpha,\beta}$ to $\mathcal{B}_{\alpha,\beta}$.

iv) $\mathcal{N}(\mathcal{U}_{\alpha,\beta}(\varepsilon_0)) \subset \mathcal{U}_{\alpha,\beta}(\varepsilon_0)$ for ε_0 small enough.

v) For ε_0 small enough, $\|\mathcal{N}(\hat{q}_1) - \mathcal{N}(\hat{q}_2)\|_{\alpha,\beta} \leq \frac{1}{2}\|\hat{q}_1 - \hat{q}_2\|_{\alpha,\beta}$, for all $\hat{q}_1$, $\hat{q}_2 \in \mathcal{U}_{\alpha,\beta}(\varepsilon_0)$.

vi) $\mathcal{N}$ has a unique fixed point in $\mathcal{U}_{\alpha,\beta}(\varepsilon_0)$ for ε_0 small enough.

Since $\mathcal{N} = \mathcal{ML}$ *iii*) follows from *i*) and *ii*). *vi*) follows from *iv*) and *v*) using the contraction mapping principle. We now prove *i*). Throughout all proofs we denote by ε a constant multiple of ε_0. This constant may depend on p and α and may be different from instance to instance. We first show that the function $\hat{u}_L$,

$$\hat{u}_L(k,t) = e^{-k^2(t-1)}\hat{u}_0(k)$$

is in $\mathcal{B}_{\alpha,0}$. Since $\hat{u}_0 \in \mathcal{A}_\alpha$ we have by definition that

$$|\hat{u}_0(k)| \leq \|\hat{u}_0\|_\alpha \ \mu_\alpha(k,1) \leq \varepsilon_0 \mu_\alpha(k,1) \ ,$$

and therefore we find using Lemma 10 that $\hat{u}_L(k,t) \leq \varepsilon\mu_\alpha(k,t)$. Therefore $u_L \in \mathcal{B}_{\alpha,0}$ and $\|\hat{u}_L\|_{\alpha,0} \leq \varepsilon$. Next let

$$\hat{u}_N(k,t) = \int_1^t e^{-k^2(t-s)} \ \hat{q}(k,s) \ ds \ .$$

Since $\hat{q} \in \mathcal{B}_{\alpha,\beta}$ we have that

$$|\hat{q}(k,t)| \leq \|\hat{q}\|_{\alpha,\beta} \, t^{-\beta}\mu_\alpha(k,t) \leq \varepsilon_0 t^{-\beta}\mu_a(k,t) \ ,$$

and therefore $\hat{u}_N \in \mathcal{B}_{\alpha,0}$ by (44) of Lemma 12, and $\|u_L\|_{\alpha,0} \leq \varepsilon$. Since $\hat{u} = \hat{u}_L + \hat{u}_N$ it now follows that $\hat{u} \in \mathcal{B}_{\alpha,0}$ as claimed. Furthermore $||\hat{u}||_{\alpha 0} \leq \varepsilon$ by the triangle inequality. We now prove *ii*). Let $\hat{u} \in \mathcal{B}_{\alpha,0}$. Applying Lemma 11 to the $p-1$ convolutions we get, since $\beta = \frac{p-1}{2}$, that $\mathcal{M}(\hat{u}) \in \mathcal{B}_{\alpha,\beta}$. Furthermore, if $\|\hat{u}\| \leq \text{const.}\varepsilon_0$, then

$$\|\mathcal{M}(\hat{u})\|_{\alpha,\beta} \leq \text{const.}\varepsilon_0^p \ .$$

Therefore, we find that for $\hat{q} \in \mathcal{U}_{\alpha,\beta}(\varepsilon_0)$,

$$\|\mathcal{N}(\hat{q})\|_{\alpha,\beta} \leq \text{const.}\varepsilon_0^p \ .$$

Now since $\text{const.}\varepsilon_0^p < \varepsilon_0$ for ε_0 small enough we find *iv*). To prove *v*) we consider, for $i = 1, 2$, the image $\hat{u}_i = \mathcal{L}(\hat{q}_i)$ of $\hat{q}_i \in \mathcal{U}_{\alpha,\beta}(\varepsilon_0)$. We have already shown that $\|\hat{u}_i\|_{\alpha,0} \leq \varepsilon$, $i = 1, 2$, and using exactly the same techniques one shows that

$$\|\hat{u}_1 - \hat{u}_2\|_{\alpha,0} \leq \text{const.} \, \|\hat{q}_1 - \hat{q}_2\|_{\alpha,\beta} \ .$$

Finally, since $\hat{u}_1^{*p} - \hat{u}_2^{*p} = P(\hat{u}_1, \hat{u}_2) * (\hat{u}_1 - \hat{u}_2)$ for a certain polynomial P of degree $p-1$ (all multiplications are convolution products here), we find that

$$\begin{aligned} \|\mathcal{N}(\hat{q}_1) - \mathcal{N}(\hat{q}_2)\|_{\alpha,\beta} &\leq \text{const.}\varepsilon_0^{p-1} \, \|\hat{q}_1 - \hat{q}_2\|_{\alpha,\beta} \\ &\leq \frac{1}{2} \, \|\hat{q}_1 - \hat{q}_2\|_{\alpha,\beta} \ , \end{aligned}$$

for ε_0 small enough. This completes the proof of Proposition 15.

We can now prove the Proposition 13. The proof is essentially as in the linear case. Let

$$\hat{U}_t(k) = e^{-k^2(t-1)} \left(\hat{u}_0(k) + \int_1^t \hat{u}_s^{*p}(k) \ ds \right) \ ,$$

and let

$$\widetilde{\hat{U}}_t(k) = U_t(\frac{k}{t^{1/2}}) \ .$$

First we note that for any fixed $k \in \mathbf{R}$,

$$\widetilde{\hat{u}}_{\text{as},p}(k) := \lim_{t\to\infty} \widetilde{\hat{U}}_t(k) = a_{\text{as},p} e^{-k^2} = \widehat{\tilde{u}}_{\text{as},p} \ ,$$

and that uniformly in $k \in \mathbf{R}$, and $t \geq 2$, say,

$$|\widetilde{\hat{U}}_t(k)| \leq \text{const.}\mu_\alpha(k,1) \ .$$

Therefore it follows from the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} ||\widetilde{\hat{U}}_t - \widetilde{\hat{u}}_{\mathrm{as},p}||_{L^1} = 0 \ .$$

Applying the Hausdorff-Young inequality to the inverse Fourier transform we find for $\tilde{U}_t = \mathcal{F}^{-1}(\widetilde{\hat{U}}_t)$ that

$$\lim_{t\to\infty} ||\tilde{U}_t - \tilde{u}_{\mathrm{as},p}||_{L^\infty} = 0 \ .$$

Inequality (50) now follows using the triangle inequality, provided

$$\lim_{t\to\infty} ||\tilde{u}_t - \tilde{U}_t||_{L^\infty} = 0 \ . \tag{53}$$

For the difference of $\hat{U}_t$ and $\hat{u}_t$ we have that

$$|\hat{U}_t(k) - \hat{u}_t(k)| \leq \varepsilon \int_1^t (e^{-k^2(t-s)} - e^{-k^2(t-1)})\mu_\alpha(k,s)\frac{ds}{s^\beta} \ ,$$

and therefore we find using (45) in Lemma 12 that

$$|\hat{U}_t(k) - \hat{u}_t(k)| \leq \frac{\varepsilon}{t^{\beta-1}}\mu_\alpha(k,t) \ ,$$

from which it follows, since $\alpha > 1$, that

$$||\widetilde{\hat{u}}_t - \widetilde{\hat{U}}_t||_{L^1} \leq \varepsilon \int_{\mathbf{R}} \frac{1}{t^{\beta-1}}\mu_\alpha(k,1)\ dk \leq \frac{\varepsilon}{t^{\beta-1}} \ .$$

Therefore $\lim_{t\to\infty} ||\widetilde{\hat{u}}_t - \widetilde{\hat{U}}_t||_{L^1} = 0$, which implies (53) by the Hausdorff-Young inequality. This completes the proof of Proposition 13.

2.7.2 The case of irrelevant perturbations, II

We now discuss the case of the nonlinearity $q = uu'$, which is relevant for the Navier-Stokes equations. Namely, we consider the equation

$$\dot{u}(x,t) = u''(x,t) - u(x,t)u'(x,t) \ . \tag{54}$$

According to the preceding power counting we find that $q \sim t^{-1/2}t^{-1} = t^{-3/2}$, *i.e.*, the nonlinearity $q = uu'$ is a priory a marginal perturbation of the heat equation. However, since $uu' = \frac{1}{2}(u^2)'$, and in contrast to the preceding nonlinearity, the present nonlinearity does not destroy the invariance property of the linear heat equation, *i.e.*, the quantity $a_0 = \int_R u_0(x)\ dx$ is preserved by the time evolution. This allows in turn to restrict the equation to the subspace of functions for which $a_0 = 0$. In this subspace the power counting for the linear heat equation is $u \sim t^{-1}$, $u' \sim t^{-3/2}$ and u'',$\dot{u} \sim t^{-2}$, and therefore $q = uu' \sim t^{-5/2}$ which is now an irrelevant perturbation. It is this case which is important for the analysis of the Navier-Stokes equations.

Proposition 16 *(universality) Let $t \mapsto \hat{u}_t$ be the solution of equation (54) for an initial condition $\hat{u}_0$ of the form $\hat{u}_0(k) = ik\hat{v}_0(k)$, with $\hat{v}_0 \in \mathcal{A}_\alpha$ and $\alpha > 3$. Let $u(x,t) = \mathcal{F}^{-1}(\hat{u}_t)(x)$, and let*

$$\tilde{u}_t(x) = t\ u(\sqrt{t}x, t) \ .$$

Define furthermore the function u_{as} by

$$\tilde{u}_{\mathrm{as}}(x) = \frac{a_{\mathrm{as}}}{\sqrt{4\pi}}\frac{x}{2}e^{-x^2/4} \ , \tag{55}$$

where

$$a_{\mathrm{as}} = \hat{v}_0(0) + \int_0^\infty \hat{u}_s^{*2}(0)\ ds \ . \tag{56}$$

Then,

$$\lim_{t\to\infty} ||\tilde{u}_t - \tilde{u}_{\mathrm{as}}||_{L^\infty} = 0 \ . \tag{57}$$

In order to prove Proposition 16 we proceed exactly as in the proof of Proposition 13. First we prove the existence of a solution, then we analyze its long time behavior. We again construct a solution of the equation (54) by solving in Fourier space the corresponding integral equation, which for the present case is

$$\hat{u}_t(k) = ike^{-k^2(t-1)}\hat{v}_0(k) + ik\int_1^t e^{-k^2(t-s)}\ \hat{q}(k,s)\ ds\ , \tag{58}$$

with

$$\hat{q}(k,t) = \hat{u}_t^{*2}(k)\ . \tag{59}$$

If $\hat{u}_t$ solves (58), (59), then the inverse Fourier transform $u_t(x) = \mathcal{F}^{-1}(\hat{u}_t)(x)$ solves (54). To prove the existence of a solution to (58), (59) for a given initial condition $\hat{v}_0$ we apply the contraction mapping principle to the map $\mathcal{N} = \mathcal{ML}$, where $\mathcal{L}\colon \hat{q} \mapsto \hat{u}$ is the map that associates to $\hat{q}$ the function $\hat{u}$ using (58), and where $\mathcal{M}\colon \hat{u} \mapsto \hat{q}$, is the map that associates to $\hat{u}$ the function $\hat{q}$ using (59).

Proposition 17 *Let $0 < \varepsilon_0$, $\alpha > 2$ and let $\beta = 3/2$. Let $\hat{v}_0 \in \mathcal{A}_\alpha$ with $||\hat{v}_0||_\alpha = \varepsilon_0$ and let $\mathcal{U}_{\alpha-1,\beta}(\varepsilon_0) = \{\hat{q} \in \mathcal{B}_{\alpha-1,\beta} \mid \|\hat{q}\|_{\alpha-1,\beta} < \varepsilon_0\}$. Then,*

i) $\mathcal{L}$ is well defined as a map from $\mathcal{B}_{\alpha-1,\beta}$ to $\mathcal{B}_{\alpha-1,1/2}$.

ii) $\mathcal{M}$ is well defined as a map from $\mathcal{B}_{\alpha-1,1/2}$ to $\mathcal{B}_{\alpha-1,\beta}$.

iii) $\mathcal{N}$ is well defined as a map from $\mathcal{B}_{\alpha-1,\beta}$ to $\mathcal{B}_{\alpha-1,\beta}$.

iv) $\mathcal{N}(\mathcal{U}_{\alpha-1,\beta}(\varepsilon_0)) \subset \mathcal{U}_{\alpha-1,\beta}(\varepsilon_0)$ for ε_0 small enough.

v) For ε_0 small enough, $\|\mathcal{N}(\hat{q}_1) - \mathcal{N}(\hat{q}_2)\|_{\alpha-1,\beta} \leq \frac{1}{2}\|\hat{q}_1 - \hat{q}_2\|_{\alpha-1,\beta}$, for all $\hat{q}_1$, $\hat{q}_2 \in \mathcal{U}_{\alpha-1,\beta}(\varepsilon_0)$.

vi) $\mathcal{N}$ has a unique fixed point in $\mathcal{U}_{\alpha-1,\beta}(\varepsilon_0)$ for ε_0 small enough.

Since $\mathcal{N} = \mathcal{ML}$, *iii*) follows from *i*) and *ii*). *vi*) follows from *iv*) and *v*) using the contraction mapping principle. We now prove *i*). Let $u(k,t) = u_t(k)$. We first show that the function $\hat{u}_L$,

$$\hat{u}_L(k,t) = ike^{-k^2(t-1)}\hat{v}_0(k)$$

is in $\mathcal{B}_{\alpha-1,1/2}$. Since $\hat{v}_0 \in \mathcal{A}_\alpha$ we have by definition that

$$|\hat{v}_0(k)| \leq \|\hat{v}_0\|_\alpha\ \ \mu_\alpha(k,1) \leq \varepsilon_0\mu_\alpha(k,1)\ .$$

Therefore, $|ik\hat{v}_0(k)| \leq \varepsilon_0|k|\mu_\alpha(k,1)$, and we find using Lemma 10 that $\hat{u}_L(k,t) \leq \varepsilon\mu_{\alpha-1}(k,t)/t^{1/2}$. Therefore $u_L \in \mathcal{B}_{\alpha-1,1/2}$ and $\|\hat{u}_L\|_{\alpha-1,1/2} \leq \varepsilon$. Next let

$$\hat{u}_N(k,t) = ik\int_1^t e^{-k^2(t-s)}\ \hat{q}(k,s)\ ds\ .$$

Since $\hat{q} \in \mathcal{B}_{\alpha-1,\beta}$ we have that

$$|\hat{q}(k,t)| \leq \|\hat{q}\|_{\alpha-1,\beta}\, t^{-\beta}\mu_{\alpha-2}(k,t) \leq \varepsilon_0 t^{-\beta}\mu_{a-2}(k,t)\ .$$

Therefore we find by (46) of Lemma 12, and using that

$$|k|\mu_a(k,t) \leq \frac{\text{const.}}{t^{1/2}}\mu_{a-1}(k,t)\ ,$$

that $u_N \in \mathcal{B}_{\alpha-1,1/2}$ with $\|u_N\|_{\alpha-1,1/2} \leq \varepsilon$. Since $\hat{u} = \hat{u}_L + \hat{u}_N$ it now follows using the triangle inequality that $\hat{u} \in \mathcal{B}_{\alpha-1,1/2}$ and that $||\hat{u}||_{\alpha-1,1/2} \leq \varepsilon$. We now prove *ii*). Let $\hat{u} \in \mathcal{B}_{\alpha-1,1/2}$. Applying Lemma 11 we find that $\hat{q} = \mathcal{M}(\hat{u}) \in \mathcal{B}_{\alpha-1,\beta}$. Furthermore, if $\|\hat{u}\|_{\alpha-1,1/2} \leq \text{const.}\varepsilon_0$, then $\|\mathcal{M}(\hat{u})\|_{\alpha-1,\beta} \leq \text{const.}\varepsilon_0^2$. Therefore, $\|\mathcal{N}(\hat{q})\|_{\alpha,\beta} \leq \text{const.}\varepsilon_0^2$ for $\hat{q} \in \mathcal{U}_{\alpha-1,\beta}(\varepsilon_0)$. Since $\text{const.}\varepsilon_0^2 < \varepsilon_0$ for ε_0 small enough we find *iv*). To prove *v*) we consider the image of $\hat{q}_i$, $i = 1,2$ in $\mathcal{U}_{\alpha-1,\beta}(\varepsilon_0)$. Let $\hat{u}_i = \mathcal{L}(\hat{q}_i)$, $i = 1,2$. We have already shown that $\|\hat{u}_i\|_{\alpha-1,1/2} \leq \varepsilon$, $i = 1,2$, and using exactly the same techniques one shows that

$$\|\hat{u}_1 - \hat{u}_2\|_{\alpha-1,1/2} \leq \text{const.}\,\|\hat{q}_1 - \hat{q}_2\|_{\alpha-1,\beta}\ .$$

Finally, since $\hat{u}_1^{*2} - \hat{u}_2^{*2} = (\hat{u}_1 + \hat{u}_2) * (\hat{u}_1 - \hat{u}_2)$, we find that

$$\begin{aligned} \|\mathcal{N}(\hat{q}_1) - \mathcal{N}(\hat{q}_2)\|_{\alpha-1,\beta} &\leq \text{const.}\varepsilon_0 \|\hat{q}_1 - \hat{q}_2\|_{\alpha-1,\beta} \\ &\leq \frac{1}{2} \|\hat{q}_1 - \hat{q}_2\|_{\alpha-1,\beta} \ , \end{aligned}$$

for ε_0 small enough. This completes the proof of Proposition 17.

