Levinson's theorem for Schrödinger operators with point interaction: a topological approach

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Abstract

In this note Levinson theorems for Schrödinger operators in $\mathbb{R}^n$ with one point interaction at 0 are derived using the concept of winding numbers. These results are based on new expressions for the associated wave operators.

1 Introduction

In [5] we proposed to look afresh at Levinson's theorem by viewing it as an identity between topological invariants, one associated with the bound state system, the other associated with the scattering system. Here we present the complete analysis for a class of solvable models in quantum mechanics which goes under the name of one point interaction at the origin (δ and δ' interactions). For these models we find novel expressions for the wave operators which allow us to prove our topological version of Levinson's theorem and to exhibit our ideas without recourse to techniques from algebraic topology.

What we prove is the following: Let H be a Schrödinger operator describing a δ interaction at 0 in $\mathbb{R}^n$ for $n \in \{1,2,3\}$ or a δ' interaction at 0 in $\mathbb{R}^1$ as discussed very carefully in [1, Chapter I]. The wave operator Ω_- for the couple $(H, -\Delta)$ can be rewritten in the form

$$\Omega_- - 1 = \varphi(A)\eta(-\Delta)P$$

where A is the generator of dilation in $\mathbb{R}^n$, P is an appropriate projection, and φ, η are continuous functions which have limits at the infinity points of the spectra $\sigma(A)$ and $\sigma(-\Delta)$, respectively, i.e. $\lim_{t\to\pm\infty} \varphi(t)$ and $\lim_{t\to 0,+\infty} \eta(t)$ exist. This allows us to define a continuous function $\Gamma : B \to \mathbb{C}^*$, from the boundary B of the square $(\sigma(-\Delta)\cup\{0,+\infty\}) \times (\sigma(A)\cup\{-\infty,+\infty\})$ by setting

$$\Gamma(\epsilon, a) = \varphi(a)\eta(\epsilon) + 1, \quad (\epsilon, a) \in B.$$

Since B is topologically isomorphic to a circle, Γ has a winding number $w(\Gamma)$ which is defined as the number of times $\Gamma(t)$ wraps (left) around $0 \in \mathbb{C}$ when t goes around B. This requires a choice of orientation for B which we fix as follows: B consists of four sides of the square one of which is $B_2 = (\sigma(-\Delta) \cup \{0, +\infty\}) \times \{+\infty\} \cong [0, +\infty]$; we choose on B the orientation which corresponds on B_2 to the natural order on $[0, +\infty]$. Our main result states a relation between this number and the number of bound states of H which is $\#\sigma_{\rm p}(H)$.

Theorem 1 (Levinson's theorem for δ and δ' interaction). *Let H be a Schrödinger operator defined by a δ-interaction at 0 in $\mathbb{R}^n$ with $n \in \{1, 2, 3\}$ or by a δ'-interaction at 0 in $\mathbb{R}^1$. Then*

$$w(\Gamma) = -\#\sigma_{\rm p}(H).$$

We prove this result in the next section by explicit verification. It is in fact a special case of an index theorem [2, 6].

Let us provide a few words of explanation for $w(\Gamma)$. Assuming differentiability it can be calculated by the integral

$$w(\Gamma) = \frac{1}{2\pi i} \int_B \Gamma^{-1} \mathrm{d}\Gamma \ .$$

To interpret this expression it is convenient to consider the four sides $B_1 = \{0\} \times (\sigma(A) \cup \{-\infty, +\infty\})$, $B_2 = (\sigma(-\Delta) \cup \{0, +\infty\}) \times \{+\infty\}$, $B_3 = \{+\infty\} \times (\sigma(A) \cup \{-\infty, +\infty\})$, $B_4 = (\sigma(-\Delta) \cup \{0, +\infty\}) \times \{-\infty\}$ of the square. Then

$$w(\Gamma) = \sum_{i=1}^{4} w_i, \quad w_i = \frac{1}{2\pi i} \int_{B_i} \Gamma_i^{-1} \mathrm{d}\Gamma_i$$

