Stochastic 3D Navier-Stokes equations in a thin domain and its α-approximation

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Abstract

In the thin domain $\mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is a two-dimensional torus, we consider the 3D Navier-Stokes equations, perturbed by a white in time random force, and the Leray α-approximation for this system. We study ergodic properties of these models and their connection with the corresponding $2D$ models in the limit $\varepsilon \to 0$. In particular, under natural conditions concerning the noise we show that in some rigorous sense the 2D stationary measure μ comprises asymptotical in time statistical properties of solutions for the 3D Navier-Stokes equations in $\mathcal{O}_\varepsilon$, when $\varepsilon \ll 1$.

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1 Introduction

In this paper we study the stochastic Navier-Stokes equations (NSE) in a thin three-dimensional domain $O_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is the torus $\mathbb{R}^2/(l_1\mathbb{Z} \times l_2\mathbb{Z})$. That is, in O_ε we consider the 3D NSE, perturbed by a random force, which is smooth as a function of the space-variable x, while as a function of time t it is a white noise. Using the Leray projection Π_ε we write the equation as

$$u' + \nu A_\varepsilon u + B_\varepsilon(u, u) = f_\varepsilon + \dot{W}_\varepsilon \,. \tag{1.1}$$

Here A_ε is the Stokes operator $-\Pi_\varepsilon \Delta$, $B_\varepsilon(u,\, u) = \Pi_\varepsilon((u \cdot \nabla)u)$, $f_\varepsilon(x)$ is a deterministic part of the force and $\dot{W}_\varepsilon(t,\, x)$ is the time-derivative of a Wiener process $W_\varepsilon(t,\, x)$ in an appropriate function space. The equations are supplemented with the free boundary conditions in the thin direction (see (2.4) below).

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This boundary value problem for the 3D NSE describes a special case of anisotropic 3D turbulence, important for the meteorology (see, e.g., [7]). The natural related question is to find out *up to what extend this anisotropic 3D turbulence can be approximated by 2D turbulence.* In our work we continue the rigorous study of this problem, initiated in [4].

The study of global existence of strong solutions for the deterministic Navier-Stokes equations in thin three-dimensional domains began with the papers of Raugel and Sell [21, 22], who proved global existence of strong solutions for large initial data and forcing terms in the case of periodic or mixed boundary conditions. After these initial results, a series of papers by different authors followed, in which the results of Raugel and Sell were sharpened and generalised in various ways, see [19, 11, 20, 1, 24, 12, 5]. In the quoted works it was also shown that for $\varepsilon \ll 1$ solutions of the 3D NSE in an ε-thin domain becomes close to solutions of the corresponding 2D NSE.

From other hand, the stochastic 2D NSE and similar to them 2D NSE, perturbed by random kick-forces, were intensively studied in recent years by many authors, see [10, 16, 2, 9, 15] and references therein. Under some mild restriction on the random force it was proved that the equation has a unique stationary measure, which governs stochastic properties of its solutions as time goes to infinity; we refer to the survey [15] for details.

In [4] the authors of this work considered the 3D NSE in $\mathcal{O}_\varepsilon$, perturbed by a random kick-force. That is, we considered the equation (1.1), when the r.h.s. $f_\varepsilon + \dot{W}_\varepsilon$ is replaced by a random kick-force. Assuming that the force is not too big and is genuinely random we proved that the equation, regarded as a random dynamical system in the H^1-space of a divergence-force vector fields, has a unique stationary measure; that all solutions converge to this measure in distribution, and that the two-dimensional part of the stationary measure (defined below) converges, when ε goes to zero, to a stationary measure for the 2D NSE on the torus $\mathbb{T}^2$. It is shown in [4] that the results obtained apply to study asymptotical properties of various physically relevant characteristics of the flow, described by the Navier-Stokes equations in the domain $\mathcal{O}_\varepsilon$.

Our goal in this work is to extend the results of [4] to the stochastic NSE (1.1). This tasks complicates by the well known difficulty: no matter how small ε is, almost every solution for the stochastic NSE (1.1) exists only finite time.[1] So we cannot study its asymptotical in time properties directly. To resolve this difficulty we apply a trick, often used in physics: we regularise the equation, study its limiting properties and next remove the regularisation. For the regularised equation we take the α-model, introduced by J. Leray in [18] for an analytical study of the NSE. Namely, we replace the nonlinearity $(u \cdot \nabla)u$ by $(G_\alpha u \cdot \nabla)\, u$, where $G_\alpha = (1 + \alpha A_\varepsilon)^{-1}$, and write thus regularised equation as

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u,\, u) = f_\varepsilon + \dot{W}_\varepsilon\,, \tag{1.2}$$

[1] More specifically, when time grows, the (strong) solution inevitably becomes very large, so due to the well known lack of a corresponding result on the 3D NSE we cannot guarantee that it keeps existing.

see Section 2.5. Analytical properties of eq. (1.2) are as good as those of the 2D NSE (in fact, they are even better). In particular, for any initial data the equation has a unique solution, existing for all t, and the techniques, developed for the stochastic 2D NSE allow to show that the equation has a stationary measure μ_ε^α, which is unique if the force $\dot{W}_\varepsilon$ is nondegenerate. In the latter case every solution $u(t,\,x)$ of (1.2) converges to this measure in distribution:

$$\mathcal{D}\,u(t) \rightharpoonup \mu_\varepsilon^\alpha \quad \text{as} \quad t \to \infty\,,$$

see Theorem 2.7. Our goal is to study behaviour of the measure μ_ε^α when $\varepsilon \to 0$ and $\alpha \to 0$.

To describe the results, we consider the operator M_ε of averaging in the thin direction x_3, which maps 3D velocity fields on $\mathcal{O}_\varepsilon$ to 2D fields on $\mathbb{T}^2$ by the formula

$$(M_\varepsilon u)(x') = \left(\frac{1}{\varepsilon}\int_0^\varepsilon u_1(x', x_3)\,dx_3,\ \frac{1}{\varepsilon}\int_0^\varepsilon u_2(x', x_3)\,dx_3\right),\ x' = (x_1, x_2) \in \mathbb{T}^2. \tag{1.3}$$

As in the previous works on the NSE in thin 3D domains, we compare $M_\varepsilon u(t)$, where $u(t)$ satisfies (1.2), with solutions for the 2D equation

$$v' + A_0 v + B_0(v,\,v) = \widetilde{f}(x') + \dot{\widetilde{W}}(t,\,x'), \quad x' \in \mathbb{T}^2\,, \tag{1.4}$$

where A_0 and B_0 are the corresponding 2D Stokes operator and the bilinear operator, and $\widetilde{f}$ and $\widetilde{W}$ are limits of $M_\varepsilon f_\varepsilon$ and $M_\varepsilon W_\varepsilon$ as $\varepsilon \to 0$ (below we assume that these limits exist). Under a mild nondegeneracy assumption on the noise $\dot{\widetilde{W}}$, the equation has a unique stationary measure μ (which is a Borel measure in the L_2-space of divergence-free vector fields on $\mathbb{T}^2$), see Theorem 2.4. We also consider the α-approximation for equation (1.4) by putting $B_0((1+\alpha A_0)^{-1}v, v)$ in (1.4) instead of $B_0(v, v)$. Under the same nondegeneracy assumptions it also has a unique stationary measure μ^α.

Our main results are presented in Theorem 3.1 and Theorem 3.2. In addition to some nondegeneracy conditions on the random forces they require that (i) the correlation operator of the Wiener process $W_\varepsilon(t)$ in the L_2-space of vector-functions on $\mathcal{O}_\varepsilon$ with respect to the normalised measure $\varepsilon^{-1}\,dx_1\,dx_2\,dx_3$ has a finite trace, bounded uniformly in ε; (ii) the L_2-norms of functions $f_\varepsilon(x)$ are bounded uniformly in ε, and (iii) the correlation operator K_0 of 2D Wiener process $\widetilde{W}(t,\,x')$ satisfies the condition $\operatorname{tr} A_0 K_0 < \infty$.

The first main result (see Theorem 3.1) states that the projection $M_\varepsilon \mu_\varepsilon^\alpha$ of a stationary measure μ_ε^α for (1.2) weakly converges as $\varepsilon \to 0$ to a unique stationary measure μ^α of the 2D Leray approximation which, in its turn, converges as $\alpha \to 0$ to the unique stationary measure μ of 2D NSE (1.4). Moreover, if $\alpha = \alpha(\varepsilon)$ is a function of ε which converges to zero as $\varepsilon \to 0$ sufficiently slow in comparison with ε, then again $M_\varepsilon \mu_\varepsilon^{\alpha(\varepsilon)}$ converges to μ as $\varepsilon \to 0$. In particular, if the 3D noise $\dot{W}_\varepsilon$ is nondegenerate, then the stationary measure μ_ε^α is unique, and for any solution $u_\varepsilon^\alpha(t)$ of (1.2) we have

$$M_\varepsilon \mathcal{D}\, u_\varepsilon^\alpha(t) \xrightarrow{t\to\infty} M_\varepsilon \mu_\varepsilon^\alpha \xrightarrow{\varepsilon\to 0} \mu^\alpha \xrightarrow{\alpha\to 0} \mu\,,$$

where the arrows indicates the weak convergence of measures.

Our next main result deals with the limit 'first $\varepsilon \to 0$, next $\alpha \to 0$'. It is easy to establish that the set of measures $\{\mu_\varepsilon^\alpha, 0 < \alpha \le 1\}$ is tight in the space of Borel measures in the corresponding L_2-space (see, e.g., Theorem 2.9 below). Let us denote by $\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha$ the set of all its limiting points as $\alpha \to 0$. We prove in Theorem 3.2 that the set $\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha$ is formed by (weakly) stationary measures for the 3D NSE (1.1), and $M_\varepsilon(\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha)$ weakly converges to 2D stationary measure μ as $\varepsilon \to 0$. This means that choosing for each $\varepsilon > 0$ any $\mu_\varepsilon \in \mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha$ we have $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$.

Jointly the two theorems show that

$$\lim_{\alpha\to 0}\lim_{\varepsilon\to 0} M_\varepsilon \mu_\varepsilon^\alpha = \lim_{\varepsilon\to 0}\mathop{\mathrm{Lim}}_{\alpha\to 0} M_\varepsilon \mu_\varepsilon^\alpha = \mu. \tag{1.5}$$

That is, in some rigorous sense the anisotropic 3D turbulence, described by eq. (1.1) with a force, having bounded normalised intensity, may be approximated by the 2D turbulence, described by eq. (1.4). In particular, the energy of the 3D flow is close to that of a corresponding 2D flow (as well as averaging of any functional of the flow, which is continuous in the L_2-norm). In the same time, we cannot prove that averaged enstrophy or enstrophy production of the 3D flow converges to that of the 2D flow, see a discussion in Section 3.

The paper is organised as follows. In Section 2 we describe the models under the consideration, quote several known results concerning statistical solutions and stationary measures and give some preliminary results on dependence of statistical characteristics on ε as $\varepsilon \to 0$. This section also contains Theorem 2.7 and Theorem 2.9 on the existence and limiting properties of stationary measures μ_ε^α and corresponding statistical solutions for fixed ε, which, as we believe, are of independent interest. In Section 3 we formulate our main results (Theorem 3.1 and Theorem 3.2). The proofs are rather technical and defer to Section 4. In this section we also prove Proposition 2.8 which makes an auxiliary step in the proof of the uniqueness of the stationary measure μ_ε^α for (1.2) in Theorem 2.7.

Notations. We denote by $\mathcal{D}(\cdot)$ the distribution of a random variable, denote by the symbol $\rightharpoonup$ the weak convergence of measures and denote by $|\cdot|_{\mathcal{L}(H)}$ the operator-norm for operators in a Hilbert space H.

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2 Models

2.1 3D Navier-Stokes equations in a thin domain

Let $\mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is the torus $\mathbb{T}^2 = \mathbb{R}^2/(l_1\mathbb{Z} \times l_2\mathbb{Z})$, $l_1, l_2 > 0$, and $\varepsilon \in (0, 1]$. Let $x = (x', x_3) = (x_1, x_2, x_3) \in \mathcal{O}_\varepsilon$, and let

$$u(x) = (u_1(x), u_2(x), u_3(x)), \quad x \in \mathcal{O}_\varepsilon,$$

stands for a vector function on $\mathcal{O}_\varepsilon$. On the domain $\mathcal{O}_\varepsilon$ we consider the Navier-Stokes equations (NSE) perturbed by the white noise

$$\partial_t u - \nu \triangle u + \sum_{j=1}^{3} u_j \partial_j u + \nabla p = f_\varepsilon + \dot{W}_\varepsilon \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.1}$$

$$\text{div } u = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.2}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon. \tag{2.3}$$

We supplement the equations with the free boundary conditions in the thin direction. Thus we impose the following boundary conditions:

$$\begin{cases} x' \in \mathbb{T}^2 \quad \text{(i.e., } u \text{ is } (l_1, l_2)\text{-periodic with respect to } (x_1, x_2)\text{)}, \\ \text{and} \\ u_3|_{x_3=\varepsilon} = 0, \quad \partial_3 u_j|_{x_3=\varepsilon} = 0, \quad j = 1, 2, \\ u_3|_{x_3=0} = 0, \quad \partial_3 u_j|_{x_3=0} = 0, \quad j = 1, 2. \end{cases} \tag{2.4}$$

Here above $f_\varepsilon = f_\varepsilon(x)$ is a deterministic time-independent force, and $\dot{W}_\varepsilon(t)$ is generalised derivative of a Wiener process with values in appropriate function space (see Section 2.2 below).