We can now prove the Proposition 16. Let

$$\hat{U}_t(k) = ike^{-k^2(t-1)} \left(\hat{v}_0(k) + \int_1^t \hat{u}_s^{*2}(k)\ ds \right) \ ,$$

and let

$$\widetilde{\hat{U}}_t(k) = t^{1/2} U_t(\frac{k}{t^{1/2}}) \ .$$

First we note that for any fixed $k \in \mathbf{R}$,

$$\widetilde{\hat{u}}_{\mathrm{as}}(k) := \lim_{t\to\infty} \widetilde{\hat{U}}_t(k) = a_{\mathrm{as}} ike^{-k^2} = \widehat{\tilde{u}}_{\mathrm{as}} \ ,$$

with $\tilde{u}_{\mathrm{as}}$ as given in (55), and that uniformly in $k \in \mathbf{R}$, and $t > 2$, say,

$$|\widetilde{\hat{U}}_t(k)| \leq \text{const.}\mu_{\alpha-1}(k,1) \ .$$

Therefore it follows from the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} ||\widetilde{\hat{U}}_t - \widetilde{\hat{u}}_{\mathrm{as}}||_{L^1} = 0 \ .$$

Applying the Hausdorff-Young inequality to the inverse Fourier transform we find for $\tilde{U}_t = \mathcal{F}^{-1}(\widetilde{\hat{U}}_t)$ that

$$\lim_{t\to\infty} ||\tilde{U}_t - \tilde{u}_{\mathrm{as}}||_{L^\infty} = 0 \ .$$

Inequality (57) now follows using the triangle inequality, provided

$$\lim_{t\to\infty} ||\tilde{u}_t - \tilde{U}_t||_{L^\infty} = 0 \ . \tag{60}$$

For the difference of $\hat{U}_t$ and $\hat{u}_t$ we have that

$$|\hat{U}_t(k) - \hat{u}_t(k)| \leq \varepsilon |k| \int_1^t (e^{-k^2(t-s)} - e^{-k^2(t-1)}) \mu_{\alpha-1}(k,s) \frac{ds}{s^\beta} \ ,$$

and therefore we find using (45) in Lemma 12 that

$$|\hat{U}_t(k) - \hat{u}_t(k)| \leq \frac{\varepsilon |k|}{t^{\beta-1}} \mu_{\alpha-1}(k,t) \leq \frac{\varepsilon}{t^{\beta-1/2}} \mu_{\alpha-2}(k,t) \ ,$$

from which it follows that

$$||\widetilde{\hat{u}}_t - \widetilde{\hat{U}}_t||_{L^1} \leq \varepsilon \int_{\mathbf{R}} \frac{1}{t^{\beta-1}} \mu_{\alpha-2}(k,1)\ dk \leq \frac{\varepsilon}{t^{\beta-1}} \ .$$

Therefore $\lim_{t\to\infty} ||\widetilde{\hat{u}}_t - \widetilde{\hat{U}}_t||_{L^1} = 0$, which implies (60) by the Hausdorff-Young inequality. This completes the proof of Proposition 16.

2.7.3 The case of marginal perturbations

According to the power counting scheme there are two critical nonlinear terms, namely $q = u^3$ and $q = uu'$. We discuss these cases here only briefly and refer to the article of Bricmont et al. (1994) for details.

For the case $q = u^3$ there are so-called logarithmic corrections to scaling, *i.e.*, the solution still converges to the Gaussian limit, however not like $1/\sqrt{t}$ but somewhat more rapidly, namely like $1/\sqrt{t \log(t)}$. In order to discuss this case the class of functions spaces $\mathcal{B}_{\alpha\beta}$ has therefore to be generalized to allow for a scaling behavior different from power laws. In addition, it is in this case not possible anymore to analyze the map $\mathcal{L}$ by separating it into a part $\hat{u}_L$ and a part $\hat{u}_N$, since the dominant terms in u_N and u_L compensate each other. But, except for these points, the proof is as above.

The case of the nonlinearity $q = uu'$ is the equation that we have discussed in the preceding subsection for initial conditions satisfying $a_0 = \int_{\mathbf{R}} u_0(x)\, dx = 0$. The same equation but for the case where $a_0 \neq 0$ is known as the dissipative Burgers equation. It has a one parameter family of nontrivial solutions and also front solutions ($a_0 = \infty$ for front solutions), and it has many nontrivial applications. We do not discuss this case here but rather refer the reader again to the papers Bricmont et al. (1994) and Bricmont et al. (1996) and to Uecker (2006) for a very interesting nontrivial application.

2.7.4 The case of relevant perturbations

The case of relevant perturbations is discussed in Bricmont and Kupiainen (1996b). In order to analyze for example the equation

$$\dot{u}(x,t) = u''(x,t) - u(x,t)^2 \tag{61}$$

one first has to identify the relevant linear problem. Indeed, there are interesting scale invariant solutions of (61), but they converge to zero like $1/t$ and not like $1/\sqrt{t}$. One therefore sets $u(x,t) = f(x/\sqrt{t})/t + u_1(x,t)$, and plugs this Ansatz into equation (61). For $u_1 = 0$ one gets a nonlinear ordinary differential equation for f. This equation has two solutions, one that decays like a modified Gaussian at infinity and one decaying like $1/x^2$ at infinity. Both of these functions can be taken as a starting point for an asymptotic analysis, and for appropriate initial conditions it can be shown that u_1 converges to zero faster than $1/t$.

2.7.5 The case of irrelevant perturbations, III

For completeness we present here also the analysis of the non-linear heat equation for the case where the nonlinearity $q = u^p$ in (48) is replaced by $q = uu''$. Formally, $q \sim t^{-1/2} t^{-3/2} = t^{-2}$, and therefore $q = uu''$ is an irrelevant perturbation of the heat equation. However, because of the second derivative its analysis requires the refined inequality (47) of Lemma 12 (maximal regularity).

Proposition 18 *Let $\alpha > 3$ and let $t \mapsto \hat{u}_t$ be the solution of equation (48) with initial condition $\hat{u}_0 \in \mathcal{A}_\alpha$. Let $u(x,t) = \mathcal{F}^{-1}(\hat{u}_t)(x)$, and*

$$\tilde{u}_t(x) = \sqrt{t} u(\sqrt{t} x, t) \ .$$

Define furthermore the function u_{as} by

$$\tilde{u}_{\text{as}}(x) = \frac{a_{\text{as}}}{\sqrt{4\pi}} e^{-x^2/4} \ , \tag{62}$$

where

$$a_{\text{as}} = \hat{u}_0(0) + \int_1^\infty \hat{q}(0,s)\ ds \ , \tag{63}$$

with

$$\hat{q}(k,t) = (\hat{u}_t * \hat{v}_t)(k) \ , \tag{64}$$

and

$$\hat{v}_t(k) = k^2 \hat{u}_t(k) \ . \tag{65}$$

Then,

$$\lim_{t\to\infty} ||\tilde{u}_t - \tilde{u}_{\text{as}}||_{L^\infty} = 0 \ . \tag{66}$$

Again, we construct a solution by solving the equations in Fourier space. The only difference is that we have to require that $\hat{u}_0 \in \mathcal{A}_\alpha$ with $\alpha \geq 3$, since the nonlinearity involves in direct space second derivatives. For $\alpha \geq 3$ we can then use (47) rather than (44) of Proposition 12, which allows us to recover the decay in k. Namely, let $t \geq 1$ and let $\hat{u}_0 \in \mathcal{A}_\alpha$ with $\alpha > 3$. Then, the equation to be solved is

$$\hat{u}_t(k) = e^{-k^2(t-1)}\hat{u}_0(k) + \int_1^t e^{-k^2(t-s)}\ \hat{q}(k,s)\ ds\ , \tag{67}$$

with $\hat{q}$ as defined in (64), (65). To prove the existence of a solution to (67), we apply as in the proof of Proposition 15 for a given initial condition $\hat{u}_0$ the contraction mapping principle to the map $\mathcal{N} = \mathcal{M}\mathcal{L}$, where $\mathcal{L}\colon \hat{q} \mapsto \hat{u}$, is the map that associates to $\hat{q}$ the function $\hat{u}$ using (67), and where $\mathcal{M}\colon \hat{u} \mapsto \hat{q}$, is the map that associates to $\hat{u}$ the function $\hat{q}$ using (64), (65).

Proposition 19 *Let $0 < \varepsilon_0$, $\alpha > 3$ and let $\beta = 3/2$. Let $\hat{u}_0 \in \mathcal{A}_\alpha$ with $\|\hat{u}_0\|_\alpha = \varepsilon_0$ and let $\mathcal{U}_{\alpha,\beta}(\varepsilon_0) = \{\hat{q} \in \mathcal{B}_{\alpha-2,\beta} \mid \|\hat{q}\|_{\alpha,\beta} < \varepsilon_0\}$. Then,*

i) $\mathcal{L}$ is well defined as a map from $\mathcal{B}_{\alpha-2,\beta}$ to $\mathcal{B}_{\alpha,0}$.

ii) $\mathcal{M}$ is well defined as a map from $\mathcal{B}_{\alpha,0}$ to $\mathcal{B}_{\alpha-2,\beta}$.

iii) $\mathcal{N}$ is well defined as a map from $\mathcal{B}_{\alpha-2,\beta}$ to $\mathcal{B}_{\alpha-2,\beta}$.

iv) $\mathcal{N}(\mathcal{U}_{\alpha-2,\beta}(\varepsilon_0)) \subset \mathcal{U}_{\alpha-2,\beta}(\varepsilon_0)$ for ε_0 small enough.

v) For ε_0 small enough, $\|\mathcal{N}(\hat{q}_1) - \mathcal{N}(\hat{q}_2)\|_{\alpha-2,\beta} \leq \frac{1}{2}\|\hat{q}_1 - \hat{q}_2\|_{\alpha-2,\beta}$, for all $\hat{q}_1$, $\hat{q}_2 \in \mathcal{U}_{\alpha-2,\beta}(\varepsilon_0)$.

vi) $\mathcal{N}$ has a unique fixed point in $\mathcal{U}_{\alpha-2,\beta}(\varepsilon_0)$ for ε_0 small enough.

The proof is again essentially identical to the proof Proposition 15. *iii*) again follows from *i*) and *ii*), and *vi*) follows from *iv*) and *v*). We now prove *i*). We have already shown that the functions $\hat{u}_L(k,t) = e^{-k^2(t-1)}\hat{u}_0(k)$ is in $\mathcal{B}_{\alpha,0}$ and and that $\|\hat{u}_L\|_{\alpha,0} \leq \varepsilon$. Next let

$$\hat{u}_N(k,t) = \int_1^t e^{-k^2(t-s)}\ \hat{q}(k,s)\ ds\ .$$

Since $\hat{q} \in \mathcal{B}_{\alpha-2,\beta}$ we have that

$$|\hat{q}(k,t)| \leq \|\hat{q}\|_{\alpha-2,\beta}\, t^{-\beta}\mu_{\alpha-2}(k,t) \leq \varepsilon_0 t^{-\beta}\mu_{\alpha-2}(k,t)\ ,$$

and therefore $\hat{u}_N \in \mathcal{B}_{\alpha,0}$ by (47) of Lemma 12, and $\|u_N\|_{\alpha,0} \leq \varepsilon$. Since $\hat{u} = \hat{u}_L + \hat{u}_N$ it now follows that $\hat{u} \in \mathcal{B}_{\alpha,0}$ as claimed. Furthermore $||\hat{u}||_{\alpha 0} \leq \varepsilon$. We now prove *ii*). Let $\hat{u} \in \mathcal{B}_{\alpha,0}$ with $||\hat{u}||_{\alpha,0} < \varepsilon$, and let $\hat{v}(k,t) = \hat{v}_t(k)$ be as defined in (65). Then,

$$|\hat{v}(k,t)| \leq \varepsilon|k|^2\mu_{\alpha,0}(k,t) \leq \frac{\varepsilon}{t}\mu_{\alpha-2}(k,t)\ ,$$

and therefore $\hat{v} \in \mathcal{B}_{\alpha-2,1}$ and $||\hat{v}|| \leq \varepsilon$. Applying Lemma 11 to the convolution in (64) we get that $\mathcal{M}(\hat{u}) \in \mathcal{B}_{\alpha-2,\beta}$ for $\hat{u} \in \mathcal{B}_{\alpha,0}$. Furthermore, if $\|\hat{u}\| \leq \text{const.}\varepsilon_0$, then $\|\mathcal{M}(\hat{u})\|_{\alpha-2,\beta} \leq \text{const.}\varepsilon_0^2$. Therefore we find that, for $\hat{q} \in \mathcal{U}_{\alpha-2,\beta}(\varepsilon_0)$, $\|\mathcal{N}(\hat{q})\|_{\alpha-2,\beta} \leq \text{const.}\varepsilon_0^2$. Now since $\text{const.}\varepsilon_0^2 < \varepsilon_0$ for ε_0 small enough we find *iv*). To prove *v*) we consider, for $i = 1,2$, the image $\hat{u}_i = \mathcal{L}(\hat{q}_i)$ of $\hat{q}_i \in \mathcal{U}_{\alpha-2,\beta}(\varepsilon_0)$. We have already shown that $\|\hat{u}_i\|_{\alpha,0} \leq \varepsilon$, $i = 1,2$, and using exactly the same techniques one shows that

$$\begin{aligned} \|\hat{u}_1 - \hat{u}_2\|_{\alpha,0} &\leq \text{const.}\, \|\hat{q}_1 - \hat{q}_2\|_{\alpha-2,\beta}\ , \\ \|\hat{v}_1 - \hat{v}_2\|_{\alpha-2,1} &\leq \text{const.}\, \|\hat{q}_1 - \hat{q}_2\|_{\alpha-2,\beta}\ , \end{aligned}$$

where $v_i(k,t) = k^2 u_i(k,t)$, $i = 1,2$. Using that

$$\begin{aligned} \hat{q}_1 - \hat{q}_2 &= u_1 * v_1 - u_2 * v_2 \\ &= (u_1 - u_2) * v_1 + u_2 * (v_1 - v_2)\ , \end{aligned}$$

we find that, for ε_0 small enough,

$$\begin{aligned} \|\mathcal{N}(\hat{q}_1) - \mathcal{N}(\hat{q}_2)\|_{\alpha-2,\beta} &\leq \text{const.}\varepsilon_0 \|\hat{q}_1 - \hat{q}_2\|_{\alpha-2,\beta} \\ &\leq \frac{1}{2} \|\hat{q}_1 - \hat{q}_2\|_{\alpha-2,\beta} \ . \end{aligned}$$

This completes the proof of Proposition 19.

We now prove Proposition 18. Let

$$\hat{U}_t(k) = e^{-k^2(t-1)} \left(\hat{u}_0(k) + \int_1^t \hat{q}(k,s) \ ds \right) \ ,$$

with $\hat{q}$ as in (64) above, and let $\widetilde{\hat{U}}_t(k) = U_t(\frac{k}{t^{1/2}})$. For fixed $k \in \mathbf{R}$ we have that $\widetilde{\hat{u}}_{\text{as}}(k) := \lim_{t\to\infty} \widetilde{\hat{U}}_t(k) = a_{0,p}e^{-k^2} = \widehat{\tilde{u}}_{\text{as}}$, with $\tilde{u}_{\text{as}}$ given by (62), and furthermore that $|\widetilde{\hat{U}}_t(k)| \leq \text{const.}\mu_{\alpha-2}(k,1)$, uniformly in $k \in \mathbf{R}$, $t \geq 2$. Therefore it follows from the Lebesgue dominated convergence theorem that $\lim_{t\to\infty} ||\widetilde{\hat{U}}_t - \widetilde{\hat{u}}_{\text{as}}||_{L^1} = 0$. Applying the Hausdorff-Young inequality to the inverse Fourier transform we find for $\tilde{U}_t = \mathcal{F}^{-1}(\widetilde{\hat{U}}_t)$ that $\lim_{t\to\infty} ||\tilde{U}_t - \tilde{u}_{\text{as}}||_{L^\infty} = 0$. It remains to be shown that

$$\lim_{t\to\infty} ||\tilde{u}_t - \tilde{U}_t||_{L^\infty} = 0 \ . \tag{68}$$

Inequality (66) then follows using the triangle inequality. Since $\alpha > 3$ we can proceed for the difference of $\hat{U}_t$ and $\hat{u}_t$ as in the proof of Proposition 13. Namely,

$$\left|\hat{U}_t(k) - \hat{u}_t(k)\right| \leq \varepsilon \int_1^t (e^{-k^2(t-s)} - e^{-k^2(t-1)})\mu_{\alpha-2}(k,s)\frac{ds}{s^\beta} \leq \frac{\varepsilon}{t^{\beta-1}}\mu_{\alpha-2}(k,t) \ ,$$

and therefore we get that

$$||\widetilde{\hat{u}}_t - \widetilde{\hat{U}}_t||_{L^1} \leq \varepsilon \int_{\mathbf{R}} \frac{1}{t^{\beta-1}}\mu_{\alpha-2}(k,1) \ dk \leq \frac{\varepsilon}{t^{\beta-1}} \ .$$

Therefore, $\lim_{t\to\infty} ||\widetilde{\hat{u}}_t - \widetilde{\hat{U}}_t||_{L^1} = 0$, and (68) now follows by using the Hausdorff-Young inequality. This completes the proof of Proposition 18.

