MULTI-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SOME NEGATIVE SPECTRUM

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1. Introduction

In this paper we consider Schrödinger operators

$$-\Delta + V(x), \quad V \in L^\infty(\mathbb{R}^d)$$

acting in the space $L^2(\mathbb{R}^d)$. If $V = 0$ then the operator has purely absolutely continuous spectrum on $(0, +\infty)$. We find conditions on V which guarantee that the absolutely continuous spectrum of both operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ is essentially supported by $[0, \infty)$. This means that the spectral projection associated to any subset of positive Lebesgue measure is not zero. Our main result is the following theorem (compare with [6]):

Theorem 1.1. *Let $V \in L^\infty(\mathbb{R}^d)$ be a real function. Assume that the negative spectrum of the operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ consists only of eigenvalues, denoted by $\lambda_n(V)$ and $\lambda_n(-V)$, which satisfy the condition*

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

Then the absolutely continuous spectra of both operators are essentially supported by $[0, +\infty)$.

Note that this theorem is proven in $d = 1$ by Damanik and Remling [6]. For $d \geq 2$ it is not a consequence of results obtained in [12], since it is still unclear whether potentials whose negative eigenvalues are in $\ell^{1/2}$ can be approximated by compactly supported functions in a proper way.

Remark. If V is periodic and one of the operators $H_\pm$ has a gap in the spectrum then we conclude that one of the operators $H_\pm$ must have a spectral band intersecting the negative half- line.

Example [14]. If $d \geq 3$ and $V \in L^{d+1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a real potential whose Fourier transform is square integrable near the origin then

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

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Typically, slow decaying potentials V must change their sign in order to force the negative eigenvalues of $-\Delta \pm V$ to be in $\ell^{1/2}$. More precisely they have the following structure (see also [5] where similar potentials were considered):

Theorem 1.2. *Let $V \in L^\infty(\mathbb{R}^d)$ be a real function. Let the essential spectrum of both operators $-\Delta + V$ and $-\Delta - V$ be either positive or empty. Assume that the negative eigenvalues of the operators $-\Delta+V$ and $-\Delta-V$ which are denoted by $\lambda_n(V)$ and $\lambda_n(-V)$ satisfy the condition*

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

Then

$$V = V_0 + \mathrm{div}(A) + |A|^2$$

where V_0 and $\mathrm{div}A$ are locally bounded, A is continuous and has locally square integrable derivatives,

$$\int (|V_0| + |A|^2)|x|^{1-d} dx < \infty.$$

The property of a.c. spectrum to be essentially supported by $\mathbb{R}_+$ is not the one which is shared by all Schrödinger operators with no negative spectrum. One can conclude very little about the a.c. spectrum from the fact that $-\Delta + V \geq 0$. Indeed, the theory of random operators gives examples of Schrödinger operators with positive V, whose spectrum is purely point. Therefore one can obtain some information about the a.c. spectrum from the behavior of the eigenvalues only by combining the information given for V and $-V$. This idea was used in [3] in dimension $d = 1$, where the authors proved the following striking

Theorem 1.3. Damanik-Killip [3] *Let $V \in L^\infty(\mathbb{R}_+)$ If the spectrum of the operators $-\frac{d^2}{dx^2} + V$ and $-\frac{d^2}{dx^2} - V$ on the half line is contained in $[0, +\infty)$ then it is purely absolutely continuous and it coincides with $[0, +\infty)$.*

Our methods are based on the estimates of the entropy of the spectral measure, whose importance in the theory of one-dimensional Schrödinger operators was discovered by P.Deift and R.Killip [7]. The main theorem of [7] is a natural culmination of the results obtained by M.Christ, A.Kiselev and C.Remling [1], M.Christ and A.Kiselev [2] and C.Remling [13]. It says that one dimensional Schrödinger operators with square integrable potentials have their a.c. spectra essentially supported by the positive real line.

