# Linear adiabatic dynamics generated by operators with continuous spectrum. I

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**Abstract**

We are interested in the asymptotic behavior of the solution to the Cauchy problem for the linear evolution equation

$$i\varepsilon\partial_t\psi = A(t)\psi, \quad A(t) = A_0 + V(t), \quad \psi(0) = \psi_0,$$

in the limit $\varepsilon \to 0$. A case of special interest is when the operator $A(t)$ has continuous spectrum and the initial data $\psi_0$ is, in particular, an improper eigenfunction of the continuous spectrum of $A(0)$. Under suitable assumptions on $A(t)$, we derive a formal asymptotic solution of the problem whose leading order has an explicit representation.

A key ingredient is a reduction of the original Cauchy problem to the study of the semiclassical pseudo-differential operator $\mathcal{M} = M(t, i\varepsilon\partial_t)$ with compact operator-valued symbol $M(t, E) = V_1(t)(A_0 - EI)^{-1}V_2(t)$ , $V(t) = V_2(t)V_1(t)$, and an asymptotic analysis of its spectral properties. We illustrate our approach with a detailed presentation of the example of the Schrödinger equation on the axis with the $\delta$-function potential: $A(t) = -\partial_{xx} + \alpha(t)\delta(x)$.

Key words: *Adiabatic evolution; Schrödinger equation; Semiclassical analysis*

# 1 Introduction.

We consider the Cauchy problem

$$i\varepsilon\partial_t\psi = A(t)\psi, \quad \psi(0) = \psi_0, \quad \psi = \psi(t) \in X, \quad t \in [0, T], \tag{1.1}$$

where $\varepsilon > 0$ is a parameter and $A(t)$ a linear self-adjoint operator in a Hilbert space $X$. We are interested in the asymptotic behavior of the solution in the limit $\varepsilon \to 0$.

The first natural question is whether the spectrum of the operator $A(t)$ has components $\sigma_j(t), j \in J$, that are uniformly separated on the interval $t \in [0, T]$. If this is the case, let $X_j(t)$ be the corresponding spectral subspaces. For small $\varepsilon$, the dynamics described by equation (1.1) is reduced with high precision to independent dynamics on the subspaces $X_j(t)$. This fact is known as the (Quantum) Adiabatic Theorem. In general, this result does not lead to the asymptotic description of the solution. Only in the case where the subspaces $X_j(t)$ are one-dimensional, i.e. the components $\sigma_j(t)$ are simple eigenvalues, equations in these subspaces can be explicitly solved, and the adiabatic theorem provides the asymptotic expansion of the solution. The corresponding formula in the leading order has the form [2]

$$\psi(t) \sim \sum_j e^{\frac{1}{i\varepsilon}\int_0^t E_j(\tau)d\tau}\psi_j(t)\langle\psi_0, \psi_j(0)\rangle. \tag{1.2}$$

Here the $\psi_j(t)$ are the normalized eigenvectors of $A(t)$ corresponding to eigenvalues $E_j(t)$. The (remaining undefined) scalar normalization of $\psi_j(t)$ is characterized by the transport equation

$$\langle\partial_t\psi_j(t), \psi_j(t)\rangle = 0. \tag{1.3}$$

A review of related results and other approaches to the adiabatic theorem is given in the very detailed presentation by Avron and Elgart [1] and in particular in Section 3 of the cited work ("A panorama of adiabatic theorems"). This allows us to avoid further discussion of connections to the adiabatic theorem and limit the list of references. In fact, the result that we present here is not intrinsically connected to the adiabatic theorem. It is not about the independent adiabatic development of subspaces, rather about the asymptotic behavior of solutions. We are interested in the situation where the spectrum of $A(t)$ is, say, purely continuous. In this case, there is no splitting of the spectral subspaces (at least, in the original direct terms) and the reduction of the problem to the adiabatic theorem is not possible.

This analysis could potentially have a lot of applications, for example in the study of asymptotic stability of solitary waves of the Nonlinear Schrödinger equations, where the linearization near a multi-soliton solution leads to a linearized time-dependent operator whose scattering properties must be studied in detail.

Our approach is based on the Duhamel formula

$$\theta(t)g(t) = -i\varepsilon \left(A(t) - i\varepsilon\partial_t I - i0I\right)^{-1} \delta(t)I, \tag{1.4}$$

where $g(t)$ is the resolving operator for the Cauchy problem:

$$i\varepsilon g_t(t) = A(t)g(t), \quad g(0) = I, \tag{1.5}$$

$\theta(t)$ is the Heaviside function and $\delta(t)$ is the delta-function.

The first step is to represent the right hand side of (1.4) in the special case where $A(t)$ does not depend on $t$ in the form of spectral resolution by means of the method of separation of variables. When $A(t)$ depends on $t$, we replace the components of this spectral resolution by their semiclassical expansion asymptotically as $\varepsilon \to 0$, and combine them to obtain the spectral resolution of $\theta(t)g(t)$.

There are two different types of spectral resolutions of $(A - i\epsilon\partial_t I - i0I)^{-1}$ when $A$ does not depend on $t$. The first one is generated by the spectral resolution of the operator $A$ itself. Suppose that the spectral resolution corresponding to $A$ has the form

$$f = \int \psi_k \langle f, \psi_k\rangle d\mu(k), \qquad f \in X. \tag{1.6}$$

Here $k$ is the index of the corresponding "eigenfunction" $\psi_k$, possibly improper, and $\mu(k)$ is the corresponding spectral measure. The spectral resolution corresponding to $(A - i\epsilon\partial_t I - i0I)^{-1}$ acquires the form:

$$f(t) = \frac{1}{2\pi\varepsilon}\int_{\mathbb{R}} d\lambda \int d\mu(k)\chi(t,\lambda,k)\int_{\mathbb{R}} dt'\langle f(t'), \chi(t',\lambda,k)\rangle, \tag{1.7}$$

where

$$\chi(t,\lambda,k) = e^{\frac{1}{i\varepsilon}\lambda t}\psi_k. \tag{1.8}$$

The corresponding semiclassical procedure was developed by Sukhanov in [3] for the study of the one-dimensional Dirac equation with a slowly varying potential. The limitations of this procedure do not allow to consider initial conditions $\psi_0$ which are eigenfunctions associated to the continuous spectrum of $A(0)$. Moreover, when applying this procedure, we met problems with the interaction of the contribution of the continuous spectrum and eigenvalues that can approach the continuous spectrum. To avoid these difficulties, we propose a different type of the spectral resolution that uses computations motivated by the scattering theory.

We reconstruct formally the right hand side of the Duhamel formula in the form

$$V_1\left(A(t) - i\varepsilon\partial_t I - i0I\right)^{-1} = \left(I + \mathcal{M}\right)^{-1} V_1 R_0(i\varepsilon\partial_t + i0). \tag{1.9}$$

In the above formula, $V(t) = V_2(t)V_1(t)$ is an appropriate factorization of $V(t)$ and $\mathcal{M}$ is a semiclassical pseudodifferential operator

$$\mathcal{M} = M(t, i\varepsilon\partial_t). \tag{1.10}$$

with a compact operator valued symbol:

$$M(t, E) = V_1(t)(A_0 - EI - i0I)^{-1}V_2(t). \tag{1.11}$$

We assume that the perturbation $V(t)$ can be treated by the methods of stationary scattering theory for any fixed $t$. We easily find the spectral resolution of $\mathcal{M}$ when $V$ does not depend on $t$, and as a result, we get the spectral resolution of the right hand side of (1.4). We then follow the general procedure to extend it to the case where the operator depends on $t$. The present work is devoted to the detailed development of this procedure.

The outline of the paper is as follows. Section 2 deals with the Duhamel formula, the derivation of (1.9) and of the full field from (1.9). In Section 3, we write the spectral resolution of the operator $\mathcal{M}$. In Section 4, we present an asymptotic expansion of the eigenfunctions of $\mathcal{M}$ and compute the leading order. In Section 5, we address the question of normalization of the eigenfunctions. In Section 6, we derive the formulas for the asymptotic expansions of the solution, and examine some special cases of initial conditions, such as the case where the initial condition is an eigenfunction associated to the discrete spectrum, and when it is an eigenfunction associated to the continuous spectrum.

To illustrate the scheme and show how it works, we consider in Sections 7 and 8, a elementary but still nontrivial example, namely the case of the Schrödinger operator on the line

$$A(t) = -\partial_{xx} + \alpha(t)\delta(x),$$

where $\alpha(t)$ is a real valued smooth function. Here the space $X$ is naturally $L_2(\mathbb{R})$. As initial condition $\psi_0$, we take an eigenfunction associated to the continuous spectrum of $A(0)$. We compute the asymptotic behavior of the solution $\psi(x, t, \varepsilon)$ for $x \in \mathbb{R}, t \in [0, T]$ in the limit $\varepsilon \to 0$.

We would like to emphasize that we only present in this paper formal calculations without attempt to plunge them into a rigorous scheme. We postpone this to a further study. We believe that the proposed analysis can be applied to cases where $A(t)$ can have not only purely continuous spectrum, but can in addition have purely point components, as well as the case of interaction between the different components of the spectrum, such as the situation where an isolated point of the spectrum is approaching the boundary of the continuous spectrum and is transformed into a resonance. This will be the subject of a further study.

# 2 Duhamel formula.

## 2.1 Statement of the problem.

We consider the problem (1.1) for $\psi(t)$ is in a Hilbert space $X$. We assume that $A(t)$ is a self-adjoint linear operator in $X$ with a domain $D$, independent on $t$. We are interested in the asymptotic behavior of the solution as $\varepsilon \to 0$. We assume that $A(t) = A_0 + V(t)$, with $A_0$ and $V(t)$ being self-adjoint operators and $V(t)$ bounded from $C^\infty([0,T])$ in the sense of norm. As for the assumptions on the spectra of $A_0$ and $A(t)$, we will not need them in an explicit form. Actually, we will use assumptions with respect to the operator-valued function $M(t,E) = V_1(t)R_0(E)V_2(t)$ , where $R_0(E) = (A_0 - EI)^{-1}$ is the resolvent operator defined for values of $E$ that do not belong to the spectrum of $A_0$, and $V_1, V_2$ are appropriate factors of $V$: $V = V_2V_1$. Our assumptions, as it will be seen, can be satisfied in very different cases such as when $A_0$ has continuous spectrum and also when $A_0$ has purely point spectrum. Roughly speaking, we assume that the analytical properties of the function $M(t,E)$ with respect to $E$ in the semi-plane $\Im E > 0$ allow analytical continuation of $M(t,E)$ on the extended semi-plane, where it can only have a finite or infinite number of uniformly separated poles and a finite number of isolated branching points where it remains continuous. Let us denote this (open) extended semi-plane by $\mathbb{C}_0$. As for the topology, we assume that this function is seen as a compact operator function. We assume that the operator $M(t,E)$ is defined as a $C^\infty$ function on the set $t \in [0,T], E \in \mathbb{C}_0$. This set of assumptions is a preliminary one that we will make more precise in the next subsection.