2.8 Appendix I: construction of a counter example

In what follows we give the details of the construction of an initial condition which satisfies Proposition 5. The example is based on Collet and Eckmann (1992b). We set, for $x > 0$, $u_0 = \sqrt{4\pi}f'$, where f is the function that we now construct. Let for $0 \leq x \leq 1$

$$h(x) = c_2 e^{-1/x} e^{-1/(1-x)} \ , \tag{69}$$

with

$$c_2 = \left(\int_0^1 e^{-1/x} e^{-1/(1-x)} dx \right)^{-1} = 142.25\ldots \ , \tag{70}$$

so that $\int_0^1 h(x) \ dx = 1$. The function h is infinitely differentiable on $[0,1]$, and satisfies $0 \leq h(x) \leq h(1/2) = h_{\max} = 2.6\ldots$. Now let, for $0 \leq x \leq 1$,

$$H(a,b)(x) = a + (b-a) \int_0^x h(y) \ dy \ . \tag{71}$$

By construction the function $H(a,b)$ interpolates between a and b on the interval $[0,1]$. For the m^{th} derivative $H^{(m)}$ of H we have for all $m \geq 1$, $H^{(m)} = (b-a)h^{(m-1)}$, and therefore we see from (69) that

$$\lim_{x\searrow 0} H^{(m)}(x) = \lim_{x\nearrow 1} H^{(m)}(x) = 0 \ . \tag{72}$$

Now let $n \geq 4$, $L_n = n!$, $l_n = 2^n$, $I_n = (L_n + l_n, L_{n+1} - l_{n+1})$ and $J_n = [L_n - l_n, L_n + l_n]$. Note that the intervals I_n are non overlapping. For the first intervals we have $I_4 = (4! + 2^4, 5! - 2^5) = (40, 88)$, and $J_4 = [8, 40]$. Now define

$$f(x) = \begin{cases} 0 \text{ for } -\infty < x \leq 8 \\ (-1)^n \text{ for } x \in I_n,\ n \geq 4 \\ H(0, 1, (x-8)/32) \text{ for } x \in J_4 \\ (-1)^n H(-1, 1, [x - (L_n - l_n)]/(2l_n)) \text{ for } x \in J_n,\ n \geq 5 \end{cases} \tag{73}$$

By construction we have

$$|f'(x)| \leq \begin{cases} 0 & \text{for } -\infty \leq x \leq 8 \\ 0 & \text{for } x \in I_n,\ n \geq 4 \\ h_{\max} 2^{-n} & \text{for } x \in J_n,\ n \geq 4 \end{cases} \tag{74}$$

Therefore, since $|J_n| = 2^{n+1}$ we have that, for $p > 1$,

$$(\|u_0\|_{L^p})^p = \int_0^\infty \left|\sqrt{4\pi} f'(x)\right|^p dx \leq (\sqrt{4\pi})^p h_{\max} \sum_{n \geq 4} 2^{n+1} 2^{-pn} < \infty \ ,$$

and therefore $u_0 = \sqrt{4\pi} f' \in L^p(\mathbf{R}, dx)$ for all $p > 1$, but by construction $u_0 \notin L^1(\mathbf{R}, dx)$. We now consider the value at zero of the scaled solution $\tilde{u}$ of the heat equation with initial condition $u_0 = \sqrt{4\pi} f'$. For $t > 0$ we have

$$\begin{aligned} \tilde{u}(0, t) = \sqrt{t} u(0, t) &= \int_{\mathbf{R}} e^{-\frac{y^2}{4t}} f'(y)\ dy = \int_0^\infty e^{-\frac{y^2}{4t}} f'(y)\ dy \\ &= \left[e^{-\frac{y^2}{4t}} f(y) \Big|_{y=0}^{y=\infty} + \int_0^\infty \frac{y}{2t} e^{-\frac{y^2}{4t}} f(y)\ dy \right] = \frac{1}{2} \int_0^\infty \frac{y}{t} e^{-\frac{y^2}{4t}} f(y)\ dy \ . \end{aligned}$$

Now let $T_n = L_n L_{n+1}$. Then, we have

$$\begin{aligned} \tilde{u}(0, T_n) = &\frac{1}{2} \int_0^{L_n + l_n} \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} f(y)\ dy \\ &+ \frac{1}{2} \int_{I_n} \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} f(y)\ dy + \frac{1}{2} \int_{L_{n+1} - l_{n+1}}^\infty \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} f(y)\ dy \ , \end{aligned}$$

and therefore

$$\begin{aligned} \tilde{u}(0, T_n) = &\frac{1}{2} \int_0^{L_n + l_n} \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} (f(y) - (-1)^n)\ dy \\ &+ \frac{(-1)^n}{2} \int_{\mathbf{R}} \frac{y}{T_n} e^{-\frac{y^2}{4T_n}}\ dy + \frac{1}{2} \int_{L_{n+1} - l_{n+1}}^\infty \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} (f(y) - (-1)^n)\ dy \ . \end{aligned}$$

The function xe^{-x^2} has its maximum value at $x = \sqrt{2}$. Therefore, since $L_n < T_n < L_{n+1}$, we find for the first term

$$\begin{aligned} &\left| \frac{1}{2} \int_0^{L_n + l_n} \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} (f(y) - (-1)^n)\ dy \right| \\ \leq &\left| \int_0^{L_n + l_n} \frac{y}{T_n}\ dy \right| = \frac{(L_n + l_n)^2}{L_n L_{n+1}} \longrightarrow_{n \to \infty} 0 \ , \end{aligned}$$

and for the third term

$$\begin{aligned} &\left| \frac{1}{2} \int_{L_{n+1} - l_{n+1}}^\infty \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} (f(y) - (-1)^n)\ dy \right| \\ \leq &\left| \int_{L_{n+1} - l_{n+1}}^\infty \frac{y}{2T_n} e^{-\frac{y^2}{4T_n}} dy \right| = e^{-\frac{(L_{n+1} - l_{n+1})^2}{4 L_n L_{n+1}}} \longrightarrow_{n \to \infty} 0 \ . \end{aligned}$$

Therefore, we conclude that, asymptotically as $n \to \infty$,

$$\tilde{u}(0, T_n) \sim \frac{(-1)^n}{2} \int_0^\infty \frac{y}{T_n} e^{-\frac{y^2}{4T_n}} \, dy = (-1)^n \ .$$

This completes the proof of Proposition 5.

2.9 Appendix II: proof of the technical lemmas

In this section we prove the Lemmas of Section 2.6. In what follows constants indicated by „const.“ may depend on α, β, γ, etc., but are independent of t and k. In addition these constants may be different from instant to instant.

2.9.1 Proof of Lemma 10

Let α, β, $\gamma \geq 0$ with $\alpha + 2\beta \geq \gamma$, and let for $k \in \mathbf{R}$ and $t \geq 1$,

$$f(k,t) = e^{-k^2(t-1)} \left(\frac{t-1}{t} \right)^\beta |k|^\gamma \mu_\alpha(k,1) \ ,$$

and

$$g(k,t) = \frac{1}{t^{\min\{\beta,\gamma/2\}}} \mu_{\alpha+2\beta-\gamma}(k,t) \ .$$

We have to prove that, uniformly in $k \in \mathbf{R}$ and $t \geq 1$, $f(k,t) \leq \text{const.} g(k,t)$. Note that $f(k,t) \geq 0$. For $1 \leq t \leq 2$ and $|k| \leq 1$ we have that

$$f(k,t) \leq \text{const.} \leq \text{const.} g(k,t) \ ,$$

and for $1 \leq t \leq 2$ and $|k| > 1$ we have that

$$\begin{aligned} f(k,t) &\leq \text{const.} e^{-k^2(t-1)} ((t-1)\, k^2)^\beta \, |k|^{\gamma - 2\beta} \, \mu_\alpha(k,1) \\ &\leq \text{const.} \mu_{\alpha+2\beta-\gamma}(k,1) \leq \text{const.} g(k,t) \ . \end{aligned}$$

For $t > 2$ we have, for $\beta \leq \gamma/2$ that

$$\begin{aligned} f(k,t) &\leq \text{const.} \frac{1}{t^\beta} e^{-k^2(t-1)} ((t-1)\, k^2)^\beta \, |k|^{\gamma-2\beta} \, \mu_\alpha(k,1) \\ &\leq \text{const.} \frac{1}{t^\beta} e^{-k^2(t-1)} ((t-1)\, k^2)^\beta \leq \text{const.} g(k,t) \ , \end{aligned}$$

and for $\beta > \gamma/2$

$$\begin{aligned} f(k,t) &\leq \text{const.} \frac{1}{t^{\gamma/2}} e^{-k^2(t-1)} ((t-1)\, k^2)^{\gamma/2} \mu_\alpha(k,1) \\ &\leq \text{const.} \frac{1}{t^{\gamma/2}} e^{-k^2(t-1)} ((t-1)\, k^2)^{\gamma/2} \\ &\leq \text{const.} \frac{1}{t^{\gamma/2}} \mu_{\alpha+2\beta-\gamma}(k,t) \leq \text{const.} g(k,t) \ . \end{aligned}$$

This completes the proof of Lemma 10.

2.9.2 Proof of Lemma 11

Let α, $\alpha' > 1$ and let $D(k) = \{k' \in \mathbf{R} \mid |k' - k| < |k|/2\}$. For $k' \in D(k)$ we have that

$$|k'| \geq |k| - |k - k'| \geq \frac{1}{2} |k| \ .$$

With this notation we get for the integral in (43)

$$\begin{aligned}
&\int_{\mathbf{R}} \mu_\alpha(k-k',t)\mu_{\alpha'}(k',t)dk' \\
&= \int_{\mathbf{R}\setminus D(k)} \mu_\alpha(k-k',t)\mu_{\alpha'}(k',t)dk' + \int_{D(k)} \mu_\alpha(k-k',t)\mu_{\alpha'}(k',t)dk' \\
&\le \sup_{k'\in\mathbf{R}\setminus D(k)} \mu_\alpha(k-k',t) \int_{\mathbf{R}\setminus D(k)} \mu_{\alpha'}(k',t)dk' \\
&+ \sup_{k'\in D(k)} \mu_{\alpha'}(k',t) \int_{D(k)} \mu_\alpha(k-k',t)dk' \\
&\le \text{const.}\mu_\alpha(k/2,t)\frac{1}{t^{1/2}} + \text{const.}\mu_{\alpha'}(k/2,t)\frac{1}{t^{1/2}} \\
&\le \text{const.}\frac{1}{t^{1/2}}\mu_{\min\{\alpha,\alpha'\}}(k,t)
\end{aligned}$$

This completes the proof of Lemma 11.

2.9.3 Proof of Proposition 12

Let $\beta > 1$ and $\alpha \ge 0$. In order to prove (44) we cut the integral over $[1,t]$ into an integral over $[1,\frac{t+1}{2}]$ and an integral over $[\frac{t+1}{2},t]$. For the first integral we have

$$\begin{aligned}
\left|\int_1^{\frac{t+1}{2}} e^{-k^2(t-s)}\frac{1}{s^\beta}\mu_\alpha(k,s)\,ds\right| &\le \text{const.}e^{-k^2\frac{t-1}{2}}\mu_\alpha(k,1)\int_1^{\frac{t+1}{2}}\frac{ds}{s^\beta} \\
&\le \text{const.}e^{-k^2\frac{t-1}{2}}\mu_\alpha(k,1) \le \text{const.}\mu_\alpha(k,t)\ , \qquad (75)
\end{aligned}$$

where we have used Proposition 10 in the last inequality. For the second integral we have

$$\begin{aligned}
\left|\int_{\frac{t+1}{2}}^t e^{-k^2(t-s)}\frac{1}{s^\beta}\mu_\alpha(k,s)\,ds\right| &\le \text{const.}\mu_\alpha(k,\frac{t+1}{2})\int_{\frac{t+1}{2}}^t\frac{ds}{s^\beta} \\
&\le \frac{\text{const.}}{t^{\beta-1}}\mu_\alpha(k,t) \le \text{const.}\mu_\alpha(k,t)\ , \qquad (76)
\end{aligned}$$

and (44) now follows using the triangle inequality. To prove (45) we note that

$$e^{-k^2(t-s)} - e^{-k^2(t-1)} = e^{-k^2(t-s)}(1-e^{-k^2(s-1)}) \le e^{-k^2(t-s)}\ .$$

Therefore we find using the bound in (76) for the second integral

$$\begin{aligned}
\left|\int_{\frac{t+1}{2}}^t (e^{-k^2(t-s)} - e^{-k^2(t-1)})\mu_\alpha(k,s)\frac{ds}{s^\beta}\right| &\le \left|\int_{\frac{t+1}{2}}^t e^{-k^2(t-s)}\frac{1}{s^\beta}\mu_\alpha(k,s)\,ds\right| \\
&\le \frac{\text{const.}}{t^{\beta-1}}\mu_\alpha(k,t)\ , \qquad (77)
\end{aligned}$$

and furthermore, since

$$\frac{1-e^{-k^2(s-1)}}{k^2(s-1)} \le \text{const.}\ ,$$

we find for the first integral that

$$\begin{aligned}
\varepsilon\int_1^{\frac{t+1}{2}} (e^{-k^2(t-s)} - e^{-k^2(t-1)})\mu_\alpha(k,s)\frac{ds}{s^\beta} &\le \varepsilon\ e^{-k^2\frac{t-1}{2}}\mu_\alpha(k,1)\int_1^{\frac{t+1}{2}} k^2(s-1)\frac{ds}{s^\beta} \\
&\le \frac{\varepsilon}{t^{\beta-2}}e^{-k^2\frac{t-1}{2}}k^2\left(\frac{t-1}{t}\right)^2\mu_\alpha(k,1) \\
&\le \frac{\varepsilon}{t^{\beta-1}}\mu_\alpha(k,t)\ . \qquad (78)
\end{aligned}$$

The proof of (46) and (47) is similar but somewhat more involved. We treat the two cases in parallel. In what follows $\delta \in \{1,2\}$ and $\alpha \geq 1$ if $\delta = 1$ and $\alpha \geq 2$ if $\delta = 2$. The only important point is that $\alpha - \delta \geq 0$. We again cut the integral into two pieces. We have

$$\int_1^{\frac{t+1}{2}} e^{-k^2(t-s)} \frac{1}{s^\beta} \mu_{\alpha-\delta}(k,s)\ ds \leq \text{const.} f_1(k,t)\ , \tag{79}$$

where

$$f_1(k,t) = e^{-k^2 \frac{t-1}{2}} \left(\frac{t-1}{t} \right) \mu_{\alpha-\delta}(k,1)\ ,$$

and

$$\int_{\frac{t+1}{2}}^{t} e^{-k^2(t-s)} \frac{1}{s^\beta} \mu_{\alpha-\delta}(k,s)\ ds \leq \text{const.} f_2(k,t)\ , \tag{80}$$

where

$$f_2(k,t) = \frac{1}{t^\beta} \mu_{\alpha-\delta}(k,t) \frac{1}{k^2} \left(1 - e^{-k^2 \frac{t-1}{2}} \right)\ .$$

We now estimate the functions f_1 and f_2 for various regimes of t and k. We again first treat the case of small times t and then the case of large times t. For $1 \leq t \leq 2$ we have for $|k| \leq 1$ and for $i = 1,2$, that

$$f_i(k,t) \leq \text{const.} \leq \text{const.} \mu_\alpha(k,t)\ .$$

Again for $1 \leq t \leq 2$, but for $|k| > 1$, we have that

$$\begin{aligned} f_1(k,t) &\leq \text{const.} e^{-k^2 \frac{t-1}{2}} (t-1) k^2 \frac{1}{k^2} \mu_{\alpha-2}(k,1) \\ &\leq \text{const.} \frac{1}{k^2} \mu_{\alpha-\delta}(k,1) \leq \text{const.} \mu_\alpha(k,1) \\ &\leq \text{const.} \mu_\alpha(k,t)\ , \end{aligned}$$

and that

$$\begin{aligned} f_2(k,t) &\leq \text{const.} \frac{1}{t^\beta} \mu_{\alpha-\delta}(k,t) \frac{1}{k^2} \\ &\leq \text{const.} \frac{1}{k^2} \mu_{\alpha-\delta}(k,1) \leq \text{const.} \mu_\alpha(k,1) \\ &\leq \text{const.} \mu_\alpha(k,t)\ . \end{aligned}$$

For $t > 2$ we have that

$$f_1(k,t) \leq \text{const.} e^{-k^2 \frac{t-1}{2}} \mu_{\alpha-\delta}(k,1) \leq \text{const.} \mu_\alpha(k,t)\ ,$$

and that

$$\begin{aligned} f_2(k,t) &\leq \text{const.} \frac{t-1}{t^\beta} \mu_{\alpha-\delta}(k,t) \frac{1 - e^{-k^2 \frac{t-1}{2}}}{k^2(t-1)} \\ &\leq \text{const.} \frac{1}{t^{\beta-1}} \mu_{\alpha-\delta}(k,t) \frac{1}{1+k^2 t} \leq \text{const.} \mu_\alpha(k,t)\ . \end{aligned}$$

This completes the proof of Lemma 12.

2.10 Bibliographic notes

The following bibliographic notes provide a chronological access to the literature on the long time asymptotics for partial differential equations in unbounded domains containing proofs that are based on the renormalization group concept outlined in the previous section. The origin of the techniques can be traced back to the article by Collet and Eckmann (1992b) and the work on the long time asymptotics of front solutions of the Ginzburg-Landau equation by Collet and Eckmann (1992a), Collet et al. (1992), Bricmont and Kupiainen (1992) and Eckmann and Gallay (1993). See also Pao (1993). Parallel to further work by Eckmann and Wayne (1994), Gallay (1994) and Bricmont and Kupiainen (1994a) the connection of the ideas underlying the analysis in these papers with the renormalization group as used in statistical mechanics, field theory and dynamical systems theory, were put forward in Bricmont et al. (1994), Bricmont and Kupiainen (1994b), and Chen et al. (1994) for the case of parabolic equations and then by Bona et al. (1994) for nonlinear dissipative wave equations. In Bona et al. (1995) higher order asymptotics of solutions are studied. The article by Bricmont et al. (1996) introduced in particular the community of mathematical physicists to this range of problems and has triggered an intense activity in this field. Important results for the Swift-Hohenberg equation were proved around the same time by Schneider (1996), and an nontrivial application of the analysis of the critical case was studied in Bricmont and Kupiainen (1996a). The case of relevant perturbations was also studied around this time in Bricmont and Kupiainen (1996b). A first formulation of a connection with invariant manifold theory can be found in Eckmann and Wayne (1997). Soon after that, Schneider (1998a) used the ideas to set up a proof of the stability of Taylor vortices in an infinite cylinder, which showed the power and robustness of the method to deal with "real world" problems. For a generalization to higher dimensions see Schneider (1998b). The method has been generalized to higher order linear operators in Eckmann and Wayne (1998a), Eckmann and Wayne (1998b) and adapted to more complicated situations in Gallay and Mielke (1998), who also considered asymptotic expansions for damped wave equations. See Gallay and Raugel (1998). A review was then written by Bricmont and Kupiainen (1998). The next wave of work was on hyperbolic fronts (Gallay and Raugel (2000)), multicomponent systems involving several length scales (van Baalen et al. (2000)), bifurcation problems (Eckmann and Schneider (2000)), and a review was written by Mielke et al. (2001). The successful introduction of the method for studying the long time asymptotics of solutions of the Navier-Stokes equations based on the analysis of the vorticity goes back to Gallay and Wayne (2002a), Gallay and Wayne (2002b). Around the same time the non-linear stability analysis of modulated fronts for the Swift-Hohenberg equations was proved in Eckmann and Schneider (2002). The renormalization group method was also successfully applied to analyze the modulational stability of quasi-steady patterns in dispersive systems in Promislow (2002). Almost global existence and transient self similar decay for Poiseuille flow at criticality for exponentially long times was proved by Schneider and Uecker (2003), and an interface model was analyzed in Gallay and Mielke (2003). Non-vanishing profiles for the Kuramoto-Sivashinsky equation were considered by van Baalen and Eckmann (2004). A review of Fourier methods and a discussion of various types of function spaces goes back to about the same time, see Guidotti (2004), as well as another review of the method by Merdan and Caginalp (2004). The question of bifurcations was reconsidered in Gallay et al. (2004). Recent work includes the discussion of the anomalous scaling for three-dimensional Cahn-Hilliard fronts by Korvola et al. (2005), of the pulse dynamics in thermally loaded optical parametric oscillators (Moore and Promislow (2005)), the discussion of the down-stream asymptotics of stationary Navier-Stokes flows, and the global stability of vortex solutions of the two-dimensional Navier-Stokes equations by Gallay and Wayne (2005).