2. Proof of Theorem 1.2

The proof of the following statement is obvious

Lemma 2.1. [Damanik-Remling] *Let ϕ be a real valued bounded function with bounded derivatives. Suppose that $-\Delta\psi + V\psi = \lambda\psi$ and the product $\phi\psi$ vanishes on the boundary of the domain $\{a < |x| < b\}$. Then*

$$\int_{a<|x|<b} \Big(|\nabla(\phi\psi)|^2 + V|\phi\psi|^2\Big)dx = \int_{a<|x|<b} \Big(|\nabla\phi|^2\psi^2 + \lambda|\phi\psi|^2\Big)dx$$

In the next statement we need the notion of the internal size (i- size) of a spherical layer $\{a < |x| < b\}$, which is by definition $b - a$.

Lemma 2.2. [Damanik-Remling] *Assume that the lowest eigenvalue $-\gamma^2$ of $H_\pm$ on the domain $\{a < |x| < b\}$ is negative. If $b - a \geq 6\gamma^{-1}$, then there exist a spherical layer $\Omega \subset \{a < |x| < b\}$ whose i-size is $d(\Omega) = 6\gamma^{-1}$ such that $H_\pm$ restricted onto Ω has an eigenvalue not bigger than $-\gamma^2/2$*

Proof. Let ψ be the eigenfunction corresponding to the eigenvalue $-\gamma^2$ for the problem on the domain $\{a < |x| < b\}$ with the Dirichlet boundary conditions. Put $L = \gamma^{-1}$ and pick a number $c > 0$ which gives the maximum to the functional $\int_{c-L<|x|<c+L} |\psi|^2 dx$

$$\phi(x) = \begin{cases} 1, & ||x| - c| < L, \\ 0, & ||x| - c| \geq 3L, \\ 3/2 - ||x| - c|/(2L), & \text{otherwise.} \end{cases} \tag{2.1}$$

Let Ω be the intersection of the support of ϕ with $\{a < |x| < b\}$. Without loss of generality we can assume that $d(\Omega) = 6\gamma^{-1}$. Now the interesting fact is that

$$\int_{a<|x|<b} |\nabla\phi|^2\psi^2 dx \leq \frac{\gamma^2}{2} \int_{a<|x|<b} |\phi\psi|^2 dx$$

which is guaranteed by the choice of c. Therefore by Lemma 2.1

$$\int_{a<|x|<b} \Big(|\nabla(\phi\psi)|^2 + V|\phi\psi|^2\Big)dx \leq -\frac{\gamma^2}{2} \int_{a<|x|<b} |\phi\psi|^2 dx.$$

That proves the statement. $\square$

Now we prove the following statement

Lemma 2.3. *Let $H_\pm \geq -\gamma^2$ on the spherical layer $\Omega = \{a < |x| < b\}$ for both indices $\pm$. Then $V + \gamma^2 = \mathrm{div}A + |A|^2$ on Ω, where A satisfies the estimate*

$$\int_{a<|x|<b} |\phi|^2|A(x)|^2 dx \leq C\Big(\gamma^2 \int_{a<|x|<b} |\phi|^2 dx + \int_{a<|x|<b} |\nabla\phi|^2 dx\Big) \tag{2.2}$$

for any $\phi \in C_0^\infty(\Omega)$ with a constant C independent of γ, Ω and ϕ.

Proof. The representation $V = -\gamma^2 + \mathrm{div}A + |A|^2$ on Ω follows from the results of [15]. Now

$$\int_{a<|x|<b} \Big(|\nabla\phi|^2 - V|\phi|^2\Big)dx \geq -\gamma^2 \int_{a<|x|<b} |\phi|^2 dx$$

which leads to (2.2). □

The main ingredients of our proof are in the following technical lemmas, which can be compared with the corresponding set of statements from [6]. Our proof is shorter because instead of functions with symmetric graphs we will use functions whose gradient on the left of a certain set is different from the one on the right of it. This will influence the choice of sets Ω_n.

