Invariant Manifolds and the Stability of Traveling Waves in Scalar Viscous Conservation Laws *

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Abstract

The stability of traveling wave solutions of scalar, viscous conservation laws is investigated by decomposing perturbations into three components: two far-field components and one near-field component. The linear operators associated to the far-field components are the constant coefficient operators determined by the asymptotic spatial limits of the original operator. Scaling variables can be applied to study the evolution of these components, allowing for the construction of invariant manifolds and the determination of their temporal decay rate. The large time evolution of the near-field component is shown to be governed by that of the far-field components, thus giving it the same temporal decay rate. We also give a discussion of the relationship between this geometric approach and previous results, which demonstrate that the decay rate of perturbations can be increased by requiring that initial data lie in appropriate algebraically weighted spaces.

*[Key words:] invariant manifolds, stability of traveling waves, scalar viscous conservation laws, weighted function spaces.

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1 Introduction

The stability of traveling wave solutions to viscous conservation laws has been extensively studied, due to an interest in both their applications and the mathematical phenomena they exhibit [1], [2], [3], [4]. One important aspect of the stability analysis is that the associated linear operator has continuous spectrum that is contained in the left half of the complex plane but touches the imaginary axis at the origin. Thus, there is no spectral gap between the stable (negative real part) and center (zero real part) parts of the spectrum. This property prevents the direct application of standard tools in stability analysis, such as invariant manifold theory.

Several different techniques have been developed in order to overcome this difficulty. For example, in the context of parabolic, scalar equations, Sattinger analyzed the evolution of perturbations in exponentially weighted spaces [5]. In these spaces, the essential spectrum of the linear operator is shifted into the left half of the complex plane, thus creating a spectral gap and resulting in exponential temporal decay of perturbations. More recently, Jones, Gardner, and Kapitula developed a method for analyzing the stability of traveling waves of scalar, viscous conservation laws in algebraically weighted spaces [1]. They directly analyze the associated semigroup using detailed estimates on the resolvent operator, which they obtain by extending the Evans function into the essential spectrum at the origin. By working in appropriate algebraically weighted spaces, they obtain an algebraic temporal decay rate of perturbations. Zumbrun and Howard obtained stability results for traveling waves of, not necessarily scalar, viscous conservation laws [4]. They utilize the scattering structure of the associated semigroup, decomposing it into "scattering" and "excited" modes that are similar to our far-field and near-field components, respectively. This allows for sharp, pointwise estimates, even inside regions of the essential spectrum, and leads to stability with respect to certain algebraically weighed spaces.

In this paper, the stability of traveling wave solutions of scalar, viscous conservation laws is investigated by decomposing perturbations into three components: two far-field components and one near-field component. We find that we can apply geometric tools like the invariant manifold theorems to obtain algebraic decay results similar to those of [1] and [4]. The main advantage to using the decomposition described below, is that it provides detailed information on the underlying structure that governs the decay of perturbations of these traveling waves. For example, the far-field analysis illustrates the importance of the speed and direction of the perturbation and its initial asymptotic spatial decay, and how these two properties interact to determine the overall temporal decay rate of solutions to the traveling wave. In addition, the method illustrates that it is possible to analyze different pieces of the perturbation in different function spaces that are appropriate for the structure of the linear operator in the corresponding regions of the spatial domain. This technique is potentially relevant for other classes of equations, as well.

The focus of this work is on equations of the form

$$\partial_t u = \partial_x^2 u - \partial_x f(u), \tag{1}$$

where $u = u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is C^2. A traveling wave solution, $\phi(\xi) = \phi(x - ct)$, of the above equation satisfies

$$0 = \phi'' + c\phi' - f'(\phi)\phi'. \tag{2}$$

In order to study the stability of the traveling wave, we will consider the evolution of perturbations in the moving coordinate frame and investigate solutions of the form $u(\xi,t) = \phi(\xi) + \tilde{v}(\xi,t)$. The evolution of the perturbation $\tilde{v}$ is given by

$$\partial_t \tilde{v} = \partial_\xi^2 \tilde{v} + [c - f'(\phi(\xi))]\partial_\xi \tilde{v} - f''(\phi(\xi))\phi'(\xi)\tilde{v} - \partial_\xi N(\tilde{v},\xi),$$

where

$$N(\tilde{v},\xi) = f(\phi + \tilde{v}) - f(\phi) - f'(\phi)\tilde{v}. \tag{3}$$

It this context, it is natural to study the evolution of the integrated form of perturbations [1]. Define

$$v(\xi,t) = \int_{-\infty}^{\xi} \tilde{v}(y,t)dy. \tag{4}$$

In order to work in reasonable function spaces, it will be required that $\int \tilde{v}(\xi,t)d\xi = 0$ for all $t \geq 0$. This need not place any additional restrictions on the allowable perturbations. To see this, notice that for any solution to equation (1), $\partial_t \int u(\xi,t)d\xi = 0$. Hence, $\int u(\xi,t)d\xi = \int u(\xi,0)d\xi$. Suppose that $\int \tilde{v}(\xi,0)d\xi = \int (u(\xi,0) - \phi(\xi))d\xi = M \neq 0$. If we were to instead study the stability of the translated wave $\phi(\xi+\delta)$, where $\int (\phi(\xi+\delta) - \phi(\xi))\, d\xi = M$, then the new initial data would have zero mass. Thus, this transformation serves to fix a particular translate of the wave, and we may assume without loss of generality that $\int \tilde{v}(\xi,t)d\xi = 0$.

We remark that this transformation removes the zero eigenvalue from the spectrum of the linear operator. It also makes the operator more amenable to the decomposition we employ below. For example, in the near-field analysis, section 4, it allows us to push the entire spectrum into the left half plane to obtain exponential decay of the associated semigroup.

The evolution of v is given by

$$\partial_t v = \partial_\xi^2 v + [c - f'(\phi(\xi))]\partial_\xi v - N(\partial_\xi v, \xi). \tag{5}$$

The associated linear operator is

$$Lv = \partial_\xi^2 v + [c - f'(\phi(\xi))]\partial_\xi v, \tag{6}$$

with asymptotic limits

$$L^{\pm} \equiv \partial_\xi^2 + \alpha^{\pm}\partial_\xi = \lim_{\xi\to\pm\infty} L. \tag{7}$$

Here we have defined $\alpha^{\pm} \equiv \lim_{\xi\to\pm\infty}(c - f'(\phi(\xi))$. It can be shown that, due to the dynamics of the traveling wave (assuming it approaches its asymptotic limits at an exponential rate), $\alpha^- < 0 < \alpha^+$. This also follows from the Lax entropy condition [1]. As a result, data will be advected to the left near $+\infty$ and to the right near $-\infty$. Because of this fact, we will use the terminology in [4] and refer to this operator as "inflowing".

Inflowing operators typically have the property that the decay rate of solutions can be increased by working in weighted function spaces. A nice intuitive explanation of this property is given in [4]. Consider an equation for which data flows in toward zero at a rate α in a weighted space: $||u||_W = ||Wu||$, where $W = W(\xi)$ is a weight function that increases as $|\xi|$ increases. Any mass that the solution has near infinity will initially experience a large weight, because $W(\xi)$ is large when $|\xi|$ is large. As information gets transported in toward zero, the weight function decreases, thus causing the norm of the solution to decay in the weighted space. This generally leads to a decay rate given roughly by $\sup_\xi W(|\xi|)/W(|\xi| + \alpha t)$. Hence, exponential weights lead to exponential decay, while algebraic weights lead to algebraic decay. These ideas are connected to the exponential decay results of [5] and the algebraic decay results of [1].

In this paper, invariant manifolds are used to provide a geometric proof that the inflowing operator in equation (6) does in fact produce algebraic decay of perturbations to the traveling wave in algebraically weighted L^2 spaces. In addition, it gives very detailed information about the way in which perturbations of the wave decay. The main ideas used in the proof are as follows. We wish to exploit the fact that the asymptotic operators $L^{\pm}$ not only determine that the operator is inflowing, but are also relatively easy to understand. We will define the functions $v^+(\xi,t)$, $v^-(\xi,t)$, and $v^n(\xi,t)$ so that v^+ represents the far-field behavior of solutions near $+\infty$, v^- represents the far-field behavior of solutions near $-\infty$, and v^n represents the near-field behavior of solutions. The evolution of v^+ and v^- will essentially be governed by equation (5), but with the operator L replaced by L^+ and L^-, respectively. The evolution of v^n will be governed by the remaining parts of the linear operator. Furthermore, in order to ensure that the asymptotic aspects of the equation are isolated in the evolution of the far-field components, the near-field equation will include all coupling between the three pieces of the perturbation.

The relation between the three components and the original perturbation v will be given by

$$v(\xi,t) = W^+(\xi)v^+(\xi,t) + W^-(\xi)v^-(\xi,t) + W^n(\xi)v^n(\xi,t), \tag{8}$$

where

$$\begin{aligned} W^+(\xi) &= e^{\frac{\alpha^+}{2}\xi}|\phi'(\xi)|^{\frac{1}{2}} \\ W^-(\xi) &= e^{\frac{\alpha^-}{2}\xi}|\phi'(\xi)|^{\frac{1}{2}} \\ W^n(\xi) &= \operatorname{sech}(\epsilon\xi), \end{aligned} \tag{9}$$

for an appropriate choice of ϵ. The weight function W^+ approaches a constant at $+\infty$ and decays to zero exponentially fast as $\xi \to -\infty$. Similarly, W^- approaches a constant at $-\infty$ and decays to zero exponentially fast as $\xi \to +\infty$. The function W^n decays to zero exponentially fast as $\xi \to \pm\infty$. Thus, the weights are chosen to isolate the appropriate component of the perturbation in various regions of the spatial variable ξ.

Effectively, we analyze each component in an appropriate exponentially weighted space. For example, suppose we were instead to define $v(\xi,t) = W^+(\xi)v^+(\xi,t)$. Then we would be analyzing the evolution of v in the exponentially weighted space defined by $1/W^+$, which requires a minimum amount of exponential decay as $\xi \to -\infty$. By using all three components simultaneously, as in equation (8), the full perturbation v need only have polynomial decay at infinity.