3 Down-stream asymptotics of stationary Navier-Stokes flows

In what follows we explain how the ideas of the preceding section can be used to analyze stationary solutions of the Navier-Stokes equations in the down-stream region of an exterior domain. The reason why this region is of particular interest is that, due to the slow decay of the vorticity in the direction of the flow and its fast decay in other directions, the dominant large distance asymptotics of the exterior flow can be entirely reconstructed from the knowledge of the asymptotic behavior of the vorticity in the down-stream region. This being said it is nevertheless interesting to study directions different from downstream with the methods introduced in Section 2. The present techniques can in particular also be used to analyze the upstream region in more detail, or - as we will discuss in the open problem section below - to analyze the asymptotic behavior of flows parallel to a wall. The original publications on which the following discussions are based are for the two dimensional case Haldi and Wittwer (2005) and for the three dimensional case Wittwer (2006). Earlier references are van Baalen (2002), Wittwer (2002) and Wittwer (2003).

Consider, in $d = 2$ or $d = 3$ dimensions, the time independent incompressible Navier-Stokes equations

$$-(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U} + \Delta\mathbf{U} - \boldsymbol{\nabla}p = \mathbf{0} \ , \tag{81}$$

$$\boldsymbol{\nabla}\cdot\mathbf{U} = 0 \ , \tag{82}$$

in a half-space $\Omega_+ = \{(x, \mathbf{y}) \in \mathbf{R} \times \mathbf{R}^{d-1} \mid x \geq 1\}$. We are interested in modeling the situation where fluid enters the half-space Ω_+ through the surface $\Sigma = \{(x, \mathbf{y}) \in \mathbf{R} \times \mathbf{R}^{d-1} \mid x = 1\}$ and where the fluid flows at infinity parallel to the x-axis at a nonzero constant speed $\mathbf{u}_\infty \equiv (1, 0)$. We therefore impose at infinity the boundary condition

$$\lim_{\substack{x^2+|\mathbf{y}|^2 \to \infty \\ x \geq 1}} \mathbf{U}(x, \mathbf{y}) = \mathbf{u}_\infty \ . \tag{83}$$

On Σ we impose the boundary condition

$$\mathbf{U}|_\Sigma = \mathbf{U}_0 \ , \tag{84}$$

with $\mathbf{U}_0$ in a class $\mathcal{U}$ of vector fields for which

$$\lim_{|\mathbf{y}| \to \infty} \mathbf{U}_0(\mathbf{y}) = \mathbf{u}_\infty \ . \tag{85}$$

We are then interested in proving the existence of solutions of (81), (82) for this setting and in particular we are interested in studying the behavior of the solutions when $x \to \infty$. Naturally, one would like to settle this question for a large class of vector fields $\mathcal{U}$. Here, we only present the results for the case of vector fields that are perturbations of the constant vector field $\mathbf{u}_\infty$, *i.e.*,

$$\mathbf{U}_0 = \mathbf{u}_\infty + \mathbf{u}_0 \ , \tag{86}$$

where $\mathbf{u}_0 = (u_0, \mathbf{v}_0)$ is small in an appropriate sense. By appropriate we mean such as to contain the cases of interest, *i.e.*, the vector fields of exterior flows at low Reynolds numbers evaluated on vertical cross sections in the down-stream region. In particular, as we will see, we can not require the functions u_0 to be integrable, and this is one of the reasons why we have discussed such cases in Section 2. We now also set

$$\mathbf{U} = \mathbf{u}_\infty + \mathbf{u} \ , \tag{87}$$

with $\mathbf{u} = (u, \mathbf{v})$. After substitution of (87) into (81), (82) we get for $\mathbf{u}$ the equations

$$-(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \partial_x \mathbf{u} + \Delta \mathbf{u} - \boldsymbol{\nabla} p = \mathbf{0} \ , \tag{88}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \ , \tag{89}$$

and the boundary conditions (84), (85) become

$$\mathbf{u}(1, \mathbf{y}) = \mathbf{u}_0(\mathbf{y}) \ , \tag{90}$$

with $\mathbf{u}_0$ satisfying

$$\lim_{|\mathbf{y}| \to \infty} \mathbf{u}_0(\mathbf{y}) = \mathbf{0} \ . \tag{91}$$

We now explain the connection of the problem (88)-(91) with the theory of large time asymptotics presented in Section 2. Basically, the idea is to analyze the equations by rewriting them as evolution equations, where the coordinate x plays the role of time. In this interpretation the vector field $\mathbf{u}_0$ is the initial condition at the “time” $x = 1$, and one is interested in studying the large time behavior (asymptotic behavior as $x \to \infty$) of the solution of this Cauchy problem.

More precisely one applies these ideas to the vorticity formulation of equation (88), *i.e.*, one considers the equation which one gets by taking the curl of equation (88), namely the equation

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) + \Delta \boldsymbol{\omega} = \mathbf{0} \ , \tag{92}$$

with the vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} \ . \tag{93}$$

The pressure in equation (88) can be determined in a second step by solving the equation which one gets by taking the divergence of equation (88), *i.e.*, the equation

$$\Delta p = \boldsymbol{\nabla} \cdot ((\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \Delta \mathbf{u}) \ , \tag{94}$$

which one solves in Ω_+, with the boundary condition on Σ,

$$\partial_x p = -(\mathbf{u} \cdot \boldsymbol{\nabla})u + \Delta u \ . \tag{95}$$

See Haldi and Wittwer (2005) and Wittwer (2006) for details. As in the preceding section the main tool for the analysis of the dynamical system that one obtains this way is the Fourier transform, *i.e.*, one studies the equations (92), (93), and (89) after taking the Fourier transform with respect to the variable $\mathbf{y}$. This leads as in Section 2 to integral equations with nonlinearities involving convolutions, and the "initial condition" $\mathbf{u}_0$ is given as an inverse Fourier transform, *i.e.*, $\mathbf{u}_0 = \mathcal{F}^{-1}(\hat{\mathbf{u}}_0)$, where

$$\mathcal{F}^{-1}(\hat{\mathbf{u}}_0)(\mathbf{y}) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbf{R}^{d-1}} e^{-i\mathbf{k}\cdot\mathbf{y}} \hat{\mathbf{u}}_0(\mathbf{k})\; d^{d-1}\mathbf{k}\ ,$$

and where $\hat{\mathbf{u}}_0$ is chosen in a certain class of vector fields such that (91) is satisfied. Note that at this point it is important that the techniques of Section 2 allow classes of initial conditions which need not be in $L^1(\mathbf{R}^{d-1}, d^{d-1}\mathbf{y})$. Indeed, as has already been mentioned above, even though it might seem a priory natural to look at initial conditions that are integrable – the quantity

$$\int_{\mathbf{R}^{d-1}} u_0(\mathbf{y})\; d^{d-1}\mathbf{y} = \int_{\mathbf{R}^{d-1}} (\mathbf{U}_0(\mathbf{y}) - \mathbf{u}_\infty) \cdot \mathbf{e}_1\; d^{d-1}\mathbf{y}$$

with $\mathbf{e}_1$ the unit vector in the x-direction measures after all the difference of flux through the surface Σ when compared to the free flow – this choice turns out to be too restrictive and does not include the case of vector fields in the downstream region of a exterior flows. In particular, the vector fields that describe the dominant asymptotic structure at large values of x turn out not to be integrable in $\mathbf{y}$.

In contrast to the example cases studied in Section 2 the evolution equations associated with equations (92), (93), and (89) consist of four equations in $d = 2$ and seven equations in $d = 3$. On a linear level these equations can be decoupled by diagonalizing the system. On the nonlinear level the equations remain coupled. A further distinction from the case that has been discussed in Section 2 is that the linear system involves in Fourier space two types of behaviors at $k = 0$. There are the equations related to the vorticity that lead to a scaling of k proportional to $x^{-1/2}$ as discussed in Section 2 (spectral branch proportional to k^2 at small values of k), but there are also the equations related to the harmonic part in the velocity field that lead to a scaling of k proportional to x^{-1} (spectral branch proportional to $|k|$ at small values of k). For these equations the appropriate scaling is x^{-1}, and therefore the power counting is different, but otherwise the theory can be developed exactly as has been discussed in Section 2. As a result of the presence of two different scaling behaviors, the velocity components of the solution of the stationary Navier-Stokes equations show a somewhat more complicated asymptotic structure than what we have seen in Section 2. This does however not affect the overall structure of the proofs in any significant way. A final distinction to the case discussed in Section 2 is the fact that, because the original system is elliptic, the evolution equation associated 92), (93), and (89) also involves unstable branches. This again does not change the structure of the proofs in any significant way since this problem can be easily solved when passing to the integral equations. Such unstable directions are simply integrated backwards in time starting with zero initial conditions at infinity. In conclusion, the interested reader will find that Section 2 provides all the necessary tools for an easy reading of the papers of Haldi and Wittwer (2005) and Wittwer (2006).

We recall again that one of the main motivations for studying the Navier-Stokes equations in the way presented here is the result that the solutions of the stationary Navier-Stokes equations admit invariant quantities with respect to the "time" x. This is related to the fact that the vector fields are divergence free, but the result is nontrivial, exactly because the invariant quantities are associated with functions that are not integrable. The invariant quantities can still be calculated though by an application of appropriate averaging procedures. Part of the result is therefore a prescription that allows the computation of the invariant quantities.

3.1 Leading order term in two dimensions

The main result in two dimensions is the following theorem (see Haldi and Wittwer (2005)).

Theorem 20 *Let Σ and Ω_+ be as defined above. Then, there exists a class of vector fields $\mathcal{C}$ (containing in particular the physically interesting case of stationary flows at low Reynolds numbers) such that for all initial conditions $\mathbf{u}_0 = \mathcal{F}^{-1}(\hat{\mathbf{u}}_0)$ with $\hat{\mathbf{u}}_0 = (\hat{u}_0, \hat{v}_0) \in \mathcal{C}$, there exist a vector field $\mathbf{u} = (u, v)$ and a* *function p satisfying the Navier-Stokes equations (88) and (89) in Ω_+ subject to the boundary conditions (90) and (91). Furthermore,*

$$\lim_{x\to\infty} x^{1/2}\left(\sup_{y\in\mathbf{R}} |(u-u_{\mathrm{as}})(x,y)|\right) = 0 \ , \tag{96}$$

$$\lim_{x\to\infty} x\left(\sup_{y\in\mathbf{R}} |(v-v_{\mathrm{as}})(x,y)|\right) = 0 \ , \tag{97}$$

where

$$u_{\mathrm{as}}(x,y) = \frac{c}{2\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-\frac{y^2}{4x}} + \frac{d}{\pi}\frac{x}{x^2+y^2} + \frac{b}{\pi}\frac{y}{x^2+y^2} \ , \tag{98}$$

$$v_{\mathrm{as}}(x,y) = \frac{c}{4\sqrt{\pi}}\frac{y}{x^{3/2}}e^{-\frac{y^2}{4x}} + \frac{d}{\pi}\frac{y}{x^2+y^2} - \frac{b}{\pi}\frac{x}{x^2+y^2} \ , \tag{99}$$

and where the (real) amplitudes b, c, and d are invariant quantities which are given in terms of the initial condition $\hat{\mathbf{u}}_0$ by,

$$b = \lim_{k\to+0}\frac{\hat{u}_0(k,t)-\hat{u}_0(-k,t)}{2i} \ , \tag{100}$$

$$d = \lim_{k\to+0}\frac{\hat{v}_0(k,t)-\hat{v}_0(-k,t)}{2i} \ , \tag{101}$$

$$c = \lim_{k\to+0}\frac{\hat{u}_0(k,t)+\hat{u}_0(-k,t)}{2} - d \ . \tag{102}$$

Remark 21 *In the next chapter we will see that the constants b, c and d have a physical interpretation and are typically different from zero. Therefore, it follows from (100)-(102) that the functions $\hat{u}_0$ and $\hat{v}_0$ are discontinuous at $k=0$. However, since b, c and d are invariant quantities, $\hat{u}_0$ (and similarly $\hat{v}_0$) can be parametrized in terms of continuous functions $u_{0,E}$ and $u_{0,O}$, i.e., $u_0(k) = u_{0,E}(k) + \mathrm{sign}(k)u_{0,O}(k)$, and this decomposition is preserved by the time evolution, so that the function spaces of the type introduced in Section 2 are sufficiently general.*

Remark 22 *As explained above, the asymptotic behavior in Theorem 20 involves the two scalings $y\sim\sqrt{x}$ and $y\sim x$. To make this explicit we can for example rewrite u_{as} as*

$$u_{\mathrm{as}}(x,y) = \frac{1}{\sqrt{x}}\tilde{u}_{\mathrm{as},1}(y/\sqrt{x}) + \frac{1}{x}\tilde{u}_{\mathrm{as},2}(y/x) \ ,$$

with

$$\tilde{u}_{\mathrm{as},1}(y) = \frac{c}{2\sqrt{\pi}}e^{-\frac{y^2}{4}} \ ,$$

and

$$\tilde{u}_{\mathrm{as},2}(y) = \frac{d}{\pi}\frac{1}{1+y^2} + \frac{b}{\pi}\frac{y}{1+y^2} \ .$$

Remark 23 *The terms proportional to b, c, and d in (98) and (99) are divergence free vector fields. This will be essential below when we use these results for the definition of artificial boundary conditions.*

Remark 24 *The functions proportional to b in (98) and proportional to d in (99) are not integrable. These and similar terms are the reason why the existence of invariant quantities is not a priory obvious. As has been explained in Section 2 working in Fourier space allows for an easy mathematical treatment of such cases. It turns out that the invariant quantities b, c and d can still be obtained by certain limiting procedures from the functions in direct space. Namely, using classical results for the Fourier transforms of functions in the spaces $\mathcal{A}_\alpha$, $\alpha > 1$ (see for example Titchmarsh (1937)), one can for example show that for all $x\geq 1$,*

$$c+b \ = \ \lim_{R\to\infty}\int_{-R}^{R}(1-\frac{|y|}{R}) \ u(x,y) \ dy \ , \tag{103}$$

$$b \ = \ -\lim_{R\to\infty}\int_{-R}^{R}(1-\frac{|y|}{R}) \ v(x,y) \ dy \ . \tag{104}$$

Remark 25 *The invariance of the constants c and d can be used together with (103), (104) to show that for all fixed $x \geq 1$ the functions $y \mapsto u(x,y) - u_{\rm as}(x,y)$ and $y \mapsto v(x,y) - v_{\rm as}(x,y)$ are integrable, and that*

$$\begin{aligned} \int_{\mathbf{R}} (u(x,y) - u_{\rm as}(x,y))\, dy &= 0 \,, \\ \int_{\mathbf{R}} (v(x,y) - v_{\rm as}(x,y))\, dy &= 0 \,. \end{aligned}$$

The estimates in Haldi and Wittwer (2005) also imply the following result for the vorticity.