For example, the approach can be applied to a pair $(A_0, A(t))$ amenable to the stationary approach to the scattering theory. In this case we assume that the spectrum of $A_0$ consists of a finite number of separated intervals and that the spectrum of $A(t)$ can have additionally a finite or infinite number of uniformly separated eigenvalues $\lambda_i(t)$. The function $M(t,E)$ has branching points on the real axis that coincide with the boundaries of the intervals of the spectrum of $A_0$.

As a main model, the reader can keep in mind the Schrödinger operator in $L_2(\mathbb{R}^d)$ with $A_0 = -\triangle$ and $V(t)$ as the multiplication by the real-valued continuous potential $v(x,t)$ . It is assumed that all the derivatives of $v$ with respect to $t$ remain continuous in both variables and are functions of $x$ exponentially decreasing at infinity: $|D_t^r v(x,t)| \leq C_r e^{-\gamma|x|}$ , $\gamma \geq 0, r \in 0, \ldots$. The continuous spectrum is the semi-axis $[0,\infty)$, the eigenvalues are negative and their set is finite.

We will suppose more precisely that in the factorization $V(t) = V_2(t)V_1(t)$, the factors may act not only in $X$. We will allow the situation when $V_1 : X \to Y$ and $V_2 : Y \to X$ where $Y$ is, in general, another Hilbert/Banach space, that is a (possibly improper) subspace of $X$, and except for the self-adjointness, both factors have the same general properties as $V$ itself. We will assume in this case that the operator $M(t,E)$ is a compact function as an operator in $Y$.

In general, for the Schrödinger operator with a fixed sign potential $v$ the operators $V_1$ and $V_2$ can be taken as $V_1 = |V_1|^{1/2}$, $V_2 = \pm|V_1|^{1/2}$, with $\pm$ depending on the sign of $V$. If $suppV_1 = suppV_2 = \mathbb{R}^d$, then $X = Y$.

An example where $X \neq Y$ is provided by a Schrödinger operator in dimension $d = 1$ with

$$V(t) = \Sigma_{k=1}^N \alpha_k(t)\delta(x - x_k). \tag{2.1}$$

We choose

$$V_1 : X \to Y = \mathbb{C}^N, \qquad V_1 f = (f(x_1), f(x_2), ..., f(x_N)), \tag{2.2}$$

and

$$V_2 : Y \to X, \qquad V_2 c = \Sigma_{k=1}^N \alpha_k(t) c_k \delta(x - x_k), \qquad c = (c_k)_1^N. \tag{2.3}$$

In this case, the space $Y$ is an improper subspace of $X$.

In the next example, $Y$ is a proper subspace of $X$. Suppose that for the one dimensional Schrödinger operator the support of $V$ is $\Delta = [a, b]$. Then, one can take $Y = L_2(\Delta)$ and $V_1, V_2$ factors of $V$ that have the same support. The operator $M(t,E)$ can then be considered as an operator in $L_2(\Delta)$. We will introduce some additional assumptions in what follows.

## 2.2 Duhamel formulation.

Consider the operator $g_0(t) = e^{\frac{1}{i\varepsilon}A_0 t}$. In what follows, we will extensively use the Duhamel formula

$$\theta(t)g_0(t) = -i\varepsilon \left(A_0 - i\varepsilon\partial_t I - i0I\right)^{-1} \delta(t)I. \tag{2.4}$$

Here $\theta$ is the Heaviside function and $\delta(t)$ is the Dirac $\delta$-function.

Let us give some short comments on the last formula. It can be seen as a consequence of the theory of distributions or of the theory of differential equations. The general solution of the operator equation

$$(i\varepsilon\partial_t I - A_0)f(t) = \delta(t)I \tag{2.5}$$

is

$$f(t) = \begin{cases} e^{\frac{1}{i\varepsilon}A_0 t} f_+, & t > 0 \\ e^{\frac{1}{i\varepsilon}A_0 t} f_-, & t < 0. \end{cases} \tag{2.6}$$

where $f_+, f_-$ are two operators satisfying the condition $i\varepsilon(f_+ - f_-) = I$. Suppose that $f_- = 0$, then one can write the solution in the form

$$f(t) = (i\varepsilon)^{-1}\theta(t)g_0(t). \tag{2.7}$$

From the above calculation, it is clear that the operator $A_0 - i\varepsilon\partial_t$ does not have a unique inverse. Notice, however, that the equation

$$(i\varepsilon\partial_t I + i\mu I - A_0)f(t) = \delta(t)I \tag{2.8}$$

with positive $\mu$ has a unique bounded (in $t \in \mathbb{R}$) solution that is equal to 0 for negative $t$. This bounded solution is

$$f(t) = (i\varepsilon)^{-1}\theta(t)e^{\frac{1}{i\varepsilon}(A_0 - i\mu)t}. \tag{2.9}$$

So in the class of bounded solutions, we have the formula

$$\theta(t)e^{\frac{1}{i\varepsilon}(A_0 - i\mu)t} = -i\varepsilon(A_0 - i\varepsilon\partial_t I - i\mu I)^{-1}\delta(t)I. \tag{2.10}$$

Formula (2.4) is the limiting form of (2.10). Actually, this limit is not an absolutely elementary procedure, but we leave out the discussion here since we restrict ourselves to a formal derivation.

A similar formula can be obtained for the resolving operator $g(t)$ of

$$i\varepsilon g_t(t) = A(t)g(t),\ \ g(0) = I. \tag{2.11}$$

It can be obtained with the same computations as the formula for $g_0$, although in this case there is no explicit expression for $g(t)$ and for solutions vanishing as $t \to \pm\infty$ we have to use a theoretical description. The formula for $g(t)$ has the form:

$$\theta(t)g(t) = -i\varepsilon R(i\varepsilon\partial_t + i0)\delta(t)I, \tag{2.12}$$

where

$$R(i\varepsilon\partial_t + i0) = \left(A(t) - i\varepsilon\partial_t I - i0I\right)^{-1}. \tag{2.13}$$

## 2.3 Computations motivated by the scattering theory.

Notice now that

$$A(t) - i\epsilon\partial_t I = A_0 + V(t) - i\epsilon\partial_t I = (A_0 - i\varepsilon\partial_t I)\left[I + \left(A_0 - i\varepsilon\partial_t I - i0I\right)^{-1} V(t)\right]. \tag{2.14}$$

Here we could use another regularization, for example, $+i0$ instead of $-i0$. Our choice is more convenient for further treatment. Let us proceed formally:

$$A(t) - i\epsilon\partial_t I = (A_0 - \partial_t I)\, V_1^{-1}(t)\left[I + V_1(t)\left(A_0 - i\varepsilon\partial_t I - i0I\right)^{-1} V_2(t)\right] V_1(t) = \tag{2.15}$$

$$= V_2(t)\left[V_1(t)R_0(i\varepsilon\partial_t + i0)V_2(t)\right]^{-1}\left[I + V_1(t)R_0(i\varepsilon\partial_t + i0)V_2(t)\right] V_1,$$

where

$$R_0(i\varepsilon\partial_t + i0) = (A_0 - i\varepsilon\partial_t I - i0I)^{-1} . \tag{2.16}$$

As a consequence, we have

$$V_1 R(i\epsilon\partial_t + i0) = (I + \mathcal{M})^{-1} \mathcal{M} V_2^{-1}, \tag{2.17}$$

or, the less restricted variant:

$$V_1 R(i\epsilon\partial_t + i0) = (I + \mathcal{M})^{-1} V_1 R_0(i\varepsilon\partial_t + i0). \tag{2.18}$$

We introduced here the pseudodifferential operator $\mathcal{M}$ with the symbol $M(t, E)$. Formally, it is equal to

$$\mathcal{M} = M(t, i\varepsilon\partial_t + i0). \tag{2.19}$$

The ordering of the operator arguments $t, i\varepsilon\partial_t$ is obvious from the definition of the symbol $M(t, E)$:

$$M(t, i\varepsilon\partial_t + i0) = V_1(t) R_0(i\varepsilon\partial_t + i0) V_2(t). \tag{2.20}$$

Now we can write the formula

$$\theta(t) V_1(t) g(t) = -i\varepsilon \left(I + \mathcal{M}\right)^{-1} V_1 R_0(i\varepsilon\partial_t + i0)\delta(t) I. \tag{2.21}$$

We have obtained two formulas for $\mathcal{D}(t) = \theta(t)V_1(t)\psi(t)$:

$$\mathcal{D}(t) = -i\varepsilon V_1(t) R(i\varepsilon\partial_t + i0)\delta(t)\psi_0. \tag{2.22}$$

and

$$\mathcal{D}(t) = -i\varepsilon \left(I + \mathcal{M}\right)^{-1} V_1 R_0(i\varepsilon\partial_t + i0)\delta(t)\psi_0. \tag{2.23}$$

What is crucially important is that both formulas have a semiclassical structure with respect to the small parameter $\varepsilon$. We will show how to use this in the next sections.

## 2.4 Recovering the field from $\mathcal{D}(t)$.

There is one more general formula that we would like to discuss. An important observation is that, in general, it is impossible to recover the asymptotic behavior of the solution $\psi(t)$ for $t > 0$ from the asymptotic behavior of $\mathcal{D}(t) = \theta(t)V_1(t)\psi(t)$ by simply applying the operator $(V_1(t))^{-1}$ to $\mathcal{D}(t)$ . For example, in the case of the Schrödinger operator $A(t)$, if the initial data $\psi_0$ is the eigenfunction of the continuous spectrum of the operator $A(0)$, the solution does not decrease as $x \to \infty$ and the asymptotic expansion of $\psi(t)$ cannot be obtain by simple dividing $\mathcal{D}(t)$ by $V_1(t)$ even if $V_1(t)$ never vanishes. Obtained in such a way, the expansion would contain powers of the secular terms such as $\varepsilon x$.

To recover the expansion of $\psi(t)$ everywhere, we will use the formula

$$\psi(t) = g_0(t)\psi_0 + \frac{1}{i\varepsilon}\int_0^t g_0(t-\tau) V_2(\tau)\mathcal{D}(\tau) d\tau. \tag{2.24}$$

# 3 Spectral resolution.

To obtain the field $\mathcal{D}(t)$ from formula (2.23), we have to invert the operator $I + \mathcal{M}$. This can be done in terms of the spectral resolution of $\mathcal{M}$.

To clarify the structure of the spectrum of the operator $\mathcal{M}$, consider first the case when $V$ does not depend on $t$. In this case the spectral equation

$$\mathcal{M}\Phi = \mu\Phi \tag{3.1}$$

allows us to use the method of separation of variables. Let $\Phi = e^{\frac{1}{i\varepsilon}Et}W$, the equation reduces to

$$M(E)W = \mu W. \tag{3.2}$$

¿From the assumptions of Section 2, the operator $M(E)$ is compact in $Y$ and, therefore, the nonzero part of its spectrum is purely point. We denote these points by $\mu_n(E), n = 1, 2, \ldots, E \in \mathbb{C}_0$ and the corresponding eigenvectors by $W_n(E)$. The eigenvalues of the adjoint operator $(M(E))^*$ are $\overline{\mu_n(E)}$, and the corresponding eigenvectors are $\widehat{W}_n(E)$.