The equations of evolution of the three components will be of the form

$$\begin{aligned} \partial_t v^+ &= \partial_\xi^2 v^+ + \alpha^+\partial_\xi v^+ - \frac{1}{W^+}N(\partial_\xi(W^+v^+),\xi) \\ \partial_t v^- &= \partial_\xi^2 v^- + \alpha^-\partial_\xi v^- - \frac{1}{W^-}N(\partial_\xi(W^-v^-),\xi) \end{aligned} \tag{10}$$

and

$$\partial_t v^n = Av^n - \mathcal{N}(v^+,v^-,v^n,\xi) + F(v^+,v^-,\xi), \tag{11}$$

where the linear operator A and the functions $\mathcal{N}$ and F will be discussed below. To determine the asymptotic temporal behavior of the three components, we first analyze that of $v^\pm$. The spectrum of the linear operators in the equations for $v^\pm$, $L^\pm$, have spectrum that lie on parabolas in the left half of the complex plane and touch the imaginary axis at the origin (see figure 1a). As a result, we can not directly use invariant manifolds to study their evolution. To overcome this, we will use a technique developed in [6] and [7] and apply scaling variables to these two equations. If we define

$$\begin{aligned} v^\pm(\xi,t) &= \frac{1}{\sqrt{t+1}}w^\pm\left(\frac{\xi+\alpha^\pm(t+1)}{\sqrt{t+1}},\log(t+1)\right) \\ \eta^\pm &= \frac{\xi+\alpha^\pm(t+1)}{\sqrt{t+1}}, \quad \tau = \log(t+1), \end{aligned} \tag{12}$$

then the evolution of $w^\pm$ is given by

$$\partial_\tau w^\pm = \mathcal{L}w^\pm - N^\pm(w^\pm,\eta^\pm,\tau). \tag{13}$$

The linear operator in the above equation is given by

$$\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}, \tag{14}$$

where $\eta = \eta^\pm$. We remark that these new spatial variables are natural in the sense that they move with the perturbation, rather than with the wave, thus capturing the inflowing nature of the linear operator. Furthermore, these scaling variable are useful because the spectrum of $\mathcal{L}$ in the weighted L^2 spaces

$$L^2(m) \equiv \{u : (1+\eta^2)^{\frac{m}{2}} u \in L^2\} \tag{15}$$

is given by (see figure 1b) [8], [7]

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \frac{1-2m}{4}\right\} \bigcup \left\{-\frac{k}{2} : k = 0, 1, 2, \ldots\right\}. \tag{16}$$

Thus, for $m > 1/2$, there is a spectral gap between the stable and center parts of the spectrum. If m is increased, the essential spectrum is pushed further into the left half plane, revealing more isolated eigenvalues. As a result, we may use invariant manifolds to determine the asymptotic decay rates of $w^\pm$.

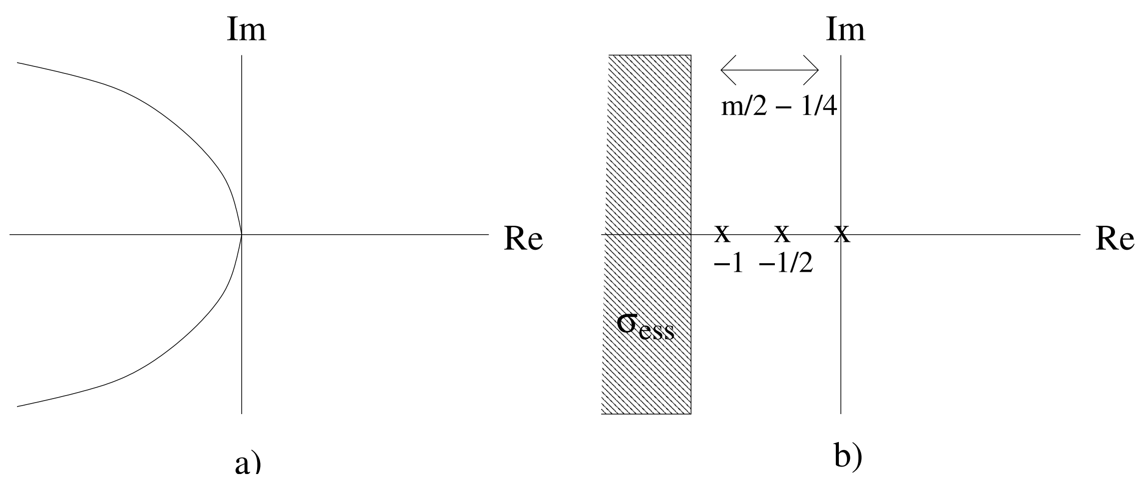

Figure 1: a) The spectrum of the operators $L^\pm$, given by $\mathrm{Re}(\lambda) = -\frac{(\mathrm{Im}(\lambda))^2}{(\alpha^\pm)^2}$. b) The spectrum of the operator $\mathcal{L}$, for $m = 3$.

The eigenfunctions associated to the eigenvalues $-\frac{k}{2}$ are given by [8], [7]

$$\varphi_0(\eta) = \frac{1}{\sqrt{4\pi}} e^{-\frac{\eta^2}{4}}, \quad \varphi_k(\eta) = \partial_\eta^k(\varphi_0(\eta)). \tag{17}$$

If we take $m > 2$, for example, then we may construct a two-dimensional center-stable manifold tangent at the origin to the subspace spanned by the eigenfunctions φ_0 and φ_1. Any solution not on this manifold will decay with a rate given by $\mathcal{O}(e^{-\frac{3}{4}\tau})$ as $\tau \to \infty$. Hence, an asymptotic expansion for $w^{\pm}$ is given by

$$w^{\pm}(\eta,\tau) = b_0^{\pm}\varphi_0(\eta) + b_1^{\pm}e^{-\frac{1}{2}\tau}\varphi_1(\eta) + h(b_0^{\pm}, b_1^{\pm}e^{-\frac{1}{2}\tau}) + \mathcal{O}(e^{-\frac{3}{4}\tau}), \tag{18}$$

where h is the function that defines the center manifold. Recall from equation (8) that in order to see how the evolution of $w^{\pm}$ effects the evolution of v, we must determine the evolution of the combined quantities $W^{\pm}v^{\pm}$. We focus on that of W^+v^+, as the other is similar, and compute only the leading order term in the above expansion. We obtain

$$\begin{aligned} W^+(\xi)v^+(\xi,t) &\sim b_0^+\frac{e^{\frac{\alpha^+}{2}\xi}|\phi'|^{\frac{1}{2}}}{\sqrt{t+1}}\varphi_0(\frac{\xi+\alpha^+(t+1)}{\sqrt{t+1}}) \\ &= b_0^+\frac{|\phi'|^{\frac{1}{2}}}{\sqrt{4\pi(t+1)}}e^{-\frac{\xi^2}{4(t+1)}}e^{-\frac{(\alpha^+)^2}{4}(t+1)}. \end{aligned}$$

Notice that the weight function has combined with the Gaussian ϕ_0 to produce exponential temporal decay. However, this is really a consequence of changing our point of view back to the frame of reference of the traveling wave, *ie* by evaluating ϕ_0 at the point $(\xi+\alpha^+(t+1))/\sqrt{t+1}$. If we work in the frame of reference traveling with the perturbation, the (η,τ) variables, then we get the detailed asymptotic expression for the form of the decay toward the (stable) traveling wave given by (18). This emphasizes the importance of the choice of function space in these stability studies.

Because each eigenfunction ϕ_k contains a Gaussian of this form, each term in the asymptotic expansion in (18) will also decay exponentially in time when evaluated in the frame of reference moving with the wave. Hence, the evolution of the far-field components of the perturbation will be governed by the higher order terms, which decay at a rate given by $\mathcal{O}((t+1)^{-\frac{5}{4}})$ in the original, unscaled variables. We remark that one must also check that the component on the center manifold that results from the function h in equation (18) also decays at this rate. We will address this issue in section 3 below.

Given the fact that the explicit terms in (18) decay much faster than is apparent at first glance, one might also wonder if the decay rate of those grouped together in the $\mathcal{O}(e^{-\frac{3}{4}\tau})$ remainder is optimal. In fact, at least at the linear level, one can compute the eigenfunctions associated with the elements of the essential spectrum of $\mathcal{L}$ (the shaded region in figure 1b) to verify that, even after combining them with the weight function W^+, they decay only algebraically in time, whether evaluated in the frame of reference moving with the perturbation or the frame of reference moving with the wave.

More generally, one can increase the decay rate of perturbations by increasing the algebraic weight, *ie* increasing m. Rather than working in $L^2(m)$, however, below we will work in

$$H^2(m) = \{u : u, \partial_\eta u, \partial_\eta^2 u \in L^2(m)\}. \tag{19}$$

The reason for this is that we will need a bit more smoothness in order to deal with the nonlinearity. Because of the transformation in equation (4), it is natural to require one derivative of the initial data. The second is used in our analysis so that, via the embedding theorems, the derivatives of the functions are defined pointwise. This property will be used in the near-field analysis in section 4. Thus, we obtain the following theorem on the asymptotic behavior of the far field components.

Theorem 1.1 *Fix any $m > 1/2$. Given any sufficiently small initial data $v^{\pm}(\xi, 0) = v_0^{\pm} \in H^2(m)$, the corresponding solution satisfies*

$$||W^{\pm}v^{\pm}(t)||_{H^2} \leq \frac{C(v_0^{\pm})}{(t+1)^{(\frac{2m+1}{4}-\epsilon)}}, \tag{20}$$

for any $\epsilon > 0$, where $C(v_0^{\pm}) \to 0$ as $||v_0^{\pm}||_{H^2(m)} \to 0$.

We remark on the presence of the small constant ϵ in the above theorem. Due to the location of the spectrum of the operator $\mathcal{L}$ (see equation (16)), one might expect decay at exactly the rate $(t+1)^{-\frac{2m+1}{4}}$. However, the estimates on the decay of the semigroup, when projected onto the stable subspace, are not quite this strong (see [7]). Furthermore, the invariant manifold theorem of Chen, Hale, and Tan [9] that we use to obtain this result guarantees decay to the center manifold at a rate arbitrarily close, but not equal, to the linear decay rates. Hence, even if the semigroup bounds held for $\epsilon = 0$, the presence of the nonlinearity could slightly weaken the result. We note that this rate is essentially the same as that found in [1] (if one equates our m with their k), if one adjusts for the fact that we work in weighted L^2 spaces, rather than weighted L^∞ spaces.

In order to analyze the evolution of the near-field component, we will work in an exponentially weighted space defined by the weight function $1/W^n$. In this space, the spectrum of the linear operator associated to the near-field component, A, is shifted off the imaginary axis into the left half plane. The resulting linear semigroup decays exponentially in time. Hence, the term in equation (11) that limits the asymptotic temporal decay of v^n is the inhomogeneity, $F(v^+, v^-, \xi)$. This term is governed by the far-field components, and so it decays algebraically in time. This results in algebraic decay of the near-field component, and we see that it is effectively slaved to the far-field pieces of the perturbation. We obtain

Theorem 1.2 *Fix any $m > 1/2$. Given any sufficiently small initial data $v^n(\xi, 0) = v_0^n \in H^2$, the corresponding solution satisfies*

$$||v^n(t)||_{H^2} \leq \frac{C(v_0^n, v_0^+, v_0^-)}{(t+1)^{(\frac{2m+1}{4}-\epsilon)}}, \tag{21}$$

for any $\epsilon > 0$*, where* $C(v_0^n, v_0^+, v_0^-) \to 0$ *as* $||v_0^+||_{H^2(m)}$*,* $||v_0^-||_{H^2(m)}$*, and* $||v_0^n||_{H^2} \to 0$*.*

By combining these results, we see that the far-field components, and hence the asymptotic limits of the linear operator, really do govern the behavior of perturbations of the traveling wave.

Theorem 1.3 *Fix any* $m > 1/2$*. Given any sufficiently small initial data* $v(\xi, 0) \equiv v_0 \in H^2(m)$*, the corresponding solution of equation (5) satisfies*

$$||v(t)||_{H^2} \leq \frac{C(v_0)}{(t+1)^{\left(\frac{2m+1}{4}-\epsilon\right)}}, \tag{22}$$

for any $\epsilon > 0$*, where* $C(v_0) \to 0$ *as* $||v_0||_{H^2(m)} \to 0$*.*

An outline for the remainder of the paper is as follows. In section 2, we present the details of the decomposition of perturbations into far-field and near-field components. Section 3 contains the analysis of the far-field components, including a proof of theorem 1.1. In the fourth section, the evolution of the near-field component is investigated, and theorem 1.2 is proven. Finally, in section 5, we explicitly carry out the decomposition for the example of Burgers equation.

