Curvature Perturbations and Stability of a Ring of Vortices.

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Abstract

Vortex modeling has a long history. Descartes (1644) used it as a model for the solar systems. J.J. Thomson (1883) used it as a model for the atom. We consider point-vortex systems, which can be regarded as "discrete" solutions of the Euler equation. Their dynamics is described by a Hamiltonian system of equations. In particular we are interested in vortex dynamics on simply connected surfaces of constant curvature K – i.e. a plane, spheres and hyperbolic surfaces . It is known that polygonal configurations of N point-vortices are relative equilibria of the system. We study the stability of such polygonal configurations, and, more specifically, how stability depends upon the number of vortices N and the curvature K of the surface. To address such a question we have to formulate the problem in a unified geometrical way. The fact that the surfaces of interest can be viewed as Kälher manifolds greatly simplify our task. Nonlinear stability is then studied by making use of the Dirichlet Criterion. Stability ranges are the K- intervals for which the Hessian of the Hamiltonian - evaluated at the equilibrium configuration – is positive or negative definite.

1 Introduction

The study of the stability of a ring of vortices has a long history started with J.J. Thomson in 1883, when considering a planar vortex model for the atom. In particular his interest was on configurations of identical vortices equally spaced along the circumference of a circle, i.e. located at the vertices of a regular polygon (see Figure 1). He proved that for six of fewer vortices the polygonal configurations are *linearly* stable, while for seven vortices he erroneously concluded that the configuration is slightly unstable. It took more than a century to make some progresses on this problem. In 1985 D.G. Dritschel succeeded in solving the heptagon mystery for what concerns its linear stability analysis, leaving open the nonlinear stability question [10]. Later one in 1993, D.G. Dritschel and L.M. Polvani generalized the techniques used in [10] to study the linear stability of a "latitudinal"

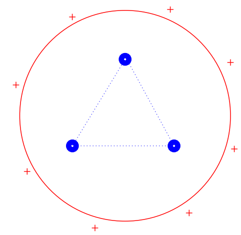


Figure 1: In the bi-dimensional atomic model of J.J. Thomson (see [19]) electrons were thought as vortices located at the vertices of a regular polygon.

ring of point vortices on the sphere [18] , as a function of the number N of vortices in the ring, and of the ring's co-latitude θ. They proved that polygonal configurations are more linearly unstable on the sphere than in the plane. Few years later, the stability analysis was extended to the case of *non-linear stability* by using a sufficient criterion due to Dirichlet, see for example Cabral & Schmidt [6], Boatto & Cabral [4], Marden & Pekarsky [17], Boatto & Simó [5]. It has to be stressed that so far the planar and the spherical case have been treated separately, and the sphere has been viewed as a surface in $\mathbb{R}^3$. We want to introduce a more geometrical formulation, aimed to answer to a more general question:

How does the stability of a ring of vortices depend upon the curvature of the surface, where the vortex motion takes place?

In particular, as Thomson did, we are going to consider a ring of N vortices on a circle of fixed radius r and we are going to study how its stability properties change when varying the curvature K of the surface (see Figure 2). Only surfaces of constant curvature are considered.

Notation: we apologize with the reader for using the symbol ω with two different meanings in this article. In Section 2, ω denotes the curl of the velocity field, while from Section 3 on it indicates the symplectic form.

We organized the article as follows: in Sec. 2 and Sec. 3 we introduce, respectively, the equations of motions and the geometrical formulation of the vortex problem on surfaces of constant curvature K. In Sec. 4 we consider the particular case of a polygonal ring of identical vortices, and in Sec. 5 we study its stability as a function of the curvature K. Detailed calculations on how to deduce the symplectic form and the corresponding vortex Hamiltonian are done in the appendix. We would like to stress that the geometrical formulation was greatly inspired to the beautiful article by Y. Kimura (see [13]) which we strongly recommend the reader to read.

Finally, we would like to dedicate this article to Carles Simó for the many stimulating discussions. Carles has been like a mentor, a figure of great inspiration both from the scientific and the human point of views.

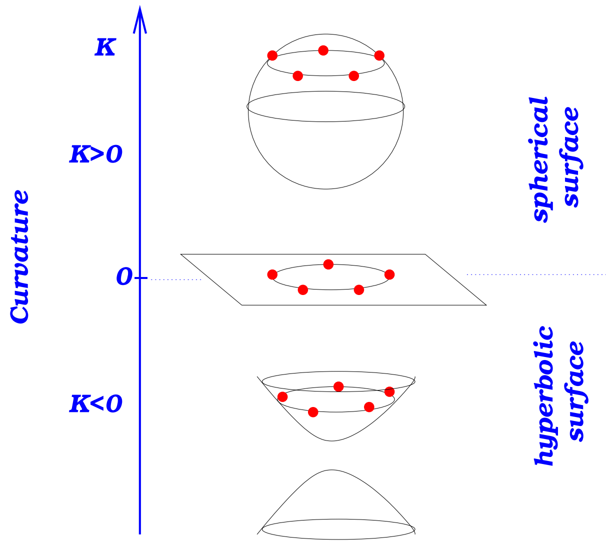


Figure 2: A ring of five identical point vortices on surface of different constant curvature (a hyperbolic plane, the plane and a sphere).

2 Equations of motion

Let's start by considering the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{f} \tag{1}$$

where $\mathbf{u}$ is the particle's velocity field, p is the pressure field, $\mathbf{f} = -\nabla U$ is a conservative force field, and restrict our attention to the two dimensional setting, for example vortex dynamics on the plane (or a sphere). We are considering only inviscid flows that are also incompressible, i.e.

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

Let's introduce the vorticity field ω, defined as

$$\omega = \nabla \wedge \mathbf{u}. \tag{3}$$

Then it is immediate that by taking the curl of Eq. (1) we obtain the evolution equation of the vorticity, i.e.

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0, \quad \text{or} \quad \frac{D\omega}{Dt} = 0, \tag{4}$$

where the operator $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is called the material derivative and describes the evolution along the flow lines. It follows from Eq. (4) that in two dimensions the vorticity is conserved as it is transported along the flow lines. Then a natural question arises: supposing the vorticity field ω is known, is it possible to deduce the velocity field

$\mathbf{u}$ generating ω? Or in other words, is it possible to solve the system of Eq. (3)-(2)? It is immediate to see that in general the solution is not unique, if some boundary conditions are not specified (see [15]). Furthermore, as already observed by Kirchhoff in 1876 [6], in two dimension we can recast the fluid equations (3)-(2) into a Hamiltonian formalism. In fact, notice that on the plane $\mathbf{u} = (\dot{x}, \dot{y})$ and Eq. (2) is still verified if we represent the velocity components as

$$\dot{x} = \frac{\partial \Psi}{\partial y}, \qquad \dot{y} = -\frac{\partial \Psi}{\partial x}, \tag{5}$$

i.e. by means of Ψ, called the *stream function*. Formally Ψ plays the rôle of a Hamiltonian for the pair of conjugate variables (x, y) and it is used to describe the dynamics of a test-particle, located at (x, y) and advected by the flow. By substituting (5) into (3) we obtain

$$\Delta\Psi(\mathbf{r}) = \omega(\mathbf{r}) \tag{6}$$

i.e. a Poisson equation with ω as a source term. Then, once we specify the vorticity field, by inverting (59) we obtain the stream function Ψ to be

$$\Psi(\mathbf{r}) \;\; = \;\; \int G(\mathbf{r}, \mathbf{r}')\omega(\mathbf{r}')d\mathbf{r}' \tag{7}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the Green's function, solution of the equation $\Delta G(\mathbf{r}) = -\delta(\mathbf{r})$. The natural question then arises: *what kind of vorticity field to consider?*

The simplest of all vorticity field is the one generated by N *point vortices*, i.e.

$$\omega(\mathbf{r}) = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) \tag{8}$$

where Γ_α, $\alpha = 1, ..., N$, is a constant and corresponds to the vorticity (or circulation) of the α-vortex, situated at $\mathbf{r}_\alpha$. Observe that a point-vortex can be thought as an entity where the vorticity field is concentrated into a point. In other words, point-vortices are singularities of the velocity field! Why point-vortices are simpler to deal with? Remember from Eq. (4) that the vortex dynamics is described by the original Euler equation (i.e. EDP),

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0.$$

Now, for point-vortices the equation above "reduces" [1] to a Hamiltonian system of equations (i.e. ODE),

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \quad \alpha = 1, ..., N, \tag{9}$$

where, as discussed in the next section (see also Appendix B), the Hamiltonian H is an autonomous Hamiltonian.

[1] About the mathematical accuracy of this reduction we suggest the reader to see the book of Marchiorro & Pulvirenti [15].

3 Geometrical formulation

For surfaces with constant curvature, K, how to derive the Darboux coordinates and the vortex Hamiltonian in a unified formalism, simply parametrized by K ?
We remind the reader that in formulating point-vortex dynamics on a surface we implicitly need:

- a Riemannian metric $\mathbf{g}$ to compute the fundamental solution of $\Delta\Psi = -\delta(\mathbf{r})$;
- a symplectic form ω [1] for the Hamiltonian formalism $\dot{q} = +\frac{\partial\Psi}{\partial p}, \quad \dot{p} = -\frac{\partial\Psi}{\partial q}$

Then it is natural to consider those structures, $\mathbf{g}$ and ω, simultaneously and require that they are *compatible*. For simply connected surfaces of constant curvature there is a natural framework where this happens, and it is the framework of Kähler manifolds. A *Kähler manifolds* is a manifold M endowed of three compatible structures: a Riemannian metric g, a symplectic form ω and a complex structure J defined on the tangent bundle TM [16]. We remind the reader that a Riemannian metric $\mathbf{g}$ and a symplectic form ω are, respectively, a symmetric and an asymmetric bilinear forms, defined on vectors of the tangent space (at a given point $p \in M$). In particular $\mathbf{g}$ is an inner product and we denote it by $\langle v, w\rangle = \mathbf{g}(v, w)$, where $v, w \in T_pM$. We say that the triplet $(\omega, J, \mathbf{g})$ is compatible if

$$\langle v, w\rangle = \omega(v, Jw) \qquad or \qquad \omega(v, w) = \langle v, Jw\rangle.$$

How are the Kähler manifold properties going to help us in the stability analysis of a vortex ring? In order to study how the stability properties depend upon the Gaussian curvature K, we would like to have a symplectic form ω which depends parametrically from the surface curvature, K, or, in other words, we would like to have at our disposal some Darboux coordinates which depend parametrically from K. Now we know that we have such an unified formalism for the metric $\mathbf{g}$ when using stereographic coordinates, namely

$$\frac{dz\, d\bar{z}}{(1 + K|z|^2)^2}$$

where $K = 1$ for the unit sphere $\mathbb{S}^2$, $K = 0$ for the plane, and $K = -1$ for the hyperbolic plane H^2. [1] Then we can infer (by the compatibility property) that there must exist an analogous way of parametrizing the symplectic form! In fact, it is proved in the Appendix A (see also the beautiful article by Kimura [13]) we have a set of Darboux coordinates parametrized by K

$$q_{\alpha K} = \frac{\Gamma_\alpha}{K} \begin{cases} (\cos(\sqrt{K} r_\alpha) - 1) & K > 0 \\ (\cosh(\sqrt{|K|} r_\alpha) - 1) & K < 0, \end{cases}, \qquad p_\alpha = \phi_\alpha, \qquad \alpha = 1, ..., N, \qquad (10)$$

[1]From now on ω will be used to indicate the symplectic form, instead of the vorticity field as in the previous section.