The crucial additional assumptions are that the spectrum is simple, the set of the eigenvectors $W_n(E)$ is complete and the following spectral resolution holds in the sense of norm in $Y$:

$$f = \sum_{n \geq 1} W_n(E)\langle f, \widehat{W}_n(E)\rangle. \tag{3.3}$$

Here $\langle \cdot, \cdot \rangle$ is the inner product in Y. As a result, the "functions" $\Phi_\sigma(t) = e^{\frac{1}{i\varepsilon}Et}W_n(E)$, $\sigma = (E, n),\ E \in \mathbb{R}, n = 1, 2, \ldots$ constitute a complete set in $L_2(\mathbb{R}) \otimes Y$ such that for the elements of this space, the following expansion

$$f(t) = \frac{1}{2\pi\varepsilon}\int_\sigma \Phi_\sigma(t)d\sigma \int_{\mathbb{R}} \langle f(t'), \widehat{\Phi}_\sigma(t')\rangle dt' \tag{3.4}$$

holds in a suitable sense, with $\widehat{\Phi_\sigma(t')} = e^{\frac{1}{i\varepsilon}Et}\widehat{W}_n(E)$. Here $\int_\sigma d\sigma = \int_{\mathbb{R}} dE \sum_{n\geq 1}$.

If $V$ depends on $t$, but the parameter $\varepsilon$ is small, the structure of the spectrum of $\mathcal{M}$ remains the same as in the case where the separation of variables is possible. Under some additional assumptions that will be clarified later, the structure of the spectral resolution is again given by formula (3.4) and $\sigma$ has the same sense as before. Of course, the explicit expressions for eigenvalues $\mu_\sigma$, eigenfunctions $\Phi_\sigma(t)$ of $\mathcal{M}$, $\mathcal{M}\Phi_\sigma = \mu_\sigma\Phi_\sigma$, and eigenfunctions $\widehat{\Phi}_\sigma(t)$ of $(\mathcal{M})^*$, $(\mathcal{M})^*\widehat{\Phi}_\sigma = \bar{\mu}_\sigma\widehat{\Phi}_\sigma$, are different. But in the semiclassical case, these eigenvalues and eigenfunctions can be still described effectively.

Let us point out in addition to the resolution theorem, the orthonormality condition

$$\int_{\mathbb{R}} \langle \Phi_\sigma(t), \widehat{\Phi}_{\sigma'}(t)\rangle dt = 2\pi\varepsilon\delta(E - E')\delta_{nn'}. \tag{3.5}$$

that is a consequence of (3.4) and will be used in the next sections.

Formulas (2.14) of the previous section for $\mathcal{D}$ can be transformed now to the form:

$$\mathcal{D}(t) = \frac{1}{2\pi i} \int_\sigma d\sigma \Phi_\sigma(t) \langle \psi_0, (V_1(t') R(t', i\varepsilon\partial_{t'} + i0))^* \, \widehat{\Phi}_\sigma(t') \rangle|_{t'=0}. \tag{3.6}$$

The second main formula (2.23) of the previous section can be analogously transformed to the form:

$$\mathcal{D}(t) = \frac{1}{2\pi i} \int_\sigma d\sigma \Phi_\sigma(t) \langle \psi_0, (V_1(t') R_0(t', i\varepsilon\partial_{t'} + i0))^* \, (I + \mathcal{M}^*)^{-1} \widehat{\Phi}_\sigma(t') \rangle|_{t'=0}, \tag{3.7}$$

or equivalently to

$$\mathcal{D}(t) = \frac{1}{2\pi i} \int_\sigma d\sigma \Phi_\sigma(t) \frac{1}{1+\mu_\sigma} \langle \psi_0, (V_1(t') R_0(t', i\varepsilon\partial_{t'} + i0))^* \, \widehat{\Phi}_\sigma(t') \rangle|_{t'=0}. \tag{3.8}$$

Clearly, we have to assume here that all the eigenvalues $\mu_\sigma$ are different from $-1$; this can always be satisfied if the contour of integration with respect to $E$ is suitably deformed. The asymptotic behavior of $\mathcal{D}(t)$ as $\varepsilon \to 0$ will be derived from effective asymptotic formulas for $\mu_\sigma, \Phi_\sigma, \widehat{\Phi}_\sigma$.

# 4 Semiclassical asymptotic analysis of eigenfunctions.

## 4.1 Eigenfunctions.

The new essential assumptions concern the spectral properties of the operator $M(t, E)$. Let us recall that this operator is compact in $Y$. We assume its spectrum is simple. Introduce the eigenvalues and eigenvectors of $M(t, E)$ and $(M(t, E))^*$:

$$M(t,E) v_n(t,E) = \nu_n(t,E) v_n(t,E), \qquad (M(t,E))^* \widehat{v}_n(t,E) = \bar{\nu}_n(t,E) \widehat{v}_n(t,E). \tag{4.1}$$

Suppose that the following spectral resolution, $f \in Y$:

$$f = \sum_{n \geq 1} v_n(t,E) \langle f, \widehat{v}_n(t,E) \rangle \tag{4.2}$$

holds (completeness). This implies, in particular, the orthonormality condition :

$$\langle v_n(t,E), \widehat{v}_{n'}(t,E) \rangle = \delta_{nn'}. \tag{4.3}$$

Let us now consider the spectral equation

$$\mathcal{M}\Phi_\sigma = \mu_\sigma \Phi_\sigma. \tag{4.4}$$

It is convenient to consider simultaneously the equation for $\widehat{\Phi}_\sigma$:

$$\mathcal{M}^* \widehat{\Phi}_\sigma = \bar{\mu}_\sigma \widehat{\Phi}_\sigma. \tag{4.5}$$

For $\varepsilon \to 0$, the eigenvectors $\Phi_\sigma$ and $\widehat{\Phi}_\sigma$ can be described by the following formulas

$$\Phi_\sigma = e^{\frac{1}{i\varepsilon}\int_0^t \omega_\sigma(\tau)d\tau} W_\sigma(t,\varepsilon), \qquad \widehat{\Phi}_\sigma = e^{\frac{1}{i\varepsilon}\int_0^t \overline{\omega}_\sigma(\tau)d\tau} \widehat{W}_\sigma(t,\varepsilon). \tag{4.6}$$

The vectors $W_\sigma$ and $\widehat{W}_\sigma$ satisfy

$$M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0) W_\sigma(t,\varepsilon) = \mu_\sigma(\varepsilon) W_\sigma(t,\varepsilon), \tag{4.7}$$

and

$$(M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0))^* \widehat{W}_\sigma(t,\varepsilon) = \bar{\mu}_\sigma(\varepsilon) \widehat{W}_\sigma(t,\varepsilon). \tag{4.8}$$

## 4.2 Formal expansions.

The vector $W_\sigma$ can be described asymptotically as $\varepsilon \to 0$ by a formal solution in the form:

$$W_\sigma(t,\varepsilon) \sim \sum_{l\geq 0} (i\varepsilon)^l w_\sigma^{(l)}(t), \qquad w_\sigma^{(0)} = w_\sigma. \tag{4.9}$$

Similarly,

$$\widehat{W}_\sigma(t,\varepsilon) \sim \sum_{l\geq 0} (i\varepsilon)^l \widehat{w}_\sigma^{(l)}(t), \qquad \widehat{w}_\sigma^{(0)} = \widehat{w}_\sigma. \tag{4.10}$$

To construct the formal solution $W_\sigma$, we have to expand the operator $M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0)$ in powers of $\varepsilon$:

$$M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0) \sim \sum_{l\geq 0} (i\varepsilon)^l M_\sigma^{(l)}(t,\partial_t), \qquad M_\sigma^{(0)}(t,\partial_t) = M_\sigma(t,\partial_t) = M(t, \omega_\sigma + i0). \tag{4.11}$$

The dependence of $M_\sigma^{(l)}$ on $\sigma$ enters through $\omega_\sigma$. For $M_\sigma^{(l)}$, we can easily find an explicit expression

$$M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0) = V_1(t)\left(A_0 - i\varepsilon\partial_t - \omega_\sigma(t) - i0\right)^{-1} V_2(t) \sim \tag{4.12}$$

$$\sim \sum_{l\geq 0} (i\varepsilon)^l V_1(t) R_0(\omega_n - i0)\left(\partial_t R_0(\omega_\sigma - i0)\right)^l V_2(t),$$

so that

$$M_\sigma^{(l)} = V_1(t) R_0 \left(\partial_t R_0\right)^l V_2(t), \qquad R_0 = R_0(\omega_\sigma - i0). \tag{4.13}$$

Now we can write the system of equations for $w_\sigma^{(l)}(t)$:

$$\sum_{m+l=r} [M_\sigma^{(m)} - \mu_\sigma^{(m)}] w_\sigma^{(l)} = 0, \qquad r = 0, 1, \ldots, \tag{4.14}$$

where $\mu_\sigma^{(l)}$ are the coefficients of the asymptotic expansion

$$\mu_\sigma \sim \sum_{l \geq 0} (i\varepsilon)^l \mu_\sigma^{(l)}, \qquad \mu_\sigma^{(0)} = \mu_\sigma. \tag{4.15}$$

Analogous formulas hold for the adjoint operator $(M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0))^*$. To construct the formal solution $\widehat{W}_\sigma(t, \varepsilon)$ we have to expand now this operator in powers of $\varepsilon$:

$$(M(t, i\varepsilon\partial_t + \omega_\sigma(t) + i0))^* \sim \sum_{l \geq 0} (i\varepsilon)^l \widehat{M}_\sigma^{(l)}(t, \partial_t). \tag{4.16}$$

For $\widehat{M}_\sigma^{(l)}$, we find the explicit expression

$$\widehat{M}_\sigma^{(l)} = V_2^* R_0^* (\partial_t R_0^*)^l V_1^*. \tag{4.17}$$

The system of equations for $\widehat{w}_\sigma^{(l)}(t)$ has the form:

$$\sum_{m+l=r} [\widehat{M}_n^{(m)} - \widehat{\mu}_\sigma^{(m)}] \widehat{w}_\sigma^{(l)} = 0, \qquad r = 0, 1, \ldots, \tag{4.18}$$

where $\widehat{\mu}_\sigma^{(l)}$ are the coefficients of the asymptotic expansion

$$\widehat{\mu_\sigma} \sim \sum_{l \geq 0} (i\varepsilon)^l \widehat{\mu}_\sigma^{(l)}, \qquad \widehat{\mu}_\sigma^{(0)} = \widehat{\mu}_\sigma, \qquad \widehat{\mu}_\sigma^{(l)} = (-1)^l \bar{\mu}_\sigma^{(l)}. \tag{4.19}$$

## 4.3 The leading order.

Equation (4.14) for $r = 0$ is

$$M(t, \omega_\sigma(t) + i0) w_\sigma = \mu_\sigma w_\sigma. \tag{4.20}$$

Comparing the last equation with (4.1) we see that

$$w_\sigma(t) = v_n(t, \omega_\sigma), \qquad \mu_\sigma = \nu_n(t, \omega_\sigma). \tag{4.21}$$

The last relation is an equation for $\omega_\sigma$. In general, there is no a priori prescribed way to choose the parameter $E$, and we propose to fix it in the following convenient way:

$$\omega_\sigma(0) = E. \tag{4.22}$$

Therefore, we write

$$\nu_n(t, \omega_\sigma) = \nu_n(0, E), \qquad \mu_\sigma = \nu_n(0, E). \tag{4.23}$$

Similarly, up to a normalization factor

$$\widehat{w}_\sigma(t) = \widehat{v}_n(t, \omega_\sigma). \tag{4.24}$$

The properties of the function $\omega_\sigma$ are quite essential for the following. We will discuss them in a subsequent work (part II). Here, we will proceed with the assumption that it is analytical function of $E$ with $\Im\omega > 0$ for $\Im E > 0$ and its analytical continuation to the lower half plane satisfies $\Im\omega < 0$. We also allow this function to have branching points of algebraic or logarithmic type at the boundary points of the continuous spectrum of $A_0$ and assume it is continuous at these points. As for the behavior of $\omega_\sigma$ at infinity, we assume that it is of order $E$. Later, we will see on examples how these properties display themselves in applications.

