Invariants of isospectral deformations and spectral rigidity

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Abstract

We introduce a notion of weak isospectrality for continuous deformations. Let us consider the Laplace-Beltrami operator on a compact Riemannian manifold with boundary with Robin boundary conditions. Given a Kronecker invariant torus Λ of the billiard ball map with a Diophantine vector of rotation we prove that certain integrals on Λ involving the function in the Robin boundary conditions remain constant under weak isospectral deformations. To this end we construct continuous families of quasimodes associated with Λ. We obtain also isospectral invariants of the Laplacian with a real-valued potential on a compact manifold for continuous deformations of the potential. As an application we prove spectral rigidity in the case of Liouville billiard tables of dimension two.

1 Introduction

This is a part of a series of papers (cf. [13, 14, 15]) concerned with spectral rigidity for compact Liouville billiard tables of dimensions $n \geq 2$. The general strategy is first to find a list of spectral invariants and then to prove for certain manifolds that these invariants imply spectral rigidity. The aim of this paper is to present a simple idea of how quasimodes can be used in inverse spectral problems. This idea works well for isospectral deformations whenever *continuous* with respect to the parameter of the deformation *quasimodes* can be constructed for the corresponding eigenvalue problem. Given a compact billiard table (X, g) with a smooth Riemannian metric g and the corresponding Laplace-Beltrami operator on it, we consider continuous deformations either of the function K in the Robin boundary condition or of a real-valued potential V on X. To construct quasimodes we assume that there is an exponent B^m, $m \geq 1$, of the corresponding billiard ball map B which admits an invariant Kronecker torus Λ with a Diophantine vector of rotation. This means that Λ is a Lagrangian submanifold of the coball bundle of the boundary which is diffeomorphic to the torus $\mathbb{T}^{n-1}$ and invariant with respect to B^m and such that the restriction of B^m to Λ is smoothly conjugated to a rotation with a constant Diophantine vector. If the deformation is isospectral we prove that certain integrals on Λ of the function K or of the potential V remain constant under the deformation. In the case of Liouville billiard tables we treat these integrals as values of a suitable Radon transform. Then the spectral rigidity follows from the injectivity of the Radon transform. Liouville billiard tables of dimension two have been studied in [13]. Liouville billiard tables of dimension $n \geq 2$ are introduced in [15], where the integrability of the corresponding billiard ball map is obtained using a simple variational principal. The injectivity of the Radon transform in higher dimensions is investigated in [14].

A billiard table (X, g) is a smooth compact Riemannian manifold of dimension $\dim X = n \geq 2$ equipped with a smooth Riemannian metric g and with a C^∞ boundary $\Gamma := \partial X \neq \emptyset$. The corresponding continuous dynamical system on it is the "billiard flow" which induces a discrete dynamical system B on an open subset of the coball bundle of Γ called billiard ball map (see Sect. 2.1). Let Δ be the "positive" Laplace-Beltrami operator on (X, g). Given a real-valued function $K \in C(\Gamma, \mathbb{R})$, we consider the operator Δ with domain

$$D := \left\{ u \in H^2(X) : \frac{\partial u}{\partial \nu}|_\Gamma = K u|_\Gamma \right\},$$

where $\nu(x)$, $x \in \Gamma$, is the inward unit normal to Γ with respect to the metric g. We denote this operator by $\Delta_{g,K}$. It is a selfadjoint operator in $L^2(X)$ with discrete spectrum

$$\mathrm{Spec}\, \Delta_{g,K} := \{\lambda_1 \leq \lambda_2 \leq \cdots\},$$

where each eigenvalue $\lambda = \lambda_j$ is repeated according to its multiplicity, and it solves the spectral problem

$$\left\{ \begin{array}{rcl} \Delta u & = & \lambda u \quad \text{in X}, \\ \frac{\partial u}{\partial \nu}|_\Gamma & = & K u|_\Gamma . \end{array} \right. \tag{1.1}$$

1.1 *Invariants of isospectral families*

Fix $\ell \in \mathbb{N}$ and consider a continuous family of C^ℓ real-valued functions K_t, $t \in [0,1]$, which means that the map $[0,1] \ni t \mapsto K_t$ is continuous in $C^\ell(\Gamma, \mathbb{R})$. To simplify the notations we denote by Δ_t the corresponding operators Δ_{g,K_t}. These operators are said to be isospectral if

$$\forall\, t \in [0,1]\,,\ \mathrm{Spec}\,(\Delta_t) \ = \ \mathrm{Spec}\,(\Delta_0)\ . \tag{1.2}$$

We are going to introduce a weaker notion of isospectrality. Fix two positive constants c and $d > 1/2$, and consider the union of infinitely many disjoint intervals

(H$_1$) $\mathcal{I} \ := \ \cup_{k=1}^{\infty} \, [a_k, b_k]\ , \quad 0 < a_1 < b_1 < \cdots < a_k < b_k < \cdots\ ,$ *such that* $\lim a_k \ = \ \lim b_k \ = \ +\infty\,,\ \lim(b_k - a_k) \ = \ 0\,,$ *and* $a_{k+1} - b_k \ \geq \ c b_k^{-d}$ *for any* $k \geq 1$.

We impose the following "weak isospectral assumption":

(H$_2$) *There is* $a \gg 1$ *such that* $\forall\, t \in [0,1]\,,\ \mathrm{Spec}\,(\Delta_t) \cap [a, +\infty) \ \subset \ \mathcal{I}$.

Using the asymptotic of the eigenvalues λ_j as $j \to \infty$ we shall see in Sect. 2 that the condition (H$_1$)-(H$_2$) is "natural" for any $d > n/2$ which means that the usual isospectral assumption implies (H$_1$)-(H$_2$) for any such d and any $c > 0$.

We suppose also that there is an integer $m \geq 1$ such that the map $P = B^m$, B being the billiard ball map, admits an invariant Kronecker torus with Diophantine vector of rotation, namely,

(H$_3$) *There exists a positive integer* m *and an embedded submanifold* Λ *of* $B^*\Gamma$ *diffeomorphic to* $\mathbb{T}^{n-1}$ *and invariant with respect to* $P = B^m$ *such that the restriction of* P *to* Λ *is* C^∞ *conjugated to the rotation* $R_{2\pi\omega}(\varphi) = \varphi - 2\pi\omega$ *(mod* 2π*) in* $\mathbb{T}^{n-1}$*, where* ω *is Diophantine.*

We take $m \geq 1$ to be the smallest positive number with this property, then $P = B^m$ is just the return map along the broken bicharacteristic flow near Λ. Recall that $\omega \in \mathbb{R}^{n-1}$ is Diophantine if there is $\kappa > 0$ and $\tau > 0$ such that

$$\forall\, (k, k_n) \in \mathbb{Z}^n,\ k = (k_1, \ldots, k_{n-1}) \neq 0\ : \quad |\langle \omega, k\rangle + k_n| \ \geq \ \frac{\kappa}{(\sum_{j=1}^{n-1} |k_j|)^\tau}\,. \tag{1.3}$$

Then $\Lambda \subset B^*\Gamma$ is Lagrangian (see [7], Sect. I.3.2). Let $\pi_\Gamma : T^*\Gamma \to \Gamma$ be the canonical projection and denote by $d\mu$ the measure associated to a Leray form at Λ. Given $(x,\xi) \in B^*\Gamma$, we denote by $\xi^+ \in T^*_x X$ the corresponding outgoing unit co-vector and by $\theta = \theta(x,\xi) \in (0, \pi/2]$ the angle between ξ^+ and $T^*_x\Gamma$ in $T^*_x X$ (see Sect. 2.1).