Let $\mathcal{W}_\varepsilon$ be the space, formed by divergence-free vector fields $u = (u_j)_{j=1,2,3}$ on $\mathcal{O}_\varepsilon$ such that

$$u \in \left[H^2(\mathcal{O}_\varepsilon)\right]^3, \quad \int_{\mathcal{O}_\varepsilon} u_j dx = 0, \ j = 1, 2,$$

and condition (2.4) is satisfied. Let V_ε (respectively, H_ε) be the closure of $\mathcal{W}_\varepsilon$ in $\left[H^1(\mathcal{O}_\varepsilon)\right]^3$ (respectively, in $\left[L^2(\mathcal{O}_\varepsilon)\right]^3$). We denote by $|\cdot|_\varepsilon$ and $(\cdot, \cdot)_\varepsilon$ the norm and the inner product in H_ε and by

$$\|u\|_\varepsilon \equiv |\nabla u|_\varepsilon = [a_\varepsilon(u, u)]^{1/2}$$

the norm in V_ε. Here and below

$$a_\varepsilon(u, v) = \sum_{j=1}^{3} \int_{\mathcal{O}_\varepsilon} \nabla u_j \cdot \nabla v_j \, dx.$$

We will also use the normalised versions of the introduced norms:

$$|\cdot|_{0,\varepsilon} = \varepsilon^{-1/2} |\cdot|_\varepsilon, \quad \|u\|_{0,\varepsilon} = \varepsilon^{-1/2} \| u\|_\varepsilon. \tag{2.5}$$

We denote by A_ε the Stokes operator defined as an isomorphism from V_ε onto the dual V_ε' by

$$(A_\varepsilon u, v)_{V,V'} = a_\varepsilon(u,v), \quad u, v \in V_\varepsilon \, .$$

The operator is extended to H_ε as a linear unbounded operator with a domain $\mathrm{D}(A_\varepsilon) = \mathcal{W}_\varepsilon$. Let Π_ε be the Leray projector on H_ε in $(L^2(\mathcal{O}_\varepsilon))^3$. Then

$$(A_\varepsilon u)(x) = (-\Pi_\varepsilon \Delta u)(x), \quad \text{for almost all } x \in \mathcal{O}_\varepsilon$$

for every $u \in \mathrm{D}(A_\varepsilon)$.

Now we consider the trilinear form

$$b_\varepsilon(u,v,w) = \sum_{j,l=1}^{3} \int_{\mathcal{O}_\varepsilon} u_j \, \partial_j v_l \, w_l \, dx, \quad u, v \in \mathrm{D}(A_\varepsilon), \; w \in (L^2(\mathcal{O}_\varepsilon))^3.$$

It defines a bilinear operator B_ε by the formula

$$(B_\varepsilon(u,v), w)_{V_\varepsilon, V_\varepsilon'} = b_\varepsilon(u,v,w), \quad u, v, w \in V_\varepsilon \, ,$$

and the system (2.1)–(2.4) can be written in the Leray form (1.1).

It is proved in the works on deterministic equations, mentioned in Introduction (see, e.g., [12, 24]), that if the random component $\dot{W}_\varepsilon$ of the force vanishes, while $f_\varepsilon \in H_\varepsilon$ and $u_0 \in V_\varepsilon$ are bounded in certain sense, then for $\varepsilon \ll 1$ the problem (1.1) has a unique strong solution. In [4] a similar result has been obtained for the 3D NSE, perturbed by a random kick-force. In this work we are concerned with forces, having non-trivial white component $\dot{W}_\varepsilon$. We begin their study with discussion of basic properties of the white forces and statistical (weak) solutions.

2.2 Noise

We assume that the Wiener process W_ε has the form

$$W_\varepsilon(t,x) = \sum_j b_j^\varepsilon \beta_j(t) \, e_{\lambda_j}(x) + \sum_j \hat{b}_j^\varepsilon \, \hat{\beta}_j(t) e_{\Lambda_j^\varepsilon}(x) \, . \tag{2.6}$$

Here b_j^ε, $\hat{b}_j^\varepsilon$ are real numbers such that

$$B_0^\varepsilon = \sum_j (b_j^\varepsilon)^2 < \infty \, , \quad \hat{B}_0^\varepsilon = \sum_j (\hat{b}_j^\varepsilon)^2 < \infty \, , \tag{2.7}$$

and $\beta_j(t)$, $\hat{\beta}_j(t)$ are standard independent Wiener process, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. So $W_\varepsilon(0) = 0$. The system of vectors $\{e_{\lambda_j}, e_{\Lambda_j^\varepsilon}; j \ge 1\}$ is the orthogonal basis of H_ε, formed by eigenfunctions of the Stokes operator, corresponding to eigenvalues $\{\lambda_j, \Lambda_j^\varepsilon\}$. They are normalised as follows:

$$|e_{\lambda_j}|_{0,\varepsilon} = |e_{\Lambda_j^\varepsilon}|_{0,\varepsilon} = 1$$

(see Appendix). So these vectors form an orthonormal basis of the space $(H_\varepsilon, |\cdot|_{0,\varepsilon})$, while the vectors $\{\varepsilon^{-1/2}e_{\lambda_j}, \varepsilon^{-1/2}e_{\Lambda_j^\varepsilon}\}$ form an orthonormal basis of $(H_\varepsilon, |\cdot|_\varepsilon)$. We also note that the eigenfunctions e_{λ_j} have the structure $e_{\lambda_j} = (\widetilde{e}_{\lambda_j}; 0)$, where $\widetilde{e}_{\lambda_j}$ are the eigenfunctions of the $2D$ Stokes operator on $\mathbb{T}^2$ which correspond the to same eigenvalues.

For any vectors $f = \sum_j f_j\, e_{\lambda_j} + \sum_j \hat{f}_j e_{\Lambda_j^\varepsilon}$ and $h = \sum_j h_j\, e_{\lambda_j} + \sum_j \hat{h}_j e_{\Lambda_j^\varepsilon}$ from H_ε, we have $\mathbf{E}(W_\varepsilon(t), f)_\varepsilon = 0$ and

$$\mathbf{E}(W_\varepsilon(t), f)_\varepsilon (W_\varepsilon(s), h))_\varepsilon = (t \wedge s)\,(K_\varepsilon f, h)_\varepsilon,$$

where the correlation operator K_ε is diagonal in the basis $\{e_{\lambda_j}, e_{\Lambda_j^\varepsilon}, j \geq 1\}$:

$$K_\varepsilon e_{\lambda_j} = \varepsilon [b_j^\varepsilon]^2 e_{\lambda_j}, \quad K_\varepsilon e_{\Lambda_j^\varepsilon} = \varepsilon [\hat{b}_j^\varepsilon]^2 e_{\Lambda_j^\varepsilon}, \quad j = 1, 2, \ldots \tag{2.8}$$

The relations above imply that

$$\mathbf{E}|W_\varepsilon(t)|_\varepsilon^2 = t \cdot \varepsilon \cdot \left[\sum_j [b_j^\varepsilon]^2 + \sum_j [\hat{b}_j^\varepsilon]^2 \right] \equiv t \cdot \operatorname{tr} K_\varepsilon < \infty.$$

Note that the correlation operator for the process W_ε with respect to the scalar product $(\cdot,\cdot)_{0,\varepsilon} \equiv \varepsilon^{-1}(\cdot,\cdot)_\varepsilon$ generated by $|\cdot|_{0,\varepsilon}$ is $\varepsilon^{-1}K_\varepsilon$.

It is well known that for a.e. ω the corresponding realisation of the process W_ε defines a continuous curve $W_\varepsilon(t) \in H_\varepsilon$, see [8].

2.3 3D statistical solutions

We recall now some results from [25] (see also [26]) concerning statistical solutions of problem (2.1)-(2.4).

Let us denote by $\mathcal{W}_\varepsilon^{-s}$ the completion of the space H_ε with respect to the norm $|A_\varepsilon^{-s}\cdot|_\varepsilon$ with some $s > 5/4$, and for any $T > 0$ let $\mathcal{Z}_T^\varepsilon$ be the space of functions $u(x,t)$ in $C(0,T;\mathcal{W}_\varepsilon^{-s})$ such that

$$|\, u\,|_{\mathcal{Z}_T^\varepsilon} \equiv \sup_{0\le t\le T} |A_\varepsilon^{-s}(u(t))|_\varepsilon + \Big(\int_0^T |u(\tau)|_\varepsilon^2 d\tau\Big)^{1/2} < \infty\,.$$

We also set

$$\mathcal{Z}^\varepsilon = \{u \in C(0,\infty;\mathcal{W}_\varepsilon^{-s}) :\, u_T := u\,|_{[0,T]} \in \mathcal{Z}_T \;\text{ for any }\; T > 0\}\,. \tag{2.9}$$

This is a complete metric space with respect to the distance

$$\operatorname{dist}_{\mathcal{Z}^\varepsilon}(u,v) = \sum_{n=1}^\infty 2^{-n} \frac{|(u-v)_n|_{\mathcal{Z}_n^\varepsilon}}{1 + |(u-v)_n|_{\mathcal{Z}_n^\varepsilon}}\,. \tag{2.10}$$

It is proved in [25] that if $f_\varepsilon \in H_\varepsilon$ and $u_0(x)$ is a random variable, independent from the force $\dot{W}_\varepsilon$ and satisfying $\mathbf{E}|u_0|_\varepsilon^{2+\eta} < \infty$ for some $\eta > 0$, then the problem

(2.1)-(2.4) has a *statistical solution* which is a Borel probability measure P_ε in $\mathcal{Z}^\varepsilon$ supported by the set of functions $u(x,t)$ in $C(\mathbb{R}_+; \mathcal{W}_\varepsilon^{-s})$ such that

$$| u |_T \equiv \sup_{0\le s,t\le T} \frac{|A_\varepsilon^{-s}(u(t)-u(s))|_\varepsilon}{|t-s|^\delta} + \Big(\int_0^T \|u(\tau)\|_\varepsilon^2 d\tau\Big)^{1/2} < \infty \tag{2.11}$$

for some $0<\delta<1/2$ and for all T. It means that there exists a new probability space and on this space there exist processes $\hat{u}(t)\in H_\varepsilon$, $t\ge 0$, and $\widehat{W}_\varepsilon(t)\in H_\varepsilon$, $t\ge 0$, such that $\mathcal{D}(\hat{u}(\cdot)) = P_\varepsilon$ and

- $\widehat{W}_\varepsilon$ is a Wiener process, distributed as the process W_ε;
- $\mathcal{D}\hat{u}(0) = \mathcal{D}u_0$ and the random variable $\hat{u}(0)$ is independent from the process $\widehat{W}_\varepsilon$;
- the process $\hat{u}(t)$, $t\ge 0$, satisfies eq. (1.1) with W_ε replaced by $\widehat{W}_\varepsilon$. That is,

$$\hat{u}(t) - \hat{u}(0) + \int_0^t \big(\nu A_\varepsilon \hat{u} + B_\varepsilon(\hat{u},\hat{u}) - f_\varepsilon\big)\, ds = \widehat{W}_\varepsilon(t) \quad \forall t\ge 0\,, \tag{2.12}$$

  almost surely (the equality (2.12) is understood in the usual sense: it holds true after we multiply it in H_ε by any function $\varphi\in V_\varepsilon\cap C^\infty(\mathcal{O}_\varepsilon)$ and replace $(B_\varepsilon(\hat{u},\hat{u}),\varphi)$ by $-(B_\varepsilon(\hat{u},\varphi),\hat{u})$).

We note that in [25, 26] the statistical solutions are defined in terms of Kolmogorov's equation. That definition is equivalent to the one above.

It is also proved in [25, 26] that eq. (1.1) has a *stationary statistical solution* which is a statistical solution, defined by a stationary Borel measure P_ε. That is, the measure P_ε is invariant under the translations

$$\mathcal{Z}^\varepsilon \to \mathcal{Z}^\varepsilon, \quad u(\cdot)\mapsto u(\tau+\cdot),\ \tau\ge 0\,.$$

The trace-measure of the measure P_ε, i.e., its image under the mapping $u(\cdot)\mapsto u(0)$, is a measure on V_ε, called a *weakly stationary measure* for eq. (1.1).

Statistical solutions for the 2D NSE and the α-approximation for the 3D NSE which we consider later in this work are defined similarly. In Theorem 2.9 below we construct stationary statistical solutions P_ε for (1.1) as limits (when $\alpha\to 0$) of the statistical solutions to the corresponding α-approximations (1.2). Due to lack of the uniqueness statement these 3D solutions P_ε may be different from the solutions constructed in [25] by the Galerkin method.

2.4 Corresponding 2D Navier-Stokes equations

Our goal is to study solutions for (2.1)–(2.4) when $\varepsilon\to 0$. Under this limit problem (2.1)–(2.4) is closely related to the 2D NSE on $\mathbb{T}^2$ (see Introduction). To describe this relation we first define the space

$$\widetilde{V} = \Big\{ u\in H^1(\mathbb{T}^2;\mathbb{R}^2) : \operatorname{div}' u = 0,\ \int_{\mathbb{T}^2} u\, dx' = 0 \Big\}\,,$$

where the prime in $div'u$ indicates that we consider the differential operation with respect to the variable $x' = (x_1, x_2)$ (in contrast with $x = (x_1, x_2, x_3) \equiv (x', x_3)$). Next we define the space $\widetilde{H}$ as a closure of $\widetilde{V}$ in $\left[L_2(\mathbb{T}^2)\right]^2$. We denote by $|\cdot|_{\mathbb{T}^2}$ and $(\cdot,\cdot)_{\mathbb{T}^2}$ the L_2-norm and L_2-inner product in $\widetilde{H}$, and denote by $\|\cdot\|_{\mathbb{T}^2} = |\nabla\cdot|_{\mathbb{T}^2}$ the norm in the space $\widetilde{V}$. The subscripts in $|\cdot|_{\mathbb{T}^2}$, $(\cdot,\cdot)_{\mathbb{T}^2}$ and $\|\cdot\|_{\mathbb{T}^2}$ will be often omitted when apparent from the context.