2 Decomposition of perturbations

We now state the details of the decomposition of perturbations. An expository explanation follows. Define the far-field components $v^+(\xi, t)$ and $v^-(\xi, t)$ to be solutions of

$$\partial_t v^+ = \partial_\xi^2 v^+ + \alpha^+ \partial_\xi v^+ - \frac{1}{W^+} N(\partial_\xi(W^+ v^+), \xi) \tag{23}$$

$$\partial_t v^- = \partial_\xi^2 v^- + \alpha^- \partial_\xi v^- - \frac{1}{W^-} N(\partial_\xi(W^- v^-), \xi), \tag{24}$$

with initial data

$$v^+(\xi, 0) = v^-(\xi, 0) = \frac{v(\xi, 0)(1 - W^n(\xi))}{W^+(\xi) + W^-(\xi)}. \tag{25}$$

In the above, $W^\pm$ and W^n are as defined in equation (9), N is as defined in equation (3), and $v(\xi, 0)$ is the initial data for the full perturbation v. We remark that, due to the dynamics of the wave, $W^+ + W^- \neq 0$ for all ξ, and the far-field weights satisfy the differential equations

$$W_\xi^+ = \frac{1}{2}[\alpha^+ - (c - f'(\phi))]W^+$$
$$W_\xi^- = \frac{1}{2}[\alpha^- - (c - f'(\phi))]W^-. \tag{26}$$

Define the near-field component $v^n(\xi, t)$ to be a solution of

$$\begin{aligned}\partial_t v^n = \partial_\xi^2 v^n + [2\frac{W_\xi^n}{W^n} + (c - f'(\phi))]\partial_\xi v^n + [\frac{W_{\xi\xi}^n}{W^n} + \frac{W_\xi^n}{W^n}(c - f'(\phi))]v^n \\ - \mathcal{N}(v^n, v^+, v^-, \xi) + F(v^+, v^-, \xi),\end{aligned} \tag{27}$$

where

$$\begin{aligned}\mathcal{N}(v^n, v^+, v^-, \xi) = \frac{1}{W^n} N(\partial_\xi(W^+ v^+ + W^- v^- + W^n v^n), \xi) - \frac{1}{W^n} N(\partial_\xi(W^+ v^+), \xi) \\ - \frac{1}{W^n} N(\partial_\xi(W^- v^-), \xi),\end{aligned} \tag{28}$$

and

$$F(v^+, v^-, \xi) = \frac{1}{W^n}[W_{\xi\xi}^+ + W_\xi^+(c - f'(\phi))]v^+ + \frac{1}{W^n}[W_{\xi\xi}^- + W_\xi^-(c - f'(\phi))]v^-. \tag{29}$$

Note that $\mathcal{N}$ may, along with F, contribute some inhomogeneous terms to the near-field equation. We will denote the linear operator in equation (27) by

$$\begin{aligned}Av^n &= \partial_\xi^2 v^n + [2\frac{W_\xi^n}{W^n} + (c - f'(\phi))]\partial_\xi v^n + [\frac{W_{\xi\xi}^n}{W^n} + \frac{W_\xi^n}{W^n}(c - f'(\phi))]v^n \\ &= \partial_\xi^2 v^n + [-2\epsilon \tanh(\epsilon\xi) + (c - f'(\phi))]\partial_\xi v^n \\ &\quad + [\epsilon^2(\tanh^2(\epsilon\xi) - \operatorname{sech}^2(\epsilon\xi)) - \epsilon \tanh(\epsilon\xi)(c - f'(\phi))]v^n.\end{aligned} \tag{30}$$

The initial data for the near-field component is

$$v^n(\xi, 0) = v(\xi, 0). \tag{31}$$

One can directly check that, if v^+, v^-, are v^n defined in the above manner, then v as given in equation (8) is a solution to equation (5) with the appropriate initial data.

This decomposition can be understood as follows. We want to define v^+ and v^- so that the linear part of their evolution will be governed by the linear operators L^+ and L^-, given in equation (7). In the far-field equations there should be no coupling with the other components. If we simply substitute $W^\pm v^\pm$ for v in equation (5), then the choice of the weight functions $W^\pm$, given in equation (9), is such that the advection coefficient in the resulting linear operator is exactly $\alpha^\pm$. There is an additional linear term of the form $[W_{\xi\xi}^\pm + W_\xi^\pm(c - f'(\phi))]v^\pm / W^n$. We do not want this term to remain in the far-field equations, because then the scaling variables in equation (12) will not transform the linear operator in the desired manner. As a result, these terms are included in the inhomogeneity F in the near-field equation. Furthermore, we retain in the corresponding far-field equation only that component of the nonlinearity that depends upon v^+ or v^-. This is to avoid coupling in the far-field equations, so that the nonlinearity is relatively easy to understand in terms of the scaling variables.

The linear part of the near-field equation is just the linear operator L in equation (6) when expressed in the exponentially weighted space defined by the function $1/W^n$. The inhomogeneity F results from the linear parts of the far-field equations that differ from the asymptotic operators $L^{\pm}$, as explained above. The remaining term in equation (27), $\mathcal{N}$, comes from the original nonlinearity $N(\partial_\xi v, \xi)$, after subtracting those parts which were included in the far-field equations.

We now turn to the choice of ϵ in the weight function W^n. The idea is to pick ϵ so that the linear operator A in equation (30) has spectrum contained entirely in the left half of the complex plane with a nonzero distance to the imaginary axis. This will lead to exponential temporal decay of the associated semigroup, which can be used to control the remaining terms in the equation that involve v^n. The inhomogeneity F is controlled by the far-field components. Therefore, if $v^{\pm}$ decay only algebraically, the inhomogeneity will limit the decay of v^n, thus determining its asymptotic behavior. Additional care must be taken in the choice of ϵ, due to the factor $1/W^n$ in the functions $\mathcal{N}$ and F. We will return to this issue in section 4, below.

3 Analysis of the far-field components

We now determine the behavior of the far-field components. The details will be carried out for v^+ only, as those of v^- are similar. The equation of evolution of v^+, as given in equation (23), is

$$\partial_t v^+ = \partial_\xi^2 v^+ + \alpha^+ \partial_\xi v^+ - \frac{1}{W^+} N(\partial_\xi (W^+ v^+), \xi). \tag{32}$$

We will use a slightly modified version of the scaling variables given in equation (12). This is to elucidate the effect of the nonlinearity on the dynamics within the center manifold, which we construct below.

Define the scaling variables (η, τ) and $w(\eta, \tau)$ to be

$$\begin{aligned} v^+(\xi, t) &= \frac{1}{(t+1)^{\frac{1}{2}-\sigma}} w\Big(\frac{\xi + \alpha^+(t+1)}{\sqrt{t+1}}, \log(t+1)\Big) \\ \eta &= \frac{\xi + \alpha^+(t+1)}{\sqrt{t+1}}, \quad \tau = \log(t+1). \end{aligned} \tag{33}$$

The equation of evolution of w is

$$\partial_\tau w = (\mathcal{L} - \sigma) w - \frac{e^{(\frac{3}{2}-\sigma)\tau}}{W^+(e^{\frac{1}{2}\tau}\eta - \alpha^+ e^\tau)} N(e^{-(1-\sigma)\tau} \partial_\eta (W^+ w), e^{\frac{1}{2}\tau}\eta - \alpha^+ e^\tau), \tag{34}$$

where the linear operator is $\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}$. As mentioned in section 1, the spectrum of this operator in the space $L^2(m)$ is known.

Proposition 3.1 *[8], [7] Fix $m \geq 0$ and let $\mathcal{L}$ be the linear operator in $L^2(m)$, defined on its maximal domain. Then the spectrum of $\mathcal{L}$ is*

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbb{C} : Re(\lambda) \leq \frac{1-2m}{4}\right\} \bigcup \left\{-\frac{k}{2} : k = 0, 1, 2, \dots\right\}.$$

Moreover, if $m > \frac{1}{2}$ and if $k = 0, 1, 2, \dots$ satisfies $k + \frac{1}{2} < m$, then $\lambda_k = -\frac{k}{2}$ is an isolated eigenvalue with multiplicity 1. (See figure 1b.) Furthermore, suppose $m > \frac{1}{2}$ is fixed. Then for $k = 0, 1, 2, \dots$, $k + \frac{1}{2} < m$, the eigenfunctions φ_k associated to the eigenvalues λ_k are

$$\varphi_0(\eta) = \frac{1}{\sqrt{4\pi}} e^{-\frac{\eta^2}{4}}, \quad \varphi_k(\eta) = \partial_\eta^k(\phi_0(\eta)). \tag{35}$$

We wish to construct, for any fixed $m > \frac{1}{2}$, a center-stable manifold with dimension given by the greatest integer less than or equal to $m + \frac{1}{2}$. To do so, we will apply the invariant manifold theorem of Chen, Hale, and Tan [9]. As mentioned above, we will work in $H^2(m)$ and use the additional smoothness to deal with the nonlinearity in section 4. As a result, we will need to satisfy the assumptions of [9] in $H^2(m)$. We remark that, in this space, the spectrum of $\mathcal{L}$ remains as in proposition 3.1.

In order to show that the hypotheses of this theorem are satisfied, we will need some properties of the linear operator $\mathcal{L}$ and the semigroup it generates.

Proposition 3.2 *[7] The linear operator $\mathcal{L}$ is the generator of a strongly continuous semigroup on the space $H^2(m)$ for any fixed $m \geq 0$. In addition, let $1 \leq p \leq q \leq \infty$, $m \geq 0$, and $T > 0$. Then for any $\alpha \in \mathbb{N}$ and $0 < \tau \leq T$ there exists a constant C such that*

$$||b^m \partial^\alpha (e^{\mathcal{L}\tau} f)||_{L^p} \leq \frac{C}{a(\tau)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})+\frac{\alpha}{2}}} ||b^m f||_{L^q}, \tag{36}$$

where $b(\eta) = (1+\eta^2)^{\frac{1}{2}}$ and $a(\tau) = 1 - e^{-\tau}$.

Note that, due to the form of the scaling variables in equation (33), $\xi = \xi(\eta, \tau)$, and hence the nonlinearity in equation (34) is dependent on the temporal variable τ. The invariant manifold theorem in [9] is directly applicable to autonomous equations, and so we define a new dependent variable $y \in [0, 1]$ via

$$\tau = \log\left(\frac{2-y}{y}\right). \tag{37}$$

Equation (34) can then be written

$$\partial_\tau w = (\mathcal{L} - \sigma)w - N^+(\partial_\eta(W^+ w), \eta, y)$$
$$\partial_\tau y = -y + \frac{1}{2}y^2, \tag{38}$$

where

$$N^+(\partial_\eta(W^+ w), \eta, y) = \frac{e^{(\frac{3}{2}-\sigma)\tau(y)}}{W^+(e^{\frac{1}{2}\tau(y)}\eta - \alpha^+ e^{\tau(y)})} N(e^{-(1-\sigma)\tau(y)}\partial_\eta(W^+ w), e^{\frac{1}{2}\tau(y)}\eta - \alpha^+ e^{\tau(y)})$$
$$\tau(y) = \log\left(\frac{2-y}{y}\right). \tag{39}$$

In order to apply the invariant manifold theorem of [9], we will need the following assumption on the nonlinearity N^+.