where Γ_i is the restriction of Γ to B_i. It can be observed for all the following examples that $\Gamma_2(-\Delta)P + P^\perp$ is equal to the scattering operator S and that $\Gamma_4(-\Delta) = 1$. This behaviour is not a coincidence but must hold in general [6]. In other words $w(\Gamma)$ contains as contribution the term

$$w_2 = \frac{1}{2\pi i} \int_0^{+\infty} \mathrm{tr}[S^*(\epsilon) S'(\epsilon)] \mathrm{d}\epsilon \ ,$$

where tr is the usual trace on the compact operators of $L^2(\mathbb{S}^{n-1})$. Now w_2 is the integral over the time delay one usually finds in Levinson's theorem. In dimension $n = 2$ one even has $\Gamma_i(-\Delta) = 1$ for all $i \neq 2$ and so there is no other contribution to $w(\Gamma)$. But for most of the other examples, the low and the high energy behaviour of the wave operator is non-trivial leading to contributions of Γ_1 and Γ_3 to the winding number.

Expressions relating w_2 to the number of bound states for one point interactions can be found in the physics literature, see e.g. [3], usually providing different arguments for the occurences of corrections. Here these corrections appear as the missing parts (w_1 and w_3) of a winding number calculation. This not only gives a full and coherent explanation of these corrections but also makes clear that Levinson's theorem is of topological nature. In a future publication [6], we shall show that a similar picture holds for general Schrödinger operators and that the corrections added in such context can also be fully explained.

The proof the theorem as well as more explanations on the underlying constructions are given in the following sections. New formulas for the wave operators are introduced successively for the δ-interaction in $\mathbb{R}^n$ for $n = 3$, 2 and 1, and then for the δ'-interaction in $\mathbb{R}^1$.

2 Schrödinger operators with point interaction

A so-called Schrödinger operator with a δ or a δ' interaction at the origin can be defined as a self-adjoint extension of the restriction of the Laplace operator to a suitable subspace of $L^2(\mathbb{R}^n)$. These operators are discussed in great detail in [1, Chapter I].

Common to all the self-adjoint extensions H we look at here is that

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ac}}(H) = [0, \infty) \ , \quad \sigma_{\text{sc}}(H) = \emptyset \ .$$

The point spectrum of H, however, depends on both the extension and the dimension. The main feature of these models is that the wave operators defined by

$$\Omega_\pm := s - \lim_{t \to \pm\infty} e^{itH} \, e^{-it(-\Delta)}$$

can be explicitly computed. We recall their explicit form, which depends on the extension and the dimension, further down. A common property shared by these wave operators is that they act non-trivially only on a small subspace of $L^2(\mathbb{R}^n)$. Denoting by P the orthogonal projection onto that subspace we have the possibility that $P = P_0$ the orthogonal projection onto the rotation invariant subspace of $L^2(\mathbb{R}^n)$, or, for $n = 1$, $P = P_1$, the orthogonal projection P_1 onto the antisymmetric functions of $L^2(\mathbb{R})$. Recall that the ranges of P_0 or P_1 are invariant under the usual Fourier transform in $L^2(\mathbb{R}^n)$.

The dilation group in $\mathbb{R}^n$ and its generator A play an important role. We recall that its action on $\psi \in L^2(\mathbb{R}^n)$ is given by $\big(U(\theta)\psi\big)(x) = e^{n\theta/2}\psi(e^\theta x)$ for all $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$, and that its self-adjoint generator A has the form $\frac{1}{2i}(Q \cdot \nabla + \nabla \cdot Q)$ on $C_0^\infty(\mathbb{R}^n)$. The group leaves the range of P_0 and P_1 invariant. We refer to [4] for more information about this group and for a detailed description of the Mellin transform $\mathcal{M}$ which is a unitary transformation diagonalizing A.