Theorem 26 *Let the vector field $\mathbf{u} = (u,v)$ and the constant c be as in Theorem 20. Let ω be the vorticity of the fluid, i.e., $\omega(x,y) = -\partial_y u(x,y) + \partial_x v(x,y)$. Let $\tilde{\omega}_x(y) = x\omega(x, y\sqrt{x})$, and let*

$$\tilde{\omega}_{\rm as}(y) = -\frac{c}{4\sqrt{\pi}} y e^{-\frac{y^2}{4}} \;.$$

Then

$$\lim_{x\to\infty} ||\tilde{\omega}_x - \tilde{\omega}_{\rm as}||_{L^\infty} = 0 \;. \tag{105}$$

Remark 27 *Theorem 26 is the exact analog of Proposition 16.*

Remark 28 *Since, as we will see, $c \neq 0$ for physically interesting cases, the result in Theorem 26 implies in particular that the vorticity decays slowly along the flow, namely only algebraically like $1/x$.*

3.2 Leading order term in three dimensions

The main result in three dimensions is the following theorem (see Wittwer (2006) for details):

Theorem 29 *Let Σ and Ω_+ be as defined above. Then, there exists a class of vector fields $\mathcal{C}$ (containing in particular the physically interesting case of stationary flows at low Reynolds numbers), such that for all initial conditions $\mathbf{u}_0 = \mathcal{F}^{-1}(\hat{\mathbf{u}}_0)$ with $\hat{\mathbf{u}}_0 = (\hat{u}_0, \hat{\mathbf{v}}_0) \in \mathcal{C}$, there exist a vector field $\mathbf{u} = (u, \mathbf{v})$ and a function p satisfying the Navier-Stokes equations (88) and (89) in Ω_+ subject to the boundary conditions (90) and (91). Furthermore,*

$$\lim_{x\to\infty} x \left(\sup_{\mathbf{y}\in\mathbf{R}^2} |\left(u - u_{\rm as}\right)(x,y)| \right) = 0 \,, \tag{106}$$

$$\lim_{x\to\infty} x^{3/2} \left(\sup_{\mathbf{y}\in\mathbf{R}^2} |\left(\mathbf{v}_1 - \mathbf{v}_{1,\rm as}\right)(x,y)| \right) = 0 \,, \tag{107}$$

$$\lim_{x\to\infty} x \left(\sup_{\mathbf{y}\in\mathbf{R}^2} |\left(\mathbf{v}_2 - \mathbf{v}_{2,\rm as}\right)(x,y)| \right) = 0 \,, \tag{108}$$

where $\mathbf{v} = \mathbf{v}_1 + v_2$, with $\mathbf{v}_1$ and $\mathbf{v}_2$ the irrotational and divergence free parts of $\mathbf{v}$, respectively, and where

$$u_{\rm as}(x,\mathbf{y}) = \frac{1}{4\pi x} e^{-\frac{y^2}{4x}}\, c + \frac{1}{2\pi}\frac{x}{r^3}\, d + \frac{1}{2\pi}\frac{\mathbf{y}\cdot\mathbf{b}}{r^3} \,, \tag{109}$$

$$\begin{aligned} \mathbf{v}_{1,\rm as}(x,\mathbf{y}) &= \frac{\mathbf{y}}{8\pi x^2} e^{-\frac{y^2}{4x}}\, c + \frac{1}{2\pi}\frac{\mathbf{y}}{r^3}\, d \\ &\quad - \frac{1}{2\pi}\frac{1}{r}\frac{1}{r+x}\left(\mathbf{1} - \frac{1}{r}\left(\frac{1}{r} + \frac{1}{r+x}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{b} \,, \end{aligned} \tag{110}$$

$$\begin{aligned} \mathbf{v}_{2,\rm as}(x,\mathbf{y}) &= \frac{1}{4\pi x} e^{-\frac{y^2}{4x}}\, \mathbf{a} \\ &\quad + \frac{1}{2\pi}\left(\frac{1}{y^2}\left(e^{-\frac{y^2}{4x}} - 1\right)\mathbf{1} - 2\frac{1}{y^4}\left(e^{-\frac{y^2}{4x}} - 1 + \frac{y^2}{4x}e^{-\frac{y^2}{4x}}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{a} \,, \end{aligned} \tag{111}$$

with $y = \sqrt{y_1^2 + y_2^2}$*, and* $(y_1, y_2) = \mathbf{y}$*, with* $r = \sqrt{x^2 + y^2}$*, with* $\mathbf{1}$ *the unit* 2×2 *matrix, with* $\mathbf{y}\mathbf{y}^T$ *the* 2×2 *matrix with entries* $\left(\mathbf{y}\mathbf{y}^T\right)_{ij} = y_i y_j$*, and where the numbers* c *and* d *and the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are invariant quantities which are given in terms of the initial condition* $\hat{\mathbf{u}}_0 = (\hat{u}_0, \hat{\mathbf{v}}_0)$ *by*

$$d = \left\langle -i\mathbf{e}^T \lim_{k \to 0} \hat{\mathbf{v}}_{0,1}(k\mathbf{e}) \right\rangle \ , \tag{112}$$

$$c = -d + \left\langle \lim_{k \to 0} \hat{u}_0(k\mathbf{e}) \right\rangle \ , \tag{113}$$

$$\mathbf{b} = \left\langle -i\mathbf{e} \lim_{k \to 0} \hat{u}_0(k\mathbf{e}) \right\rangle \ , \tag{114}$$

$$\mathbf{a} = -\mathbf{b} + \left\langle 2 \lim_{k \to 0} \hat{\mathbf{v}}_{0,2}(k\mathbf{e}) \right\rangle - \left\langle 2 \lim_{k \to 0} \hat{\mathbf{v}}_{0,1}(k\mathbf{e}) \right\rangle \ , \tag{115}$$

where $k = |\mathbf{k}|$*, where* $\hat{\mathbf{v}}_{0,1}$ *and* $\hat{\mathbf{v}}_{0,2}$*, are the irrotational and divergence free parts of* $\hat{\mathbf{v}}_0$*, respectively, i.e.,*

$$\begin{aligned} \hat{\mathbf{v}}_{0,1}(\mathbf{k}) &= \frac{\mathbf{k}\mathbf{k}^T}{k^2} \hat{\mathbf{v}}_0 \ , \\ \hat{\mathbf{v}}_{0,2}(\mathbf{k}) &= \left(1 - \frac{\mathbf{k}\mathbf{k}^T}{k^2}\right) \hat{\mathbf{v}}_0 \ , \end{aligned}$$

where $\mathbf{e} \equiv \mathbf{e}(\vartheta) = (\cos(\vartheta), \sin(\vartheta))$*, and where the average* $\langle \ . \ \rangle$ *is defined by*

$$\langle \ . \ \rangle = \frac{1}{2\pi} \int_0^{2\pi} \ . \ d\vartheta \ . \tag{116}$$

The estimates in Wittwer (2006) also imply the following result for the vorticity.

Theorem 30 *Let the vector field* $\mathbf{u} = (u, \mathbf{v})$*, the constant* c*, and the vector* $\mathbf{a} = (a_1, a_2)$ *be as in Theorem 29. Let* $\boldsymbol{\omega}$ *be the vorticity of the fluid, i.e.,* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$*. Let* $\tilde{\boldsymbol{\omega}}_x(\mathbf{y}) = x^{3/2}\boldsymbol{\omega}(x, \mathbf{y}\sqrt{x})$*, and let* $\mathbf{c} = (c, \mathbf{a})$*, and*

$$\begin{aligned} \tilde{\boldsymbol{\omega}}_{\mathrm{as}}(y) &= \mathbf{c} \times (0, \frac{1}{8\pi}\mathbf{y}e^{-\frac{y^2}{4}}) \\ &= -\frac{1}{8\pi}(a_2 y_1 - a_1 y_2, c y_2, -c y_1) e^{-\frac{y^2}{4}} \ . \end{aligned}$$

Then,

$$\lim_{x \to \infty} ||\tilde{\boldsymbol{\omega}}_x - \tilde{\boldsymbol{\omega}}_{\mathrm{as}}||_{L^\infty} = 0 \ .$$

3.3 Connection with existing results

There are many results on the large distance behavior of solutions of the stationary Navier-Stokes equations. See in particular Galdi (1998e) and references therein, where it is shown that at large distances the solution of the Navier-Stokes equations converge in certain norms to a multiple of the solution of the Oseen problem. To illustrate the connection with the above asymptotic results let $(\mathbf{E}, \mathbf{e})$ be the fundamental solution of the Oseen equation in $d = 2$, *i.e.*, of the equation

$$\begin{aligned} -\partial_x \mathbf{u} + \Delta \mathbf{u} - \nabla p &= \mathbf{f} \ , \\ \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \ . \end{aligned}$$

Namely, see Kress and Meyer (2000) or Galdi (1998d) and references therein,

$$\mathbf{E} = \begin{pmatrix} \psi - \psi_1 & -\psi_2 \\ -\psi_2 & \psi + \psi_1 \end{pmatrix} \ , \qquad \mathbf{e} = \nabla \phi \ ,$$

where, with $r = \sqrt{x^2 + y^2}$,

$$\begin{aligned} \psi(x,y) &= \frac{1}{4\pi} e^{x/2} K_0(\frac{r}{2}) \ , \\ \psi_i(x,y) &= \frac{1}{2\pi}\frac{x_i}{r}\left(\frac{1}{r} - \frac{1}{2}e^{x/2}K_1(\frac{r}{2})\right) \ , \qquad\qquad i = 1,2, \end{aligned}$$

where K_0 and K_1 are modified Bessel functions of order zero and one, and where

$$\phi(x,y) = \frac{1}{2\pi}\log(r) \ .$$

Now since

$$K_0(z) = \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{z}}\exp(-z/2) + o(1/\sqrt{z}) \ ,$$

as $z \to \infty$, we find for example that

$$\lim_{x\to\infty} \sqrt{x}\psi(x, y\sqrt{x}) = \frac{1/2}{\sqrt{2\pi}} e^{-\frac{y^2}{4}} \ , \tag{117}$$

$$\lim_{x\to\infty} \sqrt{x}\psi_1(x, y\sqrt{x}) = \frac{1/2}{\sqrt{2\pi}} e^{-\frac{y^2}{4}} \ , \tag{118}$$

and that

$$\lim_{x\to\infty} x\psi_1(x, yx) = \frac{1}{2\pi}\frac{1}{1+y^2} \ , \tag{119}$$

and similarly for the other functions, so that we recover the asymptotic terms that we have found in (98). Therefore, in the sense of the norms used in Galdi (1998e) and in view of the scaling limits (117), (118), and (119), the results of Theorem 20 are not surprising, except for the existence of invariant quantities which, as explained above is nontrivial since it is linked to non-integrable functions. As we will see below it is precisely this result which allows to use the asymptotic results as artificial boundary conditions for the numerical solution of the Navier-Stokes equations in the regimes of interest.

3.4 Higher order terms in two dimensions

The same way as Proposition 6 generalizes Proposition 1, we expect generalizations of Theorem 20 and Theorem 29 that provide higher order corrections at large x. Such results will be important for the construction of higher order artificial boundary conditions, but the corresponding results are at this point still conjectural. As has been explained in Section 2.3.4, the main input for such higher order theorems is the construction of an asymptotic expansion on a formal level. For the two dimensional case this work has been done up to second and in part to third order in Bönisch et al. (2006). The complete formal construction up to third order as well as the three dimensional case are also open problems. The reason why the construction of the corresponding asymptotic expansions is nontrivial is, first, that the asymptotic behavior involves two length scales which tends to complicate things considerably, and second, that in order for such expansions to be useful for the construction of artificial boundary conditions, it is mandatory to construct vector fields that are order by order divergence free, sufficiently regular, and which satisfy all the boundary conditions. The following proposition is one of the main results of Bönisch et al. (2006).

Proposition 31 *Let $\mathbf{u} = (u,v)$ be the vector field of Theorem 20 for an initial condition for which $c = -2d$, and assume that the Conjecture 34 below is valid. Then, we have for $N = 1,2,3$,*

$$\lim_{x\to\infty} x^{N/2}\left(\sup_{y\in\mathbf{R}}\left|\left(u - u_{\mathrm{as}}^N\right)(x,y)\right|\right) = 0 \ , \tag{120}$$

$$\lim_{x\to\infty} x^{(N+1)/2}\left(\sup_{y\in\mathbf{R}}\left|\left(v - v_{\mathrm{as}}^N\right)(x,y)\right|\right) = 0 \ , \tag{121}$$

where

$$u^N_{\rm as}(x,y) \;=\; \sum_{n=1}^{N}\sum_{m=1}^{n} u_{n,m}(x,y)\ , \tag{122}$$

$$v^N_{\rm as}(x,y) \;=\; \sum_{n=1}^{N}\sum_{m=1}^{n} v_{n,m}(x,y)\ , \tag{123}$$

with

$$\begin{aligned} u_{1,1}(x,y) &= u_{1,1,E}(x,y) - \theta(x)\frac{d}{\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-\frac{y^2}{4x}}\ , \\ v_{1,1}(x,y) &= v_{1,1,E}(x,y) - \theta(x)\frac{d}{2\sqrt{\pi}}\frac{y}{x^{3/2}}e^{-\frac{y^2}{4x}}\ , \end{aligned} \tag{124}$$

with θ the Heaviside function (i.e., $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$), with

$$\begin{aligned} u_{1,1,E}(x,y) &= \frac{d}{\pi}\frac{x}{x^2+y^2} + \frac{b}{\pi}\frac{y}{x^2+y^2}\ , \\ v_{1,1,E}(x,y) &= \frac{d}{\pi}\frac{y}{x^2+y^2} - \frac{b}{\pi}\frac{x}{x^2+y^2}\ , \end{aligned} \tag{125}$$

with

$$\begin{aligned} u_{2,1}(x,y) &= \theta(x)\frac{bd}{2}\frac{1}{(\sqrt{\pi})^3}\frac{\log(x)}{x}\frac{y}{\sqrt{x}}e^{-\frac{y^2}{4x}}\ , \\ v_{2,1}(x,y) &= \theta(x)\frac{bd}{2}\frac{1}{(\sqrt{\pi})^3}\frac{1}{x^{3/2}}\left(\log(x)\left(-1+\frac{1}{2}\frac{y^2}{x}\right)+2\right)e^{-\frac{y^2}{4x}}\ , \end{aligned} \tag{126}$$

and with

$$\begin{aligned} u_{2,2}(x,y) &= u_{2,2,E}(x,y) + \theta(x)d^2\frac{1}{x}f'(\frac{y}{\sqrt{x}}) \\ &\quad + \lambda\theta(x)f_\infty d^2\frac{3}{8}\frac{1}{x^2}\left((1+\frac{|y|}{\sqrt{x}})(1-\frac{1}{2}\frac{y^2}{x})+\frac{|y|}{\sqrt{x}}\right)e^{-\frac{y^2}{4x}}\ , \\ v_{2,2}(x,y) &= v_{2,2,E}(x,y) + \theta(x)\frac{d^2}{2}\frac{1}{x^{3/2}}\left(\left(f(\frac{y}{\sqrt{x}})-f_\infty{\rm sign}(y)\right)+\frac{y}{\sqrt{x}}f'(\frac{y}{\sqrt{x}})\right) \\ &\quad + \lambda\theta(x)f_\infty d^2\frac{3}{4}\frac{1}{x^{5/2}}\left((1+\frac{|y|}{\sqrt{x}})\frac{y}{\sqrt{x}}\left(1-\frac{1}{8}\frac{y^2}{x}\right)+\frac{1}{4}\frac{y^2}{x}{\rm sign}(y)\right)e^{-\frac{y^2}{4x}}\ , \end{aligned} \tag{127}$$

and

$$\begin{aligned} u_{2,2,E}(x,y) &= f_\infty\frac{d^2}{2}\frac{|y|}{r^2}\left(\frac{1}{r_2}-\frac{r_2}{r}\right)\ , \\ v_{2,2,E}(x,y) &= f_\infty\frac{d^2}{2}\frac{{\rm sign}(y)}{r}\left(-\frac{1}{r_2}-\frac{x}{r_2 r}+\frac{x\,r_2}{r^2}\right)\ , \end{aligned} \tag{128}$$

and with

$$\begin{aligned} u_{3,1}(x,y) &= \theta(x)\frac{b^2d}{4}\frac{1}{(\sqrt{\pi})^5}\frac{\log(x)^2}{x^{3/2}}\left(1-\frac{1}{2}\frac{y^2}{x}\right)e^{-\frac{y^2}{4x}}\ , \\ v_{3,1}(x,y) &= \theta(x)\frac{b^2d}{2}\frac{1}{(\sqrt{\pi})^5}\frac{\log(x)}{x^2}\frac{y}{\sqrt{x}}\left(\frac{\log(x)}{4}\left(3-\frac{1}{2}\frac{y^2}{x}\right)-1\right)e^{-\frac{y^2}{4x}}\ . \end{aligned} \tag{129}$$

Here, $r = \sqrt{x^2+y^2}$, $r_2 = \sqrt{2r+2x}$, $\lambda = 1$, and $f\colon \mathbf{R} \to \mathbf{R}$ is the unique solution of the third order linear inhomogeneous ordinary differential equation

$$f'''(z) + \frac{1}{2}zf''(z) + f'(z) + \frac{1}{2\pi}e^{-\frac{1}{2}z^2} = 0\ , \tag{130}$$

satisfying $f(0) = 0$, $f'(0) = -\frac{1}{2\pi}$, $f''(0) = 0$. *Explicitly,*

$$f(z) = -\frac{1}{\sqrt{2\pi}} \operatorname{erf}(\frac{z}{\sqrt{2}}) + \frac{1}{2\sqrt{\pi}} \operatorname{erf}(\frac{z}{2})\ e^{-\frac{z^2}{4}} \ . \tag{131}$$

Note that the function f *is odd, and that* f' *and* f'' *decay faster than exponential at infinity, and that*

$$f_\infty = \lim_{z\to\infty} f(z) = -\frac{1}{\sqrt{2\pi}} \ . \tag{132}$$

Remark 32 *Since* $x \geq 1$ *the functions* θ *could be suppressed from the above. However, when we specify artificial boundary conditions in Section 4, we will consider the same expressions in the domain* $\mathbf{R}^2 \backslash \mathbf{B}$*, with* $\mathbf{B}$ *a compact region with smooth boundary containing the origin. With the functions* θ *the above expressions will also be the correct asymptotic description in* $\mathbf{R}^2 \backslash \mathbf{B}$*. This avoids having to rewrite these lengthy expressions a second time in Section 5.*

Remark 33 *The terms proportional to* λ *are higher order and one might be tempted to neglect them, i.e., to set* $\lambda = 0$*. This is not possible, however, without giving up the regularity of the second order derivatives* $\partial_y^2 u$ *and* $\partial_y^2 v$ *across the positive* x*-axis.*

Proposition 31 is based on the following conjecture on the vorticity. See Bönisch et al. (2006).