One can see that the averaging operator M_ε given by (1.3) maps the spaces H_ε and V_ε in $\widetilde{H}$ and $\widetilde{V}$ respectively. The operator

$$(M_\varepsilon^* v)(x) = u(x) \quad \text{with} \quad u_j(x) = v_j(x') \text{ for } j = 1, 2 \text{ and } u_3 = 0$$

defines isometric embeddings $M_\varepsilon^* : \widetilde{V} \to (V_\varepsilon, \|\cdot\|_{0,\varepsilon})$ and $M_\varepsilon^* : \widetilde{H} \to (H_\varepsilon, |\cdot|_{0,\varepsilon})$. This operator is a right inverse to M_ε, i.e. $M_\varepsilon \circ M_\varepsilon^* = \mathrm{id}$, and is adjoint to the operator $M_\varepsilon : (H_\varepsilon, |\cdot|_{0,\varepsilon}) \to \tilde{H}$. We also define the operator $\hat{M}_\varepsilon$ in H_ε (resp. in V_ε) by the formula

$$\hat{M}_\varepsilon u = (M_\varepsilon u; 0) = M_\varepsilon^* M_\varepsilon u, \quad u \in H_\varepsilon \text{ (resp. } u \in V_\varepsilon). \tag{2.13}$$

The operator $\hat{M}_\varepsilon$ defines an orthogonal projector in H_ε and in V_ε. So

$$V_\varepsilon = \hat{M}_\varepsilon V_\varepsilon \oplus \hat{N}_\varepsilon V_\varepsilon\,, \quad \text{where } \hat{N}_\varepsilon = I - \hat{M}_\varepsilon\,. \tag{2.14}$$

By an analogy with the deterministic NSE (see, e.g., [24]) and the equation, perturbed by a kick-force [4], we can *conjecture* that if the limits

$$\tilde{f} = \lim_{\varepsilon\to 0} M_\varepsilon f \in \tilde{H} \quad \text{and} \quad b_j^0 = \lim_{\varepsilon\to 0} b_j^\varepsilon, \quad j \geq 1\,,$$

exist, then the M_ε-projections of solutions to (2.1)–(2.4) should be close (when $\varepsilon \ll 1$) to solutions of the following 2D NSE on $\mathbb{T}^2$

$$\partial_t v - \nu \triangle' v + \sum_{j=1}^{2} v_j \partial_j v + \nabla' p = \widetilde{f} + \dot{\widetilde{W}} \quad \text{in} \quad \mathbb{T}^2 \times (0, +\infty), \tag{2.15}$$

$$\int_{\mathbb{T}^2} v(t, x')\, dx' = 0\,; \quad \operatorname{div}' v = 0 \quad \text{in} \quad \mathbb{T}^2 \times (0, +\infty), \tag{2.16}$$

$$v(x', 0) = \widetilde{v}_0(x') \quad \text{in} \quad \mathbb{T}^2. \tag{2.17}$$

Here $\widetilde{W}(t, x') = \widetilde{W}(t)$ is the Wiener process in $\tilde{H}$ of the form

$$\widetilde{W}(t) = \sum_j b_j^0 \beta_j(t)\, \widetilde{e}_{\lambda_j}, \quad B_0 = \sum_j (b_j^0)^2 < \infty\,,$$

where $\widetilde{e}_{\lambda_j} \equiv M_\varepsilon e_{\lambda_j}$ are eigenfunctions of the 2D Stokes operator (see Appendix). So a.a. realisation of $\widetilde{W}$ defines a continuous trajectory in $\tilde{H}$, and the correlation operator $\tilde{K}_0$ of the process is the diagonal operator

$$K_0 \widetilde{e}_{\lambda_j} = [b_j^0]^2 \widetilde{e}_{\lambda_j}, \quad j = 1, 2, \dots\,,$$

cf. Section 2.2.

In the abstract form problem (2.15)–(2.17) can be written as

$$u' + \nu A_0 u + B_0(u,u) = \widetilde{f} + \dot{\widetilde{W}}, \quad u(0) = u_0, \tag{2.18}$$

where A_0 and B_0 are the corresponding two dimensional Stokes operator and bilinear operator.

A random field $u(t,x)$ is called a (strong) solution of the problem (2.15)-(2.17) (written in the form (2.18)) on a segment $[0,T]$ if for a.a. ω it defines a curve in $C([0,T],\widetilde{H}) \cap L_2([0,T],\widetilde{V})$, satisfying

$$u(t) - u_0 + \int_0^t \left(\nu A_0 u(\tau) + B_0(u(\tau),u(\tau)) - \widetilde{f}\right) d\tau = \widetilde{W}(t),$$

for all $0 \le t \le T$. A random field $u(t,x)$ which defines a random process with trajectories in the space

$$\mathcal{Z} = C(0,\infty;\widetilde{H}) \cap L_{2\,loc}(0,\infty;\widetilde{V})\,, \tag{2.19}$$

is a solution of (2.18) for $t \in [0,\infty)$ if it is a solution on any finite segment $[0,T]$.

We also consider the process $\widetilde{W}_\varepsilon(t) = M_\varepsilon W_\varepsilon(t)$. It has the form above with the correlation operator $\varepsilon^{-1} M_\varepsilon K_\varepsilon M_\varepsilon^*$, which is the diagonal operator in $\widetilde{H}$ with the eigenvalues $(b_i^\varepsilon)^2$. Below we will study the 2D NSE (2.18) with $\widetilde{W} = \widetilde{W}^\varepsilon$ and $f = \widetilde{f}_\varepsilon = M_\varepsilon f_\varepsilon$:

$$u' + \nu A_0 u + B_0(u,u) = \widetilde{f}_\varepsilon + \dot{\widetilde{W}}_\varepsilon, \quad u(0) = u_0^\varepsilon\,. \tag{2.20}$$

The stochastic evolution equation (2.18) was studied by many authors (see, e.g., [8, 25, 26, 14, 15] and the references therein). Here we will recall basic result on the existence and uniqueness of its solutions from [14].

Theorem 2.1 *If $f \in \widetilde{H}$ and $u_0 = u_0^\omega$ is a random variable in $\widetilde{H}$ independent from the process $\widetilde{W}(t)$ and such that $\mathbf{E}|u_0|^2 < \infty$, then eq.* (2.18) *has a unique (up to equivalence) solution $u(t)$, $t \ge 0$. If, in addition, $tr(A_0K_0) = \sum \lambda_j (b_j^0)^2 < \infty$ and*

$$E_{\beta_0}(u_0) \equiv \mathbf{E}\exp(\beta_0 \parallel u_0 \parallel^2) < \infty \tag{2.21}$$

for some $\beta_0 \in \left(0, \nu |K_0|^{-1}_{\mathcal{L}(\widetilde{H})}\right)$, then for every $\beta_1 \le \frac{1}{2}\cdot\beta_0\left(\nu - \beta_0|K_0|_{\mathcal{L}(\widetilde{H})}\right)$ we have

$$\mathbf{E}\exp\left(\beta_0\|u(t)\|^2 + \beta_1\int_0^t |\, A_0u(\tau)\,|^2\, d\tau\right) \le e^{\gamma t}\cdot E_{\beta_0}(u_0) \tag{2.22}$$

for all $t \ge 0$, where $\gamma = \frac{\beta_0}{2\nu}\left(|\,\widetilde{f}\,|^2 + \nu\, tr(A_0K_0)\right)$. Furthermore for any positive λ there exists a constant $D_{\beta_0,\lambda} > 0$ such that

$$\mathbf{E}\exp(\beta_0\|u(t)\|^2) \le D_{\beta_0,\lambda} + e^{-\lambda(t-s)}\cdot\mathbf{E}\exp(\beta_0\|u(s)\|^2), \quad t > s \ge 0. \tag{2.23}$$

Proof. The proof of the first part can be found in [14]. For the proof of relations (2.22) and (2.23) we refer to [3]. □

The solution u, constructed in this theorem, will be denoted $u(t; u_0) = u(t, x; u_0)$. It defines a Markov process in the space $\widetilde{H}$ with the transition function $P_t(v, \cdot) = \mathcal{D}u(t; v)$; see [14, 25, 8, 15].

Corollary 2.2 *Let the hypotheses of Theorem 2.1 be in force. Then for any* $c_0 > 0$ *there exists* $t_* > 0$ *such that*

$$\mathbf{E} \exp\left(c_0 \int_0^t \|u(\tau)\|^2 d\tau\right) \le e^{\gamma t} \cdot E_{\beta_0}(u_0) \tag{2.24}$$

for all $0 \le t \le t_*$*, where* $E_{\beta_0}(u_0)$ *is defined in* (2.21) *and* $\gamma > 0$ *is the same as in* (2.22).

Proof. By the Jensen inequality

$$\mathbf{E} \exp\left(c_0 \int_0^t \|u(\tau)\|^2 d\tau\right) \le \frac{1}{t} \int_0^t \mathbf{E} \exp\left(c_0 t \|u(\tau)\|^2\right) d\tau.$$

Therefore by (2.22) under the condition $c_0 t \le \beta_0$ we have

$$\mathbf{E} \exp\left(c_0 \int_0^t \|u(\tau)\|^2 d\tau\right) \le \frac{1}{t} \int_0^t e^{\gamma\tau} d\tau \mathbf{E} \exp\left(\beta_0 \|u(0)\|^2\right).$$

This implies (2.24). □

Clearly Theorem 2.1 and Corollary 2.2 remain true for problem (2.20) with the noise $\widetilde{W}_\varepsilon = M_\varepsilon W_\varepsilon$ and the force $\widetilde{f}_\varepsilon$ depending on ε. The corresponding constants β_0, β_1, γ and t_* in Theorem 2.1 and Corollary 2.2 can be chosen independent of ε provided that

$$|\widetilde{f}_\varepsilon|_{\mathbb{T}^2} \le c_1 \quad \text{and} \quad \varepsilon^{-1} \mathrm{tr}\left(A_0 M_\varepsilon K M_\varepsilon^*\right) \equiv \sum_j \lambda_j [b_j^\varepsilon]^2 \le c_2,$$

where the constants c_1 and c_2 do not depend on ε.

Below we need the following assertion on the difference of solutions for problems (2.18) and (2.20).

Proposition 2.3 *Let* u *and* u_ε *be solutions of* (2.18) *and* (2.20)*. Assume that* $tr(A_0 K_0) < \infty$ *and the initial data* u_0 *satisfies* (2.21) *with some* $\beta_0 > 0$*. Then there exists a constant* $t_* > 0$*, independent from* u_0*, such that*

$$\begin{aligned} &\mathbf{E} \max_{[0,t_*]} |u(t) - u_\varepsilon(t)|^2 \\ &\quad \le C\Big\{ \left[\mathbf{E}|u(0) - u_\varepsilon(0)|^4\right]^{1/2} + \sum_j (b_j^\varepsilon - b_j^0)^2 + |\widetilde{f} - \widetilde{f}_\varepsilon|^2 \Big\}, \end{aligned} \tag{2.25}$$

where C *depends on* t_* *and* $E_{\beta_0}(u_0)$.

Proof. Consider the difference $\bar{u} = u - u_\varepsilon$. The process $\bar{u}$ satisfies the equation

$$\bar{u}' + \nu A_0 \bar{u} + B_0(\bar{u}, u) + B_0(u_\varepsilon, \bar{u}) = \widetilde{f} - \widetilde{f}_\varepsilon + \dot{\overline{W}}_\varepsilon,$$

where $\overline{W}_\varepsilon$ is a Wiener process of the form

$$\overline{W}_\varepsilon(t) = \sum_j \left(b_j^0 - b_j^\varepsilon\right) \beta_j(t)\, \widetilde{e}_{\lambda_j}$$

with the diagonal correlation operator $\overline{K}_\varepsilon$, $\overline{K}_\varepsilon \widetilde{e}_{\lambda_j} = \left[b_j^0 - b_j^\varepsilon\right]^2 \widetilde{e}_{\lambda_j}$ for each j. Applying Ito's formula to the functional $\frac{1}{2}|\bar{u}|^2$ we obtain that

$$\begin{aligned} &\frac{1}{2}|\bar{u}(t)|^2 + \nu \int_0^t \|\bar{u}\|^2 d\tau + \int_0^t b_0(\bar{u}, u, \bar{u}) d\tau \\ &\quad = \frac{1}{2}|\bar{u}(0)|^2 + \int_0^t (\widetilde{f} - \widetilde{f}_\varepsilon, \bar{u}) d\tau + \mathcal{M}_\varepsilon(t) + \frac{1}{2} t \cdot \operatorname{tr} \overline{K}_\varepsilon, \qquad (2.26) \end{aligned}$$

where $\mathcal{M}_\varepsilon(t) = \int_0^t \left(\bar{u}(\tau), d\overline{W}_\varepsilon(\tau)\right)$. By the Doob inequality (see, e.g.,[13, 8])

$$\begin{aligned} \left[\mathbf{E} \max_{s\in[0,t]} |\mathcal{M}_\varepsilon(s)|^2\right]^{1/2} &\le 2\left[\int_0^t \mathbf{E}(\overline{K}_\varepsilon \bar{u}, \bar{u}) d\tau\right]^{1/2} \\ &\le \frac{C}{\eta} \cdot t \cdot \operatorname{tr} \overline{K}_\varepsilon + \eta \cdot \mathbf{E} \max_{s\in[0,t]} |\bar{u}(s)|^2 \qquad (2.27) \end{aligned}$$

for any $\eta > 0$.

Since $|b_0(\bar{u}, u, \bar{u})| \le C\|u\|\|\bar{u}\||\bar{u}| \le \nu\|\bar{u}\|^2 + C_\nu \|u\|^2 |\bar{u}|^2$, from (2.26) we get that

$$|\bar{u}(t)|^2 \le m(t) + C_\nu \int_0^t \|u(\tau)\|^2 |\bar{u}(\tau)|^2 d\tau,$$

where $m(t) = c_0 \left[|\bar{u}(0)|^2 + t|\widetilde{f} - \widetilde{f}_\varepsilon|^2 + |\mathcal{M}_\varepsilon(t)| + t \cdot \operatorname{tr} \overline{K}_\varepsilon\right]$. Thus by Gronwall's lemma we have that

$$\max_{s\in[0,t]} |\bar{u}(s)|^2 \le \max_{s\in[0,t]} m(s) \exp\left\{C \int_0^t \|u(\tau)\|^2 d\tau\right\}.$$

This implies that

$$\mathbf{E}\left[\max_{s\in[0,t]} |\bar{u}(s)|^2\right] \le \left[\mathbf{E} \max_{s\in[0,t]} |m(s)|^2\right]^{1/2} \left[\mathbf{E} \exp\left\{2C \int_0^t \|u(\tau)\|^2 d\tau\right\}\right]^{1/2}.$$

Consequently by Corollary 2.2 there exists $t_* > 0$ such that

$$\mathbf{E}\left[\max_{s\in[0,t]} |\bar{u}(s)|^2\right] \le \left[E_{\beta_0}(u_0)\right]^{1/2} \left[\mathbf{E} \max_{s\in[0,t]} |m(s)|^2\right]^{1/2},$$

for all $t \in [0, t_*]$. Now using (2.27) with an appropriate $\eta > 0$ we obtain the estimate desired in (2.25). □

A Borel measure μ on $\widetilde{H}$ is said to be a *stationary measure for eq.* (2.18) if it is a stationary measure for the Markov process which the equation defines in the space $\tilde{H}$. This means that

$$\int_{\widetilde{H}} \mathbf{E} f(u(t;u_0))\mu(du_0) = \int_{\widetilde{H}} f(u_0)\mu(du_0) \quad \text{for any } f \in C_b(\widetilde{H}).$$

Let us write the force $\widetilde{f}(x)$ as $\widetilde{f} = \sum_j \tilde{f}_j \widetilde{e}_{\lambda_j}$.

Theorem 2.4 *Assume that $tr(A_0K_0) < \infty$, that $b_j^0 \neq 0$ if $\tilde{f}_j \neq 0$ ($j = 1, 2, \dots$), and that $b_j^0 \neq 0$, $j = 1, \dots, N$ for N large enough. Then there exists a unique stationary measure μ for* (2.18)*, and every solution of* (2.18) *given by Theorem 2.1 converges to μ in distribution when $t \to \infty$. This measure satisfies*

$$\int_{\widetilde{H}} \exp\left\{\beta_0 \parallel u \parallel^2\right\} \mu(du) < \infty \quad \textit{for any } \beta_0 < \nu |K_0|^{-1}_{\mathcal{L}(\widetilde{H})}. \tag{2.28}$$

Proof. The existence of the stationary measure is well-known via the standard Krylov-Bogolyubov procedure (see, e.g., [14, 25]). Concerning the uniqueness of the measure under the imposed assumptions see [23, 15]. The claimed estimate (2.28) follows from (2.23) and the Fatout lemma in the standard way (see [25, 26] for similar arguments). □

The uniqueness of the measure μ implies the same property for statistical solutions. More precisely, we have thee following assertion.