Assumption 1 *Fix $T > 0$ and $m > 1/2$. For any $w \in C^0([0,T], H^2(m))$ define*

$$R(\tau) = \int_0^\tau e^{\mathcal{L}(\tau-s)} N^+(w(s))ds. \tag{40}$$

Then $R(\tau) \in C^0([0,T], H^2(m))$, and there exists a $C(m, r_0, T)$ such that, if w_1, $w_2 \in C^0([0,T], H^2(m))$ with $\sup_{0\le\tau\le T} ||w_i(\tau)||_{H^2(m)} \le r_0$, then the corresponding integral terms satisfy

$$\sup_{0\le\tau\le T} ||R_1(\tau) - R_2(\tau)||_{H^2(m)} \le C(m, T, r_0) \sup_{0\le\tau\le T} ||w_1(\tau) - w_2(\tau)||_{H^2(m)}.$$

Furthermore, the constant $C(m, T, r_0) \to 0$ as $T \to 0$ and as $r_0 \to 0$.

We remark that this assumption is less strict than, for example, requiring that the nonlinearity be Lipschitz in $H^2(m)$, as the action of the semigroup has a smoothing effect. However, in verifying this assumption for a particular nonlinearity, the semigroup cannot absorb both of the derivatives, as there would then be too many factors of the function $a(\tau)$ in the denominator of the bound in equation (36). Typically, the semigroup can absorb one derivative, and the other derivative must be absorbed by the nonlinearity itself. For example, the action of the semigroup is explicitly known [7]:

$$(e^{\mathcal{L}\tau} f)(\eta) = \frac{e^{\frac{\tau}{2}}}{\sqrt{4\pi a(\tau)}} \int_{\mathbb{R}} e^{-\frac{(\eta-y)^2}{4a(\tau)}} f(y e^{\frac{\tau}{2}})dy, \tag{41}$$

where $a(\tau) = 1 - e^{-\tau}$. Thus, using integration by parts in the above expression, one can write

$$||\partial_\eta^2 \int_0^\tau e^{\mathcal{L}(\tau-s)} N^+(w(s))ds||_{L^2(m)} = ||\int_0^\tau \left(\partial_\eta e^{\mathcal{L}(\tau-s)}\right)\left(\partial_\eta N^+(w(s))\right) ds||_{L^2(m)}, \tag{42}$$

and obtain the bound in assumption 1 using this expression and properties of N^+. (See the example in section 5, in particular lemma 5.1.)

Consider now a slightly modified version of equation (38), in which the nonlinearity is cut off outside of a small neighborhood of zero in $H^2(m)$. This is necessary so that the size of the Lipschitz constant of the nonlinearity can be made small by choosing this neighborhood to be small. Let $\chi_{r_0}(w) : H^2(m) \to \mathbb{R}^+$ be a smooth function satisfying $\chi_{r_0}(w) = 1$ if $||w||_{H^2(m)} \le r_0$ and $\chi_{r_0}(w) = 0$ if $||w||_{H^2(m)} \ge 2r_0$. We remark that such a function exists because $H^2(m)$ is a Hilbert space [10]. The equation for which a center-stable manifold will be constructed is

$$\partial_\tau W = \mathcal{L}^+ W + \mathcal{N}^+(W, \eta), \tag{43}$$

where

$$W = \begin{pmatrix} w \\ y \end{pmatrix}, \ \mathcal{L}^+ = \begin{pmatrix} \mathcal{L} - \sigma & 0 \\ 0 & -1 \end{pmatrix}, \ \mathcal{N}^+(W, \eta) = \begin{pmatrix} -\chi_{r_0}(w) N^+(w, \eta, y) \\ \frac{1}{2} y^2 \end{pmatrix}. \tag{44}$$

Proposition 3.3 *Given any sufficiently small r_0 and sufficiently small $w(\xi, 0) \in H^2(m)$, there exists a solution to equation (43) satisfying $w(\tau) \in C^0([0, \infty), H^2(m))$.*

Proof Consider the integral form of solutions to equation (43),

$$W(\tau) = e^{\mathcal{L}^+ \tau} W(0) + \int_0^\tau e^{\mathcal{L}^+(\tau - s)} \mathcal{N}^+(W(s)) ds. \tag{45}$$

Using the fact that the linear operator $\mathcal{L}$ is the generator of a strongly continuous semigroup and assumption 1, local existence can be proven via a contraction mapping argument. Global existence will then follow due to the presence of the cutoff function χ_{r_0}. More specifically, a solution can fail to exist globally only if it becomes unbounded in norm in finite time. But, if the solution were to leave a ball of radius $2r_0$ in $H^2(m)$, then the nonlinearity would become zero. The evolution would then be governed only by the linear operator, and hence the solution cannot blow up in finite time. □

In order to apply the invariant manifold theorem of [9], we will need the following proposition.

Proposition 3.4 *Let $\Phi_1^{r_0}$ be the semiflow associated to equation (43) at time $\tau = 1$. Then, if $r_0 > 0$ is sufficiently small, the semiflow can be decomposed as*

$$\Phi_1^{r_0} = \Lambda + \mathcal{R},$$

where Λ is a bounded linear map, and $\mathcal{R}$ is a globally Lipschitz map such that $Lip(\mathcal{R}) \le C(r_0)$, where $C(r_0) \to 0$ as $r_0 \to 0$. Furthermore, $\mathcal{R}$ is C^1 with $\mathcal{R}(0) = D\mathcal{R}(0) = 0$.

Proof Consider equation (45) for fixed $\tau = 1$, and define

$$\Lambda = e^{\mathcal{L}^+(1)}, \quad \mathcal{R} = \int_0^1 e^{\mathcal{L}^+(1-s)}\mathcal{N}^+(\cdot)ds.$$

By proposition 3.2, Λ is a bounded linear map on $H^2(m)$, and by assumption 1, $\mathcal{R}$ is a globally Lipschitz map with $\mathrm{Lip}(\mathcal{R}) \leq C(r_0)$, where $C(r_0) \to 0$ as $r_0 \to 0$. To see that $\mathcal{R}(0) = D\mathcal{R}(0) = 0$, note that

$$\frac{\sup_{0\leq s\leq 1} ||\mathcal{R}(W(s))||_{H^2(m)}}{\sup_{0\leq s\leq 1} ||W(s)||_{H^2(m)}} \to 0 \quad \text{as} \quad \sup_{0\leq s\leq 1} ||W(s)||_{H^2(m)} \to 0.$$

□

As a result of proposition 3.4, the hypotheses of the invariant manifold theorem in [9] are satisfied. Roughly speaking, the spectral structure and semigroup bound in propositions 3.1 and 3.2 provide the necessary linear structure, while proposition 3.4, which relies on assumption 1, provides control over the interaction between the semigroup and the nonlinearity. We refer to [9] for the details.

Thus, if we fix some $m > \frac{1}{2}$, then we are guaranteed the existence of a center-stable manifold. The dynamics on this manifold may be determined as follows. The main idea is to show that the nonlinearity does not significantly alter the dynamics that result from the linear operator. Heuristically, this is due to the fact that N^+ depends on W^+v^+ in a nonlinear fashion. In particular,

$$\frac{e^{(\frac{3}{2}-\sigma)\tau}}{W^+}N^+(e^{-(1-\sigma)\tau}\partial_\eta(W^+w)) \approx e^{(\frac{1}{2}+\sigma)\tau}W^+(e^{\frac{1}{2}\tau}\eta - \alpha^+e^\tau)||w||^2 \leq C||w||^2$$

for all $0 \leq \tau < \infty$. This is true because, for all values of η except $\eta^* \equiv \alpha^+e^{\frac{\tau}{2}}$, we have that $(e^{\frac{1}{2}\tau}\eta - \alpha^+e^\tau) \to -\infty$ as $\tau \to \infty$, and $W^+(\xi) \to 0$ as $\xi \to -\infty$. The point $\eta^* \to \infty$ as $\tau \to \infty$. This "bad" point should be taken care of by the fact that $w \in H^2(m)$, and hence decays rapidly as $\xi \to +\infty$. As as a result, for large τ the nonlinearity N^+ will be small in some sense (for small w) and will not affect the leading order dynamics on the center-stable manifold.

To see this rigorously, we determine the dynamics on the center-stable manifold. We will use the projection operators associated to the operator $\mathcal{L}$, which have an explicit form. They are defined in terms of the Hermite polynomials [7]

$$H_j(\eta) = \frac{2^j}{j!}e^{\frac{\eta^2}{4}}\partial_\eta^j(e^{-\frac{\eta^2}{4}}), \tag{46}$$

which are the eigenfunctions of the adjoint operator $\mathcal{L}^* = \partial_\eta^2 - \frac{1}{2}\eta\partial_\eta$ and satisfy

$$\int H_i(\eta)\varphi_j(\eta)d\eta = \delta_{ij}.$$

In general, for any $k < m - \frac{1}{2}$, the projection onto the k-dimensional center-stable subspace is given by

$$(P_c f)(\eta) = \sum_{j=0}^{k} \left(\int H_j(\zeta) f(\zeta) d\zeta \right)^{\frac{1}{2}} \varphi_j(\eta).$$

On this manifold, $w(\eta, \tau)$ may be written

$$w(\eta, \tau) = \sum_{i=0}^{k} \beta_i(\tau) \varphi_i(\eta) + h(\beta(\tau)), \tag{47}$$

where $h(z) = \mathcal{O}(z^2)$ is some function that defines the manifold, $\beta = (\beta_1, \ldots, \beta_k)$, and each β_i is a solution to

$$\partial_\tau \beta_i = -(\frac{i}{2} + \sigma)\beta_i + \left(\int H_i(\eta) N^+(\eta, \sum_{j=0}^{k} \beta_j(\tau)\varphi_j(\eta) + h(\beta(\tau))) d\eta \right). \tag{48}$$

Note that the assumption $f \in C^2$ in equation (1) implies that $N^+(z) \le \mathcal{O}(z^2)$ as $z \to 0$. As a result, for sufficiently small initial data,

$$\beta_i(\tau) \sim \beta_i(0) e^{-(\sigma + \frac{1}{2} i)\tau}. \tag{49}$$

(See Chapter 13, section 4, theorem 4.5 of [11].) Using equation (47), we see that the evolution of w has the form

$$w(\eta, \tau) = b_0(0) e^{-\sigma\tau} \varphi_0(\eta) + \cdots + b_k(0) e^{-(\frac{k}{2} + \sigma)\tau} \varphi_k(\eta) + h(b_0, \ldots, b_k) + \mathcal{O}(e^{(-\frac{2m-1}{4} - \sigma - \epsilon)\tau}), \tag{50}$$

for any $\epsilon > 0$. Transforming back to the original, unscaled variables, we have

$$\begin{aligned} W^+(\xi) v^+(\xi, t) &= \frac{e^{\frac{\alpha^+}{2}\xi} |\phi'|^{\frac{1}{2}}}{\sqrt{(t+1)}} p(\eta, \tau) e^{-\frac{(\xi + \alpha^+(t+1))^2}{(t+1)}} + e^{\frac{\alpha^+}{2}\xi} |\phi'|^{\frac{1}{2}} e^{-(\frac{1}{2} - \sigma)\tau} h(b_0, \ldots, b_k) \\ &\quad + e^{\frac{\alpha^+}{2}\xi} |\phi'|^{\frac{1}{2}} \mathcal{O}((t+1)^{-(\frac{2m+1}{4} - \epsilon)}) \\ &= \frac{|\phi'|^{\frac{1}{2}} e^{-\frac{\xi^2}{4(t+1)}}}{\sqrt{(t+1)}} p(\frac{\xi + \alpha^+(t+1)}{\sqrt{t+1}}, \log(t+1)) e^{-\frac{(\alpha^+)^2}{4}(t+1)} \\ &\quad + \frac{e^{\frac{\alpha^+}{2}\xi} |\phi'|^{\frac{1}{2}}}{(t+1)^{(\frac{1}{2} - \sigma)}} h(b_0, \ldots, b_k) + e^{\frac{\alpha^+}{2}\xi} |\phi'|^{\frac{1}{2}} \mathcal{O}((t+1)^{-(\frac{2m+1}{4} - \epsilon)}), \end{aligned} \tag{51}$$

where $p(\eta, \tau)$ is a polynomial in η that is bounded in τ.