Lemma 2.4. *Let $V(x) = 0$ for $|x| < 2$. There is a sequence of spherical layers Ω_n and a sequence of numbers $\epsilon_n > 0$, such that $\sum_n \epsilon_n^{1/2} < \infty$ and the i-size of Ω_n is bounded by $C\epsilon_n^{-1/2}$ with some C independent of n. The sequence of sets fulfils the condition, that $H_\pm \geq 0$ on the set $\mathbb{R}^d \setminus \cup_n \Omega_n$. Moreover*

$$H_\pm \geq -\epsilon_{j(n)}, \quad \text{on } \Omega_n$$

where $j(n)$ is the lowest number j such that $\Omega_j \cap \Omega_n \neq \emptyset$. If $\Omega_j \cap \Omega_n \neq \emptyset$ and the i-size of $\Omega_j \cap \Omega_n$ is bounded from below by $6(1-20^{-1})\epsilon_k^{-1/2}$, where $k = \min\{j, n\}$. The choice of the sequences can be done so that for each m the number of n for which $\Omega_m \cap \Omega_n \neq \emptyset$ is not bigger than 2.

Proof. In the proof we also need to construct some sequence of sets ω_n. Put $\Omega_0 = B_2$ and $\omega_0 = B_1$ where B_r denotes the ball of radius $r > 0$ about the origin; ϵ_0 can be any sufficiently large number, for example 100.

Given $\omega_n \subset \Omega_n$ and ϵ_n for $n < N$ we consider the set

$$S = \mathbb{R}^d \setminus \cup_{n<N} \Omega_n$$

and define $-\epsilon_N$ as the lowest eigenvalue of both operators $H_\pm$ on S. Let $\omega_N \subset S$ be the largest spherical layer where one of the operators $H_\pm$ has spectrum below $-\epsilon_N/2$ and the i-size of ω_N is not bigger than $L = 6\epsilon_N^{-1/2}$. In the case if the boundary of ω_N is also contained in the interior domain of S, the i-size of ω_N is equal to $L = 6\epsilon_N^{-1/2}$. The existance of this set is proven in Lemma 2.2. Denote by S_+ and S_- the right and left connected component of $S \setminus \omega_N$ correspondingly. Let $\Omega_j =: \Omega_-$ and $\Omega_k =: \Omega_+$, $j, k < N$, be the two sets which have common boundary with S_- and S_+ correspondingly. Denote $\omega_- = \omega_j$ and $\omega_+ = \omega_k$. Our construction (or induction assumptions) allow us to assume that dist$\{\omega_\pm, S_\pm\} \geq L_\pm$, where $L_- = 6\epsilon_j^{-1/2}$ and $L_+ = 6\epsilon_k^{-1/2}$ If the i-size of $S_\pm$ is not bigger than $3L$ we include $S_\pm \setminus \{x : \ \mathrm{dist}(x, \omega_\pm) \leq L_\pm/20\}$ into Ω_N by definition by demanding that $\{x:\ \mathrm{dist}(x,\omega_\pm)\le L_\pm/20\}$ and Ω_N has a non- empty piece of common boundary and $\{x:\ \mathrm{dist}(x,\omega_\pm)\le L_\pm/20\}\cap\Omega_N=\emptyset$. Otherwise

$$S_\pm\setminus\{x:\ \mathrm{dist}(x,\omega_N)<L\}=S_\pm\setminus\Omega_N.$$

Observe that the distance from the boundary of Ω_N to ω_N is not less than a specific positive number. Obviously, for any $\gamma>0$ there exist a number N such that the infimum of the spectrum of both operators $H_\pm$ on the domain

$$\mathbb{R}^d\setminus\cup_{n<N}\Omega_n$$

is higher than $-\gamma$. Assume the opposite. Then for any N one of the operators $H_\pm$ on the domain