2.1 The dimension $n = 3$

The operator $-\Delta$ defined on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ has deficiency indices $(1, 1)$ so that all its self-adjoint extensions H_α can be parametrized by an index α belonging to $\mathbb{R} \cup \{\infty\}$. This parameter determines a certain boundary condition at 0 but $-4\pi\alpha$ also has a physical interpretation as the inverse of the scattering length. The choice $\alpha = \infty$ corresponds to the free Laplacian $-\Delta$. H_α has a single bound state for $\alpha < 0$ at energy $-(4\pi\alpha)^2$ but no point spectrum for $\alpha \in [0, \infty]$. Furthermore, the action of the wave operator Ω_-^α for the couple $(H_\alpha, -\Delta)$ on any $\psi \in L^2(\mathbb{R}^3)$ is given by:

$$\big[(\Omega_-^\alpha - 1)\psi\big](x) = s - \lim_{R \to \infty} (2\pi)^{-3/2} \int_{k \le R} k^2 \, \mathrm{d}k \int_{\mathbb{S}^2} \mathrm{d}\omega \, \frac{e^{ik|x|}}{(4\pi\alpha - ik)|x|} \, \hat{\psi}(k\omega) \ ,$$

where $\hat{\psi}$ is the 3-dimensional Fourier transform of ψ.

Now, one first easily observes that $\Omega_-^\alpha - 1$ acts trivially on the orthocomplement of the range of P_0. One may also notice that it can be rewritten as a product of three operators, *i.e.* $\Omega_-^\alpha - 1 = T_2 T_1 P_0$ with

$$T_1 = \frac{2i\sqrt{-\Delta}}{4\pi\alpha - i\sqrt{-\Delta}}$$

and

$$\big[T_2\psi\big](x) = s - \lim_{R\to\infty} (2\pi)^{-1/2} \int_{k\leq R} k^2\,\mathrm{d}k\, \frac{e^{ik|x|}}{ik|x|}\,\hat{\psi}(k)\ .$$

Finally, let us observe that T_1+1 is simply equal to the scattering operator $S^\alpha := (\Omega_+^\alpha)^*\,\Omega_-^\alpha$ and that T_2 is invariant under the action of the dilation group: $U(\theta)T_2U(-\theta) = T_2$. Thus, T_2 can be diagonalized in the spectral representation of A. A direct calculation using the expression for $\mathcal{M}$ from [4] leads to the following result:

$$\Omega_-^\alpha - 1 = \Big[\frac{1}{2}\Big(1 + \tanh(\pi A) - i\big(\cosh(\pi A)\big)^{-1}\Big)\Big] \Big\{\frac{2i\sqrt{-\Delta}}{4\pi\alpha - i\sqrt{-\Delta}}\Big\}\, P_0\ .$$

So let us set

$$r(\xi) = -\tanh(\pi\xi) + i\big(\cosh(\pi\xi)\big)^{-1}\ .$$

and

$$s^\alpha(\xi) = \frac{4\pi\alpha + i\sqrt{\xi}}{4\pi\alpha - i\sqrt{\xi}}$$

As a consequence of the expression for $\Omega_-^\alpha - 1$ the functions Γ_i and their contributions to the winding number are given by

	Γ_1	Γ_2	Γ_3	Γ_4	w_1	w_2	w_3	w_4	$w(\Gamma)$
$\alpha < 0$	1	s^α	r	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1
$\alpha = 0$	r	-1	r	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
$\alpha > 0$	1	s^α	r	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\alpha = \infty$	1	1	1	1	0	0	0	0	0

and we see that $w(\Gamma)$ corresponds to minus the number of bound states of H_α.

2.2 The dimension $n = 2$

The situation for $n = 2$ parallels that of $n = 3$ in that the operator $-\Delta$ defined on $C_0^\infty(\mathbb{R}^2\setminus\{0\})$ has deficiency indices $(1,1)$ and that all its self-adjoint extensions H_α can be parametrized by an index α belonging to $\mathbb{R}\cup\{\infty\}$. Again α determines a certain boundary condition at 0 and $-2\pi\alpha$ has a physical interpretation as the inverse of the scattering length. Also here the choice $\alpha = \infty$ corresponds to the free Laplacian $-\Delta$. But in two dimensions H_α has a single eigenvalue for all $\alpha \in \mathbb{R}$. The wave operator Ω_-^α for the couple $(H_\alpha, -\Delta)$ acts on $\psi \in L^2(\mathbb{R}^2)$ as