Thus, the weight function W^+ combines with the Gaussian in the original (ξ, t) variables, resulting in exponential, temporal decay of terms corresponding to the center-stable subspace. We claim that the remaining two terms in the above expression both decay at least at the rate $(1+t)^{-(\frac{2m+1}{4}-\epsilon)}$. To show this, we will prove the following lemma.

Lemma 3.5 *The function h in equation (50) satisfies*

$$||\frac{W^+ h(b_0, \ldots, b_k)}{(t+1)^{\frac{1}{2}-\sigma}}||_{H^2} \leq \frac{C}{(1+t)^{\frac{2m+1}{4}}},$$

where the norm in the above estimate is taken in terms of the spatial variable ξ.

Proof Because h corresponds to the center manifold that was constructed in $H^2(m)$, we know that $h(\eta) \in H^2(m)$ and is bounded there. Furthermore, we may write

$$|W^+(\xi)| \leq \frac{C}{1 + e^{-\frac{1}{2}(\alpha^+ - \alpha^-)\xi}}.$$

We compute

$$\begin{aligned}
\int |W^+(\xi) h(\xi)|^2 d\xi &\leq e^{\frac{\tau}{2}} \int |W^+(e^{\frac{\tau}{2}}\eta - \alpha^+ e^{\tau}) h(\eta)|^2 d\eta \\
&\leq e^{\frac{\tau}{2}} \int \frac{|W^+(e^{\frac{\tau}{2}}\eta - \alpha^+ e^{\tau})|^2}{(1+\eta^2)^m} (1+\eta^2)^m |h(\eta)|^2 d\eta \\
&\leq C e^{\frac{\tau}{2}} \int_{-\infty}^{\alpha^+ e^{\frac{\tau}{2}} - \epsilon} \frac{(1+\eta^2)^m |h(\eta)|^2}{(1 + e^{-\frac{1}{2}(\alpha^+ - \alpha^-)(e^{\frac{\tau}{2}}\eta - \alpha^+ e^{\tau})})^2} d\eta \\
&\quad + C e^{\frac{\tau}{2}} \int_{\alpha^+ e^{\frac{\tau}{2}} - \epsilon}^{\infty} \frac{(1+\eta^2)^m |h(\eta)|^2}{(1+\eta^2)^m} d\eta \\
&\leq C e^{\frac{\tau}{2}} e^{-\epsilon(\alpha^+ - \alpha^-) e^{\frac{\tau}{2}}} ||h||^2_{L^2(m)} + C \frac{e^{\frac{\tau}{2}}}{(1 + (\alpha^+ e^{\frac{\tau}{2}} - \epsilon)^2)^m} ||h||^2_{L^2(m)} \\
&\leq \frac{C}{(t+1)^{\frac{2m-1}{2}}}.
\end{aligned}$$

Now take the square root and multiply by $(t+1)^{-(\frac{1}{2}-\sigma)}$ to obtain the desired decay rate. We note that the factor of $(t+1)^{\sigma}$ is taken care of by the fact that h is nonlinear (since $h(0) = 0$). Thus, every term contains a factor like $b_i b_j$, which decays at least as fast as $(t+1)^{-2\sigma}$. One can bound $\int |\partial_\xi^j (W^+(\xi) h(\xi))|^2 d\xi$, for $j = 1, 2$, in a similar manner. □

As a result, the seemingly higher order terms are the limiting factor in the decay of the far-field components of the perturbation, which proves theorem 1.1. Similar analysis can be carried out for the far-field component v^-. Note that the estimate in the theorem is in terms of the H^2 norm because $||W^\pm v^\pm(t)||_{H^2} \leq C||W^\pm w^\pm(\tau)||_{H^2(m)}$, and the asymptotic expansions for $w^\pm$ were carried out in the space $H^2(m)$.

4 Analysis of the near-field component

We turn now to the analysis of the near-field component. For convenience, we restate its equation of evolution:

$$\partial_t v^n = Av^n - \mathcal{N}(v^n, v^+, v^-, \xi) + F(v^+, v^-, \xi), \tag{52}$$

where A, $\mathcal{N}$, and F are defined in equations (30), (28) and (29), respectively.

In order to obtain the result stated in theorem 1.2, we will need to work in the Sobolev space $X = W^{1,p}$, rather than L^p. The reason for this is that we will use the theory of fractional Banach spaces, presented, for example, in [12]. If we work in L^p, the embedding theorems for these spaces only guarantee decay results in $W^{k,p}$ for $k < 2$. Since we ultimately want a result in H^2, this is not sufficient. Thus, we will work in the slightly smaller space $X = W^{1,p}$.

We must show that ϵ, in the definition of W^n, can be chosen so that the spectrum of A is contained entirely within the left half plane without touching the imaginary axis. In addition, ϵ must be chosen so that the presence of the factor $\frac{1}{W^n}$ in the functions F and $\mathcal{N}$ does not cause equation (52) to be ill-posed. The first issue is addressed in the following lemma.

Lemma 4.1 *For any $0 < \epsilon < \min(\alpha^+, |\alpha^-|)$, the operator A in equation (30) is sectorial on $W^{1,p}(\mathbb{R})$, $1 \leq p < \infty$. Furthermore, for a fixed ϵ, there exists some $0 < \delta < \pi/2$ and $0 < \omega < \min\left(\epsilon(\alpha^+ - \epsilon), -\epsilon(\epsilon + \alpha^-)\right)$ for which the sector*

$$S_{\delta,\omega} = \{\lambda \in \mathbb{C} : |arg(\lambda - \omega)| < \pi - \delta, \ \lambda \neq -\omega\} \tag{53}$$

is contained in the resolvent set of A.

Proof The operator A will be sectorial, regardless of the choice of ϵ, for the following reason. The operator ∂_ξ^2, with domain $D(\partial_\xi^2) = W^{3,p}(\mathbb{R})$, is sectorial in $W^{1,p}(\mathbb{R})$ for $1 \leq p < \infty$. This can be shown directly, using the explicit formula for the action of the associated resolvent operator, which is given, for example, in [12]. In addition, using the results of [13], one can show that ∂_ξ, with domain $W^{2,p}$, is ∂_ξ^2-bounded with ∂_ξ^2-bound zero. Furthermore, the coefficients in the operator A are smooth and uniformly bounded in ξ. Thus, if we consider A as a perturbation of ∂_ξ^2, we see that it is sectorial and the generator of an analytic semigroup [13].

Since analytic semigroups satisfy a spectral mapping theorem, the lemma will be proven if we show that the sector $S_{\delta,\omega}$ is contained in the resolvent set of A [13], *ie* we need not directly prove a bound on $(A - \lambda\mathbf{1})^{-1}$. We first show that the resolvent set of A on the space L^p contains the sector $S_{\delta,\omega}$ and then show how this result can be extended to the space $W^{1,p}$

Notice that $\lim_{\xi\to+\infty} A(\xi) = \partial_\xi^2 - (2\epsilon+\alpha^+)\partial_\xi + \epsilon(\epsilon-\alpha^+)$, and $\lim_{\xi\to-\infty} A(\xi) = \partial_\xi^2 + (2\epsilon-\alpha^-)\partial_\xi + \epsilon(\epsilon+\alpha^-)$. Hence, if $0 < \epsilon < \min(\alpha^+, |\alpha^-|)$, the essential spectrum of A will be contained in the left half of the complex plane and separated from the imaginary axis [12]. We must show that there are no eigenvalues that lie inside the sector $S_{\delta,\omega}$. This follows using ideas in [12] and [3].

The operator A is simply the operator L, defined in equation (6), when considered on the weighted space defined by $1/W^n$. We will analyze the point spectrum of L and relate it to that of A. The essential spectrum of L lies to the left of the parabolas $\text{Re}(\lambda) = -\text{Im}(\lambda)^2/(\alpha^\pm)^2$, which touch the imaginary axis at the origin. We will show that there are no eigenvalues that lie to the right of these parabolas.

Consider the eigenvalue equation for L,

$$u_{\xi\xi} + (c - f'(\phi(\xi)))u_\xi - \lambda u = 0. \tag{54}$$

Suppose that some λ lying to the right of the essential spectrum is an eigenvalue with eigenfunction u_λ. By analyzing the associated asymptotic equations, one can show that the function

$$v(\xi) = \frac{u_\lambda(\xi)}{|\phi'|^{\frac{1}{2}}}$$

decays exponentially to zero as $\xi \to \pm\infty$. In addition, v satisfies

$$v_{\xi\xi} + \left(\frac{\phi'''}{2\phi'} - \frac{3}{4}\frac{(\phi'')^2}{(\phi')^2} - \lambda\right) v = 0. \tag{55}$$

The linear operator in this equation is self-adjoint, and hence all eigenvalues are real. Therefore, we need only consider real eigenvalues for equation (54).

If $\lambda > 0$, then the maximum principle shows that $u \equiv 0$. This is because u cannot have a positive maximum, for which $u_{\xi\xi} \leq 0$, $u_\xi = 0$, and $u > 0$, and similarly u cannot have a negative minimum. Hence,

$$\sigma(L) = \sigma_{ess}(L) \subset \Omega \equiv \{\lambda : \text{Re}(\lambda) \leq -\frac{\text{Im}(\lambda)^2}{(\alpha^+)^2}\} \bigcup \{\lambda : \text{Re}(\lambda) \leq -\frac{\text{Im}(\lambda)^2}{(\alpha^-)^2}\}. \tag{56}$$

As mentioned above, the operator A is the operator L when considered on the exponentially weighted function space. In addition, the essential spectrum has been pushed off the imaginary axis into the left half plane. The remaining part of the spectrum consists of isolated eigenvalues of finite multiplicity. Suppose V_λ is an eigenfunction associated to such an eigenvalue. Then $u_\lambda = \text{sech}(\epsilon\xi)V_\lambda$ is a solution to equation (54). Since $\sigma(L)$ is given in equation (56), we know that any eigenvalue of A must also lie in Ω. Note that, for $\lambda = 0$, the two solutions of equation $AV = \lambda V$ are $\cosh(\epsilon\xi) = 1/W^n(\xi)$ and $\cosh(\epsilon\xi)\phi(\xi) = \phi(\xi)/W^n(\xi)$, neither of which are in L^p. Furthermore, any eigenvalue is isolated. Hence, there must exist some ω and δ such that $S_{\delta,\omega}$ is contained in the resolvent set of A.