Fix $d > 1/2$ and $\tau \geq 1$ and set $\ell = ([2d]+1)([\tau]+n)+2n+2$, where $[p]$ stands for the entire part of the real number p. In what follows d will be the exponent in (H_1), and τ the exponent in the Diophantine condition (1.3). Our main result is:

Theorem 1.1 *Let Λ be an invariant Kronecker torus of $P = B^m$ with a vector of rotation $2\pi\omega$ satisfying the Diophantine condition (1.3). Let*

$$[0,1] \ni t \mapsto K_t \in C^\ell(\Gamma, \mathbb{R}) \ ,$$

be a continuous family of real-valued functions on Γ such that Δ_t satisfy $(H_1) - (H_2)$. Then

$$\forall\, t \in [0,1], \quad \sum_{j=0}^{m-1} \int_\Lambda \frac{K_t \circ \pi_\Gamma}{\sin\theta} \circ B^j \, d\mu \ = \ \sum_{j=0}^{m-1} \int_\Lambda \frac{K_0 \circ \pi_\Gamma}{\sin\theta} \circ B^j \, d\mu \, . \qquad (1.4)$$

Before giving applications of the theorem we would like to make some comments on it. It is inspired by a result of Guillemin and Melrose [5, 6]. They consider a connected clean submanifold Λ of fixed points of $P = B^m$, $m \geq 2$, satisfying the so called "non-coincidence" condition. Let T_Λ be the common length of the closed broken geodesics with m vertexes issuing from Λ. The "non-coincidence" condition means that these geodesics are the only closed generalized geodesics in X of length T_Λ. Under this condition, Guillemin and Melrose prove that if K_j, $j = 0, 1$, are two real-valued C^∞ functions on Γ such that $\mathrm{Spec}\,(\Delta_{g,K_1}) = \mathrm{Spec}\,(\Delta_{g,K_0})$, then (1.4) holds for $t = 1$. In the case when $X \subset \mathbb{R}^2$ is the interior of an ellipse Γ they obtain an infinite sequence of confocal ellipses $\Gamma_j \subset X$ tending to Γ such that the corresponding invariant circles Λ_j of B satisfy the non-coincidence condition. In particular, (1.4) holds for $t = 1$ and $m = 1$ on each Λ_j. As a consequence they obtain in [5] spectral rigidity of (1.1) in the case of the ellipse for C^∞ functions K which are invariant with respect to the symmetries of the ellipse. The main tool in the proof is the trace formula for the wave equation with Robin boundary conditions in X (see [6]). This result was generalized in [13] for two-dimensional Liouville billiard tables of classical type.

There is no hope to apply the wave-trace formula in our situation. An invariant Kronecker torus Λ of the billiard ball map B can always be approximated with periodic points of $P = B^m$ using a variant of the Birkhoff-Lewis theorem and a "Birkhoff normal form" of P near Λ. Unfortunately, we do not know if the corresponding closed broken geodesics are non-degenerated. Moreover, it is impossible to verify in general the non-coincidence condition.

We propose a simple idea which relies on a quasimode construction. It is natural to use quasimodes for this kind of problems since quasi-eigenvalues are close to eigenvalues and they contain a lot of geometric information. In order to prove (1.4), we construct *continuous* with respect to $t \in [0,1]$ quasimodes for Δ_t of order $N = [2d]+1$, $[2d]$ being the entire part of $2d$. The quasi-eigenvalues (see Theorem 2.2) are of the form $\mu_q(t)^2$, $q \in \mathcal{M} \subset \mathbb{Z}^n$, where

$$\mu_q(t) = \mu_q^0 + c_{q,0} + c_{q,1}(t)(\mu_q^0)^{-1} + \cdots + c_{q,N}(t)(\mu_q^0)^{-N} \ ,$$

μ_q^0 and $c_{q,0}$ are independent of t, $\lim_{|q|\to\infty}\mu_q^0 = +\infty$, and $c_{q,j}$, $q\in\mathcal{M}$, is an uniformly bounded sequence of continuous functions in $t\in[0,1]$. The function $c_{q,1}$ has the form

$$c_{q,1}(t) = c'_{q,1} + c''_1 \int_\Lambda \sum_{j=0}^{m-1} \frac{K_t\circ\pi_\Gamma}{\sin\theta}\, d\mu\,,$$

where $c'_{q,1}$ and $c''_1 \neq 0$ are independent of t and c''_1 does not depend on q either. Moreover, there is $C>0$ such that for any $q\in\mathcal{M}\subset\mathbb{Z}^n$ and $t\in[0,1]$, there is $\lambda_q(t)\in\mathrm{Spec}\,(\Delta_t)$ such that

$$|\lambda_q(t) - \mu_q(t)^2| \leq C(\mu_q^0)^{-[2d]-1}\,.$$

Notice that $\mu_q(t)$ is continuous in $t\in[0,1]$ but $\lambda_q(t)$ is not continuous in general. Because of (H_2) the quasi-eigenvalues $\mu_q(t)^2$, $|q|\geq q_0\gg 1$, belong to the union of intervals $[a_k - ca_k^{-d}/4, b_k + cb_k^{-d}/4]$ which do not intersect in view of (H_1). Since $\mu_q(t)^2$ is continuous in $[0,1]$, it can not jump from one interval to another. Hence, for each $q\in\mathcal{M}$, $|q|\gg 1$, there is $k=k(q)\gg 1$ such that

$$\begin{aligned}|c_{q,1}(t) - c_{q,1}(0)| &\leq \mu_q(0)|\mu_q(t)-\mu_q(0)| + C'(\mu_q^0)^{-1} \leq C'(|\mu_q(t)^2 - \mu_q(0)^2| + (\mu_q^0)^{-1})\\ &\leq C'(b_k - a_k + ca_k^{-d} + (\mu_q^0)^{-1}) := \varepsilon_k\,,\end{aligned}$$

for any $t\in[0,1]$, where C' stands for different positive constants, and $\lim\varepsilon_{k(q)} = 0$ as $|q|\to\infty$ in view of (H_1), which proves (1.4).

We point out that if $a_k^{p/2}(b_k - a_k)\to 0$ as $k\to\infty$ for some integer $p\geq 0$ and if ℓ is sufficiently large, one can prove also that $c_{q,j}(t) = c_{q,j}(0)$ for $j\leq p+1$, which would give further isospectral invariants involving integrals of polynomials of the derivatives of K_t.

1.2 *Applications and spectral rigidity*

Kronecker invariant tori usually appear in Cantor families (with respect to the Diophantine vector of rotation ω), the union of which has positive Lebesgue measure in $T^*\Gamma$, and Theorem 1.1 applies to any single torus Λ in that family. Consider for example a strictly convex bounded domain $X\subset\mathbb{R}^2$ with C^∞ boundary Γ, and fix $\tau>1$. It is known from Lazutkin [9] that for any $0<\kappa\leq\kappa_0\ll 1$ there is a Cantor set $\Xi_\kappa\subset(0,\varepsilon_0]$, $\varepsilon_0\ll 1$, of Diophantine numbers ω satisfying (1.3) and such that for each $\omega\in\Xi_\kappa$ there is a KAM (Kolmogorov-Arnold-Moser) invariant circle $\Lambda_\omega\subset B^*\Gamma$ of B satisfying (H_3) with $m=1$ and with rotation number $2\pi\omega$. Moreover, Ξ_κ is of a positive Lebesgue measure in $(0,\varepsilon_0]$, the Lebesgue measure of $(0,\varepsilon]\setminus\Xi$, $\Xi=\cup\Xi_\kappa$, is $o(\varepsilon)$ as $\varepsilon\to 0$, and so is the Lebesgue measure of the complement to the union of the invariant circles in an ε-neighborhood of $S^*\Gamma$ in $B^*\Gamma$. More generally, the result of Lazutkin holds for any compact billiard table (X,g), $\dim X=2$, with connected boundary Γ which is locally strictly geodesically convex. Set $\ell = ([2d]+1)([\tau]+2)+6$.