Corollary 2.5 *Let the hypotheses of Theorem 2.4 be in force. Then problem* (2.18) *has a unique stationary statistical solution (in the sense of definitions in Section 2.3) as a Borel probability measure on the space $\mathcal{Z}$ given by* (2.19).

Proof. Let $u(t), t \geq 0$, be a solution of (2.18), such that $\mathcal{D}u(0) = \mu$. Its distribution is a Borel measure $\tilde{P}$ in the space $\mathcal{Z}$. This is stationary statistical solution of the equation (cf. Section 2.3). Let P' be another stationary statistical solution. Then $P' = \mathcal{D}v(\cdot)$, where $v(t)$ is a solution of (2.18) with $\widetilde{W}$ replaced by another process, distributed as $\widetilde{W}$. So $\vartheta = \mathcal{D}v(0)$ is a stationary measure for the Markov process, defined by the equation, and $\vartheta = \mu$ by the theorem above. Accordingly, P' is the distribution of trajectories of the Markov process with the initial measure μ. So $P' = \tilde{P}$; that is, the stationary statistical solution $\tilde{P}$ for the 2D NSE is unique. □

2.5 Leray α-approximation of stochastic 3D Navier-Stokes equations

It is unknown if the 3D NSE (2.1)-(2.4) has a unique solution. So to make a progress in its study we replace the equation by its *Leray α-approximation* [18], in order later to send α to zero. That is, we consider the equations

$$\partial_t u - \nu \triangle u + \sum_{j=1}^{3} v_j \partial_j u + \nabla p = f_\varepsilon + \dot{W}_\varepsilon \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.29}$$

$$\text{div } u = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty),$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon\,,$$

where the forces f_ε and $\dot{W}_\varepsilon$ are the same as in (2.1). The equations are supplemented with boundary conditions (2.4), and the vector field $v = (v_1, v_2, v_3)$ solves the elliptic problem

$$v - \alpha \triangle v = u, \ \text{ div } v = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.30}$$

and satisfies the same boundary conditions.

In the Leray representation the problem above takes the form

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u, u) = f_\varepsilon + \dot{W}_\varepsilon, \quad u(0) = u_0, \tag{2.31}$$

where $G_\alpha = (I + \alpha A_\varepsilon)^{-1}$ is the Green operator for problem (2.30) with boundary conditions (2.4). The nonlinear term $B_\varepsilon(G_\alpha u, u)$ in (2.31) possesses the properties:

$$(B_\varepsilon(G_\alpha v, u), u) = 0$$

and

$$|B_\varepsilon(G_\alpha u_1, u_1) - B_\varepsilon(G_\alpha u_2, u_2)|_\varepsilon \le C_{\alpha,\varepsilon} \left(\|u_1\|_\varepsilon + \|u_2\|_\varepsilon \right) \|u_1 - u_2\|_\varepsilon. \tag{2.32}$$

This allows to obtain for the α-model results, similar to those in the $2D$ case.

Theorem 2.6 *Assume that $W_\varepsilon(t)$ is the Wiener process in H_ε of the form* (2.6) *and relations* (2.7) *holds. Let $f \in H_\varepsilon$. Then there exists a unique (strong) solution $u(t)$ to* (2.31) *for any initial data u_0 which is independent from the noise and satisfies $\mathbf{E}|u_0|^2_\varepsilon < \infty$. Moreover, for any $n \ge 1$ we have*

$$\mathbf{E}|u(t)|^{2n}_{0,\varepsilon} + \frac{\nu n}{2} \int_0^t \mathbf{E}|u|^{2(n-1)}_{0,\varepsilon} \|u\|^2_{0,\varepsilon} d\tau \le \mathbf{E}|u(0)|^{2n}_{0,\varepsilon} + b_n t \le \infty\,, \tag{2.33}$$

where $b_n = C_n \nu^{1-n} \sigma^n_\varepsilon$ with $\sigma_\varepsilon = \varepsilon^{-1} \left(tr\, K_\varepsilon + \frac{\lambda_1}{\nu} |f_\varepsilon|^2_\varepsilon \right)$. Here above we use the notations (2.5) *and* (2.8).

Proof. Due to the regularity in (2.32) the existence and uniqueness of strong solutions can be obtained by the same argument as for 2D NSE (see, e.g., [14] or [8]).

Now we prove (2.33). Here our arguments are formal. To make them rigorous one should consider the Galerkin approximations for the problem.

Let us consider the functional $F(u(t)) = |u(t)|^{2n}_\varepsilon$. Using the Ito formula we have

$$dF = 2n|u|^{2(n-1)}_\varepsilon (u, du)_\varepsilon + n \left\{ |u|^{2(n-1)}_\varepsilon tr\, K_\varepsilon + 2(n-1)|u|^{2(n-2)}_\varepsilon (K_\varepsilon u, u)_\varepsilon \right\} dt.$$

Since $(u,du)_\varepsilon = -\nu\|u\|_\varepsilon^2 dt + (u, f_\varepsilon dt + dW_\varepsilon)$, then integrating the equality above we obtain

$$\begin{aligned}&\mathbf{E}|u(t)|_\varepsilon^{2n} + 2\nu n\int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)}\|u\|_\varepsilon^2 d\tau\\ =\ & \mathbf{E}|u(0)|_\varepsilon^{2n} + 2n\int_0^t \mathbf{E}\left(|u|_\varepsilon^{2(n-1)}(u,f_\varepsilon)_\varepsilon\right)d\tau\\ &+n\int_0^t \mathbf{E}\left(|u|_\varepsilon^{2(n-1)} tr\, K_\varepsilon + 2(n-1)|u|_\varepsilon^{2(n-2)}(K_\varepsilon u,u)_\varepsilon\right)d\tau.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbf{E}|u(t)|_\varepsilon^{2n} + \nu n\int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)}\|u\|_\varepsilon^2 d\tau\\ \le\ & \mathbf{E}|u(0)|_\varepsilon^{2n} + C_n\left(tr\, K_\varepsilon + \frac{\lambda_1}{\nu}|f_\varepsilon|_\varepsilon^2 + |K_\varepsilon|_{\mathcal{L}(H_\varepsilon)}\right)\int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)}d\tau.\end{aligned}$$

Since $|K_\varepsilon|_{\mathcal{L}(H_\varepsilon)} \le tr\, K_\varepsilon$, then the second term in the r.h.s. is bounded by

$$\begin{aligned}C_n'\varepsilon\sigma_\varepsilon\int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)}d\tau &\le C_n'\varepsilon\sigma_\varepsilon\left(\int_0^t \mathbf{E}|u|_\varepsilon^{2n}d\tau\right)^{(n-1)/n} t^{1/n}\\ &\le \frac{\nu n}{2}\lambda_1^n\int_0^t \mathbf{E}|u|_\varepsilon^{2n}d\tau + tC_n''(\varepsilon\sigma_\varepsilon)^n\nu^{1-n}\,.\end{aligned}$$

Since $\lambda_1|u|_\varepsilon^2 \le \|u\|_\varepsilon^2$, by relations (2.5) this implies the required estimate. □

Under the condition $\sigma_\varepsilon \le C$ for all $0\le\varepsilon\le e_0$ the constant b_n in (2.33) is independent from α and ε. Therefore if $\mathbf{E}|u(0)|_{0,\varepsilon}^{2n} \le C$ for all $0\le\varepsilon\le e_0$, then we have a priori uniform (with respect to α and ε) estimates for solution $u(t)$ in the theorem above. This observation is important in the limit transitions below.

Let us decompose the force $f_\varepsilon = f_\varepsilon(x)$ in (2.29) in the basis of H_ε:

$$f_\varepsilon(x) = \sum f_j e_{\lambda_j}(x) + \sum \hat{f}_j e_{\Lambda_j^\varepsilon}(x)$$

Theorem 2.7 *1) Eq.* (2.31) *has a stationary measure* μ_ε^α *in* H_ε*, satisfying*

$$\int_{H_\varepsilon} |u|_{0,\varepsilon}^{2(n-1)}\|u\|_{0,\varepsilon}^2\mu_\varepsilon^\alpha(du) \le C_n,\quad n=1,2,\dots\,, \tag{2.34}$$

where the constants C_n *are increasing functions of* $\sigma_\varepsilon = \varepsilon^{-1}\left[tr\, K_\varepsilon + \frac{\lambda_1}{\nu}|f_\varepsilon|_\varepsilon^2\right]$, *independent from* α.

2) There are constants $n(\varepsilon,\alpha)$ *and* $\hat{n}(\varepsilon,\alpha)$ *such that if* $b_j^\varepsilon \ne 0$ *for* $j\le n$ *and* $\hat{b}_j^\varepsilon \ne 0$ *for* $j\le\hat{n}$*, and if*

$$b_j^\varepsilon \ne 0 \;\; if \;\; f_j \ne 0 \quad and \quad \hat{b}_j^\varepsilon \ne 0 \;\; if \;\; \hat{f}_j \ne 0, \qquad \forall j\,, \tag{2.35}$$

then a stationary measure is unique, and every solution of (2.31) *converges to it in distribution as time goes to infinity. In particular, this conclusion holds true when* $b_j^\varepsilon \neq 0$ *and* $\hat{b}_j^\varepsilon \neq 0$ *for all* j.

3) Assume that $\varepsilon^{-1}\, tr K_\varepsilon \leq c_0$ *and* $\varepsilon^{-1}|f_\varepsilon|_\varepsilon \leq c_1$ *for all* ε. *Then* $n(\varepsilon,\alpha)$ *may be chosen independent from* ε, *and the assumption* $\hat{b}_j^\varepsilon \neq 0$ *for* $j \leq \hat{n}$ *may be dropped if* $\varepsilon \leq \varepsilon_0$, *where* $\varepsilon_0 = \varepsilon_0(\alpha) > 0$.

Proof. The first assertion follows from the Bogolyubov-Krylov arguments and the Fatout lemma in the standard way, cf. [15], Section 4.4.

The second assertion follows from the techniques, developed in recent works on the randomly forced 2D NSE, discussed in Introduction. More specifically, in [17, 23] the 2D NSE is written in the abstract form as

$$u' + Lu + B(u,u) = f + \dot{W} \tag{2.36}$$

(in [17] $f = 0$, but it is shown in [23] that the arguments of that work apply to equations with non-zero f). The proof in [17, 23] uses only basic properties of the linear operator L and the quadratic operator B. It is straightforward that the operators νA_ε and B_ε in (2.31) satisfy these properties, if we choose for the basic function space the space $(H_\varepsilon, |\cdot|_{0,\varepsilon})$. So the main theorems in the references above apply and imply the uniqueness of a stationary measure and the assertion about the convergence.

The proof in [17, 23] uses in a critical way the 'squeezing property', stating that asymptotical in time behaviour of a solution for (2.36) with a deterministic r.h.s. is determined by its finite-dimensional part, formed by first few Fourier harmonics of the solution. The validity of this property for the α-model can be checked by literal repeating of the classical arguments due to Foias-Prodi, exploited in [17, 23] (see Proposition A.1 in [17]). The finite-dimensional part corresponds to the subspace of H_ε, spanned by the vectors $e_{\lambda_j}, j \leq n(\varepsilon,\alpha)$, and $e_{\Lambda_j^\varepsilon}, j \leq \tilde{n}(\varepsilon,\alpha)$, where the corresponding eigenvalues λ_j and Λ_j^ε contain all eigenvalues of the Stokes operator A_ε, smaller than a suitable threshold N.

To prove the last assertion of the theorem we have to estimate how the numbers n and $\tilde{n}$ grow when $\varepsilon \to 0$. Let us write the spectrum of the Stokes operator A_ε, formed by the two branches $\{\lambda_j\}$ and $\{\Lambda_j^\varepsilon\}$ (see Appendix) as $\mu_1 \leq \mu_2 \leq \dots$, and denote by $\{b_{\mu_j}\}$ the corresponding coefficients in the decomposition of the Wiener process W_ε. By [17, 23] a stationary measure is unique if (2.35) holds and $b_{\mu_j} \neq 0$ for $j \leq N_\mu$. The constant $N = N_\mu$ should be so big that the assumptions (A.1) and (A.2) of Proposition A.1 in [17] imply the estimate (A.3). This can be achieved with help of the following proposition (which is an analog of Proposition A.1[17] for the case considered).

Proposition 2.8 *Let* u_i, $i = 1,2$ *be solutions to the (deterministic) problems*

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u, u) = \eta_i(t), \quad i = 1,2.$$

Assume that

$$\int_s^t \|u_1(\tau)\|_{0,\varepsilon}^2\, d\tau \leq \rho + K(t-s), \quad 0 \leq s \leq t \leq s+T, \tag{2.37}$$

where ρ, K and T are nonnegative constants. Let $P \equiv P_{n,\hat{n}}$ be the spectral orthoprojector on the subspace

$$Span\left\{e_{\lambda_j}, e_{\Lambda_i^\varepsilon} \;:\; 1 \le j \le n,\; 1 \le i \le \hat{n}\right\}, \quad n \ge 1,\; \hat{n} \ge 0$$

(if $\hat{n} = 0$, then the subspace equals Span$\{e_{\lambda_j} : 1 \le j \le n\}$), and $Q = 1 - P$. If $Pu_1(t) = Pu_2(t)$ and $Q\eta_1(t) = Q\eta_2(t)$ for all $t \in [s, s+T]$, then

$$|u_1(t) - u_2(t)|_{0,\varepsilon} \le e^{-m_{n,\hat{n}}(t-s)+\rho_n}|u_1(s) - u_2(s)|_{0,\varepsilon}, \quad t \in [s, s+T], \qquad (2.38)$$

where

$$m_{n,\hat{n}} = \frac{\nu}{2}\min\left\{\lambda_{n+1}, \Lambda^\varepsilon_{\hat{n}+1}\right\} - \frac{c_0 K}{\alpha}\left(\varepsilon^2 + \lambda_{n+1}^{-1/2}\right), \quad \rho_n = \frac{c_0\rho}{\alpha}\left(\varepsilon^2 + \lambda_{n+1}^{-1/2}\right).$$

We prove this proposition in Section 4.5.