$$\mathbb{R}^d\setminus\cup_{n<N}\Omega_n$$

has an eigenvalue which is not bigger than $-\gamma$. Then there is an eigenvalue of one of the the operators on ω_N which is not bigger than $-\gamma/2$. This implies that the negative spectrum of one of the operators $H_\pm$ is not discrete. So we come to the conclusion that $H_\pm>\ge 0$ on

$$\mathbb{R}^d\setminus\cup_n\Omega_n.$$

Now let us observe that $\sum_n\epsilon_n^{1/2}<\infty$, because the domains ω_n are disjoint. Also, it is clear that any bounded ball B_r of radius $r<\infty$ intersects only finite number of Ω_n, otherwise a Schrödinger operator on B_r would have infinite number of eigenvalues below zero. □

It follows from the proof of this theorem that one can assume that

$$\mathbb{R}^d\setminus\cup_n\Omega_n=\emptyset.$$

So we formulate

Lemma 2.5. *In Lemma 2.4 one can choose the sequences so that*

$$\mathbb{R}^d\setminus\cup_n\Omega_n=\emptyset. \tag{2.3}$$

Proof. Indeed, every time when there is a shperical layer G which has a common boundary with Ω_{n_1} and Ω_{n_2} and has the property that $H_\pm\ge 0$ on this layer, we put $\Omega_N\supset G$ and

$$\Omega_N\cap\Omega_{n_j}=\{x:\ \mathrm{dist}(x,G))<6(1-20^{-1})\epsilon_{n_j}^{-1/2}\}\cap\Omega_{n_j}$$

□

Lemma 2.6. *The sequences Ω_n and ϵ_j in Lemma 2.4 can be chosen so that* (2.3) *holds and there exists a sequence of H^1-functions $\phi_n\ge 0$ supported by Ω_n such that*

$$\sum_n\phi_n(x)=1,\qquad \sum_n\int|\nabla\phi_n(x)|^2|x|^{1-d}dx\le C\sum_n\epsilon_n^{1/2}<\infty$$

and the number of indices $j \neq n$ for which $\phi_j \phi_n \neq 0$ is 2. Moreover we can require that $V + \epsilon_{j(n)} = \mathrm{div} A_n + |A_n|^2$ on Ω_n with the bound

$$\sum_n \int_{\Omega_n} |A_n|^2 |x|^{1-d} dx \leq C\Big(1 + \sum_n \epsilon_n^{1/2}\Big) < \infty.$$

Proof. Let the functions ϕ_n be already constructed for $n < N$. Observe that the distance from the boundary of Ω_N to ω_N in the proof of Lemma 2.4 is not less than a specific positive number. That leads to the following property: let Ω_j and Ω_k, $j, k < N$, be the two sets in the construction of Lemma 2.4 which have common boundary with S_- and S_+ correspondingly, then one can always define a function ϕ_N supported on Ω_N with

$$\int_{\Omega_j} |\nabla \phi_N|^2 |x|^{1-d} dx \leq C\epsilon_j^{1/2}, \qquad \int_{\Omega_k} |\nabla \phi_N|^2 |x|^{1-d} dx \leq C\epsilon_k^{1/2},$$

$$\int_{\mathbb{R}^d \setminus \Omega_k \cup \Omega_j} |\nabla \phi_N|^2 |x|^{1-d} dx \leq C\epsilon_N^{1/2}.$$

Also $\phi_N + \phi_j = 1$ on the intersection $\Omega_N \cap \Omega_j$ and $\phi_N + \phi_k = 1$ on the intersection $\Omega_N \cap \Omega_k$. One can additionally require that

$$\sum_{j \leq N} \phi_j(x) = 1, \qquad \forall x,\ \mathrm{dist}(x, \omega_N) < L/20.$$