# 5 Normalization of $w_\sigma$ and $\widehat{w}_\sigma$ and the spectral resolution.

¿From the construction it is clear that the vector $w_\sigma(t)$ is defined up to a scalar factor that arbitrarily depends on $t, \sigma$. Since equation (3.4) has to determine the dependence of $\Phi_\sigma(t)$ on $t$, it is natural to expect that the dependence of the scalar factor on $t$ can be found from the construction of the whole formal solution. And, indeed, it can be found from the equation (4.14) for $r = 1$:

$$[M_\sigma - \mu_\sigma] w_\sigma^{(1)} = -[M_\sigma^{(1)} - \mu_\sigma^{(1)}] w_\sigma. \tag{5.1}$$

¿From this, it follows

$$\langle [M_\sigma^{(1)} - \mu_\sigma^{(1)}] w_\sigma, \widehat{w}_\sigma \rangle = 0, \tag{5.2}$$

or more explicitly,

$$\langle [V_1 R_0(\omega_\sigma + i0) \partial_t R_0(\omega_\sigma + i0) V_2 - \mu_\sigma^{(1)}] w_\sigma, \widehat{w}_\sigma \rangle = 0, \tag{5.3}$$

which can be rewritten as

$$\langle \partial_t R_0(\omega_\sigma + i0) V_2 w_\sigma(t), (R_0(\omega_\sigma + i0))^* V_1^* \widehat{w}_\sigma(t) \rangle = \mu_\sigma^{(1)} \langle w_\sigma, \widehat{w}_\sigma(t) \rangle. \tag{5.4}$$

It is not obvious that the inner product on the left hand side makes sense as these vectors might not belong to $L_2$. In such case, one has to replace the argument $\omega_\sigma$ of $R_0$ by a parameter with positive imaginary part, compute then the proper inner product and obtain the necessary improper product by the analytical continuation of the previous expression with respect to this parameter.

## 5.1 Transport equation.

Introducing two new vectors

$$\chi_\sigma = \frac{1}{\mu_\sigma} R_0(\omega_\sigma + i0) V_2 w_\sigma, \tag{5.5}$$

$$\widehat{\chi}_\sigma = \frac{1}{\bar{\mu}_\sigma} (R_0(\omega_\sigma + i0))^* V_1^* \widehat{w}_\sigma, \tag{5.6}$$

equation (5.4) becomes

$$\mu_\sigma^2 \langle \partial_t \chi_\sigma, \widehat{\chi}_\sigma \rangle = \mu_\sigma^{(1)} \langle w_\sigma, \widehat{w}_\sigma \rangle, \tag{5.7}$$

which can be seen as a transport equation. Notice that from the definition of $\chi_\sigma, \widehat{\chi}_\sigma$, we have

$$V_1 \chi_\sigma = w_\sigma, \ V_2^* \widehat{\chi}_\sigma = \widehat{w}_\sigma. \tag{5.8}$$

The analogous computations can be applied to the vector $\widehat{w}_\sigma$. Equation (4.18) for $r = 1$ gives

$$[M_\sigma^* - \bar{\mu}_\sigma] \widehat{w}_\sigma^{(1)} = -[\widehat{M}_\sigma^{(1)} + \bar{\mu}_\sigma^{(1)}] \widehat{w}_\sigma, \tag{5.9}$$

that implies

$$\langle w_\sigma, [\widehat{M}_\sigma^{(1)} + \bar{\mu}_\sigma^{(1)}] \widehat{w}_\sigma \rangle = 0, \tag{5.10}$$

or equivalently,

$$\langle w_\sigma, [V_2^* (R_0(\omega_\sigma + i0))^* \partial_t (R_0(\omega_\sigma + i0))^* V_1^* + \bar{\mu}_\sigma^{(1)}] \widehat{w}_\sigma \rangle = 0. \tag{5.11}$$

Finally,

$$\mu_\sigma^2 \langle \chi_\sigma, \partial_t \widehat{\chi}_\sigma \rangle = -\mu_\sigma^{(1)} \langle w_\sigma, \widehat{w}_\sigma \rangle. \tag{5.12}$$

Under the assumption that $\mu_\sigma \neq 0$, we have from (5.7) and (5.12)

$$\partial_t \langle \chi_\sigma, \widehat{\chi}_\sigma \rangle = 0. \tag{5.13}$$

¿From the definition of $\chi_\sigma$ and $\widehat{\chi}_\sigma$, it follows that

$$(A_0 - \frac{1}{\mu_\sigma} V - \omega_\sigma) \chi_\sigma = 0, \qquad (A_0 - \frac{1}{\bar{\mu}_\sigma} V^* - \bar{\omega}_\sigma) \widehat{\chi}_\sigma = 0. \tag{5.14}$$

It is important to note that $\chi_\sigma$ and $\widehat{\chi}_\sigma$ are not simply solutions of these "differential equations", but solutions that, due to their definitions, satisfy certain "boundary conditions". By "boundary condition" we imply a representation of the form

$$\chi_\sigma = R_0(\omega_\sigma + i0)f. \tag{5.15}$$

If $\Im\omega_\sigma$ is negative, then $R_0(\omega_\sigma + i0)f$ denotes here the analytical continuation of $R_0(\omega_\sigma + i0)f$ from the upper half-plane through the continuous spectrum to the lower half-plane. Of course, we have to continue not the element of the Hilbert space itself, but the element of the corresponding equipped space. In the case of the Schrödinger equation, $f$ must have compact support and this representation indeed implies the specific asymptotic behavior of $\chi_\sigma$ as $|x| \to \infty$.

Let us differentiate the left equation of (5.14) with respect to $E$:

$$[\partial_E(A_0 - \frac{1}{\mu_\sigma}V - \omega_\sigma)]\chi_\sigma + (A_0 - \frac{1}{\mu_\sigma}V - \omega_\sigma)\partial_E\chi_\sigma = 0. \tag{5.16}$$

Taking the inner product of the above equation with $\widehat{\chi}_\sigma$, we obtain

$$\langle w_\sigma, \widehat{w}_\sigma\rangle \frac{1}{{\mu_\sigma}^2}\mu_{\sigma E} = \langle \chi_\sigma, \widehat{\chi}_\sigma\rangle \omega_{\sigma E}. \tag{5.17}$$

We used here the notations $\mu_{\sigma E} = \partial_E \mu_\sigma$ and $\omega_{\sigma E} = \partial_E \omega_\sigma$. This means, in particular, that not only $\langle \chi_\sigma, \widehat{\chi}_\sigma\rangle$, but also $\langle w_\sigma, \widehat{w}_\sigma\rangle$ does not depend on $t$. The dependence of these scalar products on $\sigma$ has to be defined in accordance with the spectral resolution. We will find this later.

## 5.2 Normalization: Dependence of $\chi_\sigma$ and $\widehat{\chi_\sigma}$ on t.

Let us come back to the normalization rule (5.7) for $\chi_\sigma$. The analogous rule for $\widehat{\chi}_\sigma$ can be replaced by the condition (5.13). First let us choose the normalizations such that

$$\langle \partial_t \chi_\sigma, \widehat{\chi}_\sigma\rangle = 0, \qquad \langle \chi_\sigma, \partial_t \widehat{\chi}_\sigma\rangle = 0, \tag{5.18}$$

and temporarily denote the solutions satisfying these normalizations by $\chi_\sigma^0$ and $\widehat{\chi}_\sigma^0$. Let us seek $\chi_\sigma$ and $\widehat{\chi}_\sigma$ in the form $\chi_\sigma = c\chi_\sigma^0$ and $\widehat{\chi}_\sigma = \frac{1}{\bar{c}}\widehat{\chi}_\sigma^0$. Equations (5.7) and (5.17) imply

$$\frac{c_t}{c} = \mu_\sigma^{(1)} \frac{\omega_{\sigma E}}{\mu_{\sigma E}}. \tag{5.19}$$

This means

$$c = c_\sigma^0 \exp\left(t\mu_\sigma^{(1)} \frac{\omega_{\sigma E}}{\mu_{\sigma E}}\right), \tag{5.20}$$

where $c_\sigma^0$ does not depend on $t$.

In the above computations $\mu_\sigma^{(1)}$ remains undefined. It can be chosen arbitrary. Indeed, the exponential factor in $c$ could be combined with the factor $exp\frac{1}{i\varepsilon}\int_0^t\omega_\sigma(t)dt$. It would change the definition of $\omega$:

$$\omega_\sigma\to\omega_\sigma+\mu_\sigma^{(1)}\frac{\omega_{\sigma E}}{\mu_{\sigma E}}. \tag{5.21}$$

This transformation can be interpreted as a change of $E$. But we have mentioned already that the choice of $E$ is not unique by nature, and in principle, the definition of $E$ can depend on $\varepsilon$. It is actually the reason of ambiguity in the choice of $\mu_\sigma^{(1)}$. This idea allows us to put $\mu_\sigma^{(1)}=0$. We will make this choice in the following. By this, we have fixed the dependence of $\chi_\sigma$ and $\widehat{\chi}_\sigma$ on $t$ by conditions (5.18).