Conjecture 34 *Let the vector field* $\mathbf{u} = (u, v)$*, the constants* b *and* d*, and the function* f *be as in Proposition 31. Let the vorticity* ω *be given by* $\omega(x, y) = -\partial_y u(x, y) + \partial_x v(x, y)$*. Then, for* $1 \leq N \leq 3$*, we have that*

$$\lim_{x\to\infty} x^{\frac{1+N}{2}} \sup_{y\in\mathbf{R}} \left| \omega(x, y) - \sum_{n=1}^{N} \omega_n(x, y) \right| = 0 \ , \tag{133}$$

where the functions ω_n *are given by*

$$\omega_n(x, y) = \sum_{m=1}^{n} \rho_{n,m}(x) \varphi''_{n,m}(\frac{y}{\sqrt{x}}) \ , \tag{134}$$

with

$$\rho_{n,m}(x) = \frac{\log(x)^{n-m}}{x^{(1+n)/2}} \ . \tag{135}$$

with

$$\begin{aligned}
\varphi_{1,1}(z) &= d\ \operatorname{erf}(\frac{z}{2}) \ , \\
\varphi_{2,1}(z) &= bd\ \frac{1}{\pi^{3/2}} e^{-\frac{z^2}{4}} \ , \\
\varphi_{2,2}(z) &= -d^2\ f(z) + b\ c_{2,2}\ e^{-\frac{z^2}{4}} \ , \\
\varphi_{3,1}(z) &= -b^2 d\ \frac{z}{4\pi^{5/2}} e^{-\frac{z^2}{4}} \ ,
\end{aligned} \tag{136}$$

and where $\varphi_{3,2}$ *and* $\varphi_{3,3}$ *are smooth functions with derivatives decaying at infinity faster that exponential.*

It is interesting to compare the above results with similar results in the literature. For the particular case where the wake has an axial symmetry, results for the so-called center-line velocity (the velocity along the axis of symmetry in the wake region) are given on the basis of boundary layer theory up to third order in Stewartson (1957). These results have been reviewed recently in Sobey (2000). Our results show that the expansions computed from the Navier-Stokes equations differ from the ones computed from boundary layer theory already at second order, which shows that higher order results based on boundary layer theory are inadequate for modeling Navier-Stokes flows. See also Cowley (2006) for a discussion of certain insufficiencies of boundary layer theory.

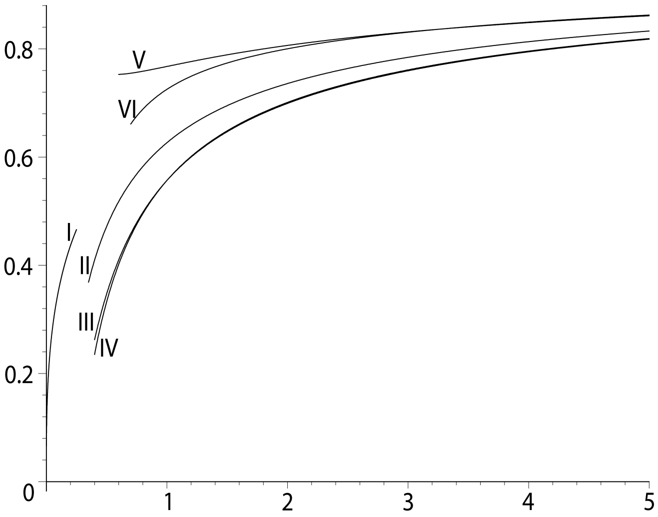

Figure 1. Center-line velocity for the far wake as given in Sobey (2000): two-term expansion, (II), three-term expansion (III), expansion with logarithmic corrections (IV). Near wake center-line velocity (I) as reviewed in Sobey (2000). Centerline velocity for the far wake based on the Navier-Stokes equations as given in Conjecture 31: expansion to first-order (V), expansion to second-order (VI).

3.5 Open problems

The following three problems are all in the reach of the techniques introduced above. Their solution will be directly useful for the definition of artificial boundary conditions in the corresponding cases.

3.5.1 The time periodic case

The asymptotic downstream behavior of time periodic data has recently been studied by van Baalen (2006). This work contains also a partial proof of some of the higher order terms presented above. The theory of periodic solutions for the Navier-Stokes equations in exterior domains is still rather incomplete so that it is not completely clear that the results of van Baalen (2006) are general enough to describe the downstream region of time periodic exterior flows, but we do expect this to be the case. The results of van Baalen (2006) have unfortunately not yet been used for the construction of artificial boundary conditions, so that this question has not yet been checked numerically either. All these questions are fascinating and the answers are in reach of the techniques described here. We therefore expect that further work will clarify the open points.

3.5.2 The case of free-falling bodies

A very interesting question that goes back to an article of Weinberger (1978) is the asymptotic description of solutions of the Navier-Stokes equations in $d = 3$ describing the steady free-fall of a rigid body. A free falling body typically also rotates along a certain axis that is aligned with the direction of the fall. What we therefore mean by a steady free fall is a solution of the Navier-Stokes equation which is stationary in a frame that is attached to the "body" and which is rotating with constant frequency around an axis that is aligned with the flow at infinity. For recent results see Galdi and Silvestre (2005). We expect the present techniques to be well adapted for a detailed study of these questions. Again this will allow the construction of artificial boundary conditions for this case which will make precise numerical solutions possible.

3.5.3 Motion in the presence of a nearby wall

Another very interesting open problem is the detailed description of the motion of a body parallel to a nearby wall. An example of recent experimental results of bubbles rising close to a wall can be found in

Takemura and Magnaudet (2003). For very slow movements this problem is discussed in the literature (a classical reference is Clift et al. (2005)) and is mathematically modeled by the Stokes equations. For higher Reynolds numbers one again needs to consider the asymptotics of the Navier-Stokes equations. We give some details of forthcoming results by Hillairet and Wittwer (2007) for the two-dimensional case. The situation can be modeled by solving equations (88), (89) in the complement of a compact region of the upper half plane. The associated downstream problem leads again essentially to the heat equation, however not on $\mathbf{R}$ but on the half-line $\mathbf{R}_+$, with Dirichlet boundary conditions at $x = 0$. This leads, instead of (98), to a dominant term for the horizontal velocity proportional to $1/x$ instead of $1/\sqrt{x}$, and therefore like $1/x^{3/2}$ for the transverse velocity and for the vorticity. This leading term decays for fixed x exponentially fast as a function of y, as y goes to infinity. The second order term for the vorticity is proportional to $1/x^2$ but the corresponding term in the expansion is a function that decays only algebraically in y, as y goes to infinity. As a result it can be shown that for fixed x the vorticity decays only like $1/y^4$, as y goes to infinity, and not exponentially fast as in the case of exterior flows without a nearby wall. This is due to the presence of a boundary layer ahead of the body, extending all the way to minus infinity in the sense that along the wall ahead of the obstacle the vorticity again only decays algebraically like $1/x^2$. Detailed results for this case as well as the artificial boundary conditions that can be derived from the results are in preparation.

4 Exterior flows at low Reynolds numbers

After the preparatory work in Section 2 and Section 3 we now consider exterior flows at low Reynolds numbers. The main goal is to explain on a heuristic level the mechanism that allows the reconstruction of the large distance asymptotics from the knowledge of the asymptotic behavior in the downstream region. We also present some details concerning the link between the invariant quantities of the preceding section and the forces that act on the "body". This is the key reason for the efficiency of the boundary conditions in Section 5.

4.1 The mathematical problem

Consider a rigid body $\tilde{\mathbf{B}}$ (a compact set with smooth boundary) of diameter R that is placed into a uniform stream of a homogeneous incompressible fluid filling up all of $\mathbf{R}^d$, $d = 2, 3$. This situation is modeled by the stationary Navier-Stokes equations

$$-\rho \left(\tilde{\mathbf{U}} \cdot \boldsymbol{\nabla}\right) \tilde{\mathbf{U}} + \mu \Delta \tilde{\mathbf{U}} - \boldsymbol{\nabla} \tilde{p} = 0 \ , \tag{137}$$

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{U}} = 0 \ , \tag{138}$$

in $\tilde{\Omega} = \mathbf{R}^d \setminus \tilde{\mathbf{B}}$, subject to the boundary conditions

$$\tilde{\mathbf{U}}\Big|_{\partial\tilde{\mathbf{B}}} = 0 \ , \tag{139}$$

$$\lim_{|\tilde{\mathbf{x}}|\to\infty} \tilde{\mathbf{U}}(\tilde{\mathbf{x}}) = \tilde{\mathbf{u}}_\infty \ . \tag{140}$$

Here, $\tilde{\mathbf{U}}$ is the velocity field, $\tilde{p}$ is the pressure and $\tilde{\mathbf{u}}_\infty$ is some constant non-zero vector field which we choose without restriction of generality to be parallel to the $\tilde{x}$-axis, *i.e.*, $\tilde{\mathbf{u}}_\infty = u_\infty \mathbf{u}_\infty$, where $\mathbf{u}_\infty = (1, 0)$ and $u_\infty > 0$. The density ρ and the viscosity μ are arbitrary positive constants. From μ, ρ and u_∞ we can form the length ℓ,

$$\ell = \frac{\mu}{\rho u_\infty} \ , \tag{141}$$

the so-called viscous length of the problem. The viscous forces and the inertial forces are quantities of comparable size if the diameter R of $\tilde{\mathbf{B}}$ is comparable with ℓ, *i.e.*, if the Reynolds number

$$\mathrm{Re} = \frac{R}{\ell} \ , \tag{142}$$

is neither very small nor very large, *i.e.*, depending on the geometry of the body, in the range from one to several thousand. Note that for bodies with a smooth boundary $\partial\tilde{\mathbf{B}}$ and for small enough Reynolds numbers (142) equation (137), (138) subject to the boundary conditions (139), (140) are known to have a unique classical solution. See Galdi (1998a), Galdi (1998b), and see Galdi (1999a) for an interesting open problem for the two dimensional case of symmetric stationary flows at arbitrary Reynolds numbers. Before proceeding any further we now rewrite the Navier-Stokes equations in dimensionless form. Let $\tilde{\mathbf{U}}$ be the velocity field and $\tilde{p}$ the pressure introduced in (137)-(140), and let ℓ be as defined in (141). Then, we define dimensionless coordinates $\mathbf{x} = \tilde{\mathbf{x}}/\ell$, and introduce a dimensionless vector fields $\mathbf{U}$ and a dimensionless pressure p through the definitions

$$\tilde{\mathbf{U}}(\tilde{\mathbf{x}}) = u_\infty \mathbf{U}(\mathbf{x}) \ , \tag{143}$$

$$\tilde{p}(\tilde{\mathbf{x}}) = (\rho u_\infty^2) p(\mathbf{x}) \ . \tag{144}$$

In the new coordinates we get instead of (137)-(140) the equations

$$-\left(\mathbf{U} \cdot \boldsymbol{\nabla}\right) \mathbf{U} + \Delta \mathbf{U} - \boldsymbol{\nabla} p = 0 \ , \tag{145}$$

$$\boldsymbol{\nabla} \cdot \mathbf{U} = 0 \ , \tag{146}$$

in $\Omega = \mathbf{R}^2 \setminus \mathbf{B}$, where $\mathbf{B} = \left\{ \mathbf{x} \in \mathbf{R}^2 \mid \ell \mathbf{x} = \tilde{\mathbf{x}} \text{ for some } \tilde{\mathbf{x}} \in \tilde{\mathbf{B}} \right\}$, and the boundary conditions

$$\mathbf{U}|_{\partial \mathbf{B}} = 0 \ , \tag{147}$$

$$\lim_{|\mathbf{x}| \to \infty} \mathbf{U}(\mathbf{x}) = (1, 0) \ . \tag{148}$$

In (145)-(146) all derivatives are with respect to the new coordinates. For convenience below we now introduce some additional notation and conventions. In practice, and in particular when solving the equations numerically, it will always be more convenient to work with zero boundary conditions at infinity. We therefore set $\tilde{\mathbf{U}} = \tilde{\mathbf{u}}_\infty + \tilde{\mathbf{u}}$ and $\mathbf{U} = \mathbf{u}_\infty + \mathbf{u}$ and consider either the dimensionfull equations

$$-\rho \left(\tilde{\mathbf{u}} \cdot \boldsymbol{\nabla}\right) \tilde{\mathbf{u}} - \rho u_\infty \partial_x \tilde{\mathbf{u}} + \mu \Delta \tilde{\mathbf{u}} - \boldsymbol{\nabla} \tilde{p} = 0 \ , \tag{149}$$

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{u}} = 0 \ , \tag{150}$$

with the boundary conditions

$$\tilde{\mathbf{u}}|_{\partial \tilde{\mathbf{B}}} = -\tilde{\mathbf{u}}_\infty \ , \tag{151}$$

$$\lim_{|\tilde{\mathbf{x}}| \to \infty} \tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = 0 \ , \tag{152}$$

or the dimensionless equations

$$-\left(\mathbf{u} \cdot \boldsymbol{\nabla}\right) \mathbf{u} - \partial_x \mathbf{u} + \Delta \mathbf{u} - \boldsymbol{\nabla} p = 0 \ , \tag{153}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \ , \tag{154}$$

with the boundary conditions

$$\mathbf{u}|_{\partial \mathbf{B}} = -(1, 0) \ , \tag{155}$$

$$\lim_{|\mathbf{x}| \to \infty} \mathbf{u}(\mathbf{x}) = 0 \ . \tag{156}$$

Finally, for Ω as defined above, we will always chose a coordinate system with origin in $\mathbf{B}$, but such that the surface $\Sigma = \{(x, \mathbf{y}) \in \mathbf{R} \times \mathbf{R}^{d-1} \mid x = 1\}$ defined in the previous section is "to the right of" $\mathbf{B}$, *i.e.*, such that $\sup_x \{x \in \mathbf{R} \mid (x, \mathbf{y}) \in \mathbf{B} \text{ for some } \mathbf{y} \in \mathbf{R}^{d-1}\} < 1$.

4.2 Consequences of incompressibility

We limit the discussion to the two-dimensional case. The three dimensional case is similar. Consider a solution $\mathbf{u} = (u, v)$ of the Navier-Stokes equations (153), (154) and consider the region $\Omega_{x_0,R} = \{(x, y) \in \Omega \mid |x| \leq x_0, |y| \leq R\}$. We assume that x_0 and R are large enough such that $\mathbf{B} \subset \Omega_{x_0,R}$. Since $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, and in view of the boundary condition (155), it follows using Gauss' theorem that

$$\int_{-R}^{R} (u(-x_0, y) - u(x_0, y)) \ dy + \int_{-x_0}^{x_0} (v(x, -R) - v(x_0, R)) \ dy = 0 \ .$$

Taking the limit $R \to \infty$ and using that for classical solutions the boundary condition (156) implies that $\lim_{R\to\pm\infty} v(x,R) = 0$, we conclude that

$$\lim_{R\to\infty} \int_{-R}^{R} (u(-x_0,y) - u(x_0,y))\ dy = 0\ .$$

Therefore, since $\lim_{R\to\infty} \int_{-R}^{R}(1-\frac{|y|}{R})\ u(x_0,y)\ dy = c+b$ by (103), it follows now that also

$$\lim_{R\to\infty} \int_{-R}^{R} (1-\frac{|y|}{R})\ u(-x_0,y)\ dy = c+b\ , \tag{157}$$

i.e., the invariant quantity $c+b$ can also be extracted from the velocity field upstream of the body. Similarly, using Stokes' theorem instead of Gauss' theorem, it can be shown that

$$-\lim_{R\to\infty} \int_{-R}^{R} (1-\frac{|y|}{R})\ v(-x_0,y)\ dy = b\ , \tag{158}$$

i.e., the invariant quantity b can also be extracted from the velocity field upstream of the body.

4.3 Connection between global and downstream asymptotics

The discussion in this section is on a heuristic level, and we consider the two dimensional case only. See Bönisch et al. (2006) and Bichsel and Wittwer (2007) for more details. See also Amick (1991) for one of the first references where the structure of the vorticity is rigorously exploited in order to obtain an improved description of the velocity field. So assume that the vorticity ω of an exterior flow is known. Then, the vector field $\mathbf{u} = (u,v)$ is given by

$$u(x,y) = \partial_y \psi(x,y)\ , \tag{159}$$
$$v(x,y) = -\partial_x \psi(x,y)\ , \tag{160}$$

with ψ solution of the Poisson equation

$$\Delta\psi(x,y) = -\omega(x,y)\ , \tag{161}$$

with the boundary conditions

$$\bar{\psi}\big|_{\partial\mathbf{B}} = \partial_{\mathbf{n}}\bar{\psi}\big|_{\partial\mathbf{B}} = 0\ , \tag{162}$$
$$\lim_{x,y\to\infty} \partial_y\psi(x,y) = \lim_{x,y\to\infty} -\partial_x\psi(x,y) = 0\ , \tag{163}$$

where $\bar{\psi} = \psi + y$. By Theorem 26 the vorticity decays slowly in the downstream direction. Assuming that it decays rapidly in the other directions, explicitly, if ω satisfies a bound of the type

$$\sup_{(x,y)\in\Omega} \left(\left|\omega(x, y\,|x|^{1/2})\right|\ e^{\delta|y|}(1+e^{-\delta x}) \right)\ , \tag{164}$$

for some $0 < \delta \ll 1$, then it follows using general results from potential theory that

$$\psi(x,y) = \psi_\omega(x,y) + h(x,y)\ , \tag{165}$$

with ψ_ω a particular solution of (161) that is independent of the geometry of $\mathbf{B}$, and h a harmonic function in Ω satisfying the bound

$$|h(x,y)| \leq \frac{\text{const.}}{r}\ , \tag{166}$$

with $r = \sqrt{x^2+y^2}$. The partial derivatives of h with respect to x and y obey analogous bounds. The function ψ_ω can be further decomposed into a part that dominates at large distances and a rest, *i.e.*,

$$\psi_\omega = \psi_0 + \psi_{\omega,1}\ . \tag{167}$$

where

$$\psi_0(x,y) = 2b\ G(x,y) + 2d\ H(x,y) + \frac{c}{2}\theta(x)(\varphi_{1,1}(\frac{y}{\sqrt{x}}) - \mathrm{sign}(y))\ , \tag{168}$$

with

$$\varphi_{1,1}(z) = \mathrm{erf}(\frac{z}{2})\ ,$$

with

$$G(x,y) = \frac{1}{4\pi}\log(x^2+y^2)\ ,$$

and with

$$H(x,y) = \frac{1}{2\pi}\arctan(\frac{y}{x}) - \frac{1}{2}\theta(x)\mathrm{sign}(y)\ .$$

From ψ_0 we get the vector field $(\partial_y\psi_0, -\partial_x\psi_0)$ which describes the solution at large distances. Furthermore, using (157), (158) it can be shown that $c = -2d$ in two and three dimensions and that $\mathbf{a} = -2\mathbf{b}$ in three dimensions. In the next subsection we now establish an additional link between the invariant quantities and drag and lift.