[1]Notice that in this case we are not considering the whole unit sphere $\mathbb{S}^2$, but $\mathbb{S}^2 - \{a\ pole\}$.

where for the spherical case ϕ_α and r_α are respectively the longitude and the geodesic distance (or spherical distance) of the αth vortex from the north pole N (see Figure 3).

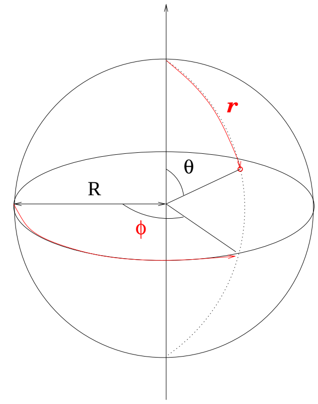


Figure 3: Let R be the radius of the sphere. A point on the sphere can be localized by specifying its longitude ϕ and its co-latitude θ, or, alternatively, by its ϕ and its geodesic distance r from the North pole. $r = \theta R = \theta/\sqrt{K}$.

The corresponding symplectic form is

$$\omega_K = \sum_{\alpha=1}^{N} dq_{\alpha\, K} \wedge dp_\alpha.$$

Notice that in the planar case, $K = 0$,

$$q_{\alpha\, o} = \lim_{K\to 0} q_{\alpha\, K} = -\frac{1}{2} r_\alpha^2,$$

which is the usual canonical action variable used in the planar case. Therefore the canonical variables (q_α, p_α), Eq. (10), are the canonical variables we were looking for! Now that we have succeeded parametrizing the symplectic form, what about the vortex Hamiltonian? As shown in the Appendix B the particle Hamiltonian is derived from the fundamental solutions of

$$\Delta \Psi_K = -\delta(\mathbf{r}). \tag{11}$$

We find (see Appendix B) that Ψ_K only depends upon the geodesic distance r and

$$\text{spherical surface:} \quad 2\pi\Psi_K = -\ln\left[\tan\left(\frac{\sqrt{K}}{2} r\right)\right], \qquad \left(r \in \left(0, \frac{\pi}{\sqrt{K}}\right)\right) \tag{12}$$

$$\text{plane:} \quad 2\pi\Psi_o = -\ln(r), \qquad (r \in (0, \infty)) \tag{13}$$

$$\text{hyperbolic surface:} \quad 2\pi\Psi_K = -\ln\left[\tanh\left(\frac{\sqrt{|K|}}{2} r\right)\right]. \tag{14}$$

Notice that when considering a compact surface Eq. (11) has to be implemented with a constant term (see [8]), or by an image vortex (see [14]). In particular for the whole sphere $\mathbb{S}^2$ we have to look for a generalized Green function, solution of

$$\Delta\Psi_{\mathbb{S}^2} = -\delta(\mathbf{r}) + \frac{1}{4\pi}, \tag{15}$$

where $1/4\pi$ can be interpreted as a constant background vorticity field. It is immediate to check that integrating (15) over $\mathbb{S}^2$ and using the Stokes theorem give the expected identity.[1] Then the Green function for $\mathbb{S}^2$ is

$$\Psi_{\mathbb{S}^2}(r) = -\frac{1}{2\pi}\ln(1-\cos(r)).$$

The corresponding Hamiltonian for a system of N vortices is

$$H_{K>0} = c\sum_{\alpha<\beta}\Gamma_\alpha\Gamma_\beta \ln\left(\frac{1-d_{\alpha\beta}}{1+d_{\alpha\beta}}\right) \quad \text{(spherical surface)}, \tag{16}$$

$$H_{K<0} = c\sum_{\alpha<\beta}\Gamma_\alpha\Gamma_\beta \ln\left(\frac{d_{\alpha\beta}-1}{d_{\alpha\beta}+1}\right) \quad \text{(hyperbolic surface)}, \tag{17}$$

$$H_{\mathbb{S}^2} = c\sum_{\alpha<\beta}\Gamma_\alpha\Gamma_\beta \ln(1-d_{\alpha\beta}) \quad \text{(sphere)}, \tag{18}$$

with $c = -1/(4\pi)$, and

$$d_{\alpha\beta} = \left(1+K\frac{q_{\alpha\,K}}{\Gamma_\alpha}\right)\left(1+K\frac{q_{\beta\,K}}{\Gamma_\beta}\right) + K\sqrt{\frac{q_{\alpha\,K}}{\Gamma_\alpha}\frac{q_{\beta\,K}}{\Gamma_\beta}\left(2+K\frac{q_{\alpha\,K}}{\Gamma_\alpha}\right)\left(2+K\frac{q_{\beta\,K}}{\Gamma_\beta}\right)}\cos(p_\alpha - p_\beta). \tag{19}$$

Remarks:

1) Notice that for $\Gamma_\alpha > 0$ ($\alpha = 1, ..., N$),

$$-2\Gamma_\alpha \le \quad Kq_{\alpha\,K} \quad \le 0 \qquad \text{for} \quad K \ge 0,$$
$$0 \le \quad Kq_{\alpha\,K} \quad < +\infty \qquad \text{for} \quad K \le 0.$$

2) In the planar case, $K = 0$:

and

$$H_o = \lim_{K\to 0} H(K) = -\frac{1}{4\pi}\sum_{\alpha\neq\beta}\Gamma_\alpha\Gamma_\beta \ln\left(\frac{q_{\alpha\,o}}{\Gamma_\alpha} + \frac{q_{\beta\,o}}{\Gamma_\beta} - 2\sqrt{\frac{q_{\alpha\,o}q_{\beta\,o}}{\Gamma_\alpha\Gamma_\beta}}\cos(p_\alpha - p_\beta)\right). \tag{20}$$

[1] Notice that expression (67) is singular for $K = 1$ and $r = \pi$ (i.e. at the South Pole).

4 A ring of vortices

The dynamics of a system of N identical vortices (i.e. $\Gamma_\alpha = \Gamma$ for $\alpha = 1, ..., N$) on a surface of constant curvature K can be described by the Hamiltonian system of equations

$$\frac{dX}{dt} = J\nabla_X H$$

where $X = (q_1, ..., q_N, p_1, ..., p_N)$ and $J = \left(\begin{array}{c|c} O & \mathbb{I} \\ \hline -\mathbb{I} & O \end{array}\right)$ and H is as in Eq. (16)-(18). There is no loss in generality to fix the vorticity $\Gamma = 1$, and therefore Eq. (10) and Eq. (19) become:

$$q_\alpha = \frac{1}{K}\begin{cases} (\cos(\sqrt{K}r_\alpha) - 1) & K > 0 \\ (\cosh(\sqrt{|K|}r_\alpha) - 1) & K < 0, \end{cases} \quad , \qquad p_\alpha = \phi_\alpha, \qquad \alpha = 1, ..., N, \tag{21}$$

$$d_{\alpha\beta} = (1 + Kq_\alpha)(1 + Kq_\beta) + K\sqrt{q_\alpha q_\beta(2 + Kq_\alpha)(2 + Kq_\beta)}\cos(p_\alpha - p_\beta). \tag{22}$$

To simplify the notation from now on we have omitted the index K in labeling the variables $q_{\alpha\,K}$, $\alpha = 1, ..., N$. Let's consider a ring configuration with initial conditions as

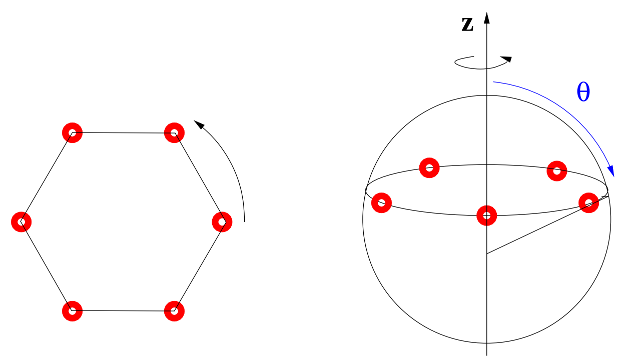


Figure 4: A polygonal configuration of identical vortices in the plane and on a sphere.

$$q_\alpha(t = 0) = q_o, \qquad p_\alpha(t = 0) = \frac{2\pi}{N}(\alpha - 1), \qquad \alpha = 1, ..., N, \tag{23}$$

where q_o is a constant (see Fig. 4). It is immediate to show that such a configuration is a polygon whose dynamics is a constant rotation, i.e.

$$q_\alpha(t) = q_o, \qquad p_\alpha(t) = \nu t + p_\alpha(0), \tag{24}$$

with

$$\nu_{\mathbb{S}^2} = -\frac{K z_o}{4\pi}\frac{(N-1)}{\rho_o^2}, \qquad , \tag{25}$$

$$\nu_{K>0} = \nu_{\mathbb{S}^2} - \frac{K z_o}{4\pi}\sum_{\alpha=2}^{N}\frac{(1-\cos(2\pi(\alpha-1)/N))}{2-\rho_o^2(1-\cos(2\pi(\alpha-1)/N))}, \tag{26}$$

$$\nu_{K<0} = \frac{1}{4\pi}\frac{K\tilde{z}_o}{\tilde{\rho}_o^2}(N-1) - \frac{K\tilde{z}_o}{4\pi}\sum_{\alpha=2}^{N}\frac{1-\cos(2\pi(\alpha-1)/N)}{2+\tilde{\rho}_o^2(1-\cos(2\pi(\alpha-1)/N))} \tag{27}$$

where

$$z_o = \cos(\sqrt{K}r_o), \qquad \rho_o = \sqrt{1-z_o^2} \qquad (K>0),$$
$$\tilde{z}_o = \cosh(\sqrt{|K|}r_o), \qquad \tilde{\rho}_o = \sqrt{\tilde{z}_o^2-1} \qquad (K<0).$$

Configurations which give rise to a dynamics of the type of Eq. (24) are called configurations of *relative equilibrium*, since they become equilibria when observed from suitable moving reference systems. In our case a relative equilibrium of the type (23) becomes an equilibrium when viewed from a reference system rotating with it, i.e. rotating at a frequency ν (as in (25)-(26)-(27)).