To see that $S_{\delta,\omega}$ is also contained in the resolvent set of A when considered as an operator on $W^{1,p}$, notice that if there were any spectral elements in $S_{\delta,\omega}$, then they must be exponentially localized. This can be seen by analyzing the asymptotic limits of the associated eigenvalue equation. Hence, they would also be in the spectrum of A on the space L^p, contradicting the above result. $\square$

We turn now to the well-posedness of equation (52). Because the operator A is the generator of an analytic semigroup, we may use the tools associated to fractional Banach spaces, given, for example, in [12]. We now state those results which will be used below. The domain of A can be taken to be $D(A) = W^{3,p}(\mathbb{R}) \subset X$, where $X = W^{1,p}(\mathbb{R})$. For any $0 < \gamma < 1$, X^γ is the fractional Banach space associated to X and A. Using a slight generalization of the embedding theorem 1.6.1 in [12], $X^\gamma \subset W^{k,q}$ for $k < 3\gamma - \frac{1}{p} + \frac{1}{q}$, and $X^\gamma \subset C^\nu$ for $\nu < 3\gamma - \frac{1}{p}$. As a result, for $\gamma \in (0, 1)$ sufficiently large and $p = 1$,

$$||e^{tA}u||_\gamma \leq \frac{C_\gamma}{t^\gamma} e^{-\omega t} ||u||_{L^1} \tag{57}$$

$$||u||_{H^2} \leq C_\gamma ||u||_\gamma \tag{58}$$

$$||u||_{C^1} \leq C_\gamma ||u||_\gamma, \tag{59}$$

where $||\cdot||_\gamma$ represents the norm associated to the fractional Banach space X^γ.

Consider the integral form of solutions,

$$v^n(t) = e^{tA} v_0^n - \int_0^t e^{(t-s)A} \mathcal{N}(v^n, v^+, v^-)(s) ds + \int_0^t e^{(t-s)A} F(v^+(s), v^-(s)) ds. \tag{60}$$

We study the properties of the two integral terms in this equation, which will subsequently be used in a standard contraction mapping argument for the existence of solutions.

Proposition 4.2 *Let $v^+(s)$ and $v^-(s)$ be the solutions to the far-field equations, constructed in section 3. If $0 < \epsilon \leq \min(\alpha^+/2, |\alpha^-|/2)$, then*

$$\begin{aligned} || \int_0^t e^{(t-s)A} F(v^+(s), v^-(s)) ds ||_\gamma \leq & \int_0^t \frac{C(v_0^+) e^{-\omega(t-s)}}{(t-s)^\gamma} \frac{e^{-\frac{(\alpha^+)^2}{4}(s+1)}}{\sqrt{s+1}} ds \\ & + \int_0^t \frac{C(v_0^-) e^{-\omega(t-s)}}{(t-s)^\gamma} \frac{e^{-\frac{(\alpha^-)^2}{4}(s+1)}}{\sqrt{s+1}} ds \\ & + \int_0^t \frac{C(v_0^\pm) e^{-\omega(t-s)}}{(t-s)^\gamma} \frac{1}{(s+1)^{(\frac{2m+1}{4} - \epsilon)}} ds, \end{aligned} \tag{61}$$

where $C(v_0^\pm) \to 0$ as $||v_0^\pm||_{H^2(m)} \to 0$.

Proof Using equation (29), this integral term may be written

$$\begin{aligned}\int_0^t e^{(t-s)A}F(v^+(s),v^-(s))ds &= \int_0^t e^{(t-s)A}\frac{1}{W^n}[W^+_{\xi\xi}+W^+_{\xi}(c-f'(\phi))]v^+ds\\ &+\int_0^t e^{(t-s)A}\frac{1}{W^n}[W^-_{\xi\xi}+W^-_{\xi}(c-f'(\phi))]v^-ds\end{aligned} \tag{62}$$

We need to determine the rate at which the terms in brackets decay to zero at either $+\infty$ or $-\infty$. Notice that equation (26) implies

$$\begin{aligned}[W^+_{\xi\xi}+W^+_{\xi}(c-f'(\phi))] &= W^+\left[\frac{(\alpha^+)^2}{4}-\frac{(c-f'(\phi))^2}{4}+\frac{1}{2}f''(\phi)\phi'\right]\equiv W^+(\xi)B^+(\xi)\\ [W^-_{\xi\xi}+W^-_{\xi}(c-f'(\phi))] &= W^-\left[\frac{(\alpha^-)^2}{4}-\frac{(c-f'(\phi))^2}{4}+\frac{1}{2}f''(\phi)\phi'\right]\equiv W^-(\xi)B^-(\xi).\end{aligned} \tag{63}$$

In order to determine the asymptotic behavior of $B^{\pm}(\xi)$, we need some details about the dynamics if ϕ. The two-dimensional system of ODEs associated to equation (2) is

$$\begin{aligned}\phi' &= \psi\\ \psi' &= -(c-f'(\phi))\psi.\end{aligned}$$

As $\xi\to+\infty$, this system is given, to leading order, by

$$\begin{aligned}\phi' &= \psi\\ \psi' &= -\alpha^+\psi+f''(0)\phi\psi.\end{aligned}$$

The integral curve containing the point $(0,0)$ is $\psi(\phi)=-\alpha^+\phi+\frac{f''(0)}{2}\phi^2$. Substituting this expression into the equation for ϕ' and solving for ϕ, we obtain

$$\begin{aligned}\phi(\xi) &\sim \frac{Ke^{-\alpha^+\xi}}{1+K\frac{f''(0)}{2\alpha^+}e^{-\alpha^+\xi}}\\ &\sim Ke^{-\alpha^+\xi}-K^2\frac{f''(0)}{2\alpha^+}e^{-2\alpha^+\xi}\end{aligned}$$

as $\xi\to+\infty$, where K is some constant. A similar analysis leads to

$$\phi(\xi)\sim\phi^-+\tilde{K}e^{-\alpha^-\xi}-\tilde{K}^2\frac{f''(\phi^-)}{2\alpha^-}e^{-2\alpha^-\xi}$$

as $\xi\to-\infty$. Using this information, we find that

$$B^+(\xi)\sim\begin{cases}e^{-2\alpha^+\xi} \text{ as } \xi\to+\infty\\ \frac{(\alpha^+)^2}{4}-\frac{(\alpha^-)^2}{4} \text{ as } \xi\to-\infty\end{cases},$$

and

$$B^-(\xi) \sim \begin{cases} \frac{(\alpha^-)^2}{4} - \frac{(\alpha^+)^2}{4} \text{ as } \xi \to +\infty \\ e^{-2\alpha^-\xi} \text{ as } \xi \to -\infty \end{cases}.$$

Consider now the term on the right hand side of equation (62) involving v^+. That involving v^- may be treated similarly. Using equation (51), we have

$$\int_0^t e^{(t-s)A} \frac{W^+B^+}{W^n} v^+ ds = \int_0^t e^{(t-s)A} \frac{B^+}{W^n} \frac{|\phi'|^{\frac{1}{2}} e^{-\frac{\xi^2}{4}}}{\sqrt{(s+1)}} g(\xi,s) e^{-\frac{(\alpha^+)^2}{4}(s+1)} ds \\ + \int_0^t e^{(t-s)A} \frac{W^+B^+}{W^n} \mathcal{O}((t+1)^{-(\frac{2m+1}{4}-\epsilon)}) ds.$$

If $0 < \epsilon \le \min(\alpha^+/2, |\alpha^-|/2)$, then

$$\frac{|\phi'|}{W^n} \le C, \quad \text{and} \quad \frac{W^+B^+}{W^n} \in L^2,$$

and the result in equation (61) follows using the estimate in equation (57). $\square$

We now obtain a similar bound on the integral term involving the function $\mathcal{N}$.

Proposition 4.3 *If $0 < \epsilon < \frac{1}{2}(\alpha^+ - \alpha^-)$, then*

$$||\int_0^t e^{(t-s)A} \mathcal{N}(v^n, v^+, v^-)(s) ds||_\gamma \le \int_0^t \frac{C(v_0^\pm)_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma} ||v_n(s)||_\gamma^2 ds \\ + \int_0^t \frac{C(v_0^\pm)_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma} \frac{||v_n(s)||_\gamma}{(1+s)^{(\frac{2m+1}{4}-\epsilon)}} ds \\ + \int_0^t \frac{C(v_0^\pm)_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma} \frac{1}{(1+s)^{(\frac{2m+1}{4}-\epsilon)}} ds, \tag{64}$$

where $C(v_0^\pm) \to 0$ as $||v_0^\pm||_{H^2(m)} \to 0$.

Proof: Note that we assumed that the function f in equation (1) is in $C^2(\mathbb{R})$. This implies that the function $\mathcal{N} \in C^2(\mathbb{R})$, as well. In addition, since $v^\pm \in H^2(m)$ and $X^\gamma \subset C^1$ for $\gamma \in (3/4, 1)$, the functions $v^\pm$, v^n, and their derivatives are defined pointwise. The main idea is to write the function $\mathcal{N}$ as

$$\mathcal{N}(v^n, v^+, v^-) = \mathcal{N}_1(v^n, v^+, v^-) + \mathcal{N}_2(v^-, v^-)(W^n v^n)_\xi + \mathcal{N}_3(v^+, v^-), \tag{65}$$

where

$$\begin{aligned}\mathcal{N}_1(v^n, v^+, v^-) &= \frac{1}{W^n} N(\partial_\xi(W^+v^+ + W^-v^- + W^nv^n)) - \frac{1}{W^n} N(\partial_\xi(W^+v^+ + W^-v^-)) \\ &\quad - \frac{1}{W^n} DN(\partial_\xi(W^+v^+ + W^-v^-))\partial_\xi(W^nv^n) \\ \mathcal{N}_2(v^+, v^-)(W^nv^n)_\xi &= DN(\partial_\xi(W^+v^+ + W^-v^-))\frac{\partial_\xi(W^nv^n)}{W^n} \\ \mathcal{N}_3(v^+, v^-) &= \frac{1}{W^n} N(\partial_\xi(W^+v^+ + W^-v^-)) - \frac{1}{W^n} N(\partial_\xi(W^+v^+)) \\ &\quad - \frac{1}{W^n} N(\partial_\xi(W^-v^-)),\end{aligned} \tag{66}$$

and bound each of the three terms separately. To deal with the first term, note that

$$|\mathcal{N}_1(v^n, v^+, v^-)| \leq C(|\partial_\xi(W^+v^+ + W^-v^-)|)\frac{|\partial_\xi(W^nv^n)|^2}{|W^n|},$$

and hence

$$||\mathcal{N}_1(v^n, v^+, v^-)||_{L^1} \leq C(||W^+v^+||_{H^2}, ||W^-v^-||_{H^2})||v^n||^2_\gamma. \tag{67}$$