### 5.3 Normalization: Dependence of $\chi_\sigma$ and $\widehat{\chi_\sigma}$ on $\sigma$.

The dependence of $\chi_\sigma$ and $\widehat{\chi_\sigma}$ on $\sigma$ has not yet been defined. Let us recall the general orthonormality condition (see (3.5))

$$\int_{\mathbb{R}}\langle\Phi_\sigma(t),\widehat{\Phi}_{\sigma'}(t)\rangle dt=2\pi\varepsilon\delta(E-E')\delta_{nn'}. \tag{5.22}$$

In a more explicit notation, this condition, rewritten at leading order, is:

$$\int dte^{\frac{1}{i\varepsilon}\int_0^t(\omega_n(t',E)-\omega_{n'}(t',E')dt')}\langle w_n(t,E),\widehat{w}_{n'}(t,E')\rangle\sim 2\pi\varepsilon\delta(E-E')\delta_{nn'} \tag{5.23}$$

Here $\omega_n(t,E)=\omega_{(E,n)}(t)$. The integrand is a fast oscillating function of $t$. Assuming that $\omega_n(t,E)=\omega_{n'}(t,E)$ only if $n=n'$ and $E=E'$, we get, at lowest order

$$\int dte^{\frac{1}{i\varepsilon}\int_0^t(\omega_n(t',E)-\omega_{n'}(t',E'))dt'}\langle w_n(t,E),\widehat{w}_{n'}(t,E')\rangle\sim\int dte^{\frac{1}{i\varepsilon}\int_0^t\omega_{nE}(t',E)(E-E')dt'}\langle w_n(t,E),\widehat{w}_n(t,E)\rangle\delta_{nn'}. \tag{5.24}$$

We have used the orthonormality condition for $v_n(t,E)$ and $\widehat{v}_n(t,E)$ and the connection between these functions and $w_n(t,E),\widehat{w}_n(t,E)$ (see (4.21)). That means that we have to choose the normalization of $w_n(t,E),\widehat{w}_n(t,E)$ such that

$$\int dte^{\frac{1}{i\varepsilon}\int_0^t\omega_{nE}(t',E)(E-E')dt'}\langle w_n(t,E),\widehat{w}_n(t,E)\rangle\sim 2\pi\varepsilon\delta(E-E'). \tag{5.25}$$

Changing the variable of integration, we can rewrite the condition in the form

$$\int d\tau e^{\frac{1}{i\varepsilon}\tau(E-E')}\frac{\langle w_n(t,E),\widehat{w}_n(t,E)\rangle}{\omega_{nE}(t,E)}\sim 2\pi\varepsilon\delta(E-E'). \tag{5.26}$$

This leads to the final normalization condition

$$\langle w_n(t,E),\widehat{w}_n(t,E)\rangle=\omega_{nE}(t,E). \tag{5.27}$$

### 5.4 Summary of results.

Let us collect now the main results of this section with respect to the normalizations of $w_\sigma, \widehat{w}_\sigma$:

$$\langle w_\sigma, \widehat{w}_\sigma \rangle = \omega_{\sigma E}, \tag{5.28}$$

$$\langle \chi_\sigma, \widehat{\chi}_\sigma \rangle = \frac{1}{{\mu_\sigma}^2} \mu_{\sigma E}, \tag{5.29}$$

and

$$\langle \partial_t \chi_\sigma, \widehat{\chi}_\sigma \rangle = 0, \qquad \langle \chi_\sigma, \partial_t \widehat{\chi}_\sigma \rangle = 0. \tag{5.30}$$

With such choice of $w_\sigma, \widehat{w}_\sigma$ the functions

$$\Phi_\sigma = e^{\frac{1}{i\varepsilon} \int_0^t \omega_\sigma(\tau) d\tau} w_\sigma(t), \qquad \widehat{\Phi}_\sigma = e^{\frac{1}{i\varepsilon} \int_0^t \overline{\omega}_\sigma(\tau) d\tau} \widehat{w}_\sigma(t) \tag{5.31}$$

satisfy at the leading order

$$\mathcal{M} \Phi_\sigma = \mu_\sigma \Phi_\sigma, \qquad \mathcal{M}^* \widehat{\Phi}_\sigma = \bar{\mu}_\sigma \widehat{\Phi}_\sigma. \tag{5.32}$$

Moreover, this choice of $w_\sigma, \widehat{w}_\sigma$ guarantees that these expressions can be taken as the leading orders of the formal solutions introduced in the previous section. They satisfy also the orthonormality condition

$$\int_{\mathbb{R}} \langle \Phi_\sigma(t), \widehat{\Phi}_{\sigma'}(t) \rangle dt = 2\pi\varepsilon \delta(E - E') \delta_{nn'}, \tag{5.33}$$

that implies the following spectral resolution:

$$f(t) = \frac{1}{2\pi\varepsilon} \int_\sigma \Phi_\sigma(t) d\sigma \int_{\mathbb{R}} \langle f(t'), \widehat{\Phi}_\sigma(t') \rangle dt'. \tag{5.34}$$

# 6 Asymptotic formulas for the wave field. Special cases.

### 6.1 Main formulas.

We can now formulate the final result of the previous constructions. Suppose we have the complete system of the functions $\Phi_\sigma(t)$ and $\widehat{\Phi}_\sigma(t)$ such that for any element $f(t)$ of $L_2 \times Y$, (5.34) holds. Then we have for $\mathcal{D}(t) = V_1(t)\theta(t)\psi(t)$, see (3.6),

$$\mathcal{D}(t) = \frac{1}{2\pi i} \int_\sigma d\sigma \Phi_\sigma(t) \langle \psi_0, (V_1(t') R(t', i\varepsilon\partial_{t'} + i0))^* \widehat{\Phi}_\sigma(t') \rangle|_{t'=0}. \tag{6.1}$$

We have chosen $\Phi_\sigma(t)$ and $\widehat{\Phi}_\sigma(t)$ as the appropriately normalized eigenfunctions of the operators $\mathcal{M}$ and $\widehat{\mathcal{M}}$. We were not able to find these functions explicitly, but have shown that they can be described asymptotically by formal solutions whose leading terms have the form (5.31). If we replace $\Phi_\sigma(t)$ and $\widehat{\Phi}_\sigma(t)$ in (5.32) by the leading terms of their explicit asymptotic representations, we obtain the leading term of asymptotic description of $\mathcal{D}(t)$:

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_\sigma d\sigma e^{\frac{1}{i\varepsilon}\int_0^t \omega_\sigma(\tau)d\tau} w_\sigma(t)\langle\psi_0, (V_1(t')R(t', i\varepsilon\partial_{t'} + \omega_\sigma(t') + i0))^* \, \widehat{w}_\sigma(t')\rangle|_{t'=0} \sim \tag{6.2}$$

$$\sim \frac{1}{2\pi i}\int_\sigma d\sigma e^{\frac{1}{i\varepsilon}\int_0^t \omega_\sigma(\tau)d\tau} w_\sigma(t)\langle\psi_0, (V_1(0)R(0, E + i0))^* \, \widehat{w}_\sigma(0)\rangle. \tag{6.3}$$

This is one of the main formulas of the present work. In a little more detailed notation

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_{\mathbb{R}} dE \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E)d\tau} w_n(t, E)\langle\psi_0, (V_1(0)R(0, E + i0))^* \, \widehat{w}_n(0, E)\rangle, \tag{6.4}$$

where $I$ is the set of indices $n$.

If instead of $w_\sigma$ and $\widehat{w}_\sigma$, we use longer germs of the corresponding formal eigenvectors, we will get more terms in the asymptotic expansion of $\mathcal{D}(t)$.

It is worth noticing that the choice of $\Phi_\sigma(t)$ and $\widehat{\Phi}_\sigma(t)$ is not unique, at least, the choice of $A_0$ is arbitrary (in the sense that we can include in $A_0$ some part of $V$ that does not depend on $t$). This remark can be useful, for example, if the spectral properties of the operator $M(t, E)$ do not fulfill the assumptions of Section 4. In principle, the choice of $\Phi_\sigma(t)$ and $\widehat{\Phi}_\sigma(t)$ can be even less restricted than the choice of $A_0$.

Let us consider also the representation (3.8) and obtain the asymptotic description of $\mathcal{D}(t)$ replacing $\Phi_\sigma, \widehat{\Phi}_\sigma$ by their asymptotic representations:

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_{\mathbb{R}} dE \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E)d\tau} w_n(t, E)\frac{\mu_n(E)}{1 + \mu_n(E)}\widetilde{c}_n(E), \tag{6.5}$$

where

$$\widetilde{c}_n(E) \sim \frac{1}{\mu_n(E)}\int_{\mathbb{R}} dt'\langle V_1(t')R_0(i\varepsilon\partial_{t'} + i0)\delta(t')\psi_0, e^{\frac{1}{i\varepsilon}\int_0^t \overline{\omega}_n(\tau,E)d\tau}\widehat{w}_n(t, E)\rangle =$$

$$= \frac{1}{\mu_n(E)}\langle\psi_0, (V_1(0)R_0(E - i0))^* \, \widehat{w}_n(0, E)\rangle,$$

or more simply

$$\widetilde{c}_n(E) \sim \langle\psi_0, \widehat{\chi}_n(0, E)\rangle. \tag{6.6}$$

## 6.2 Initial condition as an eigenfunction associated to the discrete spectrum.

Assume that the initial data $\psi_0$ is the eigenfunction $\psi_m(0)$ of the operator $A(0)$ corresponding to an eigenvalue $E_m(0)$. Assume also that the eigenvalue can be included in a curve of simple isolated eigenvalues $E_m(t)$ of $A(t)$. We denote by $\psi_m(t)$ the corresponding eigenvector somehow normalized.

It is immediately clear that in this case, the main asymptotic formula for $\mathcal{D}$ reduces to

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_{\mathbb{R}} dE \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E)d\tau} w_n(t,E)\frac{1}{E_m(0)-E-i0}\langle V_1(0)\psi_m(0), \widehat{w}_n(0,E)\rangle. \tag{6.7}$$

The symbol $i0$ here implies that the contour of integration in $E$ plane goes above the point $E_m(0)$.

Let us deform the contour of integration in the lower half plane:

$$\mathcal{D}(t) \sim \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E_m(0))d\tau} w_n(t,E_m(0))\langle V_1(0)\psi_m(0), \widehat{w}_n(0,E_m(0))\rangle +$$

$$+\frac{1}{2\pi i}\int_{\mathbb{R}} dE \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E)d\tau} w_n(t,E)\frac{1}{E_m(0)-E+i0}\langle V_1(0)\psi_m(0), \widehat{w}_n(0,E)\rangle. \tag{6.8}$$

It is obvious that $\omega_n(\tau,E), w_n(t,E)$ and $\widehat{w}_n(0,E)$ can loose their analyticity only at the boundaries of the continuous spectrum of $A_0$ and can have these points as branching points where all these functions are continuous. For $t>0$ they are the only obstacles for deforming the contour of integration to infinity in the lower half-plane. Therefore only these points contribute to the asymptotic behavior of the integral as $\varepsilon \to 0$. The corresponding contributions acquire some additional positive power of $\varepsilon$ with respect to the order of the integrand, and, as a result, the leading term of the asymptotic behavior of $\mathcal{D}(t)$ is reduced to the first term of the previous representation:

$$\mathcal{D}(t) \sim \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E_m(0))d\tau} w_n(t,E_m(0))\langle V_1(0)\psi_m(0), \widehat{w}_n(0,E_m(0))\rangle. \tag{6.9}$$

Let us now relate $w_n(t,E_m(0))$ to $\psi_m(t)$. We have

$$-V(0)\psi_m(0) = (A_0 - E_m(0))\psi_m(0), \tag{6.10}$$

and this means that there exists $\chi_m(0,E)$ solution of eq. (5.14), with the corresponding $\mu_m(E)$, for $E=E_m(0)$, equal to $-1$ and the corresponding $\omega_n(0,E)=E$. If $t$ changes,

$\mu_n(E)$ remains constant, consequently $\mu_m(E) = -1$, and the solution $\chi_m(0, E)$ enters into the one-parameter set of the solutions $\chi_m(t, E)$:

$$-V(t)\chi_m(t, E) = (A_0 - \omega_m(t, E))\chi_m(t, E). \tag{6.11}$$

This equation implies that

$$\chi_m(t, E) = c_m(t, E)\psi_m(t), \qquad \omega_m(t, E) = E_m(t), \qquad E = E_m(0), \tag{6.12}$$

where $c_m$ is some scalar factor.