Corollary 1.2 *Let (X,g), $\dim X = 2$, be a compact billiard table with C^∞-smooth connected and locally strictly geodesically convex boundary Γ. Let*

$$[0,1]\ni t\mapsto K_t\in C^\ell(\Gamma,\mathbb{R})\,,$$

be a continuous family of real-valued functions on Γ such that Δ_t satisfy (H_1) − (H_2). Then

$$\forall\omega\in\Xi\,,\ \forall t\in[0,1],\quad \int_{\Lambda_\omega}\frac{K_t\circ\pi_\Gamma}{\sin\theta}\,d\mu \;=\; \int_{\Lambda_\omega}\frac{K_0\circ\pi_\Gamma}{\sin\theta}\,d\mu\,. \tag{1.5}$$

It will be interesting to know if the relation (1.5) implies $K_t = K_0$ for generic Γ.

Another example can be obtained applying the KAM theorem to the Poincaré map of a non-degenerate elliptic periodic broken geodesic with m vertexes (in any dimension $n \geq 2$).

Theorem 1.1 can be applied also in the completely integrable case, for example for the ellipse or the ellipsoid, or more generally for Liouville billiard tables of classical type [13, 14] in any dimension $n \geq 2$. We are going to prove spectral rigidity for two dimensional Liouville billiard tables of classical type (see Sect. 5 for definition). Such billiard tables have a group of isometries $I(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ which induces a group of isometries $I(\Gamma) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ on the boundary. We denote by $\mathrm{Symm}^\ell(\Gamma)$ the space of all C^ℓ real-valued functions which are invariant with respect to $I(\Gamma)$. We show next that *any continuous weakly isospectral deformation* of K in $\mathrm{Symm}^\ell(\Gamma)$, $\ell = 3[2d] + 9$, is *trivial*. More precisely, we have

Corollary 1.3 *Let (X, g), $\dim X = 2$, be a Liouville billiard table of classical type. Let K_t, $t \in [0, 1]$, be a continuous family of real-valued functions in $C^\ell(\Gamma, \mathbb{R})$ such that Δ_t satisfy $(H_1){-}(H_2)$. Assume that $K_0, K_1 \in \mathrm{Symm}^\ell(\Gamma)$. Then $K_1 \equiv K_0$.*

It seams that even for the ellipse this result has not been known. Using Lemma 2.1 and Corollary 1.3 we obtain that *any continuous isospectral deformation* of K in the sense of (1.2) in $\mathrm{Symm}^\ell(\Gamma)$, $\ell \geq 15$, is *trivial*. We point out that the Liouville billiard tables that we consider are not analytic in general and the methods used in [5] and [13] can not be applied.

In the same way we treat the operator $\Delta_t = \Delta + V_t$ in X with fixed Dirichlet or Robin (Neumann) boundary conditions on Γ, where $V_t \in C^\ell(X)$, $t \in [0, 1]$, is a continuous family of real-valued potentials in X. The corresponding results are proved in Sect. 4. Injectivity of the Radon transform and spectral rigidity of Liouville billiard tables in higher dimensions is investigated in [14].

We point out that the method we use can be applied whenever there exists a continuous family of quasimodes of the spectral problem and if the corresponding Radon transform is injective. It can be used also for the Laplacian Δ_K in the exterior $X = \mathbb{R}^n \setminus \Omega$ of a bounded domain in $\mathbb{R}^n$ with a C^∞-smooth boundary with Robin boundary conditions on it. In this case an analogue of (H_1)-(H_2) can be formulated for the resonances of Δ_K close to the real axis replacing the intervals in the definition of $\mathcal{I}$ by boxes in the complex upper half plain. Given a Kronecker torus Λ of B we obtain quasimodes of Δ_K associated to Λ. By a result of Tang and Zworski [18] and Stefanov [16] the quasi-eigenvalues are close to resonances and one obtains an analogue of Theorem 1.1. The corresponding results will appear elsewhere.

2 Quasimodes and spectral invariants

2.1 *Billiard ball map*

We recall from Birkhoff [1] the definition of the billiard ball map B associated to the billiard table (X, g) with boundary Γ. Denote by h the Hamiltonian corresponding to the Riemannian metric g on X via the Legendre transformation. The billiard ball map B lives in an open subset of the coball bundle

$$B^*\Gamma = \{(x, \xi) \in T^*\Gamma : \, h_0(x, \xi) \leq 1\},$$

where h_0 is the Hamiltonian corresponding to the induced Riemannian metric on Γ via the Legendre transformation. The map B is defined as follows. Denote by $\overset{\circ}{B^*}\Gamma$ the interior of $B^*\Gamma$

and set

$$S^*X := \{(x,\xi) \in T^*X : h(x,\xi) = 1\}, \quad \Sigma = S^*X|_\Gamma := \{(x,\xi) \in S^*X : x \in \Gamma\},$$

$$\Sigma^\pm := \{(x,\xi) \in \Sigma : \pm\langle \xi, \nu(x)\rangle > 0\}.$$

The natural projection $\pi_\Sigma : \Sigma \to B^*\Gamma$ assigning to each $(x,\eta) \in \Sigma$ the covector $(x, \eta|_{T_x\Gamma})$ admits two smooth inverses

$$\pi^\pm_\Sigma : \overset{\circ}{B^*}\Gamma \to \Sigma^\pm, \ \pi^\pm_\Sigma(x,\xi) = (x,\xi^\pm).$$

Take $(x,\xi) \in \overset{\circ}{B^*}\Gamma$ and consider the integral curve $\exp(tX_h)(x,\xi^+)$, of the Hamiltonian vector field X_h starting at $(x,\xi^+) \in \Sigma^+$. If it intersects transversally Σ at a time $t_1 > 0$ and lies entirely in the interior $S^*\overset{\circ}{X}$ of S^*X for $t \in (0,t_1)$, we set

$$(y,\eta^-) = J(x,\xi^+) = \exp(t_1X_h)(x,\xi_+) \in \Sigma^-,$$

and define $B(x,\xi) := (y,\eta)$, where $\eta := \eta_-|_{T_y\Gamma}$. We denote by $\widetilde{B}^*\Gamma$ the set of all such points (x,ξ). In this way we obtain a smooth symplectic map $B : \widetilde{B}^*\Gamma \to B^*\Gamma$, $B = \pi_\Sigma \circ J \circ \pi^+_\Sigma$. As in [10] we can write π_Σ in an invariant form as follows. Consider the pull-back ω_0 in $T^*X|_\Gamma$ of the symplectic form ω in T^*X via the inclusion map. Then the projection along the characteristics of ω_0 induces the map $\pi_\Sigma : \Sigma \to B^*\Gamma$.

Denote by $\pi_\Gamma : T^*\Gamma \to \Gamma$ the inclusion map. Given $(x,\xi) \in B^*\Gamma$, we denote by $\theta = \theta(x,\xi) \in (0,\pi/2]$ the angle between ξ^+ and $T^*_x\Gamma$ in T^*_xX (equipped with the metric $\|\cdot\|_x = \sqrt{h(x,\cdot)}$), which is determined by $\sin\theta = \sqrt{1 - h_0(x,\xi)}$.

2.2 *Quasimodes*

First we shall show that the isospectral condition (H_1)-(H_2) is natural for any $d > n/2$. Given $c > 0$ and $a \gg 1$ we consider

$$\mathcal{I}_0 \ := \ \left\{\lambda \geq a : |\,\mathrm{Spec}\,(\Delta_{g,K}) \ - \ \lambda| \ \leq \ 2c\lambda^{-d}\right\}.$$

Let us write $\mathcal{I}_0$ as a disjoint union of connected intervals $[\overline{a}_k, \overline{b}_k]$, and then set $a_k = \overline{a}_k + c\overline{a}_k^{-d}$ and $b_k = \overline{b}_k - c\overline{b}_k^{-d}$. We have $\overline{b}_k - \overline{a}_k \geq 2c(\overline{a}_k^{-d} + \overline{b}_k^{-d})$, hence, $b_k - a_k \geq c(\overline{a}_k^{-d} + \overline{b}_k^{-d}) > 0$. Denote by $\mathcal{I} = \mathcal{I}(\Delta_{g,K})$ the union of the disjoint intervals $[a_k, b_k]$, $k \geq 1$. By construction $a_{k+1} - b_k > ca_{k+1}^{-d}$ since the intervals $[\overline{a}_k, \overline{b}_k]$ are disjoint.