How does the Hamiltonian H change when viewing the dynamics from the corotating reference system?

Notice that in addition to the Hamiltonian H, there are three first integrals of motion for the vortex system on a surface of constant curvature K[1]:

$$M_x = \frac{1}{\sqrt{|K|}}\sum_{\alpha=1}^{N}\Gamma_\alpha \begin{cases} \sin\theta_\alpha\cos\phi_\alpha & \text{(sphere)} \\ \sinh\theta_\alpha\cos\phi_\alpha & \text{(hyperbolic surface)} \end{cases}$$

$$M_y = \frac{1}{\sqrt{|K|}}\sum_{j=1}^{N}\Gamma_\alpha \begin{cases} \sin\theta_\alpha\sin\phi_\alpha & \text{(sphere)} \\ \sinh\theta_\alpha\sin\phi_\alpha & \text{(hyperbolic surface)} \end{cases}$$

$$M_z = \frac{1}{\sqrt{|K|}}\sum_{\alpha=1}^{N}\Gamma_\alpha \begin{cases} \cos\theta_\alpha & \text{(sphere)} \\ \cosh\theta_\alpha & \text{(hyperbolic surface)} \end{cases}$$

which are the components of the momentum vector $\mathbf{M} = (M_x, M_y, M_z)$.

Remarks:

[1] In the planar case the integrals of motion are $M_x = \sum_{i=1}^{N}\Gamma_i x_i$, $M_y = \sum_{i=1}^{N}\Gamma_i y_i$ and $L = \sum_{i=1}^{N}\Gamma_i(x_i^2+y_i^2)$ which correspond, respectively, to the invariance under translations and rotations of the Hamiltonian H.

i) By expressing the components of $\mathbf{M}$ with respect to the canonical variables (q_j, p_j), $j = 1, ..., N$, we obtain

$$M_x = \frac{1}{\sqrt{|K|}} \sum_{\alpha=1}^{N} \cos p_\alpha \begin{cases} \sqrt{-Kq_\alpha(Kq_\alpha + 2)}, & K > 0 \\ \sqrt{Kq_\alpha(Kq_\alpha + 2)}, & K < 0 \end{cases},$$

$$M_y = \frac{1}{\sqrt{|K|}} \sum_{\alpha=1}^{N} \sin p_\alpha \begin{cases} \sqrt{-Kq_\alpha(Kq_\alpha + 2)}, & K > 0 \\ \sqrt{Kq_\alpha(Kq_\alpha + 2)}, & K < 0 \end{cases},$$

$$M_z = \frac{1}{\sqrt{|K|}} \sum_{\alpha=1}^{N} (1 + Kq_\alpha).$$

where $\Gamma_\alpha = 1$, $\alpha = 1, ..., N$.

ii) Choosing the orientation of the z-axis parallel to the $\mathbf{M}$ we have that

$$M = |\mathbf{M}| = M_z.$$

iii) By introducing the Poisson bracket

$$[f, g] = \sum_{\alpha=1}^{N} \left(\frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha} \right) = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha} \frac{\partial g}{\partial x_\alpha} \right),$$

we can construct three integrals in involution out of the four conserved quantities M_z, M_x,M_y and H. These are M_z, $M_x^2 + M_y^2$ and H: in fact

$$[H, M_z] = 0, \qquad [H, M_x^2 + M_y^2] = 0, \qquad [M_z, M_x^2 + M_y^2] = 0.$$

It is then possible to reduce the system of equations from N to $N - 2$ degrees of freedom. A Hamiltonian system with N degrees of freedom is *integrable* whenever there are N independent integrals of motion in involution. It follows that a vortex system with $N \leq 3$ is integrable, whereas the system of equations of four identical vortices has been shown to be non-integrable in the sense that there are no other first integrals analytically depending on the coordinates and circulations, and functionally independent of H, M_x, M_y, M_z (see Kimura [13],[**?**, 12]).

To make calculations easier and to better compare with previous published results we shall rewrite the Hamiltonian as

$$H = H_1 + H_2 \tag{28}$$

where

$$H_1 = c \sum_{\alpha<\beta} \Gamma_\alpha \Gamma_\beta \log(1 - d_{\alpha\beta}), \qquad H_2 = -c \sum_{\alpha<\beta} \Gamma_\alpha \Gamma_\beta \log(1 + d_{\alpha\beta}),$$

where $c = -1/4\pi$. Notice (see Appendix B, Eq. (77)) that, for $K > 0$, H_1 is the Hamiltonian for vortices on a sphere.

5 Stability

The main considerations for our study are:

i) Change of reference frame: we view the dynamics in a frame corotating with the relative equilibrium configuration. In the corotating reference system, the Hamiltonian takes the form

$$\tilde{H} = H + \nu M, \tag{29}$$

where ν is the rotational frequency of the relative equilibrium in the original frame of reference. In the new reference frame, the relative equilibrium becomes an equilibrium, Eq. (23)

$$\begin{aligned} X^* &= (q_1(0), ..., q_N(0), p_1(0), ..., p_j(0), ..., p_N(0)), \\ &= (q_o, ..., q_o, 0, ..., (j-1)2\pi/N, ..., (N-1)2\pi/N), \end{aligned} \tag{30}$$

and standard techniques can be used to study its stability. In particular, to study *linear stability* the relevant equation is

$$\frac{d\Delta X}{dt} = J\tilde{S}\Delta X \tag{31}$$

where $X = X^* + \Delta X$, and $\tilde{S}$ is the Hessian of $\tilde{H}$ evaluated at the equilibrium X^*. Then linear (or spectral) stability is deduced by studying the eigenvalues of the matrix $J\tilde{S}$ (spectral stability). For (*nonlinear stability* we make use of a *sufficient* stability criterion due to Dirichlet (1897) [6, 9]:Hamiltonian

Theorem 5.1 (Dirichlet) *Let X^* be an equilibrium of an autonomous system of ordinary differential equations*

$$\frac{dX}{dt} = f(X), \qquad \Omega \subset \mathbb{R}^{2N}, \tag{32}$$

that is, $f(X^*) = 0$. If there exists a positive (or negative) definite integral $\mathbf{F}$ *of the system (32) in a neighborhood of the equilibrium X^*, then X^* is stable.*

In our case the Hamiltonian itself is an integral of motion. Then by studying definiteness of its Hessian, $\tilde{S}$, evaluated at X^*, we infer minimal stability intervals, as done in [6, 4] (see also [3]). In fact, notice the following:

a) Since $\tilde{S}$ is a symmetric matrix it is diagonalizable, i.e. there exists an orthogonal matrix C such that $C^T\tilde{S}C = D$, where D is a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}.$$

Furthermore, the matrix C can be chosen to leave invariant the symplectic form (or equivalently $J = C^T JC$). Then by the canonical change of variables $Y = C^T X$, Eq. (31) becomes

$$\frac{d\Delta Y}{dt} = JD\,\Delta Y \tag{33}$$

where $Y = (\tilde{q}_1, ..., \tilde{q}_N, \tilde{p}_1, ..., \tilde{p}_N)$. Eq. (33) can be rewritten as

$$\frac{d^2\Delta\tilde{q}_\alpha}{dt^2} = -\lambda_\alpha\lambda_{\alpha+N}\Delta\tilde{q}_\alpha, \qquad \alpha = 1, ..., N.$$

Therefore, computing the λ_α, $\alpha = 1, ..., N$, gives us information both on linear and nonlinear stability! [1]

b) It is immediate to prove that when evaluated at the equilibrium X^*, the Hessian S takes the block structure

$$\tilde{S} = \left(\begin{array}{c|c} Q & G \\ \hline G^T & P \end{array}\right)$$

where the matrices Q and P are symmetric circulant matrices, while G is a asymmetric circulant matrix. A *circulant* matrix is a $(N \times N)$ matrix of the form

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_N \\ a_N & a_1 & \dots & a_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_1 \end{pmatrix} \tag{34}$$

Circulant matrices are of special interest to us because we can easily compute their eigenvalues and eigenvectors for all N. In particular symmetric circulant matrices have the following properties:

[1] Nevertheless we have to stress the fact that the Dirichlet criterion is a sufficient criterion for stability.

Lemma 5.2 (symmetric circulant matrices)
Let A be a $(n \times n)$ symmetric circulant matrix; then it has n real eigenvalues of the form

$$\lambda_{Aj} = a_1 + \sum_{k=2}^{n} a_k \cos\left(\frac{2\pi(j-1)(k-1)}{n}\right), \qquad j = 1, ..., n. \tag{35}$$

The eigenvectors of $\lambda_1 = s$ form the one-dimensional subspace generated by $\mathbf{v}_1 = (1, 1, ..., 1)^T$ while in the case n even, the eigenvectors of $\lambda_{n/2}$ form the subspace generated by the vector $\mathbf{v}_{n/2} = (1, -1, 1, ..., -1)^T$. The others eigenvectors come in pairs

$$\mathbf{v}_j = \begin{pmatrix} 1 \\ \cos\frac{2\pi(j-1)}{n} \\ \cos\frac{4\pi(j-1)}{n} \\ \vdots \\ \cos\frac{2(n-1)\pi(j-1)}{n} \end{pmatrix}, \qquad \mathbf{v}'_j = \begin{pmatrix} 0 \\ \sin\frac{2\pi(j-1))}{n} \\ \sin\frac{4\pi(j-1)}{n} \\ \vdots \\ \sin\frac{2(n-1)\pi(j-1)}{n} \end{pmatrix},$$

and are associated to the double eigenvalue $\lambda_j = \lambda_{n-j+2}$ for $j = 2, ..., \left[\frac{n-1}{2}\right]$.

