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Investigation of a boundary-value problem for a sixth order equation

As is known one of the stages of the solution of a mixed problem by the contour integral method [1]-[2] is the solution of a spectral problem corresponding to the given mixed problem.

If we consider the problem for a parabolic by Petrovski equation then by means of special potentials the solution of a spectral problem is reduced to the solution of a system of regular integral equations with respect to unknown densities.

By virtue of the parabolic character of the equation, the kernels of these potentials decrease well at great values of a parameter from some infinite part of a complex plane and have weak pointwise singularity in space variable.

The goal of the paper is to study a spectral problem corresponding to the mixed problem for a sixth order weak parabolic equation. For simplicity we consider an equation with constant coefficients, namely, we consider a boundary value problem on finding of the solution of the equation

$$A_0\Delta^3 u\left(x,\lambda\right)+A_1\lambda^2\Delta^2 u\left(x,\lambda\right)+A_2\lambda^4\Delta u\left(x,\lambda\right)+\lambda^6 u\left(x,\lambda\right)=0, \quad (1)$$

satisfying the boundary conditions

$$\lim_{x\to z\in\tau} u\left(x,\lambda\right)=\varphi_0\left(z,\lambda\right);\ \lim_{x\to z\in\tau}\frac{du\left(x,\lambda\right)}{dn_z}=\varphi_1\left(z,\lambda\right)$$

$$\lim_{x\to z\in\tau}\frac{d}{dn_z}\Delta^2 u\left(x,\lambda\right)=\varphi_2\left(z,\lambda\right), \quad (2)$$

where $x=(x_1,x_2)$ is a point of domain D with boundary τ, A_k $\left(k=\overline{0,2}\right)$ are constant numbers, λ is a complex parameter, n_z is the direction of an internal normal to the boundary τ of the domain D at the point $z\in\tau$.

The fulfillment of the following conditions is assumed:

1) The roots of the characteristic equation

$$\nu^3-A_2\nu^2+A_1\nu-A_0=0, \quad (3)$$

corresponding to equation (1) are such that a real part of even if one of them equals zero, and the others are negative, i.e., $Re\nu_1 = 0,\ Re\nu_k < 0\ (k = 2, 3)$

2) The boundary functions $\varphi_s(z,\lambda)\ \left(s = \overline{0,2}\right)$ have continuous derivatives with respect to z to the 3-$s-$th order for $z \in \tau$, with analytic by λ in R_δ functions and converging to zero as $|\lambda| \to \infty$.

3) The boundary τ of the domain D is a Lyapunov line. We look for the solution of problem (1), (2) in the form of sum of potentials

$$u(x,\lambda) = w_1(x,\lambda) + w_2(x,\lambda) + w_3(x,\lambda), \tag{4}$$

where $w_k(x,\lambda)\ \left(k = \overline{1,3}\right)$ are special potentials defined by the formulae

$$w_1(x,\lambda) = \int_\tau \mathcal{P}_0(x-y;\lambda)\,\mu_1(y;\lambda)\,d\tau_y, \tag{5}$$

$$w_2(x,\lambda) = \int_\tau \mathcal{P}_1(x-y;\lambda)\,\mu_2(y;\lambda)\,d\tau_y \tag{6}$$

$$w_3(x,\lambda) = \int_\tau \mathcal{P}_2(x-y;\lambda)\,\mu_3(y;\lambda)\,d\tau_y \tag{7}$$

where $\mathcal{P}(x-y;\lambda)$ is a fundamental solution, and $\mathcal{P}_s(x-y;\lambda)\ (s = 1, 2)$ are particular solutions of equation (1) (see [3], [4])

$$\mathcal{P}_0(x-y;\lambda) = -\frac{1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{\nu_k\mathcal{K}_0\left(\frac{\lambda|x-y|}{\sqrt{-\nu_k}}\right)}{\prod\limits_{\substack{s=1\\ s\neq k}}^{3}(\nu_k-\nu_s)},$$

$$\mathcal{P}_1(x-y;\lambda) = -\frac{1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{\nu_k^2\mathcal{K}_0\left(\frac{\lambda|x-y|}{\sqrt{-\nu_k}}\right)}{\prod\limits_{\substack{s=1\\ s\neq k}}^{3}(\nu_k-\nu_s)},$$

$$\mathcal{P}_2(x-y;\lambda) = \left[\mathcal{P}_2(x-y;\lambda) - \frac{2A_0}{3\lambda^2}\frac{d^2}{dn_y^2}\mathcal{P}_3^*(x-y;\lambda)\right]$$

where

$$\mathcal{P}_3(x-y;\lambda) = -\frac{1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{\nu_k^2 - A_2\nu_k}{\prod\limits_{\substack{s=1\\ s\neq k}}^{3}(\nu_k-\nu_s)}\mathcal{K}_0\left(\frac{\lambda|x-y|}{\sqrt{-\nu_k}}\right),$$

$$\mathcal{P}_3^*(x-y;\lambda) = -\frac{1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{A_1\nu_k - A_0}{\prod\limits_{\substack{s=1\\ s\neq k}}^{3}(\nu_k-\nu_s)}\mathcal{K}_0\left(\frac{\lambda|x-y|}{\sqrt{-\nu_k}}\right),$$

where $\mathcal{K}_0(z)$ is a second genus Bessel function of zero order.

By means of asymptotic and integral representations for the function $\mathcal{K}_0(z)$ and its derivatives (see [5]), we first prove the necessary jump formulae for the potentials $W_k(x,\lambda)$ $\left(k=\overline{1,3}\right)$ and their derivatives for all the values of $\lambda \in R_\delta$, where

$$R_\delta = \left\{\lambda : |\lambda| > R; -\frac{\pi}{4} + \delta \leq \arg\lambda < \frac{\pi}{4}\right\}.$$

By direct verification we prove that at all $\lambda \in R_\delta$ for all potentials $w_k(x;\lambda)$
$\left(k=\overline{1,3}\right)$ it holds the estimation

$$\left|\frac{\partial}{\partial x_k}\Delta^m\mathcal{P}_s(x-y;\lambda)\right| \leq \frac{Ce^{-\varepsilon|\lambda||x-y|}}{|\lambda|^{4-2m}|x-y|} \tag{8}$$

$\left(s=\overline{0,2},\ m=\overline{0,2},\ k=1,2\right)$ valid for all $\lambda \in R_\delta$.

Putting (4) to the left hand sides of boundary conditions (2) and taking into account the known jump formulae for the unknown densities we obtain a system of integral equations

$$\mu(z,\lambda) = f(z,\lambda) + \int\limits_{\tau}\mathcal{K}(z;y;\lambda)\,\mu(y;\lambda)\,d\tau_y, \tag{9}$$

where $\mu(z,\lambda)$ and $f(z;\lambda)$are the columns of the functions composed of unknown densities and boundary functions, and $\mathcal{K}(x;y;\lambda)$ is a matrix of functions whose elements are fundamental solution and particular solution and their derivatives. Be means of estimates (8) it is proved that for the kernels $\mathcal{K}(x;y;\lambda)$ it holds an estimation of type (8) valid for all $\lambda \in R_\delta$. Consequently, the system of integral equations is of Fredholm property, so, we can solve it by the method of sequential approximations and the solution is an analytic, bounded function with respect to λ in the domain R_δ. So we prove the

Theorem: *Under conditions 1), 2), 3) problem (1), (2) has a unique solution $u(x,\lambda)$ represented in the form of the sum of potentials whose densities are the solutions of the system of regular integral equations (9)*

References

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