Similarly,

$$\widehat{\chi}_m(t, E) = c'_m(t, E)\chi_m(t, E), \qquad\qquad E = E_m(0), \tag{6.13}$$

$c'_m$ is another scalar factor.

Consider the coefficients $\langle V_1(0)\psi_m(0), \widehat{w}_n(0, E_n(0))\rangle = \langle w_m(0, E_m(0)), \widehat{w}_n(0, E_m(0))\rangle = c''_n\delta_{mn}$. ¿From (5.27), we know that $c''_n = 1$. Taking this formula into account, we reduce the asymptotic formula (6.9) to

$$\mathcal{D}(t) \sim e^{\frac{1}{i\varepsilon}\int_0^t E_m(\tau)d\tau} w_m(t, E_m(0))\langle V_1(0)\psi_m(0), \widehat{w}_m(0, E_m(0))\rangle =$$

$$= e^{\frac{1}{i\varepsilon}\int_0^t E_m(\tau)d\tau} V_1(t)\chi_m(t, E_m(0))\frac{1}{c_m(0, E)}\langle w_m(0, E_m(0)), \widehat{w}_m(0, E_m(0))\rangle =$$

$$= e^{\frac{1}{i\varepsilon}\int_0^t E_m(\tau)d\tau} V_1(t)\frac{\chi_m(t, E_m(0))}{c_m(0, E_m(0))}. \tag{6.14}$$

This expression obviously coincides with formula (1.2) of the Introduction. It satisfies the initial condition and the transport equation.

## 6.3 Initial condition as an eigenfunction associated to continuous spectrum.

Consider now an initial condition in the form $\psi(0) = \psi(0, E_0)$ where $\psi(t, E_0)$ is an eigenfunction of the continuous spectrum of the operator $A(t)$ corresponding to the eigenvalue $E_0$. Of course, $\psi(t, E_0)$ is an improper eigenfunction, and to consider it, we have to equip the space $X$ with some additional structure. We will not discuss them explicitly, but simply will assume that the framework necessary for a reasonable treatment of the eigenfunctions of the continuous spectrum exists. If $A(t)$ is a Schrödinger operator, such framework is well-known.

Again we rewrite the general asymptotic formula in the specific form:

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_{\mathbb{R}} dE \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E)d\tau} w_n(t, E)\frac{1}{E_0 - E - i0}\langle V_1(0)\psi(0, E_0), \widehat{w}_n(0, E)\rangle. \tag{6.15}$$

Now we repeat the computations following the analogous formula (6.9) and get the asymptotic expression:

$$\mathcal{D}(t) \sim \sum_{n\in I} e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E_0)d\tau} w_n(t,E_0)\langle V_1(0)\psi(0,E_0), \widehat{w}_n(0,E_0)\rangle. \tag{6.16}$$

The function $\psi(0,E_0)$ satisfies

$$-V(0)\psi(0,E_0) = (A_0 - E_0)\psi(0,E_0), \tag{6.17}$$

but it does not imply that $\psi(0,E_0)$ coincide with $\chi_n(0,E_0)$. These two functions obey different "boundary conditions" and in the equation for $\chi(0,E_0)$ the parameter $\mu$ is not equal to $-1$:

$$\frac{1}{\mu_n(E_0)} V_1(0)\chi_n(0,E_0) = (A_0 - E_0)\chi_n(0,E_0). \tag{6.18}$$

This implies that the sum/series representing the asymptotic behavior of the solution $\mathcal{D}(t)$ cannot in general be reduced to a shorter expression. What we have to do in practice is, first, to expand the expression $V_1(0)\psi(0,E_0)$ with respect to the solutions $\chi_n(0,E_0)$ ("resonances"), or equivalently, with respect to the $w_n(0,E_0)$, corresponding to the same eigenvalue $E_0$:

$$V_1(0)\psi(0,E_0) = \sum_{n\in I} w_n(0,E_0)\langle V_1(0)\psi(0,E_0), \widehat{w}_n(0,E_0)\rangle. \tag{6.19}$$

To get now the asymptotic expression for $\mathcal{D}(t)$ we have to replace the vectors $w_n(0,E_0)$ by their evolving expressions $e^{\frac{1}{i\varepsilon}\int_0^t \omega_n(\tau,E_0)d\tau} w_n(t,E_0)$ that remember the transport equations.

# 7 Example: the case of a potential in the form of a delta function: asymptotic behavior of $\mathcal{D}(t)$.

## 7.1 Setting of the problem.

To demonstrate the suggested approach, we consider the Schrödinger equation

$$i\varepsilon\psi_t = -\psi_{xx} + \alpha(t)\delta(x)\psi, \qquad x\in\mathbb{R}, t\in[0,T], \tag{7.1}$$

where $\psi = \psi(x,t) \in \mathbb{C}$, $\delta$ is the Dirac delta-function and $\alpha$ is a real-valued smooth function. We assume that $\alpha$ does not change the sign. This example cannot be exactly immersed into the general framework described earlier, since the domain of the definition of the operator

$$A(t) = A_0 + V(t), \qquad A_0 = -\frac{\partial^2}{\partial x^2}, \qquad V(t) = \alpha(t)\delta(x), \tag{7.2}$$

considered say in $L_2(\mathbb{R})$, depends on $t$. However, the problem is elementary in the sense of 'functional analysis' and we neglect this gap. It is even not necessary to treat the problem in terms of integral functional spaces. All solutions $\psi$ considered here will be smooth functions of $x$ and $t$ everywhere except at $x = 0$ where their first order derivatives with respect to $x$ will have jumps of the first kind $\Delta\psi(t) = \psi_x(+0,t) - \psi_x(-0,t)$. The singularities of the second derivatives of such functions at 0 are naturally interpreted as $\Delta\psi(t)\delta(x)$- function. According to this, the $\delta$ - function term in the equation is interpreted as the condition on this jump

$$\Delta\psi(t) = \alpha(t)\psi(0,t). \tag{7.3}$$

As for the behavior of solutions as $|x| \to \infty$, they remain bounded together with the derivatives. We confine the discussion of the functional properties of the solutions to these remarks. Let $X$ denote a linear space of complex-valued functions $\psi$ of $x \in \mathbb{R}$ with the properties described above.

The perturbation $V(t)$ can be rewritten in the following factorized form

$$V(t) = V_2(t)V_1. \tag{7.4}$$

Here

$$V_1\psi = \psi(0), \qquad V_1 : X \to Y, \tag{7.5}$$

where $Y = \mathbb{C}$. The factor $V_2(t) : Y \to Y$ is equal to

$$V_2(t)c(x) = c\alpha(t)\delta(x), \qquad c \in \mathbb{C}. \tag{7.6}$$

Following the ideas discussed earlier, we introduce the symbol

$$M(E,t) = V_1R_0(E)V_2(t), \qquad M(E,t) : \mathbb{C} \to \mathbb{C}. \tag{7.7}$$

where $R_0(E) = (A_0 - EI)^{-1}$. Notice that the spectrum of $A_0$ covers the semi-axis $\mathbb{R}_+ = [0,\infty)$. Consider the corresponding pseudodifferential operator

$$\mathcal{M} = M(i\varepsilon\frac{\partial}{\partial t} + i0, t). \tag{7.8}$$

The kernel $R_0(x,y,E)$ of the operator $R_0(E)$ is given by the expression

$$R_0(x,y,E) = -\frac{1}{2i\sqrt{E}}e^{i\sqrt{E}|x-y|}. \tag{7.9}$$

The function $\sqrt{E}$ is defined as a holomorphic function in the complex plane with the cut along $\mathbb{R}_+$ and is equal to $i$ at the point $-1$. We have

$$M(E,t) = -\frac{1}{2i\sqrt{E}}\alpha(t) \tag{7.10}$$

It seems, however, that the corresponding operator

$$\mathcal{M} = -\frac{1}{2i\sqrt{i\varepsilon\frac{\partial}{\partial t} + i0}}\alpha(t) \tag{7.11}$$

cannot be studied explicitly.

## 7.2 Semiclassical eigenfunctions.

Following Section 4, consider the formal solutions

$$\Phi_E(t,\varepsilon) = e^{\frac{1}{i\varepsilon}\int_0^t \omega(t,E)dt} w_E(t,\varepsilon), \qquad w_E(t,\varepsilon) = w(t,E) + i\varepsilon w_1(t,E) + \ldots. \tag{7.12}$$

of the equation

$$\mathcal{M}\Phi_E(t,\varepsilon) = \mu(E)\Phi_E(t,\varepsilon). \tag{7.13}$$

Equation (7.13) is obviously equivalent to

$$-\frac{1}{2i\sqrt{\omega(t,E) + i\varepsilon\frac{\partial}{\partial t} + i0}}\alpha(t)w_E(t,\varepsilon) = \mu(E)w_E(t,\varepsilon). \tag{7.14}$$

At the leading order

$$-\frac{1}{2i\sqrt{\omega(t,E) + i0}}\alpha(t)w(t,E) = \mu(E)w(t,E). \tag{7.15}$$

In accordance with the general outline $\omega(0,E) = E$, therefore

$$\mu(E) = -\frac{1}{2i\sqrt{E + i0}}\alpha(0). \tag{7.16}$$

As a result

$$\omega(t,E) = E\left(\frac{\alpha(t)}{\alpha(0)}\right)^2. \tag{7.17}$$

Up to a normalization factor (that is now a function of $t$ and $E$), the function $w(t,E)$ is equal to 1. This means that the dependence of the function $w(t,E)$ on $t$ is completely defined by the "transport equation" (5.18). Furthermore, the dependence on $E$ is controlled by the spectral resolution. The transport equation and the spectral resolution contain the semiclassical eigenfunction of the adjoint operator $\mathcal{M}^*$.