In the above we have used equation (58). To bound the third term, notice that for $f \in C^2(\mathbb{R})$ with $f(0) = 0$, we may write $f(x) = f'(0)x + \tilde{f}(x)$, where $\tilde{f} \in C^2(\mathbb{R})$ and $\tilde{f}(0) = \tilde{f}'(0) = 0$. Let a,$b \in \mathbb{R}$ such that $|a|$, $|b| \leq M$, and assume without loss of generality that $|a| \leq |b|$. We may write

$$\begin{aligned}|f(a+b) - f(a) - f(b)| &= |\tilde{f}(a+b) - \tilde{f}(a) - \tilde{f}(b)| \\ &= |\tilde{f}'(b)a + \frac{1}{2}\tilde{f}''(x)a^2 - \tilde{f}(a)| \\ &= |\left[\tilde{f}'(0) + \frac{1}{2}\tilde{f}''(y)b\right]a + \frac{1}{2}\tilde{f}''(x)a^2 - \left[\tilde{f}(0) + \tilde{f}'(0)a + \frac{1}{2}\tilde{f}''(z)a^2\right]| \\ &= |\frac{1}{2}\tilde{f}''(y)ab + \frac{1}{2}\left(\tilde{f}''(x) - \tilde{f}''(z)\right)a^2| \\ &\leq C_M|a||b|,\end{aligned}$$

where x, y, and $z \in (0, M)$. As a result,

$$|\mathcal{N}_3(v^+, v^-)| \leq C(|\partial_\xi(W^\pm v^\pm)|)\frac{|\partial_\xi(W^+v^+)||\partial_\xi(W^-v^-)|}{|W^n|}.$$

We can then bound

$$\begin{aligned}
||\mathcal{N}_3(v^+,v^-)||_{L^1} &\leq C\int \frac{|\partial_\xi(W^+v^+)||\partial_\xi(W^-v^-)|}{|W^n|}d\xi \\
&= C\int_{-\infty}^{0}\left(\frac{|\partial_\xi(W^+v^+)|}{|W^n|}\right)|\partial_\xi(W^-v^-)|d\xi + C\int_0^\infty\left(\frac{|\partial_\xi(W^-v^-)|}{|W^n|}\right)|\partial_\xi(W^+v^+)|d\xi \\
&\leq C||v^+||_{H^1}||W^-v^-||_{H^2(m)} + C||v^-||_{H^1}||W^+v^+||_{H^2(m)} \\
&\leq \frac{C(v_0^\pm)}{(t+1)^{(\frac{2m+1}{4}-\epsilon)}},
\end{aligned} \tag{68}$$

if $0<\epsilon<\frac{1}{2}(\alpha^+-\alpha^-)$. □

Using proposition 4.2, proposition 4.3, and a contraction mapping argument, one can prove the following.

Proposition 4.4 *Given any sufficiently small initial data in X^γ, there exists a $T>0$ and a solution to equation (52) satisfying $v^n(t)\in C^0([0,T),X^\gamma)$.*

The bound in propositions 4.2 and 4.3 will now be used to prove the main result of this section, theorem 1.2.

Proof of Theorem 1.2 Using the integral form of solutions, given in equation (60), and propositions 4.2 and 4.3, we have

$$\begin{aligned}
||v^n(t)||_\gamma \leq \frac{C}{t^\gamma}e^{-\omega t}||v_0^n||_{L^1} &+ \int_0^t \frac{C(v_0^\pm)_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}||v_n(s)||_\gamma^2 ds \\
&+ \int_0^t \frac{C(v_0^\pm)_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{||v_n(s)||_\gamma}{(1+s)^{(\frac{2m+1}{4}-\epsilon)}}ds + \int_0^t \frac{C(v_0^\pm)_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{1}{(1+s)^{(\frac{2m+1}{4}-\epsilon)}}ds \\
&+ \int_0^t \frac{C(v_0^+)e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{e^{-\frac{(\alpha^+)^2}{4}(s+1)}}{\sqrt{s+1}}ds + \int_0^t \frac{C(v_0^-)e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{e^{-\frac{(\alpha^-)^2}{4}(s+1)}}{\sqrt{s+1}}ds.
\end{aligned}$$

Define $|||v^n||| = \sup_{t_0<t<T}(t+1)^{(\frac{2m+1}{4}-\epsilon)}||v^n(t)||_\gamma$, for some $t_0>0$. Multiply the above equation by $(t+1)^{(\frac{2m+1}{4}-\epsilon)}$ to obtain

$$\begin{aligned} |||v^n||| &\leq C||v_0^n||_{L^1} + \sup_{t_0<t<T}(t+1)^{(\frac{2m+1}{4}-\epsilon)}\int_0^t \frac{C(v_0^{\pm})_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma(1+s)^{(\frac{2m+1}{2}-\epsilon)}}ds|||v^n|||^2 \\ &+ \sup_{t_0<t<T}(t+1)^{(\frac{2m+1}{4}-\epsilon)}\int_0^t \frac{C(v_0^{\pm})_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma(1+s)^{(\frac{2m+1}{2}-\epsilon)}}ds|||v^n||| \\ &+ \sup_{t_0<t<T}(t+1)^{(\frac{2m+1}{4}-\epsilon)}\int_0^t \frac{C(v_0^{\pm})_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{1}{(1+s)^{(\frac{2m+1}{4}-\epsilon)}}ds \\ &+ \sup_{t_0<t<T}(t+1)^{(\frac{2m+1}{4}-\epsilon)}\int_0^t \frac{C(v_0^{+})e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{e^{-\frac{(\alpha^+)^2}{4}(s+1)}}{\sqrt{s+1}}ds \\ &+ \sup_{t_0<t<T}(t+1)^{(\frac{2m+1}{4}-\epsilon)}\int_0^t \frac{C(v_0^{-})e^{-\omega(t-s)}}{(t-s)^\gamma}\frac{e^{-\frac{(\alpha^-)^2}{4}(s+1)}}{\sqrt{s+1}}ds. \end{aligned}$$

Therefore, we see that

$$|||v^n|||\left(1 - C(v_0^{\pm}) - M_1|||v^n|||\right) \leq M_2||v_0^n||_{L^1} + C(v_0^{\pm}).$$

If $v_0^{\pm}$ are chosen sufficiently small in $H^1(m)$ so that $C(v_0^{\pm}) \leq (1/4)/(4M_1+1)$, and T is chosen to be the maximal time such that $|||v^n||| \leq [1/4 - C(v_0^{\pm})]/M_1$, then we have that

$$|||v^n||| \leq 2M_2||v^n(0)||_{H^1} + 2C(v_0^{\pm}).$$

Therefore, if $||v^n(0)||_{H^1} \leq (1/4 - C_0^{\pm}))/(4M_1M_2)$, then the bound must hold for all $t \geq t_0$.

Finally, using the embedding $L^1 \subset L^2(m)$ and that of equation (58), we obtain the desired result. □

5 Example: Burgers Equation

We carry out the decomposition in detail for Burgers equation,

$$\partial_t u = \partial_x^2 u - \partial_x(u^2), \tag{69}$$

where $f(u) = u^2$. One can directly check that

$$\phi(\xi) = \frac{c}{1+e^{c\xi}}. \tag{70}$$

The equation of evolution for the full perturbation is

$$\partial_t v = \partial_\xi^2 v + c\tanh(\frac{c}{2}\xi)\partial_\xi v - (\partial_\xi v)^2, \tag{71}$$

with $\alpha^{\pm} = \pm c$, and the far-field weight functions are

$$W^{+}(\xi) = \frac{1}{1+e^{-c\xi}}$$
$$W^{-}(\xi) = \frac{1}{1+e^{+c\xi}}. \tag{72}$$

There is some freedom in the choice of the near-field weight function. For Burgers equation,

$$W^{n}(\xi) = \operatorname{sech}(\frac{c}{2}\xi) \tag{73}$$

is particularly convenient, due to the ease of the resulting calculations.

The equations of evolution for the three components of the perturbation are

$$\partial_t v^{+} = \partial_\xi^2 v^{+} + c\partial_\xi v^{+} - \frac{[(1+e^{-c\xi})\partial_\xi v^{+} + ce^{-c\xi}v^{+}]^2}{(1+e^{-c\xi})^3}, \tag{74}$$

$$\partial_t v^{-} = \partial_\xi^2 v^{-} - c\partial_\xi v^{-} - \frac{[(1+e^{+c\xi})\partial_\xi v^{-} - ce^{+c\xi}v^{-}]^2}{(1+e^{+c\xi})^3}, \tag{75}$$

$$\partial_t v^{n} = \partial_\xi^2 v^{n} - \frac{c^2}{4}v^{n} - \mathcal{N}(v^{+}, v^{-}, v^{n}, \xi), \tag{76}$$

with initial data

$$v^{\pm}(\xi,0) = (1-\operatorname{sech}(\frac{c}{2}\xi))v(\xi,0)$$
$$v^{n}(\xi,0) = v(\xi,0). \tag{77}$$

In equation (76),

$$\begin{aligned}\mathcal{N}(v^{+}, v^{-}, v^{n}, \xi) &= 2(a(\xi,t)+b(\xi,t))\partial_\xi v^{n} + c\tanh(\frac{c}{2}\xi)(a(\xi,t)+b(\xi,t))v^{n} \\ &+ \operatorname{sech}(\frac{c}{2}\xi)\left(-\frac{c}{2}\tanh(\frac{c}{2}\xi)v^{n} + \partial_\xi v^{n}\right)^2 + 2\cosh(\frac{c}{2}\xi)a(\xi,t)b(\xi,t),\end{aligned} \tag{78}$$

where

$$a(\xi,t) = \partial_\xi\left(\frac{v^{+}(\xi,t)}{(1+e^{-c\xi})}\right)$$
$$b(\xi,t) = \partial_\xi\left(\frac{v^{-}(\xi,t)}{(1+e^{+c\xi})}\right). \tag{79}$$

Note that for this example, F as in equation (29) is actually given by $F \equiv 0$. In order to apply the preceding analysis, we must show that assumption 1 is satisfied. We carry out the details only for the far field component v^{+}, as those of v^{-} are similar.

In terms of the scaling variables (η,τ), the equation for w is given by

$$\partial_\tau w = (\mathcal{L}-\sigma)w - N^+(\eta,\tau,w), \tag{80}$$

where

$$\begin{aligned} N^+(\eta,\tau,w) = &\frac{e^{-(\frac{1}{2}-\sigma)\tau}}{(1+e^{-ch(\eta,\tau)})}(\partial_\eta w)^2 + \frac{2ce^{\sigma\tau}e^{-ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^2} w\partial_\eta w \\ &+ \frac{c^2 e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}(w)^2, \end{aligned} \tag{81}$$

and $h(\eta,\tau) = \eta e^{\frac{\tau}{2}} - ce^\tau$.

Lemma 5.1 *Assumption 1, with*

$$R(\tau) = \int_0^\tau e^{\mathcal{L}(\tau-s)} N^+(w(s))ds, \tag{82}$$

is satisfied.