The structure of the constant $m_{n,\hat{n}}$ and the fact that $\Lambda_k^\varepsilon \ge \varepsilon^{-2}$ for all k imply the third assertion of the theorem by the same argument as in [17]. □

Let $u_\varepsilon^\alpha(t),\; t \ge 0$, be a stationary solution of (2.31), corresponding to the stationary measure μ_ε^α. Then $P_\varepsilon^\alpha = \mathcal{D}u_\varepsilon^\alpha(\cdot)$ is a stationary statistical solution of (2.31) in the space $\mathcal{Z}^\varepsilon$. For the same reason as in the 2D case (see Corollary 2.5), under the assumptions of item 2) of the theorem this equation has a unique stationary statistical solution. Other properties of P_ε^α are collected in the following assertion.

Theorem 2.9 *1) Let $P_\varepsilon^\alpha = \mathcal{D}u_\varepsilon^\alpha(\cdot)$ be a stationary statistical solution of* (2.31) *in the space $\mathcal{Z}^\varepsilon$ given by* (2.9). *Then for any fixed $\varepsilon > 0$ the set of measures $\{P_\varepsilon^\alpha, 0 < \alpha \le 1\}$ is tight in the space of Borel measures on $\mathcal{Z}^\varepsilon$ and the corresponding trace-measures μ_ε^α are tight in H_ε.*

2) Let P_ε be any limiting measure for this family as $\alpha \to 0$.[2] *Then the measure P_ε is a stationary statistical solution of the 3D NSE* (1.1) *in the space $\mathcal{Z}^\varepsilon$. Its trace-measure μ_ε (i) satisfies estimates* (2.34), *(ii) is a limiting points for μ_ε^α in H_ε as $\alpha \to 0$, and (iii) is a weakly stationary measure for* (1.1) *(see Section 2.3 for the corresponding definitions).*

Proof. 1) The tightness of the set $\{P_\varepsilon^\alpha, 0 < \alpha \le 1\}$ follows by repeating the argument from [25, Chap. IV]. Moreover, in the same way as in [25, Chap. IV] we can derive from (2.34) the estimate

$$\int_{\mathcal{Z}^\varepsilon} |u|_T^{1+\kappa} P_\varepsilon^\alpha(du) \le C_T \qquad (2.39)$$

for every $T > 0$ and for some $\kappa > 0$, where $|\cdot|_T$ is given by (2.11) and the constant C_T does not depend on α. The tightness of μ_ε^α follows from (2.34).

[2]this means that $P_\varepsilon^{\alpha_j} \rightharpoonup P_\varepsilon$ in the space of Borel measures on $\mathcal{Z}^\varepsilon$ for some sequence $\alpha_j \to 0$.

2) By the Skorokhod representation theorem (see [13]), there exists a new probability space and on this space there are random processes $\hat{u}_\varepsilon^{\alpha_j}(t) \in H_\varepsilon$, $t \geq 0$, and $\hat{u}_\varepsilon(t) \in H_\varepsilon$, $t \geq 0$, such that $\mathcal{D}(\hat{u}_\varepsilon^{\alpha_j}) = P_\varepsilon^{\alpha_l}$, $\mathcal{D}(\hat{u}_\varepsilon) = P_\varepsilon$ and

$$\hat{u}_\varepsilon^{\alpha_j} \to \hat{u}_\varepsilon \quad \text{in} \quad \mathcal{Z}^\varepsilon \tag{2.40}$$

almost surely. Since $P_\varepsilon^{\alpha_j}$ is a statistical solution, then a.s. $u = \hat{u}_\varepsilon^{\alpha_j}$ satisfies

$$u(t) - u(0) + \int_0^t \big(\nu A_\varepsilon u + B_\varepsilon(G_{\alpha_j} u, u) - f\big) ds = W_\varepsilon^{\alpha_j}(t), \quad \forall t \geq 0\,, \tag{2.41}$$

where $W_\varepsilon^{\alpha_j}(t)$ is a Wiener process, distributed as $W_\varepsilon(t)$. The validity of equation (2.41) is understood in the same way as that of (2.12).

Let e be any basis vector e_{λ_j} or $e_{\Lambda_j^\varepsilon}$, and b_e be the corresponding coefficient b_j^ε or $\hat{b}_j^\varepsilon$. Let us denote by $\xi_\varepsilon^{\alpha_j}(t)$ the H_ε-scalar product of the l.h.s. of (2.41) with $\varepsilon^{-1}e$, where $u = \hat{u}_\varepsilon^{\alpha_j}$ and we replace $b(G_{\alpha_j}u, u, e)$ with $-b(G_{\alpha_j}u, e, u)$. Then (2.41) implies that $\xi_\varepsilon^{\alpha_j}(t)$ is a scalar Wiener process with the dispersion $\mathbf{E}(\xi_\varepsilon^{\alpha_j}(t))^2 = b_e^2 t$. The convergence (2.40) and estimate (2.39) imply that

$$\xi_e^{\alpha_j}(t) \to \xi_e^0(t) \quad \text{a.s.},$$

uniformly for t in finite segments, where the process $\xi_\varepsilon^0(t)$ is obtained by replacing $\hat{u}_\varepsilon^{\alpha_j}(t)$ by $\hat{u}_\varepsilon(t)$. Therefore $\xi_\varepsilon^0(t)$ also is a Wiener process with the dispersion $b_\varepsilon^2 t$.

Now let us take any two basis vectors $e' \neq e''$. Since the Wiener processes $\xi_{e'}^{\alpha_j}(t)$ and $\xi_{e''}^{\alpha_j}(t)$ are independent, then the limiting processes $\xi_{e'}^0(t)$ and $\xi_{e''}^0(t)$ are independent as well. Therefore we see that the process $\hat{u}_\varepsilon(t)$ satisfies (2.41), where the Wiener process $W_\varepsilon^{\alpha_j}(t)$ is replaced by an equidistributed process $W_\varepsilon^0(t)$. So P_ε is a statistical solution.

The estimates on the measure μ_ε follows from the estimates (2.34) on the measures μ_ε^α, the convergence (2.40) and the Fatout lemma. The theorem is proved. □

2.6 Leray α-approximation of stochastic 2D NSE

We may also consider the α-approximation for the 2D NSE (2.18):

$$v' + \nu A_0 v + B_0(G_\alpha^0 v, v) = \widetilde{f} + \dot{\widetilde{W}}, \quad v(0) = v_0, \tag{2.42}$$

where $G_\alpha^0 = (1 + \alpha A_0)^{-1}$, $\alpha > 0$. This equation possesses the same properties as the 2D NSE: given a suitable initial condition it has a unique solution, and under the assumptions of Theorem 2.4 it has a unique stationary measure in the space $\widetilde{H}$. However we cannot guarantee bounds (2.22), (2.23) and (2.28) for exponential moments in the case when $\alpha > 0$. The point is that in the case $\alpha = 0$ the proof of these estimates in [3] involves essentially the fact that $b_0(u, u, A_0 u) = 0$. If $\alpha > 0$, then $b_0(G_\alpha^0 u, u, A_0 u)$ may be not zero and thus the argument of [3] will not apply. In the case $\alpha > 0$ we can only use the relation

$b_0(G_\alpha^0 u, u, u) = 0$. This leads to the following analogies of Theorem 2.1 and Proposition 2.3 which provide us with the boundedness of exponential moments involving weaker norms.

Theorem 2.10 *If $f \in \widetilde{H}$ and $u_0 = u_0^\omega$ is a random variable in $\widetilde{H}$ independent from the process $\widetilde{W}(t)$ and such that $\mathbf{E}|u_0|^2 < \infty$, then eq. (2.42) has a unique (up to equivalence) solution $u(t)$, $t \geq 0$. If, in addition,*

$$E_{\beta_0}^0(u_0) \equiv \mathbf{E}\exp(\beta_0|u_0|^2) < \infty \tag{2.43}$$

for some $\beta_0 \in \left(0, \nu|K_0|^{-1}_{\mathcal{L}(\widetilde{H})}\lambda_1^{-1}\right)$, where $\lambda_1 > 0$ is the first eigenvalue of the 2D Stokes operator, then for every $\beta_1 \leq \frac{1}{2}\cdot\beta_0\left(\nu - \beta_0\lambda_1|K_0|_{\mathcal{L}(\widetilde{H})}\right)$ we have

$$\mathbf{E}\exp\left(\beta_0|u(t)|^2 + \beta_1\int_0^t \|u(\tau)\|^2 d\tau\right) \leq e^{\gamma t}\cdot E_{\beta_0}^0(u_0) \tag{2.44}$$

for all $t \geq 0$, where $\gamma = \frac{\beta_0}{2\nu\lambda_1}\left(|\,\widetilde{f}\,|^2 + \nu\lambda_1\, trK_0\right)$. Furthermore for any positive λ there exists a constant $D_{\beta_0,\lambda} > 0$ such that

$$\mathbf{E}\exp(\beta_0|u(t)|^2) \leq D_{\beta_0,\lambda} + e^{-\lambda(t-s)}\cdot\mathbf{E}\exp(\beta_0|u(s)|^2), \quad t > s \geq 0. \tag{2.45}$$

Moreover, if u_ε is solution to

$$v' + \nu A_0 v + B_0(G_\alpha^0 v, v) = \widetilde{f}_\varepsilon + \dot{\widetilde{W}}_\varepsilon, \quad v(0) = v_0^\varepsilon,$$

then there exists a constant $t_* = t_*(\alpha) > 0$, independent from u_0, such that

$$\begin{aligned} &\mathbf{E}\max_{[0,t_*]}|u(t) - u_\varepsilon(t)|^2 \\ &\leq C\Big\{\left[\mathbf{E}|u(0) - u_\varepsilon(0)|^4\right]^{1/2} + \sum_j (b_j^\varepsilon - b_j^0)^2 + |\widetilde{f} - \widetilde{f}_\varepsilon|^2\Big\}, \end{aligned} \tag{2.46}$$

where C depends on t_*, α and $E_{\beta_0}^0(u_0)$.

Proof. This is a slight modification of argument given in [3] and in the proof of Proposition 2.3. To obtain (2.44) and (2.45) we apply the Ito formula to the process $F(t) = \exp\left(\beta_0|u(t)|^2 + \beta_1\int_0^t \|u(\tau)\|^2 d\tau\right)$. To establish (2.46) we use the estimate

$$|b_0(\bar{u}, u, \bar{u})| \leq \nu\|\bar{u}\|^2 + C_\nu(\alpha)|u|^2|\bar{u}|^2, \quad u, \bar{u} \in \widetilde{V},$$

which makes it possible, via the corresponding analog of Corollary 2.2, to derive (2.46) from (2.44) (cf. the proof of Proposition 2.3). □

3 Main results

In the theorem below $M_\varepsilon \mu$ stands for the image of a measure μ under the map M_ε defined in (1.3).

Theorem 3.1 *Assume that*

- *the assumptions of Theorem 2.4 hold;*
- *there exist constants c_i independent of ε such that*

$$\varepsilon^{-1} tr\, K_\varepsilon \le c_0, \quad \varepsilon^{-1}|f_\varepsilon|_\varepsilon^2 \le c_1; \tag{3.1}$$

- *we have that*

$$\lim_{\varepsilon\to 0} \sum_j [b_j^\varepsilon - b_j^0]^2 = 0 \quad and \quad \lim_{\varepsilon\to 0} |\widetilde{f} - M_\varepsilon f_\varepsilon| = 0. \tag{3.2}$$

Let μ_ε^α be a stationary measure for eq. (2.31)*, given by Theorem 2.7. Then the following assertions hold for the measures $\hat{\mu}_\varepsilon^\alpha = M_\varepsilon \mu_\varepsilon^\alpha$:*

- *$\hat{\mu}_\varepsilon^\alpha \rightharpoonup \mu^\alpha$ in $\tilde{H}$ as $\varepsilon \to 0$, where μ^α is a unique stationary measure for the corresponding 2D α-model* (2.42)*, and $\mu^\alpha \rightharpoonup \mu$ as $\alpha \to 0$, where μ is a unique stationary measure for the 2D NSE* (2.18)*.*
- *If $\alpha = \alpha(\varepsilon)$ is such that*

$$\lim_{\varepsilon\to 0} \alpha(\varepsilon) = 0 \quad and \quad \lim_{\varepsilon\to 0} \left\{\varepsilon\alpha(\varepsilon)^{-3}\right\} = 0, \tag{3.3}$$

  then $\hat{\mu}_\varepsilon^{\alpha(\varepsilon)} \rightharpoonup \mu$ in $\widetilde{H}$ as $\varepsilon \to 0$.

We note that in Theorem 3.1 we do not assume that the stationary measure μ_ε^α is unique for $\alpha, \varepsilon > 0$. However under condition (3.1) we can use the third statement of Theorem 2.7 to claim this uniqueness for $\varepsilon \le \varepsilon_0(\alpha)$ by increasing the parameter N in the hypotheses of Theorem 2.4.

We also note that an assertion similar to Theorem 3.1 can be easily established for stationary statistical solutions P_ε^α.

The following assertion give a result concerning another iterated limit: first $\alpha \to 0$, then $\varepsilon \to 0$.

Theorem 3.2 *Let $\{P_\varepsilon\}$ be the family of stationary statistical solutions constructed in Theorem 2.9. Then under conditions* (3.1) *the family $\{M_\varepsilon P_\varepsilon\}$ is tight in the space $L_2^{loc}(\mathbb{R}_+; \widetilde{H}) \cap C(\mathbb{R}_+; \widetilde{\mathcal{W}}^{-1})$, where $\widetilde{\mathcal{W}}^{-1}$ is the completion of $\widetilde{H}$ with respect to the norm $|A_0^{-1} \cdot|_{\mathbb{T}^2}$. Moreover, if* (3.2) *holds and the assumptions of Theorem 2.4 are in force, then the only limit point of the family $\{M_\varepsilon P_\varepsilon\}$ as $\varepsilon \to 0$ is the stationary statistical solution $\widetilde{P}$ of the 2D NSE* (2.18)*. Accordingly, the only limit point of the family $\{M_\varepsilon \mu_\varepsilon\}$ is the unique stationary measure μ of* (2.18)*. Here μ_ε is the trace measure for P_ε (which is weakly stationary measure for eq.* (1.1)*).*

Under the conditions of the both Theorems 3.1 and 3.2 we obviously have relation (1.5) claimed in the Introduction. Moreover, using (2.34) one can see that

$$\lim_{\varepsilon\to 0}\int_{H_\varepsilon} |\hat{N}_\varepsilon u|^2_{0,\varepsilon}\mu^\alpha_\varepsilon(du) = 0 \quad \text{uniformly in } \alpha,$$

where $\hat{N}_\varepsilon$ is the 'vertical' projection defined in Section 2.4 (see eq.2.14), and that the averaged mean energy $E(\mu^\alpha_\varepsilon) = \frac{1}{2\varepsilon}\int_{H_\varepsilon} |u|^2_{0,\varepsilon}\mu^\alpha_\varepsilon(du)$ possesses the property

$$\lim_{\alpha\to 0}\lim_{\varepsilon\to 0} E(\mu^\alpha_\varepsilon) = E(\mu) \equiv \frac{1}{2}\int_{\widetilde{H}} |u|^2_{\mathbb{T}^2}\mu(du).$$

In contrast with the kick model considered in [4] we are not able to establish similar convergence of averaged enstrophy and enstrophy production. However it should be noted that for a *fixed* $\alpha > 0$ the convergence properties of the measures μ^α_ε can be improved and the hypotheses concerning f_ε and W_ε can be relaxed. This claim is based on the fact that using additional regularity provided by α-approximation, in contrast with (2.33), we can estimate the projections $\hat{M}_\varepsilon u$ and $\hat{N}_\varepsilon u$ separately. We do not discuss this issue in details.