## 7.3 The transport equation.

The eigenfunction $\widehat{\Phi}_E(t,\varepsilon)$ of the operator $\mathcal{M}^*$ has the form

$$\widehat{\Phi}_E(t,\varepsilon) = e^{\frac{1}{i\varepsilon}\int_0^t \omega^*(t,E)dt}\widehat{w}_E(t,\varepsilon), \qquad \widehat{w}_E(t,\varepsilon) = \widehat{w}(t,E) + i\varepsilon\widehat{w}_1(t,E) + \dots \tag{7.18}$$

The function $\widehat{w}(t,E)$ is again equal to 1 up to the normalization factor that is a function of $t$ and $E$. Let us recall that the transport equation can be represented in the form

$$\langle\frac{\partial}{\partial t}\chi, \widehat{\chi}\rangle = 0, \tag{7.19}$$

where

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g}(x)dx, \tag{7.20}$$

and $\chi$, $\widehat{\chi}$ are defined by

$$\mu(E)\chi(t,x,E) = R_0(x,0,\omega+i0)\alpha(t)w(t,E), \tag{7.21}$$

and

$$\mu^*(E)\widehat{\chi}(t,x,E) = \alpha(t)(R_0(0,x,\omega+i0))^*\widehat{w}(t,E). \tag{7.22}$$

**Proposition 7.1** *The dependence of $w$ on $t$ and $E$ is determined by the transport equation (7.19) and has the form*

$$w(t,E) = \left(\frac{\alpha(t)}{\alpha(0)}\right)^{1/2}. \tag{7.23}$$

*Similarly,*

$$\widehat{w}(t,E) = \left(\frac{\alpha(t)}{\alpha(0)}\right)^{3/2}. \tag{7.24}$$

**Proof.** Equation (7.19) can be rewritten as

$$\int_{\mathbb{R}} R_0(0,x,\omega+i0)\frac{\partial}{\partial t}R_0(x,0,\omega+i0)\alpha(t)w(t,E)dx = 0. \tag{7.25}$$

Let us replace the kernel $R_0(x,y,\omega+i0)$ by its integral representation:

$$R_0(x,y,\omega+i0) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ip(x-y)}dp}{p^2-(\omega+i0)}, \tag{7.26}$$

then equation (7.25) acquires the form

$$\int_{\mathbb{R}} dx \int_{\mathbb{R}} \frac{e^{-ipx}dp}{p^2-(\omega+i0)} \frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{e^{iqx}dq}{q^2-(\omega+i0)} \alpha(t) w(t,E) = 0. \tag{7.27}$$

Integrating with respect to $x$, one obtains $\delta(p-q)$, and the integral reduces to a one-dimensional integral

$$\int_{\mathbb{R}} dp \frac{1}{p^2-(\omega+i0)} \frac{\partial}{\partial t} \frac{\alpha(t)w(t,E)}{p^2-(\omega+i0)} = 0 \tag{7.28}$$

which is equivalent to

$$\frac{1}{\alpha w}\frac{\partial}{\partial t} \int_{\mathbb{R}} dp \left( \frac{\alpha(t)w(t,E)}{p^2-(\omega+i0)} \right)^2 = 0. \tag{7.29}$$

The last integral can be easily computed

$$(\alpha w)^2 \int_{\mathbb{R}} dp \left( \frac{1}{p^2-(\omega+i0)} \right)^2 = (\alpha w)^2 \frac{\partial}{\partial \omega} \int_{\mathbb{R}} dp \frac{1}{p^2-(\omega+i0)} = (\alpha w)^2 \frac{\partial}{\partial \omega} 2\pi i \frac{1}{2\sqrt{\omega+i0}}. \tag{7.30}$$

Recalling the explicit expression for $\omega$, one arrives to the equation

$$\frac{\partial}{\partial t}(\alpha w)^2 \frac{1}{\alpha^3} = \frac{\partial}{\partial t}\frac{w^2}{\alpha} = 0. \tag{7.31}$$

This means that $w$ which satisfies the transport equation can be chosen in the form

$$w(t,E) = \left( \frac{\alpha(t)}{\alpha(0)} \right)^{1/2}. \tag{7.32}$$

We can repeat the above computations to get the function $\widehat{w}(t,E)$. More simply, we know that $w(t,E)$ and $\widehat{w}(t,E)$ have to satisfy the normalization condition

$$w(t,E)\widehat{w}^*(t,E) = \omega_E(t,E) = \left( \frac{\alpha(t)}{\alpha(0)} \right)^2. \tag{7.33}$$

This implies

$$\widehat{w}(t,E) = \left( \frac{\alpha(t)}{\alpha(0)} \right)^{3/2}. \tag{7.34}$$

We have now completed the construction of the leading semiclassical orders of the eigenfunctions $\Phi_E(t,\varepsilon)$ and $\widehat{\Phi}_E(t,\varepsilon)$:

$$\Phi_E(t,\varepsilon) \sim e^{\frac{1}{i\varepsilon}\int_0^t \omega(t,E)dt} w(t,E), \qquad \widehat{\Phi}_E(t,\varepsilon) \sim e^{\frac{1}{i\varepsilon}\int_0^t \omega^*(t,E)dt} \widehat{w}(t,E). \tag{7.35}$$

Here

$$\omega(t,E) = E\left(\frac{\alpha(t)}{\alpha(0)}\right)^2, \qquad w(t,E) = \left(\frac{\alpha(t)}{\alpha(0)}\right)^{1/2}, \qquad \widehat{w}(t,E) = \left(\frac{\alpha(t)}{\alpha(0)}\right)^{3/2}. \quad (7.36)$$

We can now write the leading term of the asymptotic behavior of $\mathcal{D}(t) = \theta(t)V_1\psi(\cdot,t) = \theta(t)\psi(0,t)$. where $\psi$ is solution of (7.1) with a given initial condition $\psi(t=0) = \psi_0$.

**Proposition 7.2** *We have for* $\mathcal{D}(t)$

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_{\mathbb{R}} dE e^{\frac{1}{i\varepsilon}E\int_0^t\left(\frac{\alpha(\tau)}{\alpha(0)}\right)^2 d\tau} w(\tau,E)\langle V_1 R(E+i0,t=0)\psi_0, \widehat{w}(0,E)\rangle, \qquad (7.37)$$

*where* $R(x,y,E,t)$ *is the kernel of the operator* $(A(t)-EI)^{-1}$*. More specifically,*

$$\mathcal{D}(t) \sim \frac{1}{2\pi i}\int_{\mathbb{R}} dE e^{\frac{1}{i\varepsilon}E\int_0^t\left(\frac{\alpha(\tau)}{\alpha(0)}\right)^2 d\tau}\left(\frac{\alpha(t)}{\alpha(0)}\right)^{1/2} R(0,\cdot,E+i0,0)\psi_0. \qquad (7.38)$$

## 7.4 Initial conditions.

Further analysis depends on the choice of the initial data $\psi_0$. The case where $\psi_0$ is the eigenfunction of the continuous spectrum of $A(0)$ is the most intricate. Let $\psi_0 = \psi_k(\cdot,t=0)$ where $\psi_k(\cdot,t)$ is an eigenfunction of the continuous spectrum of $A(t)$ corresponding to the wave vector $k$, $k\in\mathbb{R}$.

To fix $\psi_k$ we have to choose its asymptotic behavior at infinity, i.e. the outgoing part of $\psi_k$. Let us assume that it is described by the following type of oscillations:

$$\psi_k(x,t) \sim e^{ikx} + a_k(sgn(x),t)e^{i\sqrt{E_0}|x|}. \qquad (7.39)$$

Here $E_0 = k^2$ and $\sqrt{E_0} > 0$. This function satisfies the Lippman-Schwinger equation

$$\psi_k(x,t) = e^{ikx} - R_0(x,\cdot,E_0+i0)V(t)\psi_k(\cdot,t). \qquad (7.40)$$

In our special case

$$\psi_k(x,t) = e^{ikx} - R_0(x,0,E_0+i0)\alpha(t)\psi_k(0,t). \qquad (7.41)$$

Putting here $x=0$ one gets an equation for $\psi_k(0,t)$

$$\psi_k(0,t) = \left(1 - \frac{1}{2i\sqrt{E_0}}\alpha(t)\right)^{-1}. \qquad (7.42)$$

Denoting

$$g_k = \psi_k(0,0) = \left(1 - \frac{1}{2i\sqrt{E_0}}\alpha(0)\right)^{-1}, \tag{7.43}$$

we have

$$\psi_0(x) = \psi_k(x,0) = e^{ikx} - R_0(x,0,E_0+i0)\alpha(0)g_k = e^{ikx} + \frac{e^{i\sqrt{E_0}|x|}}{2i\sqrt{E_0}}\alpha(0)g_k. \tag{7.44}$$

**Proposition 7.3** *For the initial condition $\psi_0$ given by (7.44), the asymptotic behavior of $\mathcal{D}(t)$ is given by*

$$\mathcal{D}(t) \sim e^{\frac{1}{i\varepsilon}E_0\int_0^t \beta(\tau)d\tau}\beta^{1/4}g_k, \qquad \beta(\tau) = \left(\frac{\alpha(\tau)}{\alpha(0)}\right)^2. \tag{7.45}$$

**Proof.** Since $\psi_0$ is the eigenfunction, one has

$$R(0,\cdot,E+i0,0)\psi_0 = \frac{1}{E_0 - E - i0}g_k. \tag{7.46}$$

We substitute the above expression in (7.38) and get a final expression for $\mathcal{D}(t)$ in the form (7.45).

# 8 The Delta function potential example: asymptotic behavior of the solution $\psi(x,t)$.

Let us recall the general formula for the solution $\psi(x,t)$ in terms of $\mathcal{D}(t)$:

$$\psi(t) = g(t)\psi_0 + \frac{1}{i\varepsilon}\int_0^t g(t-\tau)V_2(\tau)\mathcal{D}(\tau)d\tau. \tag{8.1}$$

The kernel of the operator $g(t) = e^{\frac{1}{i\varepsilon}A_0 t}$ is $g(x,y,t) = g(x-y,t)$ and

$$g(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\frac{1}{i\varepsilon}p^2 t + ipx}dp = e^{-i\pi/4}\sqrt{\frac{\varepsilon}{4\pi t}}e^{i\frac{\varepsilon x^2}{4t}}. \tag{8.2}$$

The expression for $\psi$ can be naturally split into three terms:

$$\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)}, \tag{8.3}$$

with

$$\psi^{(0)}(x,t) = g(t)\phi_0(x) = e^{\frac{1}{i\varepsilon}k^2 t + ikx}, \qquad \phi_0(x) = e^{ikx}, \tag{8.4}$$

$$\psi^{(1)}(x,t) = \frac{\alpha(0)g_k}{2i\sqrt{E_0}} g(t)\phi_1(x), \qquad \phi_1(x) = e^{i\sqrt{E_0}|x|}, \tag{8.5}$$

$$\psi^{(2)}(x,t) = \frac{1}{i\varepsilon}\int_0^t g(x,t-\tau)\alpha(\tau)\mathcal{D}(\tau)d\tau. \tag{8.6}$$

This splitting is natural in the sense that all three terms display different kinds of oscillating behavior.

## 8.1 Standard semiclassical analysis.

We noticed earlier that the semiclassical treatment of the problem applied only to the semiclassical variable $t$ generates secular terms i.e. the expansion with respect to powers of $\varepsilon$ with each additional power of $\varepsilon$ containing an additional power of $x$. This makes natural to introduce for large $x$, the new variable $\xi = \varepsilon x$. In terms of $t$ and $\xi$ the equation for $x \neq 0$ becomes semiclassical with respect to both variables:

$$i\varepsilon \frac{\partial}{\partial t}\psi_t = \left(\frac{\varepsilon}{i}\frac{\partial}{\partial \xi}\right)^2 \psi. \tag{8.7}$$

This means that for large $x$, the oscillating behavior of the solution - or of the different parts of the solution - has to be described by different semiclassical actions corresponding to different fields of classical trajectories for the free classical hamiltonian system

$$p' = 0, \qquad \xi' = 2p. \tag{8.8}$$

In what follows, we will see how this idea works.