Lemma 2.1 *The set $\mathcal{I}(\Delta_{g,K})$ satisfies (H_1) for any $d > n/2$. In particular, the usual isospectral condition (1.2) implies* (H_2)-(H_2) *for $\mathcal{I} = \mathcal{I}(\Delta_0)$ and any $d > n/2$.*

Proof of Lemma 2.1. It remains to estimate the length of the interval $[a_k, b_k]$. Let $\lambda_p \leq \cdots \leq \lambda_r$ be the eigenvalues of $\Delta_{g,K}$ in $[\overline{a}_k, \overline{b}_k]$. Then

$$|\,\lambda_j \ - \ \lambda_{j+1}| \ \leq \ 4c\lambda_j^{-d}$$

for $p \leq j \leq r$. On the other hand, by Weyl's formula, $\lambda_j = vj^{2/n}(1 + o(1))$ as $j \to +\infty$, where $v > 0$ is a constant. Then choosing $k \gg 1$, respectively $j \gg 1$, we get $\lambda_j \geq 2^{-1}vj^{2/n}$, and

$$\overline{b}_k - \overline{a}_k \ \leq \ C\sum_{j=p}^{r} j^{-\frac{2d}{n}} \leq C\int_p^r s^{-\frac{2d}{n}}ds \ \leq C\lambda_p^{1-\frac{2d}{n}} \leq C\overline{a}_k^{1-\frac{2d}{n}},$$

where C stands for different positive constants. Hence, $b_k - a_k < \overline{b}_k - \overline{a}_k = o(1)$ for $d > n/2$, which proves the Lemma. □

Fix a positive integer N. By quasimode $\mathcal{Q}$ of $\Delta_{g,K}$ of order N we mean an infinite sequence $(\mu_q, u_q)_{q\in\mathcal{M}}$, $\mathcal{M}$ being an index set, such that μ_q are positive, $\lim \mu_q = +\infty$, $u_q \in C^2(\overline{X})$, $\|u_q\|_{L^2(X)} = 1$, and

$$\begin{cases} \left\|\Delta\, u_q \;-\; \mu_q^2\, u_q\right\| \;\leq\; C_N\, \mu_q^{-N} \quad \text{in } L^2(X)\,, \\ \|\partial u_q/\partial\nu|_\Gamma \;-\; K\, u_q|_\Gamma\| \;\leq\; C_N\, \mu_q^{-N} \quad \text{in } L^2(\Gamma)\,. \end{cases} \tag{2.6}$$

Denote by $A(\varrho)$ the action along the broken bicharacteristic starting at $\varrho \in \Lambda$ and with endpoint $P(\varrho) \in \Lambda$. Note that $2A(\varrho) > 0$ is just the length of the corresponding geodesic arc.

Theorem 2.2 *Let Λ be a Kronecker torus satisfying (H_3) with frequency given by (1.3) and exponent $\tau \geq 1$. Fix two positive integers $N \geq 2$ and $l \geq N([\tau] + n) + 2n + 2$ and let $\mathcal{B}$ be a bounded subset of $C^l(\Gamma, \mathbb{R})$. Then for any $K \in \mathcal{B}$ there is a quasimode $(\mu_q, u_q)_{q\in\mathcal{M}}$, $\mathcal{M} \subset \mathbb{Z}^n$, of $\Delta_{g,K}$ of order N satisfying (2.6) such that*

$$\mu_q = \mu_q^0 + c_{q,0} + c_{q,1}(\mu_q^0)^{-1} + \cdots + c_{q,N}(\mu_q^0)^{-N}$$

where

(i) μ_q^0 is independent of K and there is $C^0 > 0$ such that $\mu_q^0 \geq C^0|q|$ for any $q \in \mathcal{M}$,

(ii) the map $K \to c_{q,j} \in \mathbb{R}$ is continuous in $K \in C^l(\Gamma, \mathbb{R})$ and there is $C = C(\mathcal{B}) > 0$ such that $|c_{q,j}| \leq C$ for any $q \in \mathcal{M}$, $0 \leq j \leq N$, and any $K \in \mathcal{B}$,

(iii) $c_{q,0}$ is independent of K and

$$c_{q,1} = c'_{q,1} + c''_1 \sum_{j=0}^{m-1} \int_\Lambda \frac{K \circ \pi_\Gamma}{\sin\theta} \circ B^j\, d\mu\,,$$

where $c'_{q,1}$ is independent of K, and

$$c''_1 = \frac{2(2\pi)^{n-1}}{\int_\Lambda A(\varrho)\, d\mu}.$$

Moreover, the positive constant C_N in (2.6) is uniform with respect to $K \in \mathcal{B}$.

Proof of Theorem 1.1. Denote by $\mathcal{B}$ the set of K_t, $t \in [0, 1]$. Take $N = [2d] + 1 \geq 2$, the smallest positive integer bigger than $2d$, and consider the quasi-eigenvalues $\mu_q(t)^2$, $t \in [0, 1]$, given by Theorem 2.2. It is easy to see ([9], Proposition 32.1) that there is a positive constant C' depending only on C_N such that for any $q \in \mathcal{M} \subset \mathbb{Z}^n$ and $t \in [0, 1]$,

$$\left|\,\text{Spec}\,(\Delta_t) \;-\; \mu_q(t)^2\right| \;\leq\; C'\mu_q(t)^{-[2d]-1}\,.$$

Then for any $q \in \mathcal{M}$, $|q| \geq q_0 \gg 1$, and $t \in [0,1]$ there is $\lambda_{t,q} \in \mathrm{Spec}\,(\Delta_t)$ such that $\lambda_{t,q} \geq (C')^{-1}|q|$ and

$$\left|\, \lambda_{t,q} - \mu_q(t)^2 \right| \leq C' \lambda_{t,q}^{-([2d]+1)/2}$$

where $C' > 0$ depends only on C^0 and C_N. Since $([2d]+1)/2 > d$, using (H_2) we obtain that the quasi-eigenvalue $\mu_q(t)^2$ belongs to the union of the intervals $[a_k - ca_k^{-d}/4, b_k + cb_k^{-d}/4]$ for any $q \in \mathcal{M}$ with $|q| \geq q_0 \gg 1$ and any $t \in [0,1]$. These intervals do not intersect each other in view of (H_1) and since $\mu_q(t)^2$ is continuous in $[0,1]$ it can not jump from one interval to another. Hence, for each $q \in \mathcal{M}$ with $|q| \geq q_0$ there is $k = k(q)$ such that $\mu_q(t)^2 \in [a_k - ca_k^{-d}/4, b_k + cb_k^{-d}/4]$ for any $t \in [0,1]$, and we obtain

$$|c_1''| \left| \sum_{j=0}^{m-1} \int_\Lambda \frac{(K_t - K_0)\circ \pi_\Gamma}{\sin\theta} \circ B^j \, d\mu \right| = |c_{q,1}(t) - c_{q,1}(0)|$$

$$\leq \mu_q^0 |\mu_q(t) - \mu_q(0)| + C'(\mu_q^0)^{-1} \leq C' \left(\frac{\mu_q(0)}{\sqrt{a_k}} |\mu_q(t)^2 - \mu_q(0)^2| + (\mu_q^0)^{-1} \right)$$

$$\leq C' \left(b_k - a_k + ca_k^{-d} + (\mu_q^0)^{-1} \right) := \varepsilon_k \,,$$

where C' stands for different positive constants depending only on the constants C^0, C and C_N in Theorem 2.2. Hence C' depends neither on t nor on q and $\lim_{q\to+\infty} \varepsilon_{k(q)} = 0$ in view of (H_1) which proves (1.4). □

3 Construction of continuous quasimodes

3.1 *Reduction to the boundary.*

We are going to use an outgoing parametrix for the Helmholtz equation with initial conditions on Γ. In the time dependent case such a parametrix has been constructed by Guillemin and Melrose [5].