4 Proofs

In this section we provide the proofs of Theorem 3.1, Theorem 3.2 and Proposition 2.8 which we need to complete the proof of Theorem 2.7.

4.1 Preliminaries

We define the operators $\hat{M}_\varepsilon$ and $\hat{N}_\varepsilon = I - \hat{M}_\varepsilon$ as in Section 2.4 (see relations (2.13) and (2.14)). The most important property (see, e.g., [24]) of these operators is that $\hat{M}_\varepsilon$ and $\hat{N}_\varepsilon$ are spectral (orthogonal) projectors for the Stokes operator A_ε. In particular, these operators map V_ε to itself, are orthogonal in both spaces H_ε and V_ε, and commute with A_ε. Other properties of these operators which we use in the further considerations are listed below (we refer to [24] for the proofs):

(i) $\hat{M}_\varepsilon\partial_{x_i} = \partial_{x_i}\hat{M}_\varepsilon$ and $\hat{N}_\varepsilon\partial_{x_i} = \partial_{x_i}\hat{N}_\varepsilon$, $i = 1, 2$.

(ii) If one of the vector fields u, w, v lies in $\hat{N}_\varepsilon V_\varepsilon$ and two others belong to $\hat{M}_\varepsilon V_\varepsilon$, then $b_\varepsilon(u, w, v) = 0$. In particular, for all $u, w, v \in V_\varepsilon$ we have

$$b_\varepsilon(u, w, \hat{M}_\varepsilon v) = b_\varepsilon(\hat{M}_\varepsilon u, \hat{M}_\varepsilon w, \hat{M}_\varepsilon v) + b_\varepsilon(\hat{N}_\varepsilon u, \hat{N}_\varepsilon w, \hat{M}_\varepsilon v). \tag{4.1}$$

(iii) There exist positive constants ε_0 and c_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, the following inequalities hold true.

$$|\hat{N}_\varepsilon u|_\varepsilon \le \varepsilon \left|\partial_3 \hat{N}_\varepsilon u\right|_\varepsilon \quad \text{for all } u \in V_\varepsilon, \tag{4.2}$$

$$|\hat{N}_\varepsilon u|_{(L^\infty(\mathcal{O}_\varepsilon))^3} \le c_0 |\hat{N}_\varepsilon u|^{1/4}_\varepsilon \left|A_\varepsilon \hat{N}_\varepsilon u\right|^{3/4}_\varepsilon \quad \text{for all } u \in D(A_\varepsilon). \tag{4.3}$$

We use all these relations in the considerations below.

4.2 Proof of Theorem 3.2

We start with the proof of the second main result because it does not require an additional analysis of α-approximation (2.31) and thus is simpler than the proof of Theorem 3.1.

The tightness follows from estimates (2.34), which are uniform in α and ε, in the same way as in Theorem 2.9, see also arguments in [25].

Let us denote by $\tilde{\mathcal{Z}}$ the space $L_2^{loc}(\mathbb{R}_+;\widetilde{V})\cap C(\mathbb{R}_+;\widetilde{\mathcal{W}}^{-1})$ and denote by $\bar{P}$ any limiting measure as in the theorem. Then

$$M_{\varepsilon_j}P_{\varepsilon_j}\rightharpoonup \bar{P}$$

in the space of Borel measures in $\mathcal{Z}$, for some sequence $\varepsilon_j\to 0$. Using the Skorokhod representation theorem (see, e.g., [13]), we construct on a new probability space random processes $\hat{u}_{\varepsilon_j}(t)\in H_{\varepsilon_j},\ t\geq 0$, and $\hat{v}(t)\in\tilde{H},\ t\geq 0$, such that $\mathcal{D}\hat{u}_{\varepsilon_j}(\cdot)=P_{\varepsilon_j}$, $\mathcal{D}\hat{v}(\cdot)=\hat{P}$, and

$$M_{\varepsilon_j}\hat{u}_{\varepsilon_j}\to\hat{v}\quad\text{in}\quad\tilde{\mathcal{Z}},\ \text{ a.s.}\tag{4.4}$$

(cf. Lemma 5.9 in [4]). Since P_{ε_j} is a stationary statistical solution, then $\hat{u}_{\varepsilon_j}(t)$ is a stationary process, satisfying (2.12) with a suitable Wiener process $\widehat{W}_{\varepsilon_j}$. Let us denote $M_{\varepsilon_j}\hat{u}_{\varepsilon_j}(t)=\hat{v}_{\varepsilon_j}(t)$. Applying the operator M_{ε_j} to (2.12) we get that $v=\hat{v}_{\varepsilon_j}$ a.s. satisfies

$$v(t)-v(0)+\int_0^t\big(\nu A_0v+M_{\varepsilon_j}B_{\varepsilon_j}(\hat{u}_{\varepsilon_j},\hat{u}_{\varepsilon_j})-M_{\varepsilon_j}f_{\varepsilon_j}\big)ds=M_{\varepsilon_j}\widehat{W}_{\varepsilon_j}(t),\tag{4.5}$$

for any $t\geq 0$. Let us take any vector $e=\widetilde{e}_{\lambda_j}$, multiply (4.5) by e in $\tilde{H}$, and denote the corresponding l.h.s. $\xi_e^{\varepsilon_j}(t)$. Consider the 3-linear term

$$b_{\varepsilon_j}(t):=(M_{\varepsilon_j}B_{\varepsilon_j}(\hat{u},\hat{u}),e)_{\mathbb{T}^2}=\varepsilon^{-1}b_{\varepsilon_j}(\hat{u},\hat{u},M^*_{\varepsilon_j}e),$$

where $\hat{u}=\hat{u}_{\varepsilon_j}$. Due to (4.1),

$$\begin{aligned}b_{\varepsilon_j}(t)&=\varepsilon_j^{-1}b_{\varepsilon_j}(\hat{M}_{\varepsilon_j}\hat{u},\hat{M}_{\varepsilon_j}\hat{u},M^*_{\varepsilon_j}e)+\varepsilon_j^{-1}b_{\varepsilon_j}(\hat{N}_{\varepsilon_j}\hat{u},\hat{N}_{\varepsilon_j}\hat{u},M^*_{\varepsilon_j}e)\\&=b_0(\hat{v}_{\varepsilon_j},\hat{v}_{\varepsilon_j},e)+\varepsilon_j^{-1}b_{\varepsilon_j}(\hat{N}_{\varepsilon_j}\hat{u},\hat{N}_{\varepsilon_j}\hat{u},M^*_{\varepsilon_j}e).\end{aligned}$$

Using (4.2) we have that

$$\varepsilon^{-1}b_\varepsilon(\hat{N}_\varepsilon\hat{u},\hat{N}_\varepsilon\hat{u},M^*_\varepsilon e).\leq\varepsilon^{-1}C\max_{x'\in\mathbb{T}^2}\{|\nabla e(x')|\}\,|N_\varepsilon\hat{u}|^2_\varepsilon\leq C_e\varepsilon\|N_\varepsilon\hat{u}\|^2_\varepsilon.$$

Therefore

$$|b_{\varepsilon_j}(t)-b_0(\hat{v}_{\varepsilon_j},\hat{v}_{\varepsilon_j},e)|\leq C_e\varepsilon_j\|\hat{u}_{\varepsilon_j}(t)\|^2_{\varepsilon_j}.\tag{4.6}$$

Passing to the limit in (4.5), using (4.4) and the last estimate, and arguing as when proving Theorem 2.9, we get that $\bar{P}=\mathcal{D}(\hat{v})$ is a stationary statistical solution of the 2D NSE. The assertion on convergence of $M_\varepsilon\mu_\varepsilon$ follows from the uniqueness of the measure $\bar{P}$ (see Corollary 2.5) and the fact that μ is a trace-measure for $\bar{P}$. This completes the proof of Theorem 3.2.

We note that using relation (4.6) on the support of the measure $M_{\varepsilon_j} P_{\varepsilon_j}$ and also the corresponding Kolmogorov equation (see [25]) we can suggest a more direct proof of the theorem which avoids the Skorokhod theorem and statements like Lemma 5.9 in [4].

4.3 Preparations for the proof of Theorem 3.1

The following assertion on the convergence of $M_\varepsilon u_\varepsilon^\alpha(t; u_0)$ is important in our argument.

Theorem 4.1 *Under the assumptions of Theorem 3.1, let* u_ε^α *be a solution for the 3D* α*-approximation* (2.31). *Then*

- *there exist* $t_* = t_*(\alpha) > 0$ *such that*
$$\lim_{\varepsilon\to 0} \mathbf{E} \sup_{[0,t_*]} |M_\varepsilon u_\varepsilon^\alpha(t) - u^\alpha(t)|_{\mathbb{T}^2}^2 = 0 \tag{4.7}$$
for every fixed $\alpha > 0$*, provided that* $\lim_{\varepsilon\to 0} \mathbf{E}|M_\varepsilon u_\varepsilon^\alpha(0) - u^\alpha(0)|_{\mathbb{T}^2}^4 = 0$*, where* $u^\alpha(t)$ *solves the 2D* α*-approximation* (2.42)*;*
- *if* $\alpha = \alpha(\varepsilon)$ *satisfies* (3.3)*, then there exist* $t_* > 0$ *such that*
$$\lim_{\varepsilon\to 0} \mathbf{E} \sup_{[0,t_*]} |M_\varepsilon u_\varepsilon^{\alpha(\varepsilon)}(t) - u(t)|_{\mathbb{T}^2}^2 = 0, \tag{4.8}$$
provided $\lim_{\varepsilon\to 0} \mathbf{E}|M_\varepsilon u_\varepsilon^{\alpha(\varepsilon)}(0) - u_0|_{\mathbb{T}^2}^4 = 0$*, where* $u(t)$ *is a solution to the 2D NSE* (2.18)*.*

In the argument below we mainly will concentrate on the proof of the second part of Theorem 4.1 (the case $\alpha = \alpha(\varepsilon)$). The case of the fixed α is simpler.

Since
$$|M_\varepsilon u_\varepsilon^{\alpha(\varepsilon)}(t) - u(t)| \le |M_\varepsilon u_\varepsilon^{\alpha(\varepsilon)}(t) - v^\varepsilon(t)| + |v^\varepsilon(t) - u(t)|, \tag{4.9}$$
where v^ε solves the 2D problem
$$\tilde{v}' + \nu A_0 \tilde{v} + B_0(\tilde{v}, \tilde{v}) = M_\varepsilon f + M_\varepsilon \dot{W}_\varepsilon,$$
with initial data $v^\varepsilon(0) = M_\varepsilon u(0)$, then in view of Proposition 2.3 to obtain (4.8) it remains to estimate the first term in (4.9). For this end, due to estimates (2.24) and (2.33), it is sufficient to prove the following assertion.

Proposition 4.2 *Let* u *solves* (2.31)*,* $\tilde{v}$ *satisfies the 2D NSE above and* $z = M_\varepsilon u - \tilde{v} \in \tilde{H}$*. Then for a.a.* ω *we have*
$$\begin{aligned} |z(t)|_{\mathbb{T}^2}^2 \le\ & |z(0)|_{\mathbb{T}^2}^2 E(\tilde{v}; t, 0) + \frac{c_1}{\nu}\alpha^\theta \int_0^t |A_0^{1/2}\tilde{v}(\tau)|_{\mathbb{T}^2}^4 E(\tilde{v}; t, \tau) d\tau \\ & + \frac{c_2}{\nu}\left[\frac{\varepsilon}{\alpha^3}\right]^{1/2} \int_0^t \|u(\tau)\|_\varepsilon^{3/2} |u(\tau)|_\varepsilon^{5/2} E(\tilde{v}; t, \tau) d\tau, \end{aligned} \tag{4.10}$$

where $E(\tilde v;t,\tau) = \exp\left\{-\nu\lambda_1(t-\tau) + \frac{c_0}{\nu}\int_\tau^t |A_0^{1/2}\tilde v(s)|^2_{\mathbb{T}^2}ds\right\}$. *Here* $\theta \in (0,1)$ *and the constants* c_i *are independent of* ε, α *and* ν.