The estimates for A_n in this construction follow from Lemma 2.3 where instead of the function ϕ one takes functions $|x|^{(1-d)/2}\tilde{\phi_n}$, where $\tilde{\phi_n}$ and ϕ_n have similar graphs however the support of $\tilde{\phi_n}$ is bigger, so that $\tilde{\phi_n} = 1$ on Ω_n. $\square$

The end of the proof of Theorem 1.2. Let us define

$$A = \sum_n \phi_n A_n, \qquad W = -\sum_n \epsilon_{j(n)} \phi_n \quad V_1 = W + \mathrm{div}(A) + |A|^2$$

Then one can easily see that

$$V_1 = V + \sum_n A_n \nabla \phi_n + \sum_n \Big(\phi_n A_n \sum_j \phi_j A_j - \phi_n |A_n|^2\Big), \qquad \text{so that} \quad \int |V - V_1||x|^{1-d} dx < \infty$$

Put $V_0 = V - V_1 + W$. It remains to note that

$$\int |W||x|^{1-d} dx < \infty, \quad \int |A|^2 |x|^{1-d} dx < \infty.$$

and we come to the conclusion that $V = V_0 + \mathrm{div}(A) + |A|^2$, where

$$\int (|V_0| + |A|^2)|x|^{1-d} dx < \infty. \quad \square$$

Proof of Theorem 1.1 Now we apply the following technique developed in [12]. Without loss of generality we can assume that $V(x) = 0$ for $|x| < 2$. There is a probability measure μ on the real line $\mathbb{R}$ with the following properties. The essential support of the a.c. component of μ is not bigger as a set than the essential support of the a.c. spectrum of the operator H_+. Namely, one constructs an operator A_+ having the same a.c. spectrum as H_+:

$$A_+ = -\Delta + V, \qquad D(A_+) = \{u \in H^2(\mathbb{R}^d \setminus B_1) : \quad u(\theta) = 0, \ \theta \in \mathbb{S}^{d-1}\}$$

and then one sets $\mu(\delta) = (E_{A_+}(\delta)f, f)$ for a spherically symmetric function f supported in a certain spherical layer (see [12]):

$$\mathrm{supp} f \subset \{x \in \mathbb{R}^d : \quad 1 < |x| < 2\}.$$

It holds

Theorem 2.1. [12] *If V is compactly supported then*

$$\int_a^b \log \frac{1}{\mu'(\lambda)} d\lambda \leq C(\sum_j \sqrt{|\lambda_j(V)|} + \int V(x)|x|^{1-d}dx + 1) \tag{2.4}$$

where C depends on $0 < a < b < \infty$.

One of the important properties of the measure μ is that

$$V_n \to V \text{ in } L^2_{loc} \Rightarrow \mu_n \to \mu \ \text{ weakly}.$$

Let $[V_0]_+$ be the positive part of the function V_0 consider the Schrödinger operator with the potential $[V_0]_+ + \mathrm{div}(A) + |A|^2$. Let μ_1 be the corresponding measure for that operator. Then by the upper semi-continuity of the entropy (see [9]) we obtain

$$\int_a^b \log \frac{1}{\mu_1'(\lambda)} d\lambda \leq C(\int (|V_0| + |A|^2)|x|^{1-d}dx + 1). \tag{2.5}$$

Now we can extend (2.4) to the general case. Indeed let χ_n be the characteristic function of the ball of radius n and let $V_n = [V_0]_+ - \chi_n[V_0]_- + \mathrm{div}(A) + |A|^2$ then $|\lambda_j(V_n)|$ is monotonically increasing in n, so we obtain

$$\int_a^b \log \frac{1}{\mu'(\lambda)} d\lambda \leq C(\sum_j \sqrt{|\lambda_j(V)|} + \int (|V_0| + |A|^2)|x|^{1-d}dx + 1).$$

Convergence of the integral in the left hand side implies that $\mu' > 0$ almost everywhere on (a, b). This completes the proof of Theorem 1.1. $\square$

Note that this theorem leads to the following result [14] : if $d \geq 3$ and $V \in L^{d+1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a real potential whose Fourier transform is square integrable near the origin, then the a.c. spectrum of $-\Delta + V$ is essentially supported by $\mathbb{R}_+$.