## 8.2 Semiclassical treatment of $\psi^{(0)}$.

The explicit formula for $\psi^{(0)}$ rewritten in terms of $\xi$:

$$\psi^{(0)} = e^{\frac{i}{\varepsilon}(k\xi - k^2 t)} \tag{8.9}$$

becomes the most simple semiclassical solution of (8.7). For the action $k\xi - k^2 t$, the corresponding initial (lagrangian) curve on the phase plane is simply the straight line $p = k$. It remains invariant in time.

## 8.3 Computation of $\psi^{(1)}$ and geometric interpretation.

The function $\psi^{(1)}$ defined in (8.5) can be expressed in terms of the Fresnel integral. Let us first show this and then discuss the semiclassical character of the answer. Consider the explicit formula for $g(t)\phi_1(x)$:

$$g(t)\phi_1(x) = e^{-i\pi/4}\sqrt{\frac{1}{4\pi t\varepsilon}}\int_{\mathbb{R}} e^{\frac{i}{\varepsilon}(\frac{1}{4t}(\xi-\eta)^2+\sqrt{E_0}|\eta|)}d\eta = \tag{8.10}$$

$$= e^{-i\pi/4}\sqrt{\frac{1}{4\pi t\varepsilon}}\left(\int_0^\infty e^{\frac{i}{\varepsilon}(\frac{1}{4t}(\xi+\eta)^2+\sqrt{E_0}\eta)}d\eta + \int_0^\infty e^{\frac{i}{\varepsilon}(\frac{1}{4t}(\xi-\eta)^2+\sqrt{E_0}\eta)}d\eta\right) = \tag{8.11}$$

$$= e^{-i\pi/4}\left(e^{\frac{i}{\varepsilon}(|k|\xi-k^2t)}\mathcal{F}\left(\frac{2|k|t-\xi}{\sqrt{4\varepsilon t}}\right) + e^{\frac{i}{\varepsilon}(-|k|\xi-k^2t)}\mathcal{F}\left(\frac{2|k|t+\xi}{\sqrt{4\varepsilon t}}\right)\right), \tag{8.12}$$

where

$$\mathcal{F}(y) = \frac{1}{\sqrt{\pi}}\int_y^\infty e^{iz^2}dz. \tag{8.13}$$

It is clear that

$$\mathcal{F}(y) \to 0 \text{ as } y \to +\infty, \qquad \mathcal{F}(y) \to 1 \text{ as } y \to +\infty. \tag{8.14}$$

More precisely,

$$\mathcal{F}(y) \sim \frac{-1}{2iy\sqrt{\pi}}e^{iy^2} \text{ as } y \to -\infty. \tag{8.15}$$

It means that, if $\frac{2|k|t\mp\xi}{\sqrt{4\varepsilon t}} \to +\infty$, then

$$e^{\frac{i}{\varepsilon}(\pm|k|\xi-k^2t)}\mathcal{F}\left(\frac{2|k|t\mp\xi}{\sqrt{4\varepsilon t}}\right) \sim i\sqrt{\frac{\varepsilon t}{\pi}}\frac{1}{2|k|t\mp\xi}e^{i\frac{\xi^2}{4\varepsilon t}}. \tag{8.16}$$

If, oppositely, $\frac{2|k|t\mp\xi}{\sqrt{4\varepsilon t}} \to -\infty$, then

$$e^{\frac{i}{\varepsilon}(\pm|k|\xi-k^2t)}\mathcal{F}\left(\frac{2|k|t\mp\xi}{\sqrt{4\varepsilon t}}\right) \sim e^{\frac{i}{\varepsilon}(\pm|k|\xi-k^2t)} + i\sqrt{\frac{\varepsilon t}{\pi}}\frac{1}{2|k|t\mp\xi}e^{i\frac{\xi^2}{4\varepsilon t}}. \tag{8.17}$$

We can now give a geometrical interpretation of the second term $\psi^{(1)}(x,t)$ of the field $\psi(x,t)$.

In general, the field is described by the formula:

$$\psi_1(x,t) = \frac{\alpha(0)g_k}{2\sqrt{E_0}}e^{-i3\pi/4}\left(e^{\frac{i}{\varepsilon}(|k|\xi-k^2t)}\mathcal{F}\left(\frac{2|k|t-\xi}{\sqrt{4\varepsilon t}}\right) + e^{\frac{i}{\varepsilon}(-|k|\xi-k^2t)}\mathcal{F}\left(\frac{2|k|t+\xi}{\sqrt{4\varepsilon t}}\right)\right). \tag{8.18}$$

In the plane $(\xi, t), 0 \le t \le T$, consider two trajectories: $\xi = 2|k|t$ and $\xi = -2|k|t$. Outside of their expanding parabolic neighborhoods the field can be described by elementary asymptotic formulas. In the corners between the axis $\xi$ and these two trajectories (outside the parabolic neighborhoods) the field is reduced in the leading (not decreasing) order to the simplest plane waves: for $\xi > 0$:

$$\psi^{(1)}(x,t) \sim \frac{\alpha(0)g_k}{2\sqrt{E_0}} e^{-i3\pi/4} e^{\frac{i}{\varepsilon}(|k|\xi - k^2 t)}, \tag{8.19}$$

for $\xi < 0$:

$$\psi^{(1)}(x,t) \sim \frac{\alpha(0)g_k}{2\sqrt{E_0}} e^{-i3\pi/4} e^{\frac{i}{\varepsilon}(-|k|\xi - k^2 t)}. \tag{8.20}$$

The leading correction (of order $\varepsilon^{1/2}$) - which is the leading term of the field in the cone $|\xi| < 2\sqrt{E_0}t$ - is the semiclassical wave field corresponding to the trajectories starting at $t = 0$ from the point $\xi = 0$ with arbitrary momenta:

$$\psi^{(1)}(x,t) \sim e^{-i\pi/4} \frac{\alpha(0)g_k}{2\sqrt{E_0}} \sqrt{\frac{\varepsilon t}{\pi}} \frac{4|k|t}{4k^2t^2 - \xi^2} e^{i\frac{\xi^2}{4\varepsilon t}}. \tag{8.21}$$

Let us remind once again that this approximation is valid outside of the parabolic neighborhood of two trajectories with momenta $\pm k$.

## 8.4 Computation of $\psi^{(2)}$ and geometric interpretation.

Consider now

$$\psi^{(2)}(x,t) = \frac{1}{i\varepsilon} \int_0^t g(x, t-\tau)\alpha(\tau)\mathcal{D}(\tau)d\tau. \tag{8.22}$$

The asymptotic behavior of $\mathcal{D}(\tau)$ is known (see equation (7.45)). This time, it is more convenient to use the integral representation (8.2) for $g(t)$ although the explicit expression could also be used.

Substituting (8.2) and (7.45) in (8.22) one obtains

$$\psi^{(2)}(x,t) \sim J(t, \xi, E_0, \varepsilon), \tag{8.23}$$

where

$$J = \frac{1}{2\pi i\varepsilon} \int_{\mathbb{R}} dp \int_0^t d\tau e^{\frac{1}{i\varepsilon}\left[p^2(t-\tau) - p\xi + E_0 \int_0^\tau \beta(\sigma)d\sigma\right]} \beta^{1/4}(\tau) g_k. \tag{8.24}$$

The integral $J$ is some special function depending on several variables. It can be simplified for different specific locations of these variables and for different behavior of the function $\beta(t)$. We will not study it in detail here and confine ourselves to a generic situation where all parameters (except $\varepsilon$) are of order 1.

The integral $J$ contains the fast oscillating exponential factor:

$$e^{\frac{1}{i\varepsilon}\Psi(\xi,t,p,\tau)}, \text{ with } \quad \Psi = p\xi - p^2(t-\tau) - E_0 \int_0^\tau \beta(\sigma)d\sigma. \tag{8.25}$$

Generically, the asymptotic behavior of the integral $J$ is defined by the contributions of small neighborhoods (of the finite number) of the isolated critical points of the phase function $\Psi$. The contributions of the critical points of $\Psi$ restricted to the boundaries $\tau = 0$ and $\tau = T$ (that are not critical points of $\Psi$ itself) are of smaller order.

Consider the equations $d\Psi = 0$ for the critical (stationary) points :

$$-p^2 + E_0\beta(\tau) = 0, \qquad 2p(t-\tau) - \xi = 0. \tag{8.26}$$

The stationary point is defined as the intersection of these two curves in the $(\tau, p)$ plane . Suppose that $\xi > 0$. Assume that $(i)$ the function $\beta$ is such that in the strip $0 \leq t \leq T$, there is exactly one point $(\tau_*, p_*)$ of intersection of the two mentioned curves, that lies strictly inside of the strip. ¿From the picture of two curves it is easy to see the qualitative behavior of $\beta, \beta(0) = 1$, that guarantees this condition be fulfilled.

Compute the phase function $\Psi$ at the critical point $(\tau_*, p_*)$ and denote its value by $\Phi$:

$$\Phi(\xi, t) = |k|\beta^{1/2}(\tau_*)\xi - k^2\beta(\tau_*)(t-\tau_*) - k^2 \int_0^{\tau_*} \beta(\sigma)d\sigma. \tag{8.27}$$

Compute now the second differential of $\Psi$:

$$d^2\Psi = -2(t-\tau_*)d^2p + 4p_*dpd\tau - E_0\beta'(\tau_*)d^2\tau. \tag{8.28}$$

Consider the Hessian $\det H$ of $\Psi$ at the point $(\tau_*, p_*)$:

$$\det H = 2E_0[(t-\tau_*)\beta'(\tau_*) - 2\beta(\tau_*)], \qquad H = \begin{pmatrix} -2(t-\tau_*) & 2p_* \\ 2p_* & -E_0\beta'(\tau_*) \end{pmatrix} \tag{8.29}$$

Assume further that $(ii)$ $\det H \neq 0$. Then the asymptotic behavior of $\psi^{(2)}(x,t)$ is given by the formula

$$\psi^{(2)}(x,t) \sim \frac{e^{-i\delta(H)\frac{\pi}{2}}}{|detH(\xi,t)|^{1/2}} g_k \beta^{1/4}(\tau_*) e^{\frac{i}{\varepsilon}\Phi(\xi,t)}. \tag{8.30}$$

Here $\delta(H)$ is the number of the negative eigenvalues of the matrix $H(\xi, t)$.

The last asymptotic formula has again a semiclassical character. The field of trajectories corresponding to the phase $\Psi$ has a geometric interpretation. The trajectories start at point $\xi = 0$, travel in the phase plane along the axis $\xi = 0$ with the time-dependent momentum $p = |k|\beta^{1/2}(\tau)$ and at time $\tau_*$ abandon the $\xi = 0$ axis and proceed in the phase plane $(\xi, p)$ as free trajectories of system (8.8) with constant momentum $p = |k|\beta^{1/2}(\tau_*)$. The instant $\tau_*$ is defined by the condition of the critical value of the action. What is important is that this term contributes to a new type of oscillation in the wave field that is determined by the function $\beta$.

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