For *skew-symmetric* circulant matrices a similar lemma holds with the eigenvalues given by

$$\lambda_{Aj} = i \sum_{k=2}^{N} a_k \sin\left(\frac{2\pi(j-1)(k-1)}{n}\right), \qquad j = 1, ..., n. \tag{36}$$

It is manifest that the eigenvectors $\mathbf{v}_j$ $(j = 1, ..., n)$ of Lemma 5.2 are orthogonal to each other. To simplify the calculations, we shall consider the corresponding orthonormal basis

$$\mathbf{e}_1 = \sqrt{\frac{1}{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}; \mathbf{e}_2 = \sqrt{\frac{2}{n}} \begin{pmatrix} 1 \\ \cos\left(\frac{2\pi}{n}\right)) \\ \vdots \\ \cos\left(\frac{2(n-1)\pi}{n}\right) \end{pmatrix}; \mathbf{e}_3 = \sqrt{\frac{2}{n}} \begin{pmatrix} 0 \\ \sin\left(\frac{2\pi}{n}\right) \\ \vdots \\ \sin\left(\frac{2(n-1)\pi}{n}\right) \end{pmatrix}; .... \tag{37}$$

and $\mathbf{e}_n = \sqrt{\frac{1}{n}}(1, -1, 1, ..., -1)^T$ if n is even.

ii) It is preferable to work in the full system (as opposed to the reduced system). We know that, when evaluated at the equilibrium, the Hessian $\tilde{S}$ of the Hamiltonian $\tilde{H}$ has a zero eigenvalue, due to the invariance under rotation (or equivalently $J\tilde{S}$ has a double zero eigenvalue). Then if all the remaining eigenvalues have the same sign we shall consider the Hessian as definite. In our case, the Hessian when definite is positive definite, and stability is guaranteed when the minimum eigenvalue is positive.

5.1 Calculating the Hessian at the equilibrium

In what follows we shall detail the calculations done for the spherical case – i.e. for the sphere without the South pole, $\mathbb{S}^2 - \{(00,-1)\}$, and the whole sphere, $\mathbb{S}^2$. The calculations for the hyperbolic case are done is a similar way. The Hessian of $\tilde{H}$, Eq. (73), evaluated at the equilibrium X^* (Eq. (30)) is

$$\tilde{S} = \left(\begin{array}{c|c} \mathcal{Q} & \mathcal{G} \\ \hline \mathcal{G}^T & \mathcal{P} \end{array}\right) = \left(\begin{array}{c|c} \dfrac{\partial^2 \tilde{H}_1}{\partial q_i \partial q_j} & \dfrac{\partial^2 \tilde{H}_1}{\partial q_i \partial p_j} \\ \hline \dfrac{\partial^2 \tilde{H}_1}{\partial p_i \partial q_j} & \dfrac{\partial^2 \tilde{H}_1}{\partial p_i \partial p_j} \end{array}\right)|_{X^*} \tag{38}$$

where

$$\begin{aligned} \mathcal{Q} &= c\frac{K^2}{r_o^4}[(t_1\mathbb{I} - A_1) + r_o^2(t_2\mathbb{I} - A_2)], \qquad \mathcal{P} = c[(-s_1\mathbb{I} + A_1) + (1 - z_o^2)(s_2\mathbb{I} - A_2)], \\ \mathcal{G} &= c2Kz_oG, \end{aligned}$$

with, $z_o = \cos(\sqrt{K})$, $r_o = \sqrt{1 - z_o^2}$,

$$\begin{aligned} G_{ij} &= \frac{sin(\phi_{io} - \phi_{jo})}{(1 + d_{ij}|_{X^*})^2}, \\ (A_1)_{ij} &= \frac{1}{1 - \cos(\phi_{io} - \phi_{jo})}, \quad \text{for} \quad i \neq j, \qquad (39) \\ (A_1)_{ii} &= (A_2)_{ii} = 0 \qquad \text{for } i = 1, ..., N, \\ d_{ij}|_{X^*} &= z_o^2 + (1 - z_o^2)\cos(\phi_{io} - \phi_{jo})), \quad (A_2)_{ij} = \frac{1 - z_o^2 + (1 + z_o^2)\cos(\phi_{io} - \phi_{jo})}{(1 + d_{ij}|_{X^*})^2} \\ s_1 &= \frac{N^2 - 1}{6}, \quad s_2 = \sum_{j=2}^{N}(A_2)_{1j}, \quad t_1 = s_1 - (N-1)(1 + z_o^2), \qquad (40) \\ t_2 &= \sum_{j=2}^{N}(D_2)_{1j}\, d_{1j}|_{X^*}, \quad \phi_{jo} = (j-1)\frac{2\pi}{N}, \; j = 1, ..., N. \end{aligned}$$

Remarks:

1) The matrices $\mathcal{Q}$, $\mathcal{P}$ and $\mathcal{G}$ are circulant matrices, i.e. they have the structure as in Eq. (39),

$$\mathcal{Q} = \begin{pmatrix} \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 & \dots & \tilde{q}_{N-1} & \tilde{q}_N \\ \tilde{q}_N & \tilde{q}_1 & \tilde{q}_2 & \dots & \tilde{q}_{N-2} & \tilde{q}_{N-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \tilde{q}_2 & \tilde{q}_3 & \tilde{q}_4 & \dots & \tilde{q}_N & \tilde{q}_1 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \dots & \tilde{p}_{N-1} & \tilde{p}_N \\ \tilde{p}_N & \tilde{p}_1 & \tilde{p}_2 & \dots & \tilde{p}_{N-2} & \tilde{p}_{N-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 & \dots & \tilde{p}_N & \tilde{p}_1 \end{pmatrix}.$$

Furthermore $\mathcal{Q}$ and $\mathcal{P}$ are symmetric matrices, i.e.

$$\tilde{q}_j = \tilde{q}_{N-j+2}, \qquad \tilde{p}_j = \tilde{p}_{N-j+2}, \quad j = 2, ..., N,$$

while the matrix $\mathcal{G}$ is skew-symmetric, i.e.

$$\tilde{g}_k = -\tilde{g}_{N-k+2} \quad k = 2, ..., N, \qquad \tilde{g}_1 = 0,$$

and if N is even $\tilde{g}_{N/2+1} = 0$.

2) From Eq. (10)

$$q_\alpha = \frac{\cos(\sqrt{K} r_\alpha) - 1}{K}$$

r_α being the geodesic distance of the αth vortex from the center of the ring (the north pole in the spherical case). In the planar limit ($K \to 0$)

$$\lim_{K \to 0} q_\alpha = -\frac{r_\alpha^2}{2}.$$

Since we are interested in studying the stability of the relative equilibrium of a ring,

$$q_\alpha = q_o = \frac{1}{K} \begin{cases} \cos(\sqrt{(K)} r_o) - 1 & (K > 0) \\ \cosh(\sqrt{|K|} r_o) - 1 & (K < 0) \end{cases}, \qquad p_\alpha = \frac{2\pi\alpha}{N}, \quad \alpha = 1, ..., N,$$

and how the stability properties change with the curvature of the surface K, there is no loss in generality in fixing $r_o = 1$. Then from now on

$$q_o = \frac{\cos(\sqrt{K}) - 1}{K}.$$

3) An alternative, approach would be to do the reduction right away. This corresponds to consider the canonical change of variables

$$\begin{aligned} \tilde{q}_j &= q_j - q_N, \qquad \tilde{p}_j = p_j \qquad \text{for} \quad j = 1, ..., N-1 \\ \tilde{q}_N &= q_N, \qquad \tilde{p}_N = \sum_{j=1}^{N} p_j \end{aligned}$$

where $\tilde{p}_N = M$, the integral of motion. In the new variables

$$\begin{aligned} \tau_{ij} &= (1 - K\tilde{p}_i)(1 - K\tilde{p}_j) + \sqrt{K^2\tilde{p}_i\tilde{p}_j}\sqrt{(2 - K\tilde{p}_i)(2 - K\tilde{p}_j)}\cos(\tilde{q}_i - \tilde{q}_j), \quad i, j = 1, ..., N-1 \\ \tau_{iN} &= (1 - K\tilde{p}_i)[1 - K(M - \sum_{j=1}^{N-1}\tilde{p}_j)] \\ &+ \sqrt{K^2\tilde{p}_i(M - \sum_{j=1}^{N-1}\tilde{p}_j)}\sqrt{(2 - K\tilde{p}_i)[2 - K(M - \sum_{j=1}^{N-1}\tilde{p}_j)]}\cos(\tilde{q}_i), \quad i = 1, ..., N-1. \end{aligned}$$

It follows from Lemma 5.2 that the matrices $\mathcal{Q}$, $\mathcal{P}$ and $\mathcal{G}$ have eigenvalues

$$\lambda_{\mathcal{Q}\,j} = \tilde{q}_1 + \sum_{k=2}^{N} \tilde{q}_k \cos\left(\frac{2\pi(j-1)(k-1)}{N}\right) \qquad j = 1, ..., N, \tag{41}$$

$$\lambda_{\mathcal{P}\,j} = \tilde{p}_1 + \sum_{k=2}^{N} \tilde{p}_k \cos\left(\frac{2\pi(j-1)(k-1)}{N}\right) \qquad j = 1, ..., N, \tag{42}$$

$$\lambda_{\mathcal{G}\,j} = i\sum_{k=2}^{N} \tilde{g}_k \sin\left(\frac{2\pi(j-1)(k-1)}{N}\right) \qquad j = 1, ..., N, \tag{43}$$

with the symmetry

$$\lambda_{\mathcal{Q}\,j} = \lambda_{\mathcal{Q}\,N-j+2}, \quad \lambda_{\mathcal{P}\,j} = \lambda_{\mathcal{P}\,N-j+2}, \quad \lambda_{\mathcal{G}\,j} = -\lambda_{\mathcal{G}\,N-j+2}, \quad j = 1, ..., \left[\frac{N+1}{2}\right].$$