4.4 Drag, lift, and torque

We again limit the discussion to the two-dimensional case. Let $\mathbf{U}$, p be a solution of the Navier-Stokes equations (145), (146) subject to the boundary conditions (147), (148), and let $\mathbf{e}$ be some arbitrary unit vector in $\mathbf{R}^2$. Multiplying (145) with $\mathbf{e}$ leads to

$$-(\mathbf{U}\cdot\boldsymbol{\nabla})(\mathbf{U}\cdot\mathbf{e}) + \Delta(\mathbf{U}\cdot\mathbf{e}) - \boldsymbol{\nabla}\cdot(p\mathbf{e}) = 0\ . \tag{169}$$

Equation (169) can be written as $\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{e}) = 0$, where

$$\mathbf{P}(\mathbf{e}) = -(\mathbf{U}\cdot\mathbf{e})\,\mathbf{U} + [\boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^T]\cdot\mathbf{e} - p\mathbf{e}\ , \tag{170}$$

i.e., the vector field $\mathbf{P}(\mathbf{e})$ is divergence free. Therefore, applying Gauss' theorem to the region $\Omega_{x_0,R}$ of the preceding subsection we find that

$$\int_{\partial\mathbf{B}} \mathbf{P}(\mathbf{e})\cdot\mathbf{n}\ d\sigma = \int_S \mathbf{P}(\mathbf{e})\cdot\mathbf{n}\ d\sigma\ , \tag{171}$$

with $S = \partial\Omega_{x_0,R}\setminus\partial\mathbf{B}$, and with $\mathbf{n}$ outside unit normal vectors. We have that $\mathbf{P}(\mathbf{e}_1)\cdot\mathbf{e}_2 = \mathbf{P}(\mathbf{e}_2)\cdot\mathbf{e}_1$ for any two unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, and therefore, since the vector $\mathbf{e}$ in (171) is arbitrary, it follows that

$$\int_{\partial\mathbf{B}} \mathbf{P}(\mathbf{n})\ d\sigma = \int_S \mathbf{P}(\mathbf{n})\ d\sigma\ . \tag{172}$$

Since $\mathbf{u} = 0$ on $\partial\mathbf{B}$, one finally get from (172) and (170) that the total force which the fluid exerts on the body is

$$\mathbf{F} = \int_{\partial\mathbf{B}} \Sigma(\mathbf{U},p)\mathbf{n}\ d\sigma = \int_S \Big(-(\mathbf{U}\cdot\mathbf{n})\,\mathbf{U} + [\boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^T]\mathbf{n} - p\mathbf{n}\Big)\ d\sigma\ ,$$

with $\Sigma(\mathbf{U},p) = \boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^T - p$ the Stress tensor. The force $\mathbf{F}$ is traditionally decomposed into a component F parallel to the flow at infinity called drag and a component L perpendicular to the flow at infinity called lift. Note that $\mathbf{F}$ is independent of the choice of the surface S. This has the important consequence that F and L can be computed from the dominant terms of the velocity field and the pressure (the dominant large distance behavior of the pressure can also be recovered together with the velocity field starting from the information in the downstream region (see again Bönisch et al. (2006) for further details). In particular, since $\lim_{|y|\to\infty}\mathbf{u}(x,y) = (1,0)$ and with the normalization $\lim_{|y|\to\infty} p(x,y) = 0$, one can again first take the limit $R\to\infty$, and use then the invariance properties of the equations. This leads to the following theorem.

Theorem 35 *If a stationary solution of the stationary Navier-Stokes equations satisfies the conditions of Theorem 20 or Theorem 29 then we have that*

$$d = \frac{1}{2\rho\ell^{d-1}u_\infty^2}\tilde{F} \ , \tag{173}$$

$$\mathbf{b} = \frac{1}{2\rho\ell^{d-1}u_\infty^2}\tilde{\mathbf{L}} \ , \tag{174}$$

with $\tilde{F}$ is the drag and $\tilde{L}$ the lift (dimension-full quantities).

Remark 36 *We are not aware of any work that links the additional constants that appear at higher order in the asymptotic expansion of the wake to other physical quantities like the torque acting on the body. This is another problem worth pursuing. Its solution will allow to obtain a complete prescription of third order artificial boundary conditions (see Section 5).*

Expressions similar to (173) and (174) can be found in many textbooks but are typically based on boundary layer theory (see for example Batchelor (1967), Landau and Lipschitz (1989) or Berger (1971)). In the context of the boundary layer approach the quantities d and $\mathbf{b}$ are however not invariant so that the relations therefore appear to be valid only asymptotically, in the sense that d and $\mathbf{b}$ are supposed to be computed from the flow far downstream of the body. Our results show that in the context of the Navier-Stokes equations the quantities can be computed on any transversal section.

4.5 Open problems

Using the results in van Baalen (2006) one can get expressions similar to (173) and (174) also in the time periodic case, but the right hand side contains an average over the period of the periodic motion. Therefore, either the period of this motion needs to be known in order to link the forces to the invariant quantities, or a procedure like averaging over sufficiently large intervals of time needs to be used in order to extract the constant part of the periodic signal. The theory is however still rather incomplete and the corresponding artificial boundary conditions have not been numerically tested yet. This is an other interesting open problem, in particular also in view of the study of the transition from stationary to time periodic flows. See for example Pipe and Monkewitz (2005) for recent experimental work. Finally, for the case of a falling body that rotates (see Weinberger (1978)), even the downstream asymptotic behavior of the solution is not yet known. We hope that the present work will stimulate further activity in this direction.

5 Artificial boundary conditions

In what follows we define artificial boundary conditions for stationary exterior flows in two and three dimensions. The section is basically self-contained which allows the interested reader to use the boundary conditions without having to work through all of the material of the preceding sections. We expect that beyond the stationary case discussed here similar boundary conditions will be developed in the near future for the case of time periodic flows, for the case of free-falling bodies, for the case of bodies that move close to a wall and for the case of bodies undergoing collisions. One of the main benefits of the artificial boundary conditions defined here is their simplicity. No additional differential equations need to be solved. The boundary conditions are simply explicit Dirichlet boundary conditions on the artificial boundary, depending on parameters which are updated as part of the solution process.

In order to solve the exterior problem described in Section 4.1 numerically, one typically uses the formulation with zero boundary conditions at infinity , *i.e.*, one considers the equations

$$\begin{aligned} -\rho u_\infty \partial_x \tilde{\mathbf{u}} - \rho\left(\tilde{\mathbf{u}} \cdot \boldsymbol{\nabla}\right)\tilde{\mathbf{u}} + \mu\Delta\tilde{\mathbf{u}} - \boldsymbol{\nabla}\tilde{p} &= 0 \ , \\ \boldsymbol{\nabla} \cdot \tilde{\mathbf{u}} &= 0 \ , \end{aligned} \tag{175}$$

in $\tilde{\Omega} = \mathbf{R}^d \setminus \tilde{\mathbf{B}}$, subject to the boundary conditions

$$\tilde{\mathbf{u}}|_{\partial\tilde{\mathbf{B}}} = -\tilde{\mathbf{u}}_\infty \ , \tag{176}$$

$$\lim_{|\tilde{\mathbf{x}}|\to\infty} \tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = 0 \ . \tag{177}$$

Here $\tilde{\mathbf{B}}$, is "the body", *i.e.*, a compact set of diameter R, and $\tilde{\mathbf{u}}_\infty$ is a constant vector field. When restricting for numerical purposes the equations from the exterior infinite domain $\tilde{\Omega}$ to (a sequence of) bounded domains $\tilde{\mathbf{D}} \subset \tilde{\Omega}$, one is confronted with the necessity of finding appropriate boundary conditions on the surface $\tilde{\Gamma} = \partial\tilde{\mathbf{D}} \setminus \partial\tilde{\mathbf{B}}$ of the truncated domain. We now use the results of the proceeding sections to define appropriate boundary conditions. Namely, we set

$$\tilde{\mathbf{u}}|_{\tilde{\Gamma}} = \tilde{\mathbf{u}}_{ABC} \ , \tag{178}$$

with $\tilde{\mathbf{u}}_{ABC}$ the vector fields that are explicitly given below. The vector fields $\tilde{\mathbf{u}}_{ABC}$ depend on the invariant quantities discussed in the previous section and these quantities are computed as part of the solution process.

Note that, choosing instead of (178) for example simply $\tilde{\mathbf{u}}|_{\tilde{\Gamma}} = 0$, forces the mass flux through a vertical line in $\tilde{\mathbf{D}}$ to be zero. This corresponds to invariant quantities equal to zero. However, because of the link that exists between the invariant quantities and the forces acting on the body, any choice of boundary conditions that does not respect the correct mass flux through vertical lines produces significant changes to the forces, unless extremely large computational domains are used. The adaptive boundary conditions (178) eliminate this problem. More details can be found in the sections below and in Bönisch et al. (2005) and Bönisch et al. (2006).

5.1 Stationary flows in two dimensions

For the two dimensional case the artificial boundary conditions on $\tilde{\Gamma} = \partial\tilde{\mathbf{D}} \setminus \partial\tilde{\mathbf{B}}$ are

$$\tilde{\mathbf{u}}_{ABC}(\tilde{\mathbf{x}}) = u_\infty \ \mathbf{u}_{ABC}(\frac{\tilde{\mathbf{x}}}{\ell}) \ , \tag{179}$$

where $\ell = \rho u_\infty/\mu$, and where

$$\mathbf{u}_{ABC} = (u_{ABC}, v_{ABC}) \ , \tag{180}$$

with

$$u_{ABC}(x,y) = -\theta(x)\frac{d}{\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-\frac{y^2}{4x}} + \frac{d}{\pi}\frac{x}{x^2+y^2} + \frac{b}{\pi}\frac{y}{x^2+y^2} \ , \tag{181}$$

$$v_{ABC}(x,y) = -\theta(x)\frac{d}{2\sqrt{\pi}}\frac{y}{x^{3/2}}e^{-\frac{y^2}{4x}} + \frac{d}{\pi}\frac{y}{x^2+y^2} - \frac{b}{\pi}\frac{x}{x^2+y^2} \ , \tag{182}$$

with θ the Heaviside function (*i.e.*, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$), and with

$$d = \frac{1}{2}\frac{1}{\rho\ell u_\infty^2}\tilde{F} \ , \tag{183}$$

$$b = \frac{1}{2}\frac{1}{\rho\ell u_\infty^2}\tilde{L} \ , \tag{184}$$

where $\tilde{F}$ and $\tilde{L}$ are respectively the drag and the lift acting on the body (dimensionfull quantities). The drag $\tilde{F}$ and the lift $\tilde{L}$ are computed (by an evaluation of the stress tensor for example) as part of the solution process (see Section 6), which in turn allows to update the boundary conditions on $\tilde{\Gamma} = \partial\tilde{\mathbf{D}} \setminus \partial\tilde{\mathbf{B}}$ using (183) and (184).

Using the results in Section 3 the boundary conditions can be improved without the introduction of constants other than b and d. Namely one sets

$$\tilde{\mathbf{u}}_{ABC}(\tilde{\mathbf{x}}) = u_\infty \ \mathbf{u}_N(\frac{\tilde{\mathbf{x}}}{\ell}) \ ,$$

where

$$\mathbf{u}_N(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{n} \mathbf{u}_{n,m}(x,y) \ ,$$

with $\mathbf{u}_{n,m}$ the vector fields that have been defined in Conjecture 31. The case $N = 0$ corresponds to choosing homogeneous Dirichlet data on $\tilde{\Gamma} = \partial\tilde{\mathbf{D}} \setminus \partial\tilde{\mathbf{B}}$, the case $N = 1$ is identical to (181), (182) (first order adaptive boundary conditions (see Bönisch et al. (2005)), and $N = 2$ and $N = 3$ are second, respectively third order adaptive boundary conditions (see Bönisch et al. (2006)).

5.2 Stationary flows in three dimensions

For the three dimensional case the artificial boundary conditions on $\tilde{\Gamma} = \partial\tilde{\mathbf{D}} \setminus \partial\tilde{\mathbf{B}}$ are

$$\tilde{\mathbf{u}}_{ABC}(\tilde{\mathbf{x}}) = u_\infty \mathbf{u}_{ABC}(\frac{\tilde{\mathbf{x}}}{\ell}) \ , \tag{185}$$

where $\ell = \rho u_\infty/\mu$, and where

$$\mathbf{u}_{ABC} = (u_{ABC}, \mathbf{v}_{ABC}) \ , \tag{186}$$

$$\mathbf{v}_{ABC} = \mathbf{v}_{1,ABC} + \mathbf{v}_{2,ABC} \ , \tag{187}$$

with

$$u_{ABC}(x,\mathbf{y}) = 1 - \frac{\theta(x)}{2\pi x} e^{-\frac{y^2}{4x}} \ d + \frac{1}{2\pi}\frac{x}{r^3} \ d + \frac{1}{2\pi}\frac{\mathbf{y}\cdot\mathbf{b}}{r^3} \ , \tag{188}$$

$$\begin{aligned} \mathbf{v}_{1,ABC}(x,\mathbf{y}) = {} & -\frac{\mathbf{y}}{4\pi x^2}\theta(x) e^{-\frac{y^2}{4x}} \ d + \frac{1}{2\pi}\frac{\mathbf{y}}{r^3} \ d \\ & - \frac{1}{2\pi}\frac{1}{r}\frac{\mathrm{sign}(x)}{r+|x|}\left(\mathbf{b} - \frac{1}{r}\left(\frac{1}{r} + \frac{1}{r+|x|}\right)(\mathbf{y}\cdot\mathbf{b}) \ \mathbf{y}\right) \ , \end{aligned} \tag{189}$$

$$\begin{aligned} \mathbf{v}_{2,ABC}(x,\mathbf{y}) = {} & -\frac{\theta(x)}{2\pi x}\left(e^{-\frac{y^2}{4x}} + \frac{1}{2}\frac{e^{-\frac{y^2}{4x}} - 1}{\frac{y^2}{4x}}\right) \mathbf{b} \\ & + \frac{\theta(x)}{2\pi x}\left(\frac{e^{-\frac{y^2}{4x}} - 1}{\frac{y^2}{4x}} + e^{-\frac{y^2}{4x}}\right)\frac{(\mathbf{y}\cdot\mathbf{b})}{y^2}\mathbf{y} \ , \end{aligned} \tag{190}$$

with θ the Heaviside function (*i.e.*, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \le 0$), with $\mathrm{sign}(x) = -1 + 2\theta(x)$, and with $\mathbf{y}\cdot\mathbf{b} = y_1 b_1 + y_2 b_2$, and with

$$d = \frac{1}{2\rho\ell^2 u_\infty^2}\tilde{F} \ , \tag{191}$$

$$\mathbf{b} = \frac{1}{2\rho\ell^2 u_\infty^2}\tilde{\mathbf{L}} \ , \tag{192}$$

where $\tilde{F}$ and $\tilde{\mathbf{L}}$ are respectively the drag and the lift acting on the body (dimensionfull quantities). The drag $\tilde{F}$ and the lift $\tilde{\mathbf{L}}$ are computed (by an evaluation of the stress tensor for example) as part of the solution process (see Section 6), which in turn allows to update the boundary conditions on $\tilde{\Gamma} = \partial\tilde{\mathbf{D}} \setminus \partial\tilde{\mathbf{B}}$ using (191) and (192).

5.3 Bibliographic notes

The following notes give access to the literature on artificial boundary conditions in a chronological order. Indeed, even so different problems require different solutions, the basic ideas and techniques are pretty much independent of the specific problem. The literature on the topic is very numerous and we have therefore made no attempt to be exhaustive. The main goal has rather been to compile a list of references from a broad set of applications. Large additional lists of references can be found in the various reviews that we mention.

The problem of artificial boundary conditions can be traced back all the way back to the beginnings of scientific computing. Originally, a popular way of handling exterior problems was to map the exterior domain to some finite domain and to discretize the resulting equations in this finite domain. We do not discuss this approach here, mainly, because in its essence this is not different from choosing artificial boundary conditions, since it corresponds simply to a particular way of specifying such conditions.

The problem with artificial boundaries is most obvious for wave-like equations, since scattering on the artificial boundary obviously produces unphysical reflections of waves back into the computational domain. This has lead to the development of so-called non-reflecting or absorbing boundary conditions. Early work which addresses the subject is due to Engquist and Majda (1979), Bayliss and Turkel (1980), and Sochacki et al. (1986). In Peterson (1988) absorbing boundary conditions for the vector wave equation are discussed. See also Luebbers et al. (1991) for another early reference with a somewhat different discussion of the subject.

Artificial boundary conditions for incompressible viscous flows can be found in Halpern and Schatzman (1989). Another classic reference is Heywood et al. (1992). This work discusses in particular questions related to the mass flux.

The work of Grote (1995), Grote and Keller (1995) on nonreflective boundary conditions has stimulated the work on artificial boundary conditions for the hyperbolic case. Higher order radiation boundary conditions have been proposed in Hagstrom (1995) and absorbing boundary conditions for the Schrödinger equation can be found in Fevens and Jiang (1995). Absorbing boundary conditions for the linearized Euler equations have been proposed in Hu (1996) on the basis of a so-called perfectly matched layers. See also Hesthaven (1997). Griffiths (1997) contains a proof (in $d = 1$) of the effectiveness of well chosen boundary conditions when compared to traditional boundary conditions. A first article containing artificial boundary conditions for the computation of oscillating external flows is due to Tsynkov (1997). Ryaben'kii and Tsynkov (1997) contains a first review of a method that allows the construction of artificial boundary conditions for exterior problems in computational fluid dynamics, and Hagstrom and Hariharan (1998) contains a formulation of asymptotic and exact boundary conditions using local operators. A two dimensional treatment of transonic flow around an airfoil is discussed in Coclici et al. (1998), and transparent boundary conditions for the two dimensional Helmholtz equation are discussed in Schmidt (1998). In Tsynkov and Vatsa (1998) an improved treatment of external boundary conditions for three-dimensional flow computations is discussed, and Tsynkov (1998) contains a major review of the numerical solution of problems in unbounded domains.