Example. For each $n \in \mathbb{N} = \{1, 2, \dots\}$ we introduce the characteristic function χ_n of the interval $[n-1, n) \subset \mathbb{R}$ and for each $j \in \mathbb{Z}^d$ we introduce the characteristic function ξ_j of the cube $j + [0,1)^d \subset \mathbb{R}^d$. We put

$$f_{n,j}(x) := \chi_n(|x|)\xi_j(n^s x/|x|), \qquad n \in \mathbb{N},\, j \in \mathbb{Z}^d,$$

with a parameter $s > 1$ to be specified later. Note that $\sum_{n,j} f_{n,j} \equiv 1$. Suppose that $\omega_{n,j}$, $n \in \mathbb{N}, j \in \mathbb{Z}^d$, are bounded, independent and identically distributed random variables with zero expectations,

$$\mathbb{E}[\omega_{n,j}] = 0,$$

and consider the (bounded) random potential

$$V_\omega(x) = \sum_{n,j} n^{-\alpha}\omega_{n,j} f_{n,j}(x), \qquad x \in \mathbb{R}^d.$$

where $\alpha > d/(d+1)$. Note that for $|x| > 1$ the absolute value of this potential can be estimated from below by $c|x|^{-\alpha}$ with some $c > 0$.

Theorem 2.2. *Let $d \geq 3$, $s > 2 + \frac{(1-2\alpha)}{(d-1)}$. Then the negative eigenvalues of the operator $-\Delta + V_\omega$ almost surely satisfy the condition*

$$\sum_n \sqrt{|\lambda_n(V_\omega)|} < \infty.$$

Proof. We consider the Fourier transform

$$\hat{V}_\omega(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi x} V_\omega(x)\, dx = \sum_n \sum_j \omega_{n,j} n^{-\alpha} \hat{f}_{n,j}(\xi)$$

and note that $|\hat{\chi}_{n,j}(\xi)| \leq C_1 n^{(d-1)(1-s)}$ for all ξ with a constant C_1 independent of j, n. Hence for any compact set K

$$\mathbb{E}\left[\int_K |\hat{V}_\omega(\xi)|^2\, d\xi\right] = \mathbb{E}[\omega_{0,0}] \sum_n n^{-2\alpha} \sum_j \int_K |\hat{\chi}_{n,j}(\xi)|^2\, d\xi \leq$$

$$\leq C_2|K| \sum_n n^{(d-1)(2-s)-2\alpha},$$

and the sum on the right hand side converges. Since the Fourier transform of V_ω belongs almost surely to $L^{2,loc}$ and $V_\omega \in L^{d+1}$, the assertion follows from the main result of [14]. $\square$

3. One question about the Lieb-Thirring estimates

1. Let us note that for any potential V there is the following eigenvalue estimate for $d \leq 4$

$$\sum_j |\lambda_j(V)|^\gamma \leq C(r)\Big(\int_{|\xi|<r} |\hat{V}(\xi)|^2 d\xi + \int |V(x)|^4 dx\Big), \tag{3.1}$$

with $\gamma = 2 - d/2 \geq 0$ (see [14]). This leads to the following very simple observation. Let $f_{n,j}$ and $\omega_{n,j}$ be the same as in Theorem 2.2. Consider the potential

$$V_\omega := \sum_{n,j} v_{n,j}\omega_{n,j} f_{n,j}$$

where $v_{n,j}$ are fixed real coefficients satisfying the condition

$$||v||_{4,s}^4 := \sum_{n,j} |v_{n,j}|^4 \, n^{(1-s)(d-1)} < \infty.$$

For the ball K of radius r about the origin we have

$$\mathbb{E}\left[\int_K |\hat{V}_\omega(\xi)|^2 \, d\xi\right] = \mathbb{E}[\omega_{0,0}] \sum_n \sum_j |v_{n,j}|^2 \int_K |\hat{\chi}_{n,j}(\xi)|^2 \, d\xi \leq$$