Furthermore, if N is even

$$\lambda_{\mathcal{Q}\,[\frac{N}{2}+1]} = \sum_{k=1}^{N}(-1)^{k+1}q_k, \qquad \lambda_{\mathcal{P}\,[\frac{N}{2}+1]} = \sum_{k=1}^{N}(-1)^{k+1}p_k, \qquad \lambda_{\mathcal{G}\,[\frac{N}{2}+1]} = 0.$$

The eigenvalues of the Hessian $\tilde{S}$, Eq. (38), are

$$\lambda_j = \frac{1}{2}[(\lambda_{\mathcal{Q}\,j} + \lambda_{\mathcal{P}\,j}) - \sqrt{(\lambda_{\mathcal{Q}\,j} - \lambda_{\mathcal{P}\,j})^2 + 4|\lambda_{\mathcal{G}\,j}^2|}], \tag{44}$$

$$\lambda_{j+N} = \frac{1}{2}[(\lambda_{\mathcal{Q}\,j} + \lambda_{\mathcal{P}\,j}) + \sqrt{(\lambda_{\mathcal{Q}\,j} - \lambda_{\mathcal{P}\,j})^2 + 4|\lambda_{\mathcal{G}\,j}^2|}], \tag{45}$$

where $j = 1, ..., N$.

Remark: Note from Eq. (44) and Eqs (41)-(43) that for $j = 1$,

$$\lambda_{\mathcal{G}\,1} = 0 \quad \& \quad \lambda_{\mathcal{Q}_1} = 0 \quad \longrightarrow \quad \lambda_1 = 0.$$

This corresponds to the invariance of the Hamiltonian with respect to rotations about the z-axis. Then by the Dirichlet's criterion (see 5.1), nonlinear stability is guaranteed when all the other eigenvalues have the same sign. For this purpose, we (numerically) monitor the maximum and the minimum eigenvalues of $\tilde{S}$. It is found that when definite the Hessian S is positive definite. The maximum eigenvalues is always found to be positive. Then stability is assured when the minimum eigenvalue of S is positive. For a given N, Figure 5 shows that stability is guaranteed for sufficiently low K. In particular for $N \geq 7$, we can only assure stability for negative K.

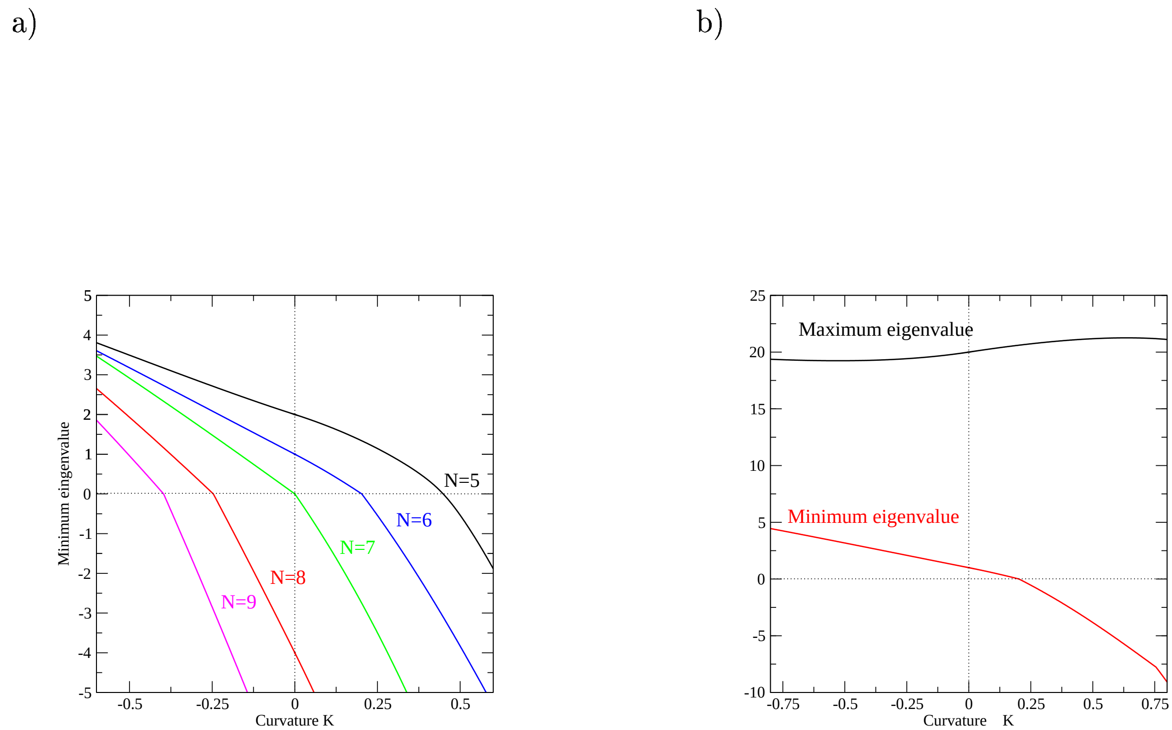


Figure 5: a) For different values of N, the number of vortices in the polygonal ring, we plot the minimum eigenvalue of the Hessian S versus the curvature K of the surface, where the vortex dynamics takes place. Only surfaces of constant curvature were considered. b) For $N = 6$, the curves of the maximum and the minimum eigenvalues of S.

Regarding stability, what is the difference between a punctured sphere and a whole sphere?

As done in [5], in the case of a sphere the Hessian simplifies to

$$\tilde{S} = \tilde{S}_1 \quad = \quad \left(\begin{array}{c|c} \mathcal{Q}_1 & O \\ \hline O & \mathcal{P}_1 \end{array} \right) = \left(\begin{array}{c|c} \dfrac{\partial^2 \tilde{H}_1}{\partial q_i \partial q_j} & \dfrac{\partial^2 \tilde{H}_1}{\partial q_i \partial p_j} \\ \hline \dfrac{\partial^2 \tilde{H}_1}{\partial p_i \partial q_j} & \dfrac{\partial^2 \tilde{H}_1}{\partial p_i \partial p_j} \end{array} \right) |_{X^*},$$

where

$$\mathcal{Q}_1 \quad = \quad c\frac{K^2}{r_o^4}(t_1 \mathbb{I} - A_1), \qquad \mathcal{P}_1 = c(-s_1 \mathbb{I} + A_1);$$

with A_1 as in Eq. (39), s_1 and t_1 as in Eq. (40). In this case the eigenvalues of $\tilde{S}$ are

$$\begin{cases} \lambda_{Qj} = \frac{K^2}{4\pi r_o^4}[(N-1)(1+z_o^2) - (j-1)(N-j+1)] \\ \lambda_{Pj} = \frac{1}{4\pi}(j-1)(N-j+1) \geq 0, \end{cases} \qquad j = 1, ..., N. \tag{46}$$

Notice that the λ_{Pj} eigenvalues are always positive, so that the definiteness of $\tilde{S}$ depends upon the sign of the smallest eigenvalues of Q ($\lambda_{Q\,min} = \lambda_{[\frac{N}{2}]+1}$)! Then we have the following (see Boatto and Simó [5]):

Theorem 5.3 **(Stability for $\mathbb{S}^2$)**
The equilibrium (30) is (linearly and nonlinearly) stable if

$$4(N-1)(11-N) - 24(N-1)(1-z_o^2) - 2N^2 - 1 - 3(-1)^N > 0.$$

and it is unstable if the inequality is reversed.

It follows from the theorem above that for a given N, we have stability for

$$0^o \leq \theta_o \leq \theta^* \qquad \text{and} \qquad \theta^* \leq \theta_o \leq 180^o,$$

where $\theta_o = \arccos(z_o)$, $\theta^* = \arccos(z^*)$ and

$$z^* = \begin{cases} \frac{\sqrt{N-3}}{2} & \text{for } N \text{ odd}, \\ \frac{N-2}{2\sqrt{N-1}} & \text{for } N \text{ even}. \end{cases}$$

a) b)

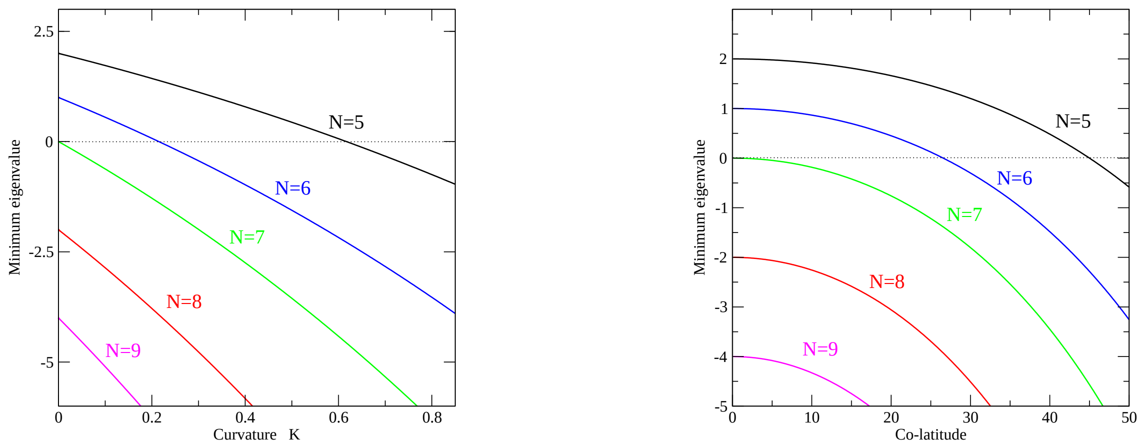


Figure 6: For a sphere of radius $R = 1/\sqrt{K}$ and different values of N, the number of vortices on the ring, we plot *a*) $\tilde{\lambda}$, Eq. (47) versus the curvature K; *b*) $\tilde{\lambda}$ versus the co-latitude θ ($\theta \in [0, \pi]$), where here $\theta = \sqrt{K}$.