Set $\Lambda_j = B^j(\Lambda)$, $j = 0, 1, \ldots, m$, where $\Lambda_m = P(\Lambda) = \Lambda$, $m \geq 1$. Since ω is Diophantine, P acts transitively on each Λ_j, hence, $\Lambda_i \cap \Lambda_j = \emptyset$ if $0 < |i-j| < m$ and $m \geq 2$. Choose neighborhoods $U_j \subset \widetilde{B^*}\Gamma$ of Λ_j, $0 \leq j \leq m$, such that U_{j+1} is a neighborhood of the closure of $B(U_j)$ for $j = 0, \ldots, m-1$, $m \geq 1$, and such that $U_i \cap U_j = \emptyset$ if $0 < |i-j| < m$ and $m \geq 2$. We denote by $(\widetilde{X}, \widetilde{g})$ a C^∞ extension of (X, g) across Γ such that any integral curve γ of the Hamiltonian vector field $X_{\widetilde{h}}$, $\widetilde{h}$ being the corresponding Hamiltonian, starting at $\pi_\Sigma^+(U_j)$, $j = 0, \ldots, m-1$, satisfies

$$\gamma \cap T^*\widetilde{X}|_\Gamma \subset \pi_\Sigma^+(U_j) \cup \pi_\Sigma^-(U_{j+1}). \tag{3.7}$$

Then γ intersects transversally $T^*X|_\Gamma$ and for each $\varrho \in U_j$ there is an unique $T_j(\varrho) > 0$ such that

$$\exp(T_j(\varrho)X_{\widetilde{h}})(\pi_\Sigma^+(\varrho)) \in \pi_\Sigma^-(B(U_j)) \,.$$

Let $\psi_j(\lambda)$, $j = 0, 1, \ldots, m$, be classical λ-pseudodifferential operators (λ-PDOs) of order 0 on Γ with a large parameter λ and compactly supported amplitudes in U_j [12] such that

$$\mathrm{WF}'(\mathrm{Id} - \psi_j) \cap \Lambda_j = \emptyset \,,$$

and

$$\mathrm{WF}'(\psi_{j+1}) \subset B(U_j)\,,\ \mathrm{WF}'(\mathrm{Id}-\psi_{j+1}) \cap B(\mathrm{WF}'(\psi_j)) = \emptyset \text{ for } j=0,\dots m-1\,. \tag{3.8}$$

Hereafter $\mathrm{WF}'(\psi_j)$ stands for the frequency set of ψ_j [12], and by a "classical" λ-PDO we mean that in any local coordinates the corresponding distribution kernel is of the form (A.1) where the amplitude has an asymptotic expansion $q(x,\xi,\lambda) \sim \sum_{k=0}^{\infty} q_k(x,\xi)\lambda^{-k}$ and q_k are C^∞ smooth and uniformly compactly supported. In particular the distribution kernel $\mathrm{OP}_\lambda(q)(\cdot,\cdot)$ is smooth for each λ fixed. We take λ in a complex strip

$$\mathcal{D} := \{z \in \mathbf{C} :\ |\mathrm{Im}\, z| \le D_0,\ \mathrm{Re}\, z \ge 1\}\,,$$

$D_0 > 0$ being fixed.

We are looking for a microlocal outgoing parametrix $H_j : L^2(\Gamma) \to C^\infty(\widetilde{X})$, of the Dirichlet problem for the Helmholtz equation with "initial data" concentrated in U_j such that

$$(\Delta - \lambda^2)H_j(\lambda) = O_M(|\lambda|^{-M}) \tag{3.9}$$

in a neighborhood of X in $\widetilde{X}$. Hereafter,

$$O_M(|\lambda|^{-M})\ :\ L^2(\Gamma) \longrightarrow L^2_{\mathrm{loc}}(\widetilde{X})$$

stands for any family of continuous operators depending on λ with norms $\le C_{M,F}(1+|\lambda|)^{-M}$, $C_{M,F} > 0$, on any compact $F \subset \widetilde{X}$. We shall denote also by

$$O_M(|\lambda|^{-M})\ :\ L^2(\Gamma) \longrightarrow L^2(\Gamma)\ ,$$

any family of continuous operators depending on λ with norms $\le C_M(1+|\lambda|)^{-M}$, $C_M > 0$.

The operator H_j is a Fourier integral operator of order 1/4 with a large parameter $\lambda \in \mathcal{D}$ (λ-FIO) the distribution kernel of which is an oscillatory integral in the sense of Duistermaat [4] (see also [12]). In any local coordinates its amplitude is C^∞ smooth, it is uniformly compactly supported for $\lambda \in \mathcal{D}$ and it has an asymptotic expansion in powers of λ up to any negative order. In particular, $H_j(\lambda)u$ is a C^∞ smooth function for any fixed λ and $u \in L^2(\Gamma)$. The corresponding canonical relation lies in $T^*\Gamma \times T^*\widetilde{X}$ and it is given by

$$\mathcal{C}_j\ :=\ \left\{\left(\varrho\,;\, \exp(sX_{\widetilde{h}})(\pi^+_\Sigma(\varrho))\right)\ :\ \varrho \in U_j\,,\ -\varepsilon < s < T_j + \varepsilon\right\}\ ,\ \varepsilon > 0\,.$$

We parameterize it by (ϱ, s). Consider the operator of restriction $\imath^*_\Gamma : C^\infty(\widetilde{X}) \to C^\infty(\Gamma)$, $\imath^*_\Gamma(u) = u_{|\Gamma}$, as a λ-FIO of order 0, the canonical relation $\mathcal{R}$ of which is just the inverse of the canonical relation given by the conormal bundle of the graph of the inclusion map $\imath : \Gamma \to \widetilde{X}$. Notice that the composition $\mathcal{R} \circ \mathcal{C}_j$ is transversal for any j and it is a disjoint union of the diagonal in $U_j \times U_j$ (for $s = 0$) and of the graph of the billiard ball map $B : U_j \to U_{j+1}$ (for $s = T_j$). Let $\Psi_j(\lambda)$ be a λ-PDO of order 0 such that $\mathrm{WF}'(\Psi_j - \mathrm{Id}) \cap \mathrm{WF}'(\psi_j) = \emptyset$. Taking $\Psi_j(\lambda)$ as initial data at Γ for $s = 0$ and solving the corresponding transport equations, we obtain an operator $H_j(\lambda)$ satisfying (3.9) and such that

$$\imath^*_\Gamma H_j(\lambda) = \Psi_j(\lambda) + G_j(\lambda) + O_M(|\lambda|^{-M})\,, \tag{3.10}$$

where $G_j(\lambda)$ is a λ-FIO of order 0, the canonical relation of which is the graph of the billiard ball map $B : U_j \to U_{j+1}$. Moreover, its principal symbol is equal to 1 in a neighborhood of $\mathrm{WF}'(\psi_j)$

modulo Maslov's factor times the Liouville factor $\exp(i\lambda A_j(\varrho))$, where $A_j(\varrho) = \int_{\gamma_j(\varrho)} \xi dx$ is the action along the integral curve $\gamma_j(\varrho)$ of the Hamiltonian vector field $X_{\widetilde{h}}$ starting at $\varrho \in U_j$ and with endpoint $B(\varrho) \in U_{j+1}$. In particular, the frequency set WF' of $G_j(\lambda)$ is contained in $U_j \times U_{j+1}$ for any $j = 0, \dots, m-1$. Note that $2A_j(\varrho)$ is just the length $T_j(\varrho)$ of the corresponding geodesic $\widetilde{\gamma}_j(\varrho)$ in X and we have

$$\pi_\Sigma \left(\exp(2A_j(\varrho)X_{\widetilde{h}})(\pi_\Sigma^+(\varrho))\right) = B(\varrho)\,, \ \varrho \in U_j\,.$$

Fix a bounded set $\mathcal{B}$ in $C^l(\Gamma, \mathbb{R})$ and take $K \in \mathcal{B}$. Consider the operator $\mathcal{N} = \partial/\partial\widetilde{\nu} - \widetilde{K}$ in a neighborhood of Γ in $\widetilde{X}$, where $\widetilde{\nu}$ is a normal vector field to Γ and $\widetilde{K}$ is a C^l-smooth extension of K with compact support contained in a small neighborhood of Γ. To construct $\widetilde{K}$ we extend K as a constant on the integral curves of $\widetilde{\nu}$ and then multiply it with a suitable cut-off function. In this way we obtain a continuous map $K \to \widetilde{K}$ from $C^l(\Gamma, \mathbb{R})$ to $C_0^l(\widetilde{X}, \mathbb{R})$.