$$\leq C_2|K| \sum_{n,j} |v_{n,j}|^2 n^{(d-1)(2-2s)}.$$

Let us introduce the following norm of the sequence v:

$$||v||_{2,s}^2 := \sum_{n,j} |v_{n,j}|^2 n^{(d-1)(2-2s)}.$$

We conclude from (3.1) that

Theorem 3.1. *For any $s > 1$, $d \leq 4$ and $\gamma = 2 - d/2$*

$$\mathbb{E}\Big(\sum_j |\lambda_j(V_\omega)|^\gamma\Big) \leq C(||v||_{2,s}^2 + ||v||_{4,s}^4).$$

In particular, if the right hand side is finite, then $\sum_j |\lambda_j(V_\omega)|^\gamma < \infty$ almost surely.

Note that if $d = 4$ then the number of negative eigenvalues is finite almost surely.

2. Let us study the problem in a more general setting. First of all for any self adjoint operator T and $s > 0$ we define

$$n \pm (s, T) = \mathrm{rank} E_{\pm T}(s, +\infty),$$

where $E_T(\cdot)$ denotes the spectral measure of T. We also put $n(s,T) = n(s^2, T^*T)$ for a non-selfadjoint operator T. Recall the following relations

$$n_\pm(s+t, T+S) \le n_\pm(s,T) + n_\pm(t,S);$$
$$n(st, TS) \le n(s,T) + n(t,S).$$

We can always represent the Lieb-Thirring sum as an integral

$$\sum_j |\lambda_j(V)|^\gamma = \gamma \int_0^\infty s^{\gamma-1} n_-(1, (H_+ + s)^{-1/2} V (H_+ + s)^{-1/2})\, ds$$

Let $E = E_{H_+}(0,\delta)$ where $\delta > 0$ and let $P = I - E$. Then

(3.2)

$$(H_+ + s)^{-1/2} V (H_+ + s)^{-1/2} = (H_+ + s)^{-1/2} EVE (H_+ + s)^{-1/2} + T(s)$$

where $T(s)$ satisfies the estimate

$$\gamma \int_0^\infty s^{\gamma-1} n_-(1/2, T(s))\, ds \le C \int |V|^{2\gamma+d}(x) dx. \tag{3.3}$$

To prove the relations (3.2), (3.3) one has to follow the proof in [14] given for the case $\gamma = 1/2$. Now let us define a real valued function V_0 so that

$$\hat V_0(\xi) = \chi_{B_{2\sqrt{\delta}}}(\xi) \hat V(\xi) \tag{3.4}$$

where $\chi_{B_{2\sqrt{\delta}}}$ is the characteristic function of the ball of radius $2\sqrt{\delta}$ about the origin. Then

$$EV_0E = EVE.$$

Therefore

$$\gamma \int_0^\infty s^{\gamma-1} n_-(1/2, (H_+ + s)^{-1/2} EVE (H_+ + s)^{-1/2})\, ds \le \sum_j |\lambda_j(2V_0)|^\gamma$$

$$\le C_0 \int |V_0|^{\gamma+d/2} dx$$

Finally we obtain

Proposition 3.1. *Let V be a real valued function on $\mathbb{R}^d$ and let V_0 be defined in* (3.4)*. Then for $\gamma \ge 1/2$*

$$\sum_j |\lambda_j(V)|^\gamma \le C_\delta \Big(\int |V_0|^{d/2+\gamma} dx + \int |V|^{d+2\gamma} dx \Big).$$