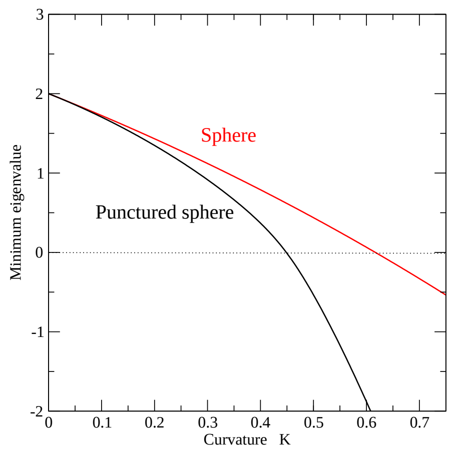
In Figure 6, for different values of N, we plot the value of

$$\begin{aligned} \tilde{\lambda} &= 4\pi \, \lambda_{Q\,min}, \\ &= \frac{K^2}{r_o^4}[4(N-1)(11-N) - 24(N-1)(1-z_o^2) - 2N^2 - 1 - (-3)^N], \end{aligned} \tag{47}$$

respectively versus the curvature K (Fig. 6.a) and versus the co-latitude θ (Fig. 6.b). Then, for a given N, we compare the curves of the minimum eigenvalue versus K – or, equivalently versus the co-latitude θ– for the sphere and the punctured sphere. It is manifest that the two curves have a good agreement near the north pole, but for large K (resp. θ) they give different intervals of guaranteed stability. In particular a whole sphere seems to assure a bigger interval of stability than a punctured sphere, see Fig. 7 for the case $N = 5$.

a)

b)



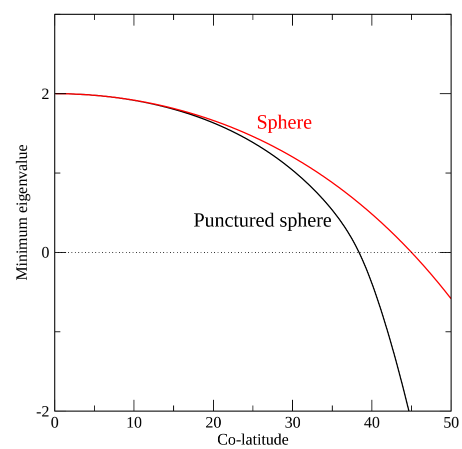


Figure 7: For $N = 5$ we plot the minimum eigenvalue of the Hessian S (rescaled by $1/4\pi$) for the sphere and the punctured sphere in a) versus the curvature K and in b) versus the colatitude θ. Notice that the assured stability interval is larger on the whole sphere.

6 Conclusions

The study of the stability of a ring of N point-vortices has a long history. In its vortex model for the atom (1883), J.J. Thomson studied how, in the planar case, the *linear stability* of the ring depends upon de number of vortices N. He could prove linear stability for $N \leq 6$ and conjectured instability for $N > 7$. D. Dritschel in his PhD thesis (1985) completed Thomson analysis by proving linear instability for $N > 7$ and showing that for $N = 7$ the ring is neutrally stable [10]. Then in 1993 D. Dritschel and L.M. Polvani generalized the linear analysis to the case of a ring on a sphere – which is a model more of interest from the atmospheric point of view. In particular they determined the ranges of linear stability in the colatitude θ, as a function of N. Few years later, J.E. Marsden, S. Pekarsky, H.E. Cabral and D. S. Schmidt were the first to address the issue of the *non linear stability* of a ring of N vortices. In particular Marsden and Pekarsky considered the stability properties of a ring of three vortices of arbitrary vorticities on the surface of a sphere [17]; while Cabral and Schmidt studied how the stability of a ring of N identical vortices is affected by the presence of a central vortex of vorticity Γ_c [6]. Then S. Boatto

and H.E. Cabral (2003) extended the nonlinear stability analysis of Masden and Pekarsky to the non-integrable case of a ring of $N > 3$ identical vortices. Still on the sphere, F. Laurent-Polz considered the case of the nonlinear stability of two rings of N vortices each but opposite vorticities [14]. In all the cited cases, the spherical and the planar cases were treated separately and the sphere was viewed as a surface in $\mathbb{R}^3$. In 1999, the beautiful article of Y. Kimura [13] finally addressed the issue of viewing the vortex dynamics in a more geometrical way by deriving the vortex Hamiltonian both for the unit sphere $\mathbb{S}^2$ ($K = +1$) and the Hyperbolic plane ($K = -1$) as a function of intrinsic geometrical quantities (such as geodesic length and curvature). This was of help in addressing our particular question:
How the stability of a ring of N identical point-vortices depends upon the curvature K of the surface?
We observed that to be able to do vortex dynamics on a two dimensional surface we need a metric $\mathbf{g}$ and a symplectic form ω (see Sec. 3). It is then desirable that the two forms be compatible! This is achieved in the framework of Kälher manifold theory. In fact it is known that a plane, punctured spheres and hyperbolic surfaces the metric can be expressed in a unified way

$$\mathbf{g} = \frac{dz\, d\bar{z}}{(1 + K|z|^2)^2}.$$

We determine a corresponding generalized symplectic form $\omega_K \sum_{\alpha=1}^{N} dq_{\alpha K} \wedge dp_\alpha$ (see Appendix A). Then by using the symmetry properties of the surface we derive the vortex Hamiltonian (see Appendix B). The key point of our stability analysis is to determine the eigenvalues of the Hessian S of the Hamiltonian - evaluated at the equilibrium. In fact by knowing the eigenvalues of S we have information on both linear and non linear stability. Linear stability is determined by solving the system

$$\frac{d\Delta X}{dt} = JS\Delta X$$

where ΔX represents the deviation from the equilibrium configuration X^*. Nonlinear stability is determined by using a sufficient criterion due a Dirichlet [9] for which stability is guaranteed when the Hessian of an integral on the motion – in our case the Hamiltonian – is positive or negative when evaluated at the equilibrium. Therefore when all the eigenvalues of S have the same sign we both have linear and non linear stability! From our analysis as K decreases we can assure stability for a larger number of N. In particular a ring of $N \geq 7$ can only be assured to be nonlinearly stable for sufficiently negative values of K.

Acknowledgments

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Appendix

A How to deduce the symplectic form ω for $K > 0$ and $K < 0$?

Let's consider the plane: we can view it as $\mathbb{R}^2$ or $\mathbb{C}$. It follows that the associated Riemannian metric g can be expressed as

$$g = dx^2 + dy^2, \qquad \text{or} \qquad g = dz\, d\bar{z},$$

where $(x, y) \in \mathbb{R}^2$ and $z = x + iy$, $\bar{z} = x - iy$. Furthermore, let's introduce the almost complex structure J [7, 16] :

a) $J = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)$ if acting on elements of $\mathbb{R}^2$;

b) $J = i$ is acting on elements of $\mathbb{C}$.

Let's consider $\mathbb{R}^2$: there is a unique symplectic form compatible with g and J, i.e. the form defined by

$$\omega(v, u) = g(v, Ju) \quad \text{for} \quad \forall u, v \in \mathbb{R}^2.$$

In fact for $v = (v_1, v_2)$ and $u = (u_1, u_2)$ we have

$$g(v, u) = v_1 u_1 + v_2 u_2 \quad \text{and} \quad J = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}.$$

and therefore

$$\omega(v, u) = g(v, Ju) = v_1 u_2 - v_2 u_1,$$

i.e.

$$\omega = dx \wedge dy \quad \text{or} \quad \omega = \frac{2}{i}(dz \wedge d\bar{z}). \tag{48}$$

Summarizing on the plane, through the given J, we have the following correspondence:

$$g = dz\, d\bar{z} \qquad \Longleftrightarrow \qquad w = \frac{2}{i} dz \wedge d\bar{z}.$$

How to generalize this procedure to the sphere? What is the associated symplectic form?

Following Kimura [13], we shall proceed by steps.

1) For simplicity let's consider the case of the unit sphere $\mathbb{S}^2$. A point on a sphere is uniquely specified by the coordinates (ϕ, θ) which are the usual longitudinal and co-latitudinal variables (see Fig. 8.a)). We start by "complexifying" the sphere through

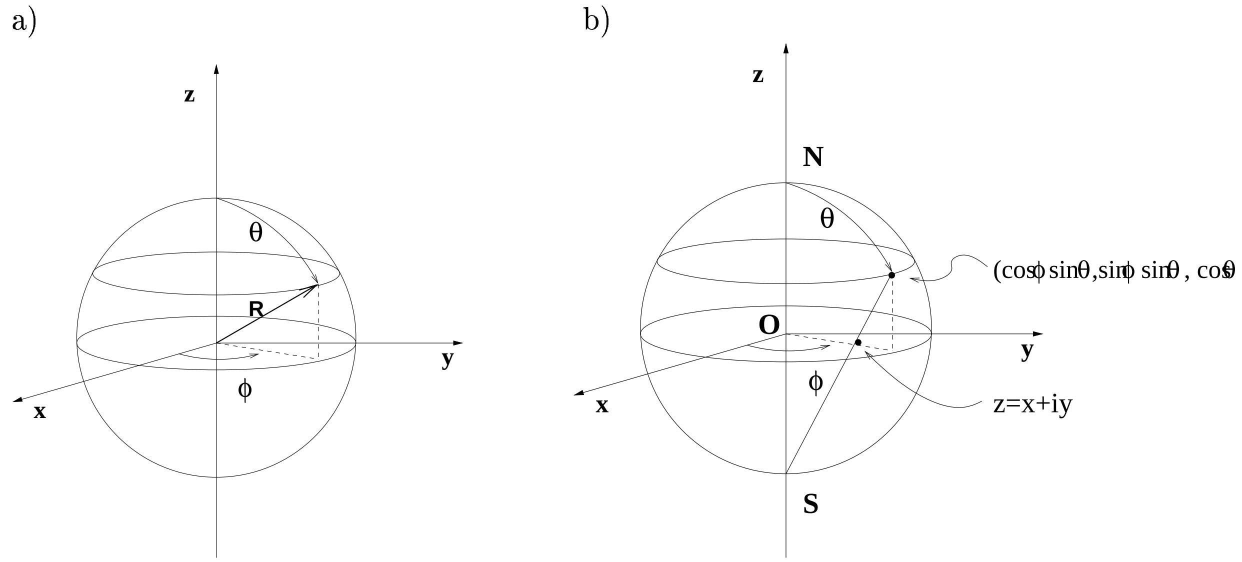

Figure 8:

steoreographic projection by the South pole which associates to each pair of spherical coordinates (ϕ, θ) a point z=x+i y on $\mathbb{C}$ (see Fig. 8.b)). The transformation Ψ is given by

$$\begin{aligned} \Psi : \mathbb{S}^2 \quad &\longrightarrow \quad \mathbb{C} \\ (\phi, \theta) \quad &\longrightarrow \quad z = \tan\left(\frac{1}{2}\theta\right) e^{i\phi} \end{aligned} \tag{49}$$

where $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$.