Proof Using equation (81), we have

$$\begin{aligned} ||R_1(\tau)-R_2(\tau)||_{H^2(m)} \leq &\int_0^\tau ||e^{\mathcal{L}(\tau-s)}\left(\frac{e^{-(\frac{1}{2}-\sigma)\tau}}{(1+e^{-ch(\eta,\tau)})}\right)\left((\partial_\eta w_1)^2-(\partial_\eta w_2)^2\right)||_{H^2(m)}ds \\ &+\int_0^\tau ||e^{\mathcal{L}(\tau-s)}\left(\frac{2ce^{\sigma\tau}e^{-ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^2}\right)(w_1\partial_\eta w_1 - w_2\partial_\eta w_2)||_{H^2(m)}ds \\ &+\int_0^\tau ||e^{\mathcal{L}(\tau-s)}\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\right)\left((w_1)^2-(w_2)^2\right)||_{H^2(m)}ds. \end{aligned} \tag{83}$$

We first present the details of the bound in $L^2(m)$ for the last term on the right hand side only, to indicate how one can deal with the factor $e^{(\frac{1}{2}+\sigma)\tau}$. The rest of the terms are similar. Using proposition 3.2, we have

$$\begin{aligned} &\int_0^\tau ||e^{\mathcal{L}(\tau-s)}\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\right)\left((w_1)^2-(w_2)^2\right)||_{L^2(m)}ds \leq \\ &\qquad\int_0^\tau \frac{C}{a(\tau-s)^{\frac{1}{4}}}||\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\right)(w_1+w_2)(w_1-w_2)||_{L^1(m)}ds \\ &\quad\leq \int_0^\tau \frac{C}{a(\tau-s)^{\frac{1}{4}}}||\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\right)(w_1+w_2)||_{L^2}||w_1-w_2||_{L^2(m)}ds. \end{aligned}$$

Notice that

$$||\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\right)(w_1+w_2)||_{L^2} = ||\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}\mathrm{sech}(\frac{c}{2}h(\eta,\tau))}{(1+e^{+ch(\eta,\tau)})}\right)(w_1+w_2)||_{L^2}$$
$$\leq C||e^{(\frac{1}{2}+\sigma)\tau}\mathrm{sech}(\frac{c}{2}h(\eta,\tau))(w_1+w_2)||_{L^2}.$$

We may bound this term using the following estimate. Define $\mathcal{B} = \{ce^{\frac{\tau}{2}} - \delta < \eta < ce^{\frac{\tau}{2}} + \delta\}$. Then, for example, considering the first term in $(w_1+w_2)^2 = w_1^2 + 2w_1w_2 + w_2^2$,

$$\int e^{(1+2\sigma)\tau}\mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))(w_1)^2 d\eta =$$
$$\int_{\mathcal{B}} e^{(1+2\sigma)\tau}\mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))(w_1)^2 d\eta + \int_{\mathbb{R}\setminus\mathcal{B}} e^{(1+2\sigma)\tau}\mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))(w_1)^2 d\eta$$
$$\leq \int_{\mathcal{B}} \frac{e^{(1+2\sigma)\tau}}{(1+\eta^2)^m}(1+\eta^2)^m(w_1)^2 d\eta + C\int_{\mathbb{R}\setminus\mathcal{B}} e^{(1+2\sigma)\tau}e^{-\delta e^{\frac{\tau}{2}}}(w_1)^2 d\eta$$
$$\leq C\frac{e^{(1+2\sigma)\tau}}{(1+(ce^{\frac{\tau}{2}}-\delta)^2)^m}||w_1||^2_{L^2(m)} + C||w_1||^2_{L^2}$$
$$\leq C||w_1||^2_{L^2(m)},$$

if $m > 1$ and $0 < \sigma < (m-1)/2$. A similar calculation can be made to bound

$$\int_0^\tau ||\partial_\eta e^{\mathcal{L}(\tau-s)}\left(\frac{c^2e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\right)\left((w_1)^2-(w_2)^2\right)||_{L^2(m)}ds,$$

by taking $\alpha = 1$ in proposition 3.2.

To bound the second derivative, we present the details for the first term on the right hand side of equation (83), since that term has the most derivatives and is therefore potentially problematic. Using equation (42), we have

$$\int_0^\tau ||\partial_\eta^2 e^{\mathcal{L}(\tau-s)}\left(\frac{e^{-(\frac{1}{2}-\sigma)\tau}}{(1+e^{-ch(\eta,\tau)})}\right)\left((\partial_\eta w_1)^2-(\partial_\eta w_2)^2\right)||_{L^2(m)}ds$$
$$\leq \int_0^\tau \frac{C}{a(\tau-s)^{\frac{3}{4}}}||\partial_\eta\left[\frac{e^{-(\frac{1}{2}-\sigma)\tau}}{(1+e^{-ch(\eta,\tau)})}\left((\partial_\eta w_1)^2-(\partial_\eta w_2)^2\right)\right]||_{L^1(m)}$$
$$\leq \int_0^\tau \frac{C}{a(\tau-s)^{\frac{3}{4}}}||e^{\sigma\tau}\mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))(\partial_\eta w_1+\partial_\eta w_2)(\partial_\eta w_1-\partial_\eta w_2)||_{L^1(m)}$$
$$+\int_0^\tau \frac{C}{a(\tau-s)^{\frac{3}{4}}}||e^{-(\frac{1}{2}-\sigma)\tau}\mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))(\partial_\eta w_1+\partial_\eta w_2)(\partial_\eta^2 w_1-\partial_\eta^2 w_2)||_{L^1(m)}$$
$$+\int_0^\tau \frac{C}{a(\tau-s)^{\frac{3}{4}}}||e^{-(\frac{1}{2}-\sigma)\tau}\mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))(\partial_\eta w_1-\partial_\eta w_2)(\partial_\eta^2 w_1+\partial_\eta^2 w_2)||_{L^1(m)}.$$

We may now complete the bound in a manner similar to that above, by splitting the region of integration in the $L^1(m)$ norm into $\mathcal{B}$ and its complement. In addition, we must use the fact that $w_{1,2} \in C^0([0,T], H^2(m))$ in order to deal with the second derivatives that appear in the above expression.

Therefore, we have shown that

$$\begin{aligned}\sup_{0\le\tau\le T} ||R_1(\tau) - R_2(\tau)||_{H^2(m)} &\le C\left(\sup_{0\le\tau\le T} ||w_1(\tau)||_{H^2(m)}\right)\left(\sup_{0\le\tau\le T} ||w_2(\tau)||_{H^2(m)}\right) \\ &\quad \times \left(\int_0^T \frac{1}{a(T-s)^{\frac{3}{4}}} ds\right)\left(\sup_{0\le\tau\le T} ||w_1(\tau) - w_2(\tau)||_{H^2(m)}\right) \\ &\le C(m, r_0, T) \sup_{0\le\tau\le T} ||w_1(\tau) - w_2(\tau)||_{H^2(m)},\end{aligned}$$

which proves the proposition. □

In order to illustrate the details of the center manifold calculation, we present them in the context of this example. First, we show that the nonlinearity in equation (48) does not affect the leading order dynamics on the manifold. In particular, we show that the nonlinearity N^+ satisfies

$$|\left(\int H_i(\eta) N^+(\eta, \sum_{j=0}^{k} \beta_j(\tau)\varphi_j(\eta) + h(\beta(\tau)))d\eta\right)| = \mathcal{O}(|\beta|^2) \tag{84}$$

uniformly in $\tau \ge 0$ for small $|\beta|$. We prove the estimate in equation (84) for only the last term in the definition of N^+. The rest are similar. We have

$$\begin{aligned}&|\int H_i(\eta)\left[\frac{c^2 e^{(\frac{1}{2}+2\sigma)\tau} e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\left(\sum_{j=1}^{k} \beta_j(\tau)\varphi_j(\eta) + h(\beta(\tau))\right)^2\right] d\eta| = \\ &|\sum_{m,n\le k} \int \left[\frac{2^i}{i!} e^{\frac{\eta^2}{4}} \partial_\eta(e^{\frac{\eta^2}{4}}) \frac{c^2 e^{(\frac{1}{2}+\sigma)\tau} \mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))}{4(1+e^{ch(\eta,\tau)})} C_{m,n}\beta_m\beta_n\varphi_m(\eta)\varphi_n(\eta)\right] d\eta| + \mathcal{O}(|\beta|^4) \\ &\le C \sum_{m,n\le k} \beta_m\beta_n \int |\left[p_i(\eta) e^{(\frac{1}{2}+\sigma)\tau} \mathrm{sech}^2(\frac{c}{2}h(\eta,\tau))\varphi_m(\eta)\varphi_n(\eta)\right]| d\eta + \mathcal{O}(|\beta|^4),\end{aligned}$$

where $p_i(\eta)$ is a polynomial of degree i and we have used the fact that $h(\beta) = \mathcal{O}(|\beta|^2)$. The integral in the last line may be bound independently of τ by splitting the region of integration into two parts, $\mathcal{B}$ and $\mathbb{R} \setminus \mathcal{B}$, as in the proof of lemma 5.1.

Next, we compute the dynamics on the center manifold and the resulting leading order expansion, for $m > 1/2$. Equation (48), for $i = 0$, is given by

$$\partial_\tau \beta_0 = -\sigma\beta_0 - G(\tau)\beta_0^2 + \mathcal{O}(|\beta|^4),$$

where

$$G(\tau) = \int \frac{e^{(\sigma-\frac{1}{2})\tau}}{1+e^{-ch(\eta,\tau)}}(\varphi_0'(\eta))^2 d\eta + \int \frac{2ce^{\sigma\tau}e^{-ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^2}\varphi_0'(\eta)\varphi_0(\eta)d\eta$$
$$+ \int \frac{c^2 e^{(\frac{1}{2}+\sigma)\tau}e^{-2ch(\eta,\tau)}}{(1+e^{-ch(\eta,\tau)})^3}\varphi_0^2(\eta)d\eta.$$

Using techniques similar to those above, one can show that $|G(\tau)| \leq M$ for all $\tau \geq 0$. Therefore, for $m > 1$ we find that

$$\beta_0(\tau) = \beta_0(0)e^{-\sigma\tau} + \mathcal{O}(e^{-(\sigma+\epsilon)\tau}),$$

for some $\epsilon > 0$, and so the asymptotic expansion of the far-field component at $+\infty$ is given by

$$W^+(\xi)v^+(\xi,t) = \frac{\beta_0(0)}{\sqrt{4\pi(t+1)}(1+e^{-c\xi})}e^{-\frac{(\xi+c(t+1))^2}{4(t+1)}} + \mathcal{O}((t+1)^{(\frac{2m+1}{4}-\epsilon)})$$
$$= \frac{\beta_0(0)}{\sqrt{4\pi(t+1)}(1+e^{+c\xi})}e^{-\frac{\xi^2}{4(t+1)}}e^{-\frac{c^2}{4}(t+1)} + \mathcal{O}((t+1)^{(\frac{2m+1}{4}-\epsilon)}).$$

We remark that, for the nonlinearity in Burgers equation, it is not necessary to work in $H^2(m)$ when analyzing the far-field components. One can check that the estimates in the proofs of propositions 4.2 and 4.3 can be made even if only $v^{\pm} \in H^1$. Therefore, for some nonlinearities, one can get by with slightly less smoothness in the initial data.

6 Summary

We have investigated the stability of traveling waves to scalar, viscous conservation laws by decomposing perturbations into three parts: two far-field components and one near-field component. The linear operators associated to the far-field components were determined by the asymptotic spatial limits of the original operator. By applying scaling variables to these operators, a spectral gap was created, thus allowing for the use of invariant manifold theory to determine the temporal decay rate of the far-field components. The linear operator associated to the near-field component had spectrum contained entirely within the left half plane, and so the associated semigroup decayed exponentially in time. The inhomogeneity in the equation was shown to be governed by the far-field components and determine the decay rate of the near-field component. As a result, the full perturbation was shown to decay at the same rate as the far-field components. This algebraic decay could be increased by requiring that the initial data lie in appropriate algebraically weighted spaces.

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