Suppose first that $m = 1$ and set $G(\lambda) = H_0(\lambda)\psi_0(\lambda)$. Then $(\Delta - \lambda^2)H_j(\lambda) = O_M(|\lambda|^{-M})$ in a neighborhood of X in $\widetilde{X}$, in view of (3.9). Moreover, using the symbolic calculus and (3.8) we obtain

$$i_\Gamma^* \, \mathcal{N}\, G(\lambda) = \psi_1(\lambda)(\lambda R_0^+ + K)\, \psi_0(\lambda) + \psi_1(\lambda)(\lambda R_1^- + K)\, G_0(\lambda)\psi_0(\lambda) + O_M(|\lambda|^{-M})\,.$$

Here, $R_0^+(\lambda)$ is a classical λ-PDO of order 0 on Γ independent of K, with a C_0^∞-symbol in any local coordinates, and with principal symbol

$$\sigma(R_0^+)(\varrho) \ = \ i\sqrt{1 - h_0(\varrho)}\,, \ \varrho \in U_0\,,$$

and R_1^- is a classical λ-PDO of order 0 on Γ independent of K with principal symbol

$$\sigma(R_1^-)(\varrho) \ = \ -i\sqrt{1 - h_0(\varrho)}\,, \ \varrho \in U_1\,.$$

We consider the following equation with respect to Q_1

$$\psi_1 \left[\lambda R_1^- + K + (\lambda R_0^+ + K)Q_1(\lambda)\right] = O_{\mathcal{B}}(|\lambda|^{-M}) \ , \tag{3.11}$$

which we solve using the classes $\mathrm{PDO}_{l,2,M-1}(\Gamma; \mathcal{B}; \lambda)$ defined in the Appendix. Hereafter, $O_{\mathcal{B}}(|\lambda|^{-M}) \ : \ L^2(\Gamma) \to L^2(\Gamma)$ denotes any family of continuous operators depending on $K \in \mathcal{B}$ and on $\lambda \in \mathcal{D}$ with norms uniformly bounded by $C_{\mathcal{B}}(1 + |\lambda|)^{-M}$, where $C_{\mathcal{B}} > 0$ is a constant independent of $K \in \mathcal{B}$. We cover U_1 by finitely many local charts, and in each of them we write the complete symbol of Q_1 of the form (A.2). Then using a suitable C^∞ partition of the unity in the phase space, we put them together and obtain an operator

$$Q_1 = Q_1^0 + \lambda^{-1} Q_1^1$$

which is well defined modulo $O_{\mathcal{B}}(|\lambda|^{-M})$. Here Q_1^0 is a classical λ-PDOs of order 0 independent of K and with a C^∞ symbol, and $Q_1^1 \in \mathrm{PDO}_{l,2,M-1}(\Gamma; \mathcal{B}; \lambda)$. The corresponding principal symbols are

$$\sigma_0(Q_1^0)(x, \xi) = 1 \ , \quad \sigma_0(Q_1^1)(x, \xi) = \frac{2iK(x)}{\sqrt{1 - h_0(x, \xi)}} = \frac{2iK(x)}{\sin\theta(x, \xi)}$$

in a neighborhood of $\mathrm{WF}'(\psi_1)$ in U_1. In this way the equation

$$i_\Gamma^* \, \mathcal{N}\, G(\lambda) v = O_M(|\lambda|^{-M}) v$$

reduces to $(W(\lambda) - \mathrm{Id})\psi_0(\lambda)v \ = \ O_{\mathcal{B}}(|\lambda|^{-M})v$, where $W(\lambda) := Q_1(\lambda)G_0(\lambda)$.

Suppose now that $m \geq 2$. In order to satisfy the boundary conditions at U_{j+1}, $0 \leq j \leq m-2$, we are looking for a λ-PDO $Q_{j+1}(\lambda)$ such that

$$\psi_{j+1}(\lambda)\imath_\Gamma^* \mathcal{N} H_{j+1}(\lambda)Q_{j+1}(\lambda)G_j(\lambda) \ + \ \psi_{j+1}(\lambda)\imath_\Gamma^* \mathcal{N} H_j(\lambda) = O_{\mathcal{B}}(|\lambda|^{-M})\,. \tag{3.12}$$

Using the symbolic calculus we write

$$\psi_{j+1}(\lambda)\imath_\Gamma^* \ \mathcal{N} H_{j+1}(\lambda)Q_{j+1}(\lambda)G_j(\lambda) = \psi_{j+1}(\lambda)(\lambda R_{j+1}^+(\lambda) + K)Q_{j+1}(\lambda)G_j(\lambda) + O_M(|\lambda|^{-M})$$

where $R_{j+1}^+(\lambda)$ is a classical λ-PDO of order 0 on Γ independent of K, with a C_0^∞-symbol in any local coordinates, and with principal symbol

$$\sigma(R_{j+1}^+)(\varrho) \ = \ i\sqrt{1-h_0(\varrho)}\,, \ \ \varrho \in U_{j+1}\,.$$

In the same way we obtain

$$\psi_{j+1}(\lambda)\imath_\Gamma^* \mathcal{N} H_j(\lambda) = \psi_{j+1}(\lambda)(\lambda R_{j+1}^- + K)\, G_j(\lambda) + O_M(|\lambda|^{-M}),$$

where R_{j+1}^- is a classical λ-PDO of order 0 on Γ independent of K with principal symbol

$$\sigma(R_{j+1}^-)(\varrho) \ = \ -i\sqrt{1-h_0(\varrho)}\,, \ \ \varrho \in U_{j+1}\,.$$

Then (3.12) reduces into the equation

$$\psi_{j+1}(\lambda)\left[(\lambda R_{j+1}^+ + K)Q_{j+1} + \lambda R_{j+1}^- + K\right] = O_{\mathcal{B}}(|\lambda|^{-M}) \tag{3.13}$$

on U_{j+1}, which we solve as above in the classes $\mathrm{PDO}_{l,2,M-1}(\Gamma;\mathcal{B};\lambda)$. More precisely, we obtain an operator

$$Q_{j+1} = Q_{j+1}^0 + \lambda^{-1}Q_{j+1}^1$$

which is well defined modulo $O_{\mathcal{B}}(|\lambda|^{-M})$, where Q_{j+1}^0 is a classical λ-PDOs of order 0 independent of K and with a C^∞ symbol, and $Q_{j+1}^1 \in \mathrm{PDO}_{l,2,M-1}(\Gamma;\mathcal{B};\lambda)$. The corresponding principal symbols are

$$\sigma_0(Q_{j+1}^0)(x,\xi) = 1\ , \quad \sigma_0(Q_{j+1}^1)(x,\xi) = \frac{2iK(x)}{\sqrt{1-h_0(x,\xi)}} = \frac{2iK(x)}{\sin\theta(x,\xi)}$$

in a neighborhood of $\mathrm{WF}'(\psi_{j+1})$ in U_{j+1}.