2) By expressing the Riemannian metric of the sphere in complex coordinates and by iterating the procedure followed above for the planar case, we obtain

$$g = 4\frac{dz\, d\bar{z}}{(1+|z|^2)^2} \quad \Longleftrightarrow \quad \omega = \frac{8\, dz \wedge d\bar{z}}{i(1+|z|^2)} \tag{50}$$

3) To obtain the corresponding symplectic form on the sphere, we pull-back the symplectic form ω (50) on the sphere by means of the stereographic transformation (49)

$$\omega_{\mathbb{S}^2} = d\Psi^* \omega = -\sin\theta\, d\theta \wedge d\phi = d(\cos\theta) \wedge d\phi.$$

It follows that the corresponding Darboux coordinates are

$$q = \cos\theta \quad \text{and} \quad p = \phi. \tag{51}$$

Generalizing to the case of a sphere of radius R or, equivalently, of curvature $K = 1/R^2$ we obtain

$$q = \frac{\cos\theta}{K}, \qquad p = \phi.$$

The expression above can be rewritten in terms of the geodesic distance r of the particle from the north pole (see Fig. 3) as

$$q = \frac{\cos(\sqrt{K}r)}{K}, \qquad p = \phi. \tag{52}$$

Remarks:

a) The above procedure can be iterated for a particle on the Hyperbolic plane ($K = -1$). In this case the steorographic map Ψ is

$$\begin{aligned} \Psi : \mathbb{H}^2 \quad &\longrightarrow \quad \mathbb{C} \\ (\phi, \theta) \quad &\longrightarrow \quad z = \tanh\left(\frac{1}{2}\theta\right) e^{i\phi}. \end{aligned} \tag{53}$$

where $\theta \in (0, \infty)$ and $\phi \in (0, 2\pi)$. We obtain

$$g = 4\frac{dz\, d\bar{z}}{(1-|z|^2)^2} \quad \Longleftrightarrow \quad \omega = \frac{8\, dz \wedge d\bar{z}}{i(1-|z|^2)}. \tag{54}$$

As done above, it is immediate to derive that the corresponding Darboux coordinates are

$$q = -\cosh\theta \quad \text{and} \quad p = \phi$$

which for the case of a hyperbolic surface of curvature $K < 0$ generalize to

$$q_K = \frac{\cosh(\sqrt{|K|}r)}{K}, \qquad p = \phi. \tag{55}$$

b) By comparing (52) with (55) we observe that the spherical Darboux coordinates are mapped into the hyperbolic ones, as the curvature K is made varying from positive to negative values. What about the planar case? Do we recover it from the expressions above, (52) and (55), in the limit $K \to 0$? Yes, if we renormalize the q_{jK} as

$$\begin{aligned} q_K &= \frac{\cos(\sqrt{K}r) - 1}{K}, \qquad (K > 0) \\ &= \frac{\cosh(\sqrt{|K|}r) - 1}{K}, \qquad (K < 0).^1 \end{aligned} \tag{56}$$

In fact, by using polar coordinate, we observe that the symplectic form for the plane can be rewritten as

$$w = dx \wedge dy = -d\left(\frac{1}{2}r^2\right) \wedge d\phi$$

where $x = r\cos\phi$ and $y = r\sin\phi$, and r is the distance from the origin. Therefore, in the planar case ($K = 0$), the polar Darboux coordinates are

$$q_o = -\frac{r^2}{2}, \qquad p = \phi. \tag{57}$$

Finally, for fixed r, let's consider the limit of (56) for $K \to 0$:

$$\lim_{K\to 0} q_K = \lim_{K\to 0} \frac{\cos(\sqrt{K}r) - 1}{K} = \lim_{K\to 0} \frac{1}{K}(1 - K\frac{r^2}{2} + O(K^2r^4) + 1) = -\frac{r^2}{2} = q_o.$$

Therefore the coordinates

$$\begin{aligned} q_K &= \frac{\cos(\sqrt{K}r) - 1}{K}, \qquad p = \phi \qquad (K > 0) \\ &= \frac{\cosh(\sqrt{|K|}r) - 1}{K}, \qquad p = \phi \qquad (K < 0). \end{aligned} \tag{58}$$

are the Darboux coordinates which can be analytically continued from the sphere to the plane, up to the hyperbolic plane! Just one set of coordinates parametrized by the curvature as we wished.

c) We can generalize what done above for a particle to the case of a system of N point vortices of vorticities Γ_j, $j = 1, ..., N$. In the plane $\mathbb{R}^2$ the corresponding symplectic form is

$$\omega = \sum_{j=1}^{N} \Gamma_j dx_j \wedge dy_j = \sum_{j=1}^{N} \Gamma_j d\left(-\frac{r_j^2}{2}\right) \wedge d\phi_j$$

where as before $x_j = r_j\cos\phi_j$ and $y_j = r_j\sin\phi_j$. Then a set of canonical variable is

$$q_{jo} = -\Gamma_j\frac{r_j}{2}, \quad \text{and} \quad p_j = \phi_j$$

where $x_j = r_j\cos\theta_j$ and $y_j = r_j\sin\theta_j$. Or, more in general, for N point-vortices on a surface of constant curvature K a set of Darboux coordinates is

$$\begin{aligned} q_{jK} &= \Gamma_j\frac{\cos(\sqrt{K}r_j) - 1}{K}, \qquad p_j = \phi_j, \quad (K > 0), \\ &= \Gamma_j\frac{\cosh(\sqrt{|K|}r_j) - 1}{K}, \qquad p_j = \phi_j, \quad (K < 0), \end{aligned}$$

where r_j is the geodesic distance of the jth vortex from the "north pole". As before

$$\begin{aligned}\lim_{K\to 0} q_{jK} &= \lim_{K\to 0}\Gamma_j \frac{\cos(\sqrt{K}r_j)-1}{K} = \lim_{K\to 0}\Gamma_j \frac{1}{K}\left(1-K\frac{r_j^2}{2}+O(K^2r_j^4)+1\right)\\ &= -\Gamma_j\frac{r_j^2}{2} = q_{jo}.\end{aligned}$$

B How to compute the stream-function Ψ?

Let's outline the main steps to obtain the stream function Ψ of particles in the velocity field of N point-vortices on the plane, $\mathbb{R}^2$, on the unit sphere, $\mathbb{S}^2$ and on the Hyperbolic plane $\mathbb{H}^2$.

I) Planar case, $\mathbb{R}^2$. We want to find a fundamental solution of

$$\Delta\Psi = -\Gamma\delta(\mathbf{x}), \tag{59}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$. A solution of Eq. (59), Ψ, represents the streamfunction of a particle in the velocity field of a single point-vortex of vorticity Γ, located at the origin. Let's integrate (59) over a domain D encircling the origin,

$$\int_D \Delta\Psi d\mathbf{x} = -\Gamma. \tag{60}$$

Following Tokieda [20] we observe that:

i) the stream function has to be radially symmetric, and therefore we choose D to be a circular domain.

ii) $\Delta\Psi = \nabla\cdot\nabla\Psi$ and, therefore, by using Stokes Theorem the left hand side of Eq. (60) becomes

$$\int_D \Delta\Psi d\mathbf{x} = \oint_C \nabla\Psi\cdot n\, d\phi = \oint_C \frac{\partial\Psi(r)}{\partial r}d\phi = 2\pi r\,\frac{\partial\Psi(r)}{\partial r}, \tag{61}$$

where $\phi\in[0,2\pi]$, n is the outward normal to the circle of radius r and $\nabla\Psi\cdot n = \partial\Psi/\partial r$ (see Fig. 9.a).

It follows from Eq. (60) and (61) that

$$\Psi_{\mathbb{R}^2}(r) = -\frac{\Gamma}{2\pi}\log(r). \tag{62}$$

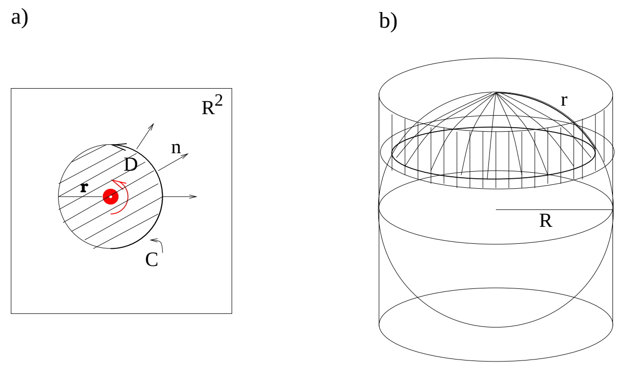

Figure 9: a) In the plane, we consider a circular domain D, encircling the vortex. $C = \partial D$. a) planar case; b) the case of a sphere of radius R. By Archimedian projection, the area of the spherical cap of radius r is equal to the area of the cylindrical band of height $h = 2R\sin^2[r/(2R)]$ (i.e. $Area_{cap} = 4\pi R^2 \sin^2[r/(2R)]$).