$$\big[(\Omega_-^\alpha - 1)\psi\big](x) = s - \lim_{R\to\infty} (2\pi)^{-1} \int_{k\leq R} k\,\mathrm{d}k \int_{\mathbb{S}^1} \mathrm{d}\omega\, \frac{i\pi/2}{2\pi\alpha - \Psi(1) + \ln(k/2i)} H_0^{(1)}(k|x|)\,\hat{\psi}(k\omega)\ ,$$

where $\hat{\psi}$ is the 2-dimensional Fourier transform of ψ, $H_0^{(1)}$ denotes the Hankel function of the first kind and order zero, and Ψ is the digamma function. A similar calculation as above yields that this wave operator can be rewritten as

$$\Omega_-^\alpha - 1 = \left[\frac{1}{2}\big(1+\tanh(\pi A/2)\big)\right]\left\{\frac{i\pi}{2\pi\alpha - \Psi(1) + \ln(\sqrt{-\Delta}/2) - i\frac{\pi}{2}}\right\} P_0 \ .$$

Thus we get the following results for the functions Γ_i and their contribution to the winding number. Let us set

$$r(\xi) = -\tanh(\pi\xi/2)$$

and

$$s^\alpha(\xi) = \frac{2\pi\alpha - \Psi(1) + \ln(\sqrt{\xi}/2) + i\pi/2}{2\pi\alpha - \Psi(1) + \ln(\sqrt{\xi}/2) - i\pi/2} \ .$$

Then

	Γ_1	Γ_2	Γ_3	Γ_4	w_1	w_2	w_3	w_4	$w(\Gamma)$
$\alpha \in \mathbb{R}$	1	s^α	1	1	0	-1	0	0	-1
$\alpha = \infty$	1	1	1	1	0	0	0	0	0

and we see that $w(\Gamma)$ corresponds to minus the number of bound states of H_α.

2.3 The dimension $n = 1$ with δ-interaction

The classification of self-adjoint extensions defining point interactions is more complicated in one dimension. Also here one starts with the Laplacian restricted to a subspace of functions which vanish at 0 but there are more possibilities. We refer the reader to [1] for the details considering in this section the family of extensions H_α called δ-interactions. Here the parameter $\alpha \in \mathbb{R}\cup\{\infty\}$ of the extension describes the boundary condition $\psi'(0_+) - \psi'(0_-) = \alpha\psi(0)$ which can be formally interpreted as arising from a potential $V = \alpha\delta$ where δ is the Dirac δ-function at 0. The extension for $\alpha = 0$ is the free Laplace operator and the extension $\alpha = \infty$ the Laplacian (or rather the direct sum of two Laplacians) with Dirichlet boundary conditions at 0. The extensions H_α have a single eigenvalue if $\alpha < 0$ and do not have any eigenvalue if $\alpha \in [0,\infty]$. Furthermore, the wave operator Ω_-^α for the couple $(H_\alpha, -\Delta)$ acts on $\psi \in L^2(\mathbb{R})$ as

$$\big[(\Omega_-^\alpha - 1)\psi\big](x) = s - \lim_{R\to\infty} (2\pi)^{-1/2} \int_{k\le R} \mathrm{d}k \int_{\mathbb{S}^0} \mathrm{d}\omega \, \frac{-i\alpha}{2k+i\alpha} e^{ik|x|}\, \hat{\psi}(k\omega) \ ,$$

where $\hat{\psi}$ denotes the 1-dimensional Fourier transform of ψ. By rewriting this operator in terms of $-\Delta$ and A one obtains:

$$\Omega_-^\alpha - 1 = \left[\frac{1}{2}\Big(1+\tanh(\pi A) + i\big(\cosh(\pi A)\big)^{-1}\Big)\right]\left\{\frac{-2i\alpha}{2\sqrt{-\Delta}+i\alpha}\right\} P_0 \ .$$

Thus we get the following results for the functions Γ_i and their contribution to the winding number. Let us set

$$r(\xi) = -\tanh(\pi\xi) - i\big(\cosh(\pi\xi)\big)^{-1}$$

and

$$s^\alpha(\xi) = \frac{2\sqrt{\xi} - i\alpha}{2\sqrt{\xi} + i\alpha} \ .$$

Then

	Γ_1	Γ_2	Γ_3	Γ_4	w_1	w_2	w_3	w_4	$w(\Gamma)$
$\alpha < 0$	r	s^α	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1
$\alpha = 0$	1	1	1	1	0	0	0	0	0
$\alpha > 0$	r	s^α	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\alpha = \infty$	r	-1	r	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0

and we see that $w(\Gamma)$ corresponds to minus the number of bound states of H_α.