Consider the operator $G(\lambda) : C^\infty(\Gamma) \to C^\infty(\widetilde{X})$ defined by

$$G(\lambda) = H_0(\lambda)\psi_0(\lambda) + \sum_{k=2}^{m} H_{k-1}(\lambda)\Pi_{j=0}^{k-2}\left(Q_{j+1}(\lambda)G_j(\lambda)\right)\psi_0(\lambda)\,.$$

Using (3.8) - (3.10) and (3.12) we obtain

$$\begin{cases} (\Delta - \lambda^2)G(\lambda) & = & O_{\mathcal{B}}(|\lambda|^{-M})\,, \\ \imath_\Gamma^* \mathcal{N} G(\lambda) & = & \psi_m(\lambda)(\lambda R_0^+ + K)\,\psi_0(\lambda) + \psi_m(\lambda)(\lambda R_m^- + K)\,\widetilde{W}(\lambda)\psi_0(\lambda) + O_{\mathcal{B}}(|\lambda|^{-M})\,, \end{cases}$$

where

$$\widetilde{W}(\lambda) = \imath_\Gamma^* H_{m-1}(\lambda)\Pi_{j=0}^{m-2}\left(\psi_{j+1}(\lambda)Q_{j+1}(\lambda)G_j(\lambda)\right) ,$$

and R_0^+ and R_m^- are defined as above. As in (3.11) we find $Q_m = Q_m^0 + \lambda^{-1}Q_m^1$ such that

$$\psi_m(\lambda)\left[\lambda R_m^- + K + (\lambda R_0^+ + K)Q_m(\lambda)\right] = O_{\mathcal{B}}(|\lambda|^{-M}) ,$$

where Q_m^k, $k = 0, 1$, are as above. In this way we reduce the equation $\imath_\Gamma^* \mathcal{N}\, G(\lambda)v = O_{\mathcal{B}}(|\lambda|^{-M})v$ to the following one

$$(W(\lambda) - \mathrm{Id}\,)\psi_0(\lambda)v \;=\; O_{\mathcal{B}}(|\lambda|^{-M})v\,, \tag{3.14}$$

where

$$W(\lambda) := Q_m(\lambda)\widetilde{W}(\lambda) = \Pi_{j=0}^{m-1}\left(\psi_{j+1}(\lambda)Q_{j+1}(\lambda)G_j(\lambda)\right) .$$

Set $S(\lambda) := \Pi_{j=0}^{m-1}G_j(\lambda)$. By construction $G_j(\lambda)$ is elliptic on $\mathrm{WF}'(\psi_j Q_j)$, and using Lemma A.2 we commute $G_j(\lambda)$ with $\psi_j Q_j$. Since $\mathrm{PDO}_{l,2,M-1}(\Gamma;\mathcal{B};\lambda)$ is closed under multiplication (see Remark A.1), we obtain another λ-PDO of the same class which we commute with $G_{j+1}(\lambda)$ and so on. Finally, for any $m \geq 1$ we obtain

$$W(\lambda) = \psi_m(\lambda)\left(Q^0(\lambda) + \lambda^{-1}Q^1(\lambda)\right)S(\lambda)\psi_0(\lambda) + O_{\mathcal{B}}(\lambda^{-M})\,.$$

Here, $Q^0(\lambda)$ is a classical λ-PDOs on Γ with a C^∞ symbol independent of K and with principal symbol 1 in a neighborhood of Λ, and $Q^1 \in \mathrm{PDO}_{l,2,M-1}(\Gamma;\mathcal{B};\lambda)$. By Egorov's theorem (see Lemma A.2) the principal symbol of $Q^1(\lambda)$ is

$$\sigma_0(Q^1)(x,\xi) = 2i\sum_{j=0}^{m-1}\frac{K(\pi_\Gamma(x^j,\xi^j))}{\sin\theta(x^j,\xi^j)}\,, \quad (x^j,\xi^j) = B^{-j}(x,\xi)\,,$$

in $P(U_0)$. The operator $S(\lambda)$ does not depend on K, and it is a classical λ-FIO of order 0 with a large parameter $\lambda \in \mathcal{D}$. The canonical relation of $S(\lambda)$ is given by the graph of the map $P = B^m : U_0 \to U_m$, and the principal symbol of $S(\lambda)$ equals one modulo a Maslov's factor times the Liouville factor $\exp(i\lambda A(x,\xi))$, $(x,\xi) \in P(U_0)$, where $A(x,\xi) = \sum_{j=0}^{m-1} A_j(x^j,\xi^j)$.

3.2 *Birkhoff normal form of P.*

First we find a symplectic Birkhoff normal form of P in a neighborhood Λ using [9], Proposition 9.13. We choose a basis of cycles γ_j, $j = 1,\ldots,n-1$, of the first homology group $H_1(\Lambda,\mathbb{Z})$, and set $I^0 = (I_1^0,\ldots,I_{n-1}^0)$, where $I_j^0 = (2\pi)^{-1}\int_{\gamma_j}\xi dx$. Using Proposition 9.13, [9], we obtain an exact symplectic transformation χ mapping a neighborhood of $\mathbb{T}^{n-1}\times\{I^0\}$ in $T^*\mathbb{T}^{n-1}$ to a neighborhood of Λ in $\overset{\circ}{B}{}^*\Gamma$ such that

(i) $\chi(\mathbb{T}^{n-1}\times\{I^0\}) = \Lambda$,

(ii) the symplectic map $P^0 := \chi^{-1}\circ P\circ\chi$ has a generating function of the form

$$\Phi(x,I) = \langle x,I\rangle + L(I) + R(x,I)\,,\ x\in\mathbb{R}^{n-1}\,,\ |I - I_0| \ll 1\,,$$

i.e. $P^0(\nabla_I\Phi, I) = (x,\nabla_x\Phi)$, where R is 2π-periodic in x,

(iii) $\nabla L(I^0) = 2\pi\omega$ and $\partial_I^\alpha R(x,I^0) = 0$, $x\in\mathbb{R}^{n-1}$, for each $\alpha\in\mathbb{N}^{n-1}$.

In particular, we obtain

$$\forall p \in \mathbb{N}\,, \quad P^0(\varphi, I) = (\varphi - \nabla L(I), I) + O_p(|I - I_0|^p)\,. \tag{3.15}$$

We choose the constant $L(I^0)$ as follows. Consider the "flow-out" $\mathcal{T} \cong \mathbb{T}^n$ of Λ by the broken bicharacteristic flow of h in T^*X. Let $\rho^0 = \chi(\varphi^0, I^0) \in \Lambda$. We denote by $\gamma_{n1}(\rho^0)$ the broken bicharacteristic arc in $\mathcal{T}$ issuing from ρ^0 and having endpoint at $P(\rho^0)$, and by $\gamma_{n2}(\rho^0) := \chi(\varphi^0 + (s-1)2\pi\omega), I^0)$, $s \in [0,1]$, the arc connecting $P(\rho^0)$ and ρ^0 in Λ. Let γ_n be the union of the two arcs. We denote by $L(I^0)$ the action along γ_n, i.e.