The work of Rols et al. (1998) introduces the idea of fractal absorbing boundary conditions in electromagnetic simulations by an application of the spectral moments method. In Rowley and Colonius (2000) so-called discrete non-reflecting boundary conditions are discussed for linear hyperbolic systems and in Schmidt (2000) such boundary conditions are discussed for the Helmholtz equation.

The review of Bruneau (2000) discusses boundary conditions not only for the incompressible but also for compressible Navier-Stokes equations, and Huan and Thompson (2000) discuss boundary conditions for the time-dependent wave equation. The question of adequate artificial boundary conditions for the computation of external flows with jets is reviewed in Tsynkov et al. (2000), and a further review of external boundary conditions for three-dimensional problems of aerodynamics is given in Tsynkov (2000). Thompson and Huan (2000) contains exact nonreflecting boundary conditions and Ryaben'kii and et al. (2001) discrete artificial boundary conditions for the time-dependent wave equation. Higher order artificial boundary conditions for nonlinear wave propagation with backscattering are introduced in Fibich and Tsynkov (2001), and Lions et al. (2002) discusses the questions of the well posedness of an absorbing layer for hyperbolic problems. Nazarov and Specovius-Neugebauer (2003) introduced nonlinear artificial boundary conditions for the exterior three dimensional Navier-Stokes problem, together with pointwise error estimates. An adaptive finite element method with perfectly matched absorbing layers for the computation of wave scattering by periodic structures is introduced in Chen and Wu (2003), and in Novak and Bonazzola (2004) absorbing boundary conditions for the simulation of gravitational waves with spectral methods are discussed. In Nataf (2005), a new construction of perfectly matched layers for the linearize Euler equations is presented.

6 Summary of numerical results

In what follows we give some details concerning the discretization procedure and the algorithms that we have used to solve (175), (176), (178) numerically. See Bönisch et al. (2005), Bönisch et al. (2006) for details. To unburden the notation we suppress throughout this section the “tildes”.

6.1 Galerkin finite element discretization

In order to solve equation (175), we have considered a discretization based on conforming mixed finite elements with continuous pressure. This discretization starts from a variational formulation of the system of equations (175).

For a bounded domain $\mathbf{D} \subset \mathbf{R}^2$, we denote by $L^2(\mathbf{D})$ the Lebesgue space of square-integrable functions on $\mathbf{D}$ equipped with the inner product and the associated norm

$$(f,g)_{\mathbf{D}} = \int_{\mathbf{D}} fg \; d\mathbf{x} \;, \qquad ||f||_{\mathbf{D}} = (f,f)_{\mathbf{D}}^{1/2} \;.$$

The pressure is assumed to be an element of the space $L^2_0(\mathbf{D}) := \{q \in L^2(\mathbf{D}) \mid \int_{\mathbf{D}} q \; d\mathbf{x} = 0\}$, which defines it uniquely. The $L^2(\mathbf{D})$ functions with generalized (in the sense of distributions) first-order derivatives in $L^2(\mathbf{D})$ form the Sobolev space $H^1(\mathbf{D})$, and we define $H^1_0(\mathbf{D}) = \{v \in H^1(\mathbf{D}) \mid v|_{\partial\mathbf{D}} = 0\}$. Now let $W = [H^1_0(\mathbf{D})]^2 \times L^2_0(\mathbf{D})$. For $\mathbf{w} = \{\mathbf{v}, p\} \in W$ and $\boldsymbol{\phi} = \{\boldsymbol{\varphi}, q\} \in W$, we define the semi-linear form

$$\mathcal{A}(\mathbf{w};\boldsymbol{\phi}) = \rho \; (((\mathbf{v}+\mathbf{u}_\infty)\cdot\boldsymbol{\nabla})\mathbf{v},\boldsymbol{\varphi})_{\mathbf{D}} - (p,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_{\mathbf{D}} + 2\mu\int_{\mathbf{D}} \mathcal{D}(\mathbf{v}) : \mathcal{D}(\boldsymbol{\varphi}) \; d\mathbf{x} - (\boldsymbol{\nabla}\cdot\mathbf{v}, q)_{\mathbf{D}} \;, \tag{193}$$

which is obtained by testing the equations (175) with $\boldsymbol{\phi} \in W$ and by integration by parts of the diffusive term and the pressure gradient (see *e.g.* Rannacher (2000); Galdi (1998d,f); Turek (1999); Heywood et al. (1992) for more details). $\mathcal{D}(\mathbf{v})$ denotes the deformation tensor, *i.e.*, $\mathcal{D}(\mathbf{v}) = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T)$. Then, a weak form of the equations (175) can be formulated as: find $\mathbf{w} = \{\mathbf{v}, p\} \in W$, such that

$$\mathcal{A}(\mathbf{w};\boldsymbol{\phi}) = 0 \;, \quad \forall \boldsymbol{\phi} \in W \;. \tag{194}$$

The discretization of problem (194) uses a conforming finite element space $W_h \subset W$ defined on quasi-uniform triangulations $\mathcal{T}_h = \{K\}$ consisting of quadrilateral cells K covering the domain $\mathbf{D}$. We have used the standard Hood-Taylor finite elements (Hood and Taylor (1973)) for the trial and test spaces, *i.e.*, we used

$$W_h = \left\{(\mathbf{v},p) \in [C(\overline{\mathbf{D}})]^3 \mid \mathbf{v}|_K \in [Q_2]^2, \; p|_K \in Q_1\right\} \;,$$

where Q_r describes the space of iso-parametric tensor-product polynomials of degree r (for a detailed description of this standard construction process see for example Brenner and Scott (1994)). This choice for the trial and test functions guarantees a stable approximation of the pressure since the Babuska-Brezzi inf-sup stability condition is satisfied uniformly in $\mathbf{D}$ (see Brezzi and Falk (1991) and references therein). The advantage, when compared to equal order function spaces for the pressure and the velocity, is that no additional stabilization terms are needed. The discrete counterpart of the problem (194) then reads: find $\mathbf{w}_h = \{\mathbf{v}_h, p_h\} \in \mathbf{w}_{b,h} + W_h$, such that

$$\mathcal{A}(\mathbf{w}_h;\boldsymbol{\phi}_h) = 0 \;, \quad \forall \boldsymbol{\phi}_h \in W_h \;. \tag{195}$$

Here, $\mathbf{w}_{b,h}$ describes the prescribed Dirichlet data on the boundary $\boldsymbol{\Gamma}$ of the domain $\mathbf{D}$.

The artificial boundary conditions of Section 5 are independent of the details of the geometry of the body, but they depend explicitly on drag and lift. The accurate determination of these forces is therefore a key issue in this context. We have used the approach proposed in Giles et al. (1997) which is based on a reformulation of the expressions for drag and lift in terms of volume integrals. This formulation allows to attain the full order of convergence for the values of drag and lift.

6.2 The solver

We have solved the nonlinear algebraic system (195) in a fully coupled manner by means of a damped Newton method. Denoting the derivative of $\mathcal{A}(\cdot,\cdot)$ taken at a discrete function $\mathbf{w}_h \in W_h$ by $\mathcal{A}'(\mathbf{w}_h,\cdot)(\cdot)$, the linear system arising at the Newton step number k has the following form,

$$\mathcal{A}'(\mathbf{w}_h^k,\boldsymbol{\phi}_h)(\hat{\mathbf{w}}_h^k) = (\mathbf{r}_h^k,\boldsymbol{\phi}_h) \;, \qquad \forall \boldsymbol{\phi}_h \in W_h \;, \tag{196}$$

where $\mathbf{r}_h^k$ is the equation residual of the current approximation $\mathbf{w}_h^k$, and where $\hat{\mathbf{w}}_h^k$ corresponds to the needed correction. The updates $\mathbf{w}_h^{k+1} = \mathbf{w}_h^k + \alpha^k\hat{\mathbf{w}}_h^k$ with a relaxation parameter α^k chosen by means of Armijo's rule are carried out until convergence. In practice, the Jacobian involved in (196) is directly derived from the analytical expression for the derivative of the variational system (195).

It is well known that the rapid convergence of Newton iterations greatly depends on the quality of the initial approximation (see *e.g.* Kelley (1995)). In order to find such an initial approximation, we consider a mesh hierarchy $\mathcal{T}_{h_l}$ with $\mathcal{T}_{h_l} \subset \mathcal{T}_{h_{l+1}}$, and the corresponding system of equations (195) is successively solved by taking advantage of the previously computed solution, *i.e.*, the nonlinear Newton steps are embedded in a nested iteration process.

More precisely, the linear subproblems (196) are solved by the generalized minimal residual method (GMRES), see Saad (1996), preconditioned by means of multigrid iterations. See Wesseling and Oosterlee (2001) and Wesseling (1992) and references therein for a description of the different multigrid techniques for flow simulations. Our preconditioner is based on a new multigrid scheme which is optimized for conformal higher order finite element methods. It is a key ingredient of the overall solution process. Two specific features characterizing the scheme are: varying order of the finite element Ansatz on the mesh hierarchy and a Vanka type smoother (Vanka (1986)) adapted to higher order discretization. This somewhat technical part of the solver is described in full detail in Heuveline (2003). Its implementation is part of the HiFlow project (see Heuveline (2000)).

6.3 Numerical results

The following figure summarizes some of the results in two dimensions. See Bönisch et al. (2005), Bönisch et al. (2006) for details. Similar work in three dimensions is in preparation and will be published elsewhere.

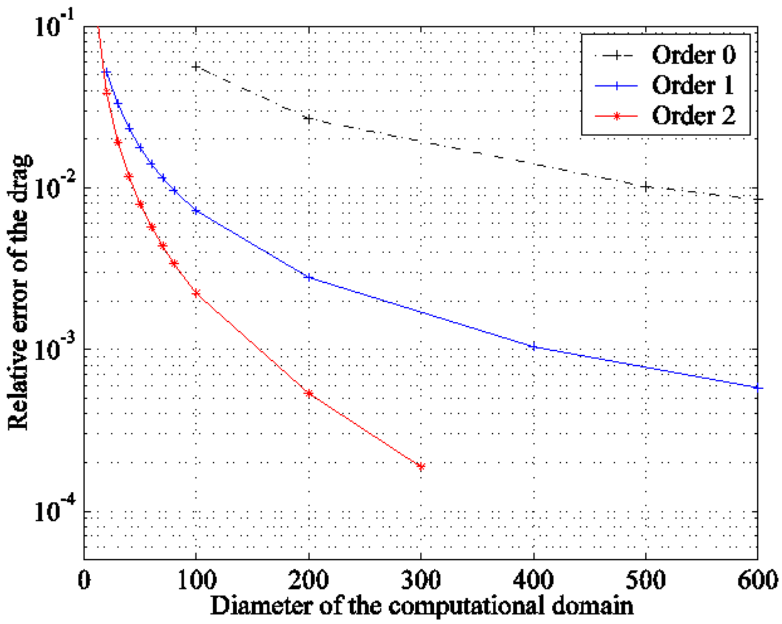

Figure 2. The figure shows the size of the relative error for the drag as a function of the diameter of the computational domain for a test configuration consisting of a flat plate of diameter one at Reynolds number Re $= 1$. A reference value for the drag has been computed with a very large scale computation on a domain of size 5000. To compute the drag with an error of about one percent, a domain with 500 times the body size is needed with naive boundary conditions, with about 100 times the body size with first order and with about 50 times the body size with second order artificial boundary conditions.

7 Bibliography

The following notes provide entry points to the literature on low Reynolds number flows inasmuch as they have not yet been provided in the more specific bibliographic sections above or within the text. The main goal is again to provide a list of references from a broad range of applications. The big number of recent references shows that the subject of low Reynolds numbers is in spite of its long history still, or maybe, again, a very active topic of research.

7.1 Books

A very interesting early reference are the lectures on fluid mechanics by Goldstein (1957) and a classic reference to boundary layer theory is the book by Schlichting and Gersten (1999). A classic reference for perturbation theory in fluid mechanics is van Dyke (1975). An nicely written and easy to read introduction to the problem of exterior flows is the booklet by Ockendon and Ockendon (1995). A recent book that contains an important section on low Reynolds number flows is Guyon et al. (2001). A book on viscous incompressible flows at (very) low Reynolds numbers is Kohr and Pop (2004).

7.2 Boundary layer theory, wakes

The computation of the forces on bodies has only made significant progress after the introduction of boundary layer theory, see Blasius (1908), which has allowed to explain and resolve the d'Alambert paradox, *i.e.*, the fact that the Euler equations lead to a no drag theorem in two dimensions and a no drag and no lift theorem in three dimensions. See also Goldshtik (1990) for a review of viscous-flow paradoxes. Extensive computations of wakes based on boundary layer theory can be found in Stewartson (1957). Questions concerning the uniqueness of solutions are discussed for example in Smith (1984). The computation of axisymmetric flows for slender bodies goes back to the paper of Bodonyi et al. (1985). More recently boundary layer computations have been reviewed in Cole (1994), Anderson Jr (2005), Tulapurkara (2005), and Cowley (2006). In this general context it is also useful to consult the publications by Lamb et al. (2003) which discusses bifurcation theory from periodic solutions with spatiotemporal symmetry.

7.3 Expansion techniques

Expansion techniques have played a very important role for the computation of the forces that act on bodies that move through liquids. See in particular also Keller and Ward (1996) for a reference concerning low Reynolds number flows, and Boyd (1999) for a discussion of expansion techniques. See also Schwartz (2002) for a discussion of the work of van Dyke. Recent references are Vorobieff et al. (2002) and the publication by Kohr (2004) where the method of matched asymptotic expansions for low Reynolds number flow past a cylinder of arbitrary cross section is discussed.

7.4 Flows around plates, cylinders and spheres

With the introduction of boundary layer theory many authors have computed the drag on simple geometric obstacles like plates, cylinders and spheres. For (semi-infinite) flat plates interesting references are Alden (1948), Imai (1957), Olmstead and Hector (1966), Olmstead (1975) and Lagerstrom (1975). See also Bichsel and Wittwer (2007) for a recent review of the semi-infinite flat plate problem.

Interesting references discussing flows around cylinders are Dennis and Shimshoni (1965), Kropinski et al. (1995), and Titcombe et al. (1999). For a recent reference to experimental techniques, including the case of time periodic flows see Fujisawa et al. (2005).

Flows around spheres, including the question of the stability of such flows are discussed in Shirayama (1992) and Cliffe et al. (2000). An analytical solution of low Reynolds number slip flow past a sphere can be found in Barber and Emerson (2000), and the drag on a sphere moving horizontally in a stratified fluid is discussed in Greenslade (2000). Jayaraman and Belmonte (2003) contains the observation of the oscillations of a solid sphere falling through a wormlike micellar fluid.

7.5 Numerical studies

An early reference containing a numerical study of the drag on a sphere at low and intermediate Reynolds number is Le Clair et al. (1970). A recent reference for the flow around a cylinder is Padrino and Joseph (2006). Oscillatory flows are discussed in Testik et al. (2005). An other reference to the time dependent case is Bönisch and Heuveline (2006). Interesting work on the bases of the lattice Boltzmann method is Verberg and Ladd (2000) and Latt et al. (2006). The application of the lattice Boltzmann method to the simulation of particle-fluid suspensions can be found in Ladd and Verberg (2001).

7.6 Linearized problems (Stokes, Oseen)

Linearized problems play an important role. On one hand they are directly used as approximations to the full equations, on the other hand their study is the basis of most work on the nonlinear problems. The Stokes equations are quantitatively useful at very low Reynolds numbers (less than one). Quantitatively, the Oseen equations are less successful and have played a less important role for direct computations. See for example Weisenborn and Bosch (1995). In spite of these shortcomings the Oseen equation captures much better than the Stokes equations the asymptotic behavior of the flows in the regime of Reynolds numbers above one to several hundred. Interesting aspects of the Oseen equation are discussed in Kress and Meyer (2000). The hydrodynamic forces on submerged rigid bodies and its relation to the far field behavior are discussed in Guenther et al. (2002). A well-posedness analysis for the so-called Oseen coupling method for exterior flows is discussed in He et al. (2004). A recent discussion of the Oseen problem in the whole space is due to Boulmezaoud and Razafison (2005). Guenther and Thomann (2005) contains a new discussion of the fundamental solutions of the Stokes and Oseen problem in two spatial dimensions, and Thomann and Guenther (2006) contains the fundamental solutions, including the time dependent case, for the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions. Girault et al. (1992) contains a stream-function-vorticity variational formulation for the exterior Stokes problem in weighted Sobolev spaces, and Amrouche and Razafison (2006) provide weighted estimates for the Oseen problem in three dimensions.

7.7 Other references on exterior flows

The standard reference concerning the existence of solutions for exterior flows is Leray (1934). The question of the behavior of the solutions at infinity was for longtime an open problem. See Finn (1960) and Finn (1965). Another early reference concerning solutions of the stationary and non stationary Navier-Stokes equations in exterior domains is Chen (1993). The problem is reviewed in Galdi (1998c). See also Galdi (1999b) for the description of an important open problem. More recent references are Farwig (1998), Cerejeiras and Kähler (2000), Galdi and Rabier (2000), and Giga et al. (2001). Stability questions are discussed in Biler et al. (2004). The approximation of three dimensional stationary flows by flows in bounded domains are discussed in Deuring and Kračmar (2004). Nazarov (1999) contains the discussion of the Navier-Stokes problem in a two-dimensional domain with angular outlets to infinity, and Shibata and Yamazaki (2005) provides uniform estimates for the velocity at infinity for stationary solutions. Geissert et al. (2004) reviews the theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle, and Galdi (2006) discusses modes, nodes and volume elements for stationary solutions of the Navier-Stokes problem past a three-dimensional body.

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