2.4 The dimension $n = 1$ with δ'-interaction

Let us finally consider the family of extensions called δ'-interaction. As in [1] we use $\beta \in \mathbb{R} \cup \{\infty\}$ for the parameter of the self-adjoint extension which describes the boundary condition $\psi(0_+) - \psi(0_-) = \beta\psi'(0)$. This can be formally interpreted as arising from a potential $V = \beta\delta'$. The extension for $\beta = 0$ is the free Laplace operator and the extension $\beta = \infty$ the Laplacian (or rather the direct sum of two Laplacians) with Neumann boundary conditions at 0. The operator H_β possesses a single eigenvalue if $\beta < 0$ of value $-4\beta^{-2}$ but no eigenvalue if $\beta \in [0, \infty]$. The wave operator Ω_-^β for the couple $(H_\beta, -\Delta)$ acts on any $\psi \in L^2(\mathbb{R})$ as

$$\big[(\Omega_-^\beta - 1)\psi\big](x) = s - \lim_{R\to\infty} (2\pi)^{-1/2} \int_{k\leq R} \mathrm{d}k \int_{\mathbb{S}^0} \mathrm{d}\omega \, \frac{i\beta k\omega}{2 - i\beta k} \vartheta(x,k)\, \hat{\psi}(k\omega) \ ,$$

where $\hat{\psi}$ denotes the 1-dimensional Fourier transform of ψ and with $\vartheta(x,k) = e^{ikx}$ for $x > 0$ and $\vartheta(x,k) = -e^{-ikx}$ for $x < 0$.

It is easily observed that the action of $\Omega_-^\beta - 1$ on any symmetric (*i.e.* even) function is trivial. Moreover, this operator can be rewritten as

$$\Omega_-^\beta - 1 = \left[\frac{1}{2}\Big(1 + \tanh(\pi A) - i\big(\cosh(\pi A)\big)^{-1}\Big)\right] \left\{\frac{2i\beta\sqrt{-\Delta}}{2 - i\beta\sqrt{-\Delta}}\right\} P_1 \ .$$

We get the following results for the functions Γ_i and their contribution to the winding number. Let us set

$$r(\xi) = -\tanh(\pi\xi) + i\big(\cosh(\pi\xi)\big)^{-1}$$

and

$$s^\beta(\xi) = \frac{2 + i\beta\sqrt{\xi}}{2 - i\beta\sqrt{\xi}} \ .$$

Then

	Γ_1	Γ_2	Γ_3	Γ_4	w_1	w_2	w_3	w_4	$w(\Gamma)$
$\beta < 0$	1	s^β	r	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1
$\beta = 0$	1	1	1	1	0	0	0	0	0
$\beta > 0$	1	s^β	r	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\beta = \infty$	r	-1	r	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0

which verifies again the theorem.

Remark 1. It can be observed that the fonctions φ and η for the operators $\Omega_-^\alpha - 1$ are always given by $\frac{1}{2}(1-r)$ and $s^\alpha - 1$ respectively. A straightforward calculation shows that the operators $\Omega_+ - 1$ can also be rewritten in the form $\varphi(A)\eta(-\Delta)P$ with $\varphi = \frac{1}{2}(1+r)$ and $\eta = \overline{s^\alpha - 1}$.

The explicit formulae obtained for the wave operator allow us to observe a symmetry among the models in one and three dimensions. We see exactly the same formulas for φ and η in the case of the δ-interaction in $n = 3$ and the δ'-interaction in $n = 1$, provided we set $2\pi\alpha = \beta^{-1}$. From the C^*-algebraic point of view the fact that φ and η are the same means that the wave operators for the models are just two different representations of the same element of an abstract C^*-algebra.

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