II) The calculations for the spherical case are quite similar to the planar one. We shall distinguish two case:
a) a sphere of radius R; b) a punctured sphere.
In both cases we locate the point vortex at the north pole and we denote with r the geodesic distance of a point from the north pole. The circumference of the circle is

$$Cir_{cap} = 2R\pi \sin(r/R), \tag{63}$$

and, by applying Archimede Theorem, the area of the polar cap is

$$Area_{cap} = 2\pi R^2(1 - \cos(r/R)) = 4\pi R^2 \sin^2[r/(2R)], \tag{64}$$

see Fig. 9.b. Why to distinguish between the whole unit-sphere and the sphere without a point? Because in the first case we are dealing with a compact object while in the second case we are not. If we want to consider some analytical continuation results from the Hyperbolic Plane (a non compact surface) to the sphere (a compact surface) we have to at least remove a point from the sphere. We shall see that we obtain different stream functions for the two cases.

1) The unit spherical shell. By Eq. (60) and (61), generalized to the sphere, we obtain

$$\begin{aligned} -\Gamma &= \int_D \Delta\Psi dx = \oint_C \nabla\Psi \cdot n\, d\phi \\ &= \oint_C \frac{\partial\Psi(r)}{\partial r} d\phi = Cir_{cap}\frac{\partial\Psi(r)}{\partial r} = 2R\pi\sin(r/R)\frac{\partial\Psi(r)}{\partial r}. \end{aligned}$$

It follows that

$$\Psi_{shell\ K>0}(r) = -\frac{\Gamma}{2\pi}\ln\left(\tan\frac{\sqrt{K}r}{2}\right), \tag{65}$$

where $R = 1/\sqrt{K}$.

2) Notice from Eq. (65) that $\Psi_{shell\ K>0}(r)$ is singular at $r = \pi R$ (i.e. at the South pole). This reflects the fact that Eq. (59) does not have a solution if we are considering a sphere (the sphere has not boundary!). In fact for a sphere we have to consider a modified version of (59) (see [8, 13]):

$$\Delta\Psi(r,\phi) = -\Gamma\delta(r,\phi) + \frac{1}{4\pi R^2}. \tag{66}$$

Then integrating (66) and using Eq. (64) gives

$$Cir_{cap}\frac{\partial\Psi}{\partial r} = \Gamma\left(-1 + \frac{Area_{cap}}{4\pi R^2}\right) = \Gamma(-1 + \sin^2[r/(2R)]).$$

Finally, by substituting the expression for Cir_{cap} (Eq. (63)) and integrating once more, we obtain

$$\Psi_{sphere}(r) = -\frac{\Gamma}{2\pi}\ln\left(\sin\frac{\sqrt{K}r}{2}\right) \tag{67}$$

where $r \in (0, \pi/\sqrt{K})$.

III) Calculations for a surface of constant negative curvature K are quite similar to the ones for a spherical shell. In fact integrating Eq. (59) gives

$$-\Gamma = \int_D \Delta\Psi dx = \oint_C \nabla\Psi\cdot n\, d\phi = \oint_C \frac{\partial\Psi(r)}{\partial r}d\phi = \frac{2\pi\sinh(\sqrt{|K|}r)}{\sqrt{|K|}}\frac{\partial\Psi(r)}{\partial r}, \tag{68}$$

where r is the geodesic distance from the north "pole" N (see Fig. 10). Integrating (72) gives

$$\Psi_{K<0}(r) = -\frac{\Gamma}{2\pi}\ln\left(\tanh\frac{\sqrt{|K|}r}{2}\right) \tag{69}$$

where $r \in (0, \infty)$.

Remarks:

i) When considering the limit $K \to 0$ for $\Psi_{K>0}$ and $\Psi_{K<0}$ we would like to recover the planar case ($K = 0$), $\Psi_o = \Psi_{\mathbb{R}^2}$. To do that we have to "renormalize" $\Psi_{K>0}$ and

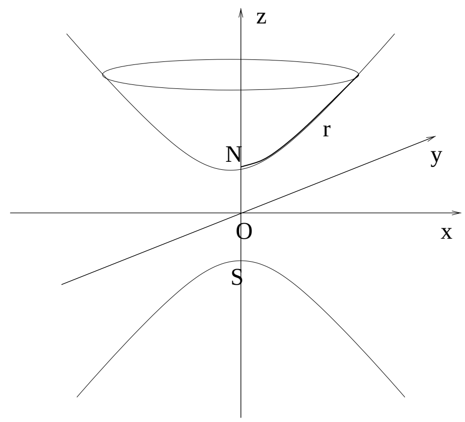

Figure 10: The Hyperbolic plane $\mathbb{H}^2$.

$\Psi_{K<0}$ by adding a suitable constant. In fact it is immediate to check (by Taylor expansion) that:

$$
\begin{aligned}
2\pi\Psi_{K>0} &= -\ln\left[\tan\left(\sqrt{K}r/2\right)\right] + \ln(\sqrt{K}/2) \qquad \sim -\ln(r) - \frac{K}{12}r^2 + ... \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
2\pi\Psi_o &= -\ln(r) \qquad\qquad\qquad\qquad\qquad\qquad \text{small } K \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
2\pi\Psi_{K<0} &= -\ln\left[\tanh\left(\sqrt{|K|}r/2\right)\right] + \ln(\sqrt{|K|}/2) \quad \sim -\ln(r) + \frac{|K|}{12}r^2 + ...
\end{aligned}
$$

ii) The previous result can be generalized to obtain the Hamiltonian describing the dynamics of a system of N point-vortices, whose canonical coordinates are

$$
q_{jK} = \frac{\cos(\sqrt{K}r_j) - 1}{K}, \qquad p_j = \phi_j, \qquad j = 1, ..., N \quad (K \geq 0) \tag{70}
$$

$$
q_{jK} = \frac{\cosh(\sqrt{|K|}r_j) - 1}{K}, \quad p_j = \phi_j, \qquad j = 1, ..., N \quad (K \leq 0), \tag{71}
$$

as seen in the previous section. The stream functions for a particle can be rewritten as

$$
\begin{aligned}
\Psi_{K>0} &= -\frac{1}{4\pi}\sum_{j=1}^{N}\Gamma_j \ln\frac{1-\cos(\sqrt{K}r_\alpha)}{1+\cos(\sqrt{K}r_\alpha)}, \\
\Psi_{K<0} &= -\frac{1}{4\pi}\sum_{\alpha}^{N}\Gamma_\alpha \ln\frac{\cosh(\sqrt{|K|}r_\alpha)-1}{\cosh(\sqrt{|K|}r_\alpha)+1},
\end{aligned}
$$

and it can be shown that the associated vortex Hamiltonians are

$$H_{K>0} = -\frac{1}{4\pi}\sum_{\alpha\neq\beta}^{N}\Gamma_\beta\Gamma_\beta \ln\frac{1-\cos\tau_{\alpha\beta}}{1+\cos\tau_{\alpha\beta}}, \quad (72)$$

$$H_{K<0} = -\frac{1}{4\pi}\sum_{\alpha\neq\beta}^{N}\Gamma_\alpha\Gamma_\beta \ln\frac{\cosh\tau_{\alpha\beta}-1}{\cosh\tau_{\alpha\beta}+1}, \quad (73)$$

where $\tau_{\alpha\beta}$ is the geodesic distance between the αth and βth vortex. In particular, for $K \geq 0$,

$$\begin{aligned}\cos\tau_{\alpha\beta} &= \cos(\theta_\alpha)\cos(\theta_\beta)+\sin(\theta_\alpha)\sin(\theta_\beta)\cos(\phi_\alpha-\phi_\beta),\\ &= \cos(\sqrt{K}r_\alpha)\cos(\sqrt{K}r_\beta)+\sin(\sqrt{K}r_\alpha)\sin(\sqrt{K}r_\beta)\cos(\phi_\alpha-\phi_\beta),\end{aligned}$$

while for $K < 0$,

$$\begin{aligned}\cosh\tau_{\alpha\beta} &= \cosh(\theta_\alpha)\cosh(\theta_\beta)-\sinh(\theta_\alpha)\sinh(\theta_\beta)\cos(\phi_\alpha-\phi_\beta),\\ &= \cosh(\sqrt{|K|}r_\alpha)\cosh(\sqrt{|K|}r_\beta)-\sinh(\sqrt{|K|}r_\alpha)\sinh(\sqrt{|K|}r_\beta)\cos(\phi_\alpha-\phi_\beta).\end{aligned}$$

Using (70)-(71), we obtain for $K \geq 0$

$$\begin{aligned}\cos\tau_\alpha(q_\alpha,q_\beta,p_\alpha,p_\beta) &= (1+Kq_\alpha)(1+Kq_\beta)\\ &\quad+\sqrt{K^2q_\alpha q_\beta(2+Kq_\alpha)(2+Kq_\beta)}\cos(p_\alpha-p_\beta), \quad (74)\end{aligned}$$

and for $K < 0$

$$\begin{aligned}\cosh\tau_{\alpha\beta}(q_\alpha,q_\beta,p_\alpha,p_\beta) &= (1+Kq_\alpha)(1+Kq_\beta)\\ &\quad-\sqrt{K^2q_\alpha q_\beta(2+Kq_\alpha)(2+Kq_\beta)}\cos(p_\alpha-p_\beta). \quad (75)\end{aligned}$$

Notice that in both cases, $K > 0$ and $K < 0$, $\cos\tau_{\alpha\beta} = d_{\alpha\beta}$ and $\cosh\tau_{ij} = d_{\alpha\beta}$ where

$$\begin{aligned}d_{\alpha\beta} &= (1+Kq_\alpha)(1+Kq_\beta)\\ &\quad+K\sqrt{q_\alpha q_\beta(2+Kq_\alpha)(2+Kq_\beta)}\cos(p_\alpha-p_\beta). \quad (76)\end{aligned}$$

Finally, notice that the Hamiltonian for the whole sphere is

$$H_{sphere} = -\frac{1}{4\pi}\sum_{i\neq j}^{N}\Gamma_i\Gamma_j\ln(1-\cos\tau_{ij}). \quad (77)$$

Summarizing: As functions of the variables (q_i, p_i), $i = 1, ..., N$,
i) $\cos\tau_{ij}$ and $\cosh\tau_{ij}$ have the same expression Eq. (76), when expressed in suitable Darboux coordinates;
ii) the Hamiltonians $H_{K>0}$ and $H_{K<0}$ have also a quite similar expression as functions of $d_{\alpha\beta}$, the only difference is the difference in sign in the numerator of the logarithm.

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