Proof. The function $z = M_\varepsilon u - \tilde v$ satisfies

$$\tilde z' + \nu A_0 z + M_\varepsilon B_\varepsilon(G_\alpha u, u) - B_0(\tilde v, \tilde v) = 0,$$

Multiplying this relation in $\widetilde H$ by z we get that

$$\frac{1}{2}\frac{d}{dt}|z|^2_{\mathbb{T}^2} + \nu|A_0^{1/2}z|^2_{\mathbb{T}^2} = F(u,\tilde v, z), \tag{4.11}$$

where $F(u,\tilde v,z) = -(M_\varepsilon B_\varepsilon(G_\alpha u,u) - B_0(\tilde v,\tilde v), z)_{\mathbb{T}^2}$. We rewrite $F(u,\tilde v,z)$ in the form

$$F(u,\tilde v,z) = -\frac{1}{\varepsilon}b_\varepsilon(G_\alpha u, u, M_\varepsilon^* z) + b_0(\tilde v,\tilde v, z).$$

By (4.1) we have that

$$\frac{1}{\varepsilon}b_\varepsilon(G_\alpha u,u,M_\varepsilon^* z) = \frac{1}{\varepsilon}b_\varepsilon(\hat N_\varepsilon G_\alpha u, \hat N_\varepsilon u, M_\varepsilon^* z) + b_0(M_\varepsilon G_\alpha u, M_\varepsilon u, M_\varepsilon^* z)$$

and the symmetry $b_0(M_\varepsilon G_\alpha u, z, z) = 0$ yields

$$\frac{1}{\varepsilon}b_\varepsilon(G_\alpha u,u,M_\varepsilon^* z) = \frac{1}{\varepsilon}b_\varepsilon(\hat N_\varepsilon G_\alpha u, \hat N_\varepsilon u, M_\varepsilon^* z) + b_0(M_\varepsilon G_\alpha u, \tilde v, z).$$

Therefore using the fact that $M_\varepsilon G_\alpha = G_\alpha^0 M_\varepsilon$ we obtain

$$\begin{aligned} F(u,\tilde v,z) &= -\frac{1}{\varepsilon}b_\varepsilon(\hat N_\varepsilon G_\alpha u, \hat N_\varepsilon u, M_\varepsilon^* z) + b_0(\tilde v,\tilde v,z) - b_0(M_\varepsilon G_\alpha u,\tilde v,z) \\ &= -\frac{1}{\varepsilon}b_\varepsilon(\hat N_\varepsilon G_\alpha u, \hat N_\varepsilon u, M_\varepsilon^* z) + b_0(\tilde v - M_\varepsilon G_\alpha u,\tilde v,z) \\ &\equiv b_1 + b_2 + b_3, \end{aligned} \tag{4.12}$$

where

$$b_1 = -\frac{1}{\varepsilon}b_\varepsilon(\hat N_\varepsilon G_\alpha u, \hat N_\varepsilon u, M_\varepsilon^* z),$$
$$b_2 = -b_0(G_\alpha^0 z, \tilde v, z), \quad b_3 = b_0(\left[I - G_\alpha^0\right]\tilde v, \tilde v, z).$$

Estimate for b_1: Using the symmetry of the trilinear form b_ε we obtain that

$$\varepsilon b_1 = -b_\varepsilon(\hat N_\varepsilon G_\alpha u, \hat N_\varepsilon u, M_\varepsilon^* z) = b_\varepsilon(\hat N_\varepsilon G_\alpha u, M_\varepsilon^* z, \hat N_\varepsilon u).$$

Thus by (4.3) we have that

$$\begin{aligned} \varepsilon|b_1| &\le |\nabla M_\varepsilon^* z|_\varepsilon \max_{x\in\mathcal{O}_\varepsilon}\left\{|\hat N_\varepsilon G_\alpha u(x)|\right\}|\hat N_\varepsilon u|_\varepsilon \\ &\le c\sqrt{\varepsilon}|A_0^{1/2}z|_{\mathbb{T}^2}|A_\varepsilon \hat N_\varepsilon G_\alpha u|_\varepsilon^{3/4}|\hat N_\varepsilon u|_\varepsilon^{5/4} \\ &\le c\sqrt{\varepsilon}|A_0^{1/2}z|_{\mathbb{T}^2}\|A_\varepsilon G_\alpha\|^{3/4}_{\mathcal{L}(H_\varepsilon)}|N_\varepsilon u|^2_\varepsilon. \end{aligned}$$

Since $\|A_\varepsilon G_\alpha\|_{\mathcal{L}(H_\varepsilon)} \le \alpha^{-1}$ and by (4.2)

$$|\hat{N}_\varepsilon u|_\varepsilon^2 = |\hat{N}_\varepsilon u|_\varepsilon^{3/4} |\hat{N}_\varepsilon u|_\varepsilon^{5/4} \le \varepsilon^{3/4} \|\hat{N}_\varepsilon u\|_\varepsilon^{3/4} |\hat{N}_\varepsilon u|_\varepsilon^{5/4},$$

we obtain that

$$\begin{aligned} |b_1| &\le c \left[\frac{\varepsilon}{\alpha^3}\right]^{1/4} |A_0^{1/2} z|_{\mathbb{T}^2} \|\hat{N}_\varepsilon u\|_\varepsilon^{3/4} |\hat{N}_\varepsilon u|_\varepsilon^{5/4} \\ &\le \delta |A_0^{1/2} z|_{\mathbb{T}^2}^2 + \frac{c}{\delta} \left[\frac{\varepsilon}{\alpha^3}\right]^{1/2} \|\hat{N}_\varepsilon u\|_\varepsilon^{3/2} |\hat{N}_\varepsilon u|_\varepsilon^{5/2} \end{aligned} \tag{4.13}$$

for every $\delta > 0$.

Estimate for b_2: It is clear that

$$|b_2| \le c |A_0^{1/2} \tilde{v}|_{\mathbb{T}^2} \|G_\alpha^0 z\|_{L_{2p_1}(\mathbb{T}^2)} \|z\|_{L_{2p_2}(\mathbb{T}^2)},$$

where $p_1^{-1} + p_2^{-1} = 1$ and $1 < p_1, p_2 < \infty$. Since

$$\mathcal{D}(A_0^{s/2}) \subset \left[H^s(\mathbb{T}^2)\right]^2 \subset \left[L_q(\mathbb{T}^2)\right]^2 \quad \text{for} \quad s = 1 - \frac{2}{q}, \ \ q \ge 2, \tag{4.14}$$

we obtain that

$$\begin{aligned} \|G_\alpha^0 z\|_{L_{2p_1}(\mathbb{T}^2)} \|z\|_{L_{2p_2}(\mathbb{T}^2)} &\le c |A_0^{\theta/2} G_\alpha^0 z|_{\mathbb{T}^2} |A_0^{(1-\theta)/2} z|_{\mathbb{T}^2} \\ &\le c |A_0^{\theta/2} z|_{\mathbb{T}^2} |A_0^{(1-\theta)/2} z|_{\mathbb{T}^2}, \end{aligned}$$

where $\theta = 1 - p_1^{-1} = p_2^{-1} \in (0,1)$. Thus by interpolation we have that

$$\|G_\alpha^0 z\|_{L_{2p_1}(\mathbb{T}^2)} \|z\|_{L_{2p_2}(\mathbb{T}^2)} \le c |A_0^{1/2} z|_{\mathbb{T}^2} |z|_{\mathbb{T}^2}.$$

Therefore

$$|b_2| \le c |A_0^{1/2} \tilde{v}|_{\mathbb{T}^2} |A_0^{1/2} z|_{\mathbb{T}^2} |z|_{\mathbb{T}^2} \le \delta |A_0^{1/2} z|_{\mathbb{T}^2}^2 + \frac{c}{\delta} |z|_{\mathbb{T}^2}^2 |A_0^{1/2} \tilde{v}|_{\mathbb{T}^2}^2$$

for every $\delta > 0$.

Estimate for b_3: We obviously have that

$$b_3 = b_0(\left[I - G_\alpha^0\right] \tilde{v}, \tilde{v}, z) = -b_0(\left[I - G_\alpha^0\right] \tilde{v}, z, \tilde{v}).$$

Let $p_1^{-1} + p_2^{-1} = 1$ and $1 < p_1, p_2 < \infty$. As above, using the embedding in (4.14) we obtain

$$\begin{aligned} |b_3| &\le c |A_0^{1/2} z|_{\mathbb{T}^2} \|\tilde{v}\|_{L_{2p_1}(\mathbb{T}^2)} \| \left[I - G_\alpha^0\right] \tilde{v}\|_{L_{2p_2}(\mathbb{T}^2)} \\ &\le c |A_0^{1/2} z|_{\mathbb{T}^2} \Big| A_0^{\theta/2} \tilde{v}|_{\mathbb{T}^2} |A_0^{(1-\theta)/2} \left[I - G_\alpha^0\right] \tilde{v}\Big|_{\mathbb{T}^2} \\ &\le c \Big| A_0^{-\theta/2} \left[I - G_\alpha^0\right] \Big|_{\mathcal{L}(\widetilde{H})} |A_0^{1/2} z|_{\mathbb{T}^2} |A_0^{\theta/2} \tilde{v}|_{\mathbb{T}^2} |A_0^{1/2} \tilde{v}|_{\mathbb{T}^2}, \end{aligned}$$

where $\theta = 1 - p_1^{-1} = p_2^{-1} \in (0,1)$. One can see that

$$|A_0^{-\theta/2}\left[I - G_\alpha^0\right]|_{\mathcal{L}(\widetilde{H})} \le \max_{\lambda>0} \frac{\alpha\lambda^{1-\theta/2}}{1+\alpha\lambda} \le \alpha^{\theta/2}.$$

Therefore

$$|b_3| \le c\,\alpha^{\theta/2}|A_0^{1/2}z|_{\mathbb{T}^2}|A_0^{1/2}\tilde{v}|^2_{\mathbb{T}^2}, \le \delta|A_0^{1/2}z|^2_{\mathbb{T}^2} + \frac{c}{\delta}\,\alpha^\theta|A_0^{1/2}\tilde{v}|^4_{\mathbb{T}^2}$$

for every $\delta > 0$ and $\theta \in (0,1)$.

Final step: Substituting the estimates for b_1, b_2 and b_3 in (4.12) and choosing appropriative $\delta > 0$ from (4.11) we obtain that

$$\begin{aligned} \frac{d}{dt}|z|^2_{\mathbb{T}^2} + \nu|A_0^{1/2}z|^2_{\mathbb{T}^2} \quad &\le \quad \frac{c_0}{\nu}|z|^2_{\mathbb{T}^2}|A_0^{1/2}\tilde{v}|^2_{\mathbb{T}^2} \\ &+\frac{c_1}{\nu}\alpha^\theta|A_0^{1/2}\tilde{v}|^4_{\mathbb{T}^2} + \frac{c_2}{\nu}\left[\frac{\varepsilon}{\alpha^3}\right]^{1/2}\|\hat{N}_\varepsilon u\|_\varepsilon^{3/2}|\hat{N}_\varepsilon u|_\varepsilon^{5/2}. \end{aligned} \tag{4.15}$$

Since $|A_0^{1/2}z|^2_{\mathbb{T}^2} \ge \lambda_1|z|^2_{\mathbb{T}^2}$ we can apply Gronwall's lemma to obtain (4.10) and complete the proof of Proposition 4.2. □

For the proof the first statement of Theorem 4.1 instead of Proposition 4.2 we use the following assertion.

Proposition 4.3 *Let u solves* (2.31) *and $\tilde{v}$ solves the 2D α-approximation*

$$\tilde{v}' + \nu A_0\tilde{v} + B_0(G_\alpha^0\tilde{v}, \tilde{v}) = M_\varepsilon f + M_\varepsilon\dot{W}_\varepsilon, \quad u(0) = u_0. \tag{4.16}$$

Then the difference $z = M_\varepsilon u - \tilde{v}$ satisfy the relation

$$\begin{aligned} |z(t)|^2_{\mathbb{T}^2} \quad &\le \quad |z(0)|^2_{\mathbb{T}^2}E^\alpha(\tilde{v};t,0) \\ &+\frac{c_1}{\nu}\left[\frac{\varepsilon}{\alpha^3}\right]^{1/2}\int_0^t \|u(\tau)\|_\varepsilon^{3/2}|u(\tau)|_\varepsilon^{5/2}E^\alpha(\tilde{v};t,\tau)d\tau \end{aligned} \tag{4.17}$$

for a.a. ω, where $E^\alpha(\tilde{v};t,\tau) = \exp\left\{-\nu\lambda_1(t-\tau) + \frac{c_0}{\nu\alpha}\int_\tau^t|\tilde{v}(s)|^2_{\mathbb{T}^2}ds\right\}$. Here the constants c_i are independent of ε, α and ν.

Proof. The same argument as in Proposition 4.2 shows that relation (4.11) holds with

$$F(u,\tilde{v},z) = -\frac{1}{\varepsilon}b_\varepsilon(\hat{N}_\varepsilon G_\alpha u, \hat{N}_\varepsilon u, M_\varepsilon^* z) - b_0(G_\alpha^0 z, \tilde{v}, z) \equiv b_1 + b_2.$$

We note that b_1 and b_2 have the same structure as in the proof of Proposition 4.2. However to apply exponential estimates from Theorem 2.10 we need some modification in the estimating of b_2.

Since $\|z\|_{L_\infty(\mathbb{T}^2)} \le C|z|_{\mathbb{T}^2}^{1/2}|A_0 z|_{\mathbb{T}^2}^{1/2}$ (see, e.g., [6]) and $|A_0 G_\alpha^0|_{\mathcal{L}(\tilde{H})} \le \alpha^{-1}$, one can see that

$$\begin{aligned}
|b_2| &\equiv |b_0(G_\alpha^0 z, \tilde{v}, z)| = |b_0(G_\alpha^0 z, z, \tilde{v})| \\
&\le C \sup_{x'\in\mathbb{T}^2} \left\{ |[G_\alpha^0 z](x')| \right\} |\tilde{v}|_{\mathbb{T}^2} |A_0^{1/2} z|_{\mathbb{T}^2} \\
&\le C|G_\alpha^0 z|_{\mathbb{T}^2}^{1/2} |A_0 G_\alpha^0 z|_{\mathbb{T}^2}^{1/2} |\tilde{v}|_{\mathbb{T}^2} |A_0^{1/2} z|_{\mathbb{T}^2} \\
&\le \frac{C}{\sqrt{\alpha}} |\tilde{v}|_{\mathbb{T}^2} |z|_{\mathbb{T}^2} |A_0^{1/2} z|_{\mathbb{T}^2} \le \delta |A_0^{1/2} z|_{\mathbb{T}^2}^2 + \frac{C}{\delta\alpha} |\tilde{v}|_{\mathbb{T}^2}^2 |z|_{\mathbb{T}^2}^2
\end{aligned}$$

for any $\delta > 0$. Therefore using (4.11) and (4.13) for the case considered we obtain that

$$\frac{d}{dt}|z|_{\mathbb{T}^2}^2 + \nu\lambda_1 |z|_{\mathbb{T}^2}^2 \le \frac{c_0}{\nu\alpha} |\tilde{v}|_{\mathbb{T}^2}^2 |z|_{\mathbb{T}^2}^2 + \frac{c_1}{\nu} \left[\frac{\varepsilon}{\alpha^3}\right]^{1/2} \|u\|_\varepsilon^{3/2} |u|_\varepsilon^{5/2}.$$

Therefore Gronwall's lemma implies (4.17). This completes the proof of Proposition 4.3. □

Now we are in position to complete the proof of Theorem 4.1.