$$L(I^0) \;=\; \int_{\gamma_n} \xi dx\,. \tag{3.16}$$

Note that the integral above depends only on the homotopy class of the loop γ_n in the Lagrangian torus $\mathcal{T}$. We can give now a geometric interpretation of L which will be needed later. The Poincaré identity gives

$$P^*(\xi dx) = \xi dx + dA,$$

where ξdx is the fundamental one form on $T^*\Gamma$ and $A(\rho)$, $\rho = \chi(\varphi, I)$, $|I - I^0| \ll 1$, stands for the action along the broken bicharacteristic $\gamma_{n1}(\rho)$. Since χ is exact symplectic we have $\chi^*(\xi dx) = I d\varphi + d\Psi$ with a suitable smooth function $\Psi \in C^\infty(T^*\mathbb{T}^{n-1})$. Combining the two equalities we obtain

$$(P^0)^*(Id\varphi) - Id\varphi = d((A \circ \chi) + \Psi - \Psi \circ P^0).$$

In view of (3.15) this implies

$$L(I) - \langle I, \nabla L(I)\rangle = A(\chi(\varphi, I)) + \Psi(\varphi, I) - \Psi(P^0(\varphi, I)) + O_p(|I - I^0|^p) \tag{3.17}$$

for any $p \in \mathbb{N}$ modulo a constant $C \in \mathbb{R}$. Notice that C should be zero since for $I = I^0$ and $\omega = \nabla L(I^0)/2\pi$ we obtain using (3.16)

$$\begin{aligned} &L(I^0) - \langle I^0, \nabla L(I^0)\rangle = L(I^0) - 2\pi\langle I^0, \omega\rangle = \int_{\gamma_{n1}^0} I^0 d\varphi \\ &= \int_{\gamma_{n1}(\rho^0)} \xi dx + \Psi(\varphi^0, I^0) - \Psi(\varphi^0 - 2\pi\omega, I^0) = A(\chi(\varphi^0, I^0)) + \Psi(\varphi^0, I^0) - \Psi(P^0(\varphi^0, I^0))\,, \end{aligned}$$

where $\gamma_{n1}^0 := \chi^{-1}(\gamma_{n1}(\rho^0))$.

Set $\varrho^j = P^j(\varrho^0) = \chi(\varphi^0 - 2\pi j\omega, I^0)$. The measure $d\mu = \chi_*(d\varphi)$ on Λ is invariant with respect to the map $P : \Lambda \to \Lambda$ which is ergodic since $2\pi\omega$ is Diophantine, and we get

$$L(I^0) - 2\pi\langle I^0, \omega\rangle = \lim_{j\to\infty} \frac{1}{j} \sum_{k=0}^{j-1} A(\varrho^k) = (2\pi)^{1-n} \int_\Lambda A(\varrho)\, d\mu \;>\; 0\,. \tag{3.18}$$

3.3 *Quantum Birkhoff normal form.*

Using the restriction of χ to $\mathbb{T}^{n-1} \times \{I^0\}$, we identify the first cohomology groups $H^1(\Lambda, \mathbb{Z}) = H^1(\mathbb{T}^{n-1}, \mathbb{Z}) = \mathbb{Z}^{n-1}$, and we denote by $\vartheta_0 \in \mathbb{Z}^{n-1}$ the Maslov class of the invariant torus Λ. As in [3] we consider the flat Hermitian line bundle $\mathbb{L}$ over $\mathbb{T}^{n-1}$ which is associated to the class ϑ_0. The sections f in $\mathbb{L}$ can be identified canonically with functions $\widetilde{f} : \mathbb{R}^{n-1} \to \mathbb{C}$ so that

$$\widetilde{f}(x + 2\pi p) \;=\; e^{i\frac{\pi}{2}\langle\vartheta_0, p\rangle}\,\widetilde{f}(x) \tag{3.19}$$

for each $x \in \mathbb{R}^{n-1}$ and $p \in \mathbb{Z}^{n-1}$. An orthonormal basis of $L^2(\mathbb{T}^{n-1}, \mathbb{L})$ is given by $e_k,\ k \in \mathbb{Z}^{n-1}$, where

$$\widetilde{e}_k(x) \;=\; \exp\left(i\langle k + \vartheta_0/4, x\rangle\right).$$

We quantize the canonical transformation χ as in [3]. More precisely we find a classical λ-FIO $T(\lambda) : C^\infty(\mathbb{T}^{n-1}, \mathbb{L}) \to C^\infty(\Gamma)$ the canonical relation of which is just the graph of χ and such that $\mathrm{WF}'(T(\lambda)T(\lambda)^* - \mathrm{Id}_\Gamma) \cap B(U_m) = \emptyset$. We suppose that the principal symbol of $T(\lambda)$ is equal to one in $\mathbb{T}^{n-1} \times D^0$ modulo the Liouville factor $\exp(i\lambda\Psi(\varphi, I))$, where D^0 is a small neighborhood of I^0. Conjugating $W(\lambda)$ with $T(\lambda)$ and using Lemma A.2 and Remark A.3 we obtain

$$\begin{aligned} T(\lambda)^* W(\lambda) T(\lambda) &= \left[T(\lambda)^* \left(Q^0(\lambda) + \lambda^{-1} Q^1(\lambda)\right) T(\lambda)\right] \left[T(\lambda)^* S(\lambda) T(\lambda)\right] \\ &= e^{i\pi\vartheta/4} W_1(\lambda) + O_{\mathcal{B}}(|\lambda|^{-M}) \end{aligned}$$

where $\vartheta \in \mathbb{Z}$ is a Maslov's index and $W_1(\lambda)$ is a λ-FIO operator of the form

$$\widetilde{W_1(\lambda)} u(x) = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{i\lambda(\langle x-y, I\rangle + \Phi(x,I))}\, w(x, I, \lambda)\, \widetilde{u}(y)\, dI dy\,, \tag{3.20}$$

$u \in C^\infty(\mathbb{T}^{n-1}, \mathbb{L})$. The symbol $w(x, I, \lambda)$, $(x, I) \in \mathbb{R}^{n-1} \times D$, is 2π-periodic with respect to x and uniformly compactly supported in $I \in D$, where D is a small neighborhood of I^0, and it is obtained by the stationary phase method. We have $w = w_0 + \lambda^{-1} w^0$, where $w_0 \in C^\infty(\mathbb{R}^{n-1} \times D)$, $w_0(x, I) = 1$ for $(x, I) \in \mathbb{R}^{n-1} \times D^0$, D^0 being a neighborhood of I^0, and

$$w^0 = \sum_{j=0}^{M-2} w_j^0(x, I)\lambda^{-j} \in S_{l,2,M-1}(\mathbb{T}^{n-1} \times D; \mathcal{B}; \lambda)\,.$$

Moreover,

$$w_0^0(x, I) = i w_0'(x, I) + 2i \sum_{j=0}^{m-1} \left(\frac{K \circ \pi_\Gamma}{\sin\theta}\right) \left(B^{-j}\chi(\pi_0(x), I)\right)\right)\,,$$

where w_0' is a C^∞ real valued function independent of K and $\pi_0 : \mathbb{R}^{n-1} \to \mathbb{T}^{n-1}$ is the canonical projection. The phase function is given by $\Phi(x, I) = L(I) + R(x, I) + C$, where C is a constant, since the canonical relation of $W_1(\lambda)$ is just the graph of P^0. Comparing the Liouville factors in the principal symbols of $W_1(\lambda)$ and $W(\lambda)$ and using (3.16) and (3.17), we obtain as in [12] that $C = 0$.

The frequencies I of the quasimode we are going to construct satisfy $I - I^0 \sim \lambda^{-1}$, where λ^2 are the corresponding quasi-eigenvalues. For that reason we consider the Taylor polynomials of the symbols at $I = I^0$ up to certain order. Let $\psi \in C_0^\infty(D)$ and $\psi = 1$ in a neighborhood of I^0. For any positive integers $l, \widetilde{l} \geq 2$, $s \geq 2$ and $N \geq 1$ such that $\widetilde{l} \geq sN + 2n$ and for any bounded set $\mathcal{B} \subset C^l(\Gamma)$ we denote by $\widetilde{S}_{\widetilde{l},s,N}(\mathbb{T}^{n-1} \times D; \mathcal{B}; \lambda)$ the class of symbols

$$\left\{ \begin{array}{lcl} a(\varphi, I, \lambda) &=& \sum_{j=0}^{N-1} a_j(\varphi, I)\lambda^{-j}\,, \\ a_j(\varphi, I) &=& \psi(I) \sum_{|\alpha| \leq N-j-1} (I - I^0)^\alpha a_{j,\alpha}(\varphi) \end{array} \right. \tag{3.21}$$

where $a_{j,\alpha} = \partial_I^\alpha