Now let us give an application of this estimate to the theory of random operators. Let ω_n be independent bounded identically distributed random variables and let χ_n be the characteristic functions of disjoint sets Q_n. Consider the potential

$$V_\omega := \sum_n v_n \omega_n \chi_n$$

where v_n are fixed real coefficients satisfying the condition

$$||v||_{d+2\gamma,1}^{d+2\gamma} := \sum_n |v_n|^{d+2\gamma} |Q_n| < \infty.$$

Consider first the case $d/2 + \gamma = 2$. For the ball K of radius r about the origin we have

$$\mathbb{E}\left[\int_K |\hat{V}_\omega(\xi)|^2 \, d\xi\right] = C \sum_n \sum_j |v_n|^2 \int_K |\hat{\chi}_n(\xi)|^2 \, d\xi \leq$$

$$\leq C_1 |K| \sum_n |v_n|^2 |Q_n|^2.$$

Let us introduce the following norm of the sequence $v = \{v_n\}$:

$$\|v\|_{p,2}^p := \sum_n |v_n|^p |Q_n|^2.$$

and consider the map

$$\mathfrak{T} : v \mapsto (V_\omega)_0$$

where $(V_\omega)_0$ is constructed from V_ω in the same way as V_0 from V in (3.4), but instead of the characteristic function $\chi_{B_{2\sqrt{\delta}}}$ we multiply $\hat{V}$ by a real C_0^∞-function in (3.4). If the coefficients are bounded then $(V_\omega)_0$ is bounded; if the norm $\|v\|_{p,2}$ is finite, then $(V_\omega)_0$ is in $L^2(\mathbb{R}^d \times \Omega)$, where Ω is the set of all points ω. Since the map is linear, we can apply the interpolation:

$$\mathbb{E}\Big(||(V_\omega)_0||_{L^p}^p\Big) \leq C||v||_{p,2}^p, \qquad p \geq 2.$$

We conclude from Proposition 3.1 that

Theorem 3.2. *For* $\gamma \geq (2 - d/2)_+ =$*max*$(2 - d/2, 0)$

$$\mathbb{E}\Big(\sum_j |\lambda_j(V_\omega)|^\gamma\Big) \leq C(||v||_{p,2}^p + ||v||_{2p,1}^{2p}), \qquad p = \gamma + d/2.$$

In particular, if the right hand side is finite, then $\sum_j |\lambda_j(V_\omega)|^\gamma < \infty$ *almost surely.*

Note that if $d \geq 4$ then the number of negative eigenvalues is finite almost surely provided that the right hand side is finite for $p = d/2$.

Example. Let χ_n be the characteristic function of an interval Δ_n with the length d_n where $n \in \mathbb{N}$. Assume that the intervals are not only disjoint but the left edge of the next interval Δ_{n+1} is the right edge of Δ_n. Let ω_n be bounded, independent and identically distributed random variables with $\mathbb{E}[\omega_n] = 0$ and define

$$V_\omega(x) := \sum_n \omega_n \chi_n(x), \qquad x \in \mathbb{R}.$$

Theorem 3.3. *Assume that*

$$\sum_n d_n \sum_{j=1}^n d_j^2 < \infty.$$

Then

$$\sum_j \sqrt{|\lambda_j(V_\omega)|} < \infty$$

almost surely.

Proof. Indeed $V_\omega = W'_\omega$, where $W_\omega = \int_0^x V_\omega(x)dx$. Now observe that

$$\mathbb{E}\Big(\int W_\omega^2(x)dx\Big) = C\int \sum_n (\int_0^x \chi_n(y)dy)^2 dx < \infty.$$

Thus $W_\omega \in L^2$ almost surely. It remains to note that the operator $-d^2/dx^2 + 2V_\omega + 4W_\omega^2$ is positive, therefore

$$\sum_j \sqrt{|\lambda_j(V_\omega)|} \le 4L\int W_\omega^2\, dx,$$

with the Lieb- Thirring constant L. □

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