Completion of the proof of Theorem 4.1: It follows from Proposition 4.2 that

$$\begin{aligned}
\mathbf{E} \max_{[0,t_*]} |z(t)|_{\mathbb{T}^2}^2 &\le \left[\mathbf{E}|z(0)|_{\mathbb{T}^2}^4\right]^{1/2} \left[\mathbf{E}E(\tilde{v}; t_*)^2\right]^{1/2} \\
&+ \frac{c_1}{\nu} \alpha^\theta \left[\int_0^{t_*} \mathbf{E}|A_0^{1/2}\tilde{v}(\tau)|_{\mathbb{T}^2}^8 d\tau\right]^{1/2} \left[t_* \mathbf{E}E(\tilde{v}; t_*)^2\right]^{1/2} \\
&+ \frac{c_2}{\nu} \left[\frac{\varepsilon}{\alpha^3}\right]^{1/2} \left[\int_0^{t_*} \mathbf{E}\|u(\tau)\|_\varepsilon^2 |u(\tau)|_\varepsilon^{10/3} d\tau\right]^{3/4} \left[t^* \mathbf{E}E(\tilde{v}; t_*)^4\right]^{1/4},
\end{aligned}$$

where we denote $E(\tilde{v}; t) = \exp\left(\frac{c_0}{\nu}\int_0^t |A_0^{1/2}\tilde{v}(s)|_{\mathbb{T}^2}^2 ds\right)$. Therefore Proposition 2.3 and estimates (2.24) and (2.33) allow us to conclude the proof of the second part of Theorem 4.1.

As for the first part of this theorem, arguments are based on Theorem 2.10 and Proposition 4.3 and involve the same ideas.

4.4 Proof of Theorem 3.1

We restrict ourselves to the second assertion of the theorem since a proof of the first one is similar and a bit simpler. By (2.34) with $n = 1$ and the Prokhorov theorem the family of measures $\{\hat{\mu}_\varepsilon^{\alpha(\varepsilon)}\}$ is tight in $\mathcal{P}(\tilde{H})$. Let us take any converging subsequence

$$\mu_j := \hat{\mu}_{\varepsilon_j}^{\alpha(\varepsilon_j)} \rightharpoonup \tilde{\mu} \quad \text{in} \quad \tilde{H}.$$

To prove the assertion we have to show that $\tilde{\mu} = \mu$.

By the Skorokhod representation theorem (see [13]), on a new probability space for which we take the segment [0,1], given the Borel sigma-algebra and the Lebesgue measure, we can construct $\widetilde{H}$-valued random variables $\tilde{v}$ and $\{v_j\}$ such that

$$\mathcal{D}(v_j) = \mu_j, \quad \mathcal{D}(\tilde{v}) = \tilde{\mu} \quad \text{and} \quad v_j \to \tilde{v} \quad \text{a.s. in } \widetilde{H}.$$

Moreover, $E_{\beta_0}(\tilde{v}) < \infty$, where E_{β_0} is defined in (2.21) and, due to (2.34), we have that $\mathbf{E}|v_j - \tilde{v}|^4 \to 0$ as $j \to \infty$.

We will view $\tilde{v}$ and $\{v_j\}$ as random variables in the probability space $\Omega_{new} = [0,1] \times [0,1]$, depending only on the first factor. For each j we find an H_{ε_j}-valued random variable u_j on Ω_{new}, satisfying

$$M_\varepsilon u_j = v_j \quad \text{and} \quad \mathcal{D} u_j = \mu_{\varepsilon_j}^{\alpha(\epsilon_j)},$$

see Lemma 5.9 in [4]. Next on the space Ω_{new} we construct independent standard Wiener processes $\{\beta_{j\ new}, \hat{\beta}_{j\ new}\}$, independent from the random variables u_j, v_j and $\tilde{v}$, and consider equations (2.31) and (2.18), where the processes β_j and $\hat{\beta}_j$ are replaced by their 'new' replicas. Let $u_{\varepsilon_j}(t; u_j)$ and $u_0(t; \tilde{v})$ be solutions for these equations with the initial data u_j and $\tilde{v}$, respectively. These processes have the same distributions as solutions for the equations with the non-modified forces and with the initial data, distributed as μ_j and $\tilde{\mu}$, respectively. In particular,

$$\mathcal{B}_t^* \tilde{\mu} = \mathcal{D} u_0(t; \tilde{v}),$$

where $\{\mathcal{B}_t^*, t \geq 0\}$, is the Markov semigroup in measures, corresponding to the 2D NSE, see [8, 15]. Let g be any bounded Lipschitz function on $\widetilde{H}$. Then

$$\left|\langle g, \mathcal{B}_t^* \mu_j \rangle - \mathbf{E} g(M_{\varepsilon_j} u_{\varepsilon_j}(t; u_j))\right| \leq \mathbf{E}\left|g(u_0(t; v_j)) - g(M_{\varepsilon_j} u_{\varepsilon_j}(t; u_j))\right|.$$

Since $M_{\varepsilon_j} u_j = v_j$, then choosing $t \leq t_*$ and applying Theorem 4.1 we get that the r.h.s. goes to zero with ε_j. Since $\mathcal{D} u_j$ is a stationary measure, then

$$\mathbf{E} g(M_{\varepsilon_j} u_{\varepsilon_j}(t; u_j)) = \langle g, M_{\varepsilon_j} \circ \mu_{\varepsilon_j}^{\alpha(\epsilon_j)} \rangle = \langle g, \mu_j \rangle.$$

That is, $\left|\langle g, \mathcal{B}_t^* \mu_j \rangle - \langle g, \mu_j \rangle\right| \to 0$. Since $\mu_j \to \tilde{\mu}$ and the linear operators $\mathcal{B}_t^*$ are continuous in the $*$-weak topology in the space of measures, then passing to the limit in the last relation we get that $\langle g, \mathcal{B}_t^* \tilde{\mu} \rangle = \langle g, \tilde{\mu} \rangle$ for arbitrary bounded Lipschitz function g. Hence, $\mathcal{B}_t^* \tilde{\mu} = \tilde{\mu}$ for any $0 \leq t \leq t^*$. So μ is a stationary measure for (2.18), and $\tilde{\mu} = \mu$ by the uniqueness. The theorem is proved.

4.5 Proof of Proposition 2.8

Now we complete the proof of Theorem 2.7 on the uniqueness of the stationary measure for the 3D α-approximation (2.31). To do this we need to establish Proposition 2.8 only.

For $u = u_1 - u_2$ we have that

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u, u_1) + B_\varepsilon(G_\alpha u_2, u) = \eta_1 - \eta_2.$$

Since $u = Qu$, we obtain that

$$\frac{1}{2}\frac{d}{dt}|Qu|_{0,\varepsilon}^2 + \nu|A_\varepsilon^{1/2}Qu|_{0,\varepsilon}^2 + \varepsilon^{-1}b_\varepsilon(G_\alpha u, u_1, Qu) = 0. \tag{4.18}$$

The projector P can be written in the form $P = P_n^1 + P_{\hat{n}}^2$, where P_n^1 and $P_{\hat{n}}^2$ are the spectral orthoprojectors on the subspaces Span $\{e_{\lambda_j} : 1 \le j \le n\}$ and Span $\{e_{\Lambda_i^\varepsilon} : 1 \le i \le \hat{n}\}$. Therefore

$$|A_\varepsilon^{1/2}Qu|_{0,\varepsilon}^2 \ge \min\{\lambda_{n+1}, \Lambda_{\hat{n}+1}^\varepsilon\}\,|Qu|_{0,\varepsilon}^2. \tag{4.19}$$

Now we estimate the nonlinear term in (4.18). Since

$$u = Qu = (1 - P_n^1)\hat{M}_\varepsilon u + (1 - P_{\hat{n}}^2)\hat{N}_\varepsilon u \equiv Q_n^1\hat{M}_\varepsilon u + Q_{\hat{n}}^2\hat{N}_\varepsilon u,$$

we have that

$$b_\varepsilon(G_\alpha u, u_1, Qu) = b_\varepsilon(\hat{N}_\varepsilon G_\alpha Q_{\hat{n}}^2 u, u_1, Qu) + b_\varepsilon(\hat{M}_\varepsilon G_\alpha Q_n^1 u, u_1, Qu). \tag{4.20}$$

To estimate the first term in r.h.s. of (4.20) we use the relation

$$|b_\varepsilon(\hat{N}_\varepsilon u, w, v)| \le c\varepsilon^2|A_\varepsilon\hat{N}_\varepsilon u|_{0,\varepsilon} \cdot \|w\|_{0,\varepsilon} \cdot |v|_{0,\varepsilon},$$

see [24] and Lemma 6.3 in [4]. Since $|A_\varepsilon^{1/2}G_\alpha|_{\mathcal{L}(H_\varepsilon)} \le \alpha^{-1/2}$, this inequality implies that

$$\begin{aligned}
|b_\varepsilon(\hat{N}_\varepsilon G_\alpha Q_{\hat{n}}^2 u, u_1, Qu)| &\le C\varepsilon^2|A_\varepsilon\hat{N}_\varepsilon G_\alpha Q_{\hat{n}}^2 u|_{0,\varepsilon}\|u_1\|_{0,\varepsilon}|Qu|_{0,\varepsilon} \\
&\le C\varepsilon^2|A_\varepsilon^{1/2}G_\alpha|_{\mathcal{L}(H_\varepsilon)}\|Q_{\hat{n}}^2 u\|_{0,\varepsilon}\|u_1\|_{0,\varepsilon}|Qu|_{0,\varepsilon} \\
&\le \delta\varepsilon\|Qu\|_{0,\varepsilon}^2 + \frac{C_\delta\varepsilon^3}{\alpha}\|u_1\|_{0,\varepsilon}^2|Qu|_{0,\varepsilon}^2
\end{aligned}$$

for any $\delta > 0$. To estimate the second term in r.h.s. of (4.20) we note that

$$|b_\varepsilon(\hat{M}_\varepsilon G_\alpha Q_n^1 u, u_1, Qu)| \le C\varepsilon \max_{x'\in\mathbb{T}^2}\{|(M_\varepsilon G_\alpha Q_n^1 u)(x')|\}\,\|u_1\|_{0,\varepsilon}|Qu|_{0,\varepsilon},$$

where

$$\max_{x'\in\mathbb{T}^2}\{|(M_\varepsilon G_\alpha Q_n^1 u)(x')|\} \le C|M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2}^{1/2}|A_0 M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2}^{1/2}. \tag{4.21}$$

Since $|M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2} \le C\lambda_{n+1}^{-1/2}|A_0^{1/2}Q_n^1 u|_{\mathbb{T}^2}$ and

$$|A_0 M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2} \le |A_0^{1/2}G_\alpha^0|_{\mathcal{L}(\widetilde{H})} \cdot |A_0^{1/2}Q_n^1 u|_{\mathbb{T}^2},$$

then the r.h.s. in (4.21) is $\le C(\lambda_{n+1}^{1/2}\alpha)^{-1/2}\|Q_n^1 u\|_{0,\varepsilon}$. Therefore

$$|b_\varepsilon(\hat{M}_\varepsilon G_\alpha Q_n^1 u, u_1, Qu)| \le \delta\varepsilon\|Qu\|_{0,\varepsilon}^2 + \frac{C_\delta\varepsilon}{\alpha\lambda_{n+1}^{1/2}}\|u_1\|_{0,\varepsilon}^2|Qu|_{0,\varepsilon}^2$$

for every $\delta > 0$. Thus

$$|b_\varepsilon(G_\alpha u, u_1, Qu)| \le \frac{\nu}{2}\varepsilon |A_\varepsilon^{1/2} Qu|^2_{0,\varepsilon} + c_0 \left(\frac{\varepsilon^3}{\alpha} + \frac{\varepsilon}{\alpha \lambda_{n+1}^{1/2}} \right) \|u_1\|^2_{0,\varepsilon} |Qu|^2_{0,\varepsilon}$$

and we get from (4.18) an (4.19) that

$$\frac{d}{dt}|Qu|^2_{0,\varepsilon} + \nu \min\left\{\lambda_{n+1}, \Lambda^\varepsilon_{\hat{n}+1}\right\} |Qu|^2_{0,\varepsilon} \le c_0 \left(\frac{\varepsilon^2}{\alpha} + \frac{1}{\alpha \lambda_{n+1}^{1/2}} \right) \|u_1\|^2_{0,\varepsilon} |Qu|^2_{0,\varepsilon}.$$

Now the desired relation in (2.38) follows from Gronwall's lemma.

5 Appendix: spectral problem for the Stokes operator

The spectral boundary value problem which corresponds to operator A_ε has the form

$$\begin{cases} -\Delta w = \lambda w, \quad \operatorname{div} w = 0 \quad \text{in } \mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon), \\[1ex] w(x', x_3) \text{ is } (l_1, l_2)\text{-periodic with respect to } x', \\[1ex] w_3\big|_{x_3=\varepsilon} = 0, \quad \partial_3 w_j\big|_{x_3=\varepsilon} = 0, \quad j = 1, 2, \\ w_3\big|_{x_3=0} = 0, \quad \partial_3 w_j\big|_{x_3=0} = 0, \quad j = 1, 2. \\[1ex] \int_{\mathcal{O}_\varepsilon} w_j dx = 0, \ j = 1, 2. \end{cases}$$

Using the spectral decomposition (2.14) one can see that the spectrum consists of two branches. Recalling estimate (4.2) we find that these branches are: (i) the spectrum of the 2D Stokes operator A_0, $0 < \lambda_1 \le \lambda_2 \le \ldots$, and (ii) series of eigenvalues $0 < \Lambda_1^\varepsilon \le \Lambda_2^\varepsilon \le \ldots$, depending on ε and greater than ε^{-2} . We denote the corresponding eigenfunctions e_{λ_j} and $e_{\Lambda_j^\varepsilon}$. We have

$$\hat{M}_\varepsilon e_{\lambda_j} = e_{\lambda_j}\,, \quad \hat{M}_\varepsilon e_{\Lambda_j^\varepsilon} = 0\,,$$

where the (spectral) projector $\hat{M}_\varepsilon$ is defined by (2.13). One can also see that $e_{\lambda_j} = (\widetilde{e}_{\lambda_j}; 0)$, where $\widetilde{e}_{\lambda_j} \equiv M_\varepsilon e_{\lambda_j}$ is the eigenfunction for the 2D Stokes operator on $\mathbb{T}^2$ which correspond the to the eigenvalue λ_j. The eigenvalues λ_j are properly ordered numbers $\left(s_1 \frac{2\pi}{l_1}\right)^2 + \left(s_2 \frac{2\pi}{l_2}\right)^2$, $s = (s_1, s_2) \in \mathbb{Z}^2 \setminus \{0\}$, so that $C^{-1} j \le \lambda_j \le Cj$ for all j, with some $C > 1$ (see, e.g., [6]). We normalise the eigenfunctions as follows:

$$|e_{\lambda_j}|_\varepsilon = |e_{\Lambda_j^\varepsilon}|_\varepsilon = \sqrt{\varepsilon} \quad \forall\, j\,.$$

It is also obvious that $|\widetilde{e}_{\lambda_j}|_{\mathbb{T}^2} = 1$ and $\|\widetilde{e}_{\lambda_j}\|_{\mathbb{T}^2} = \sqrt{\lambda_j}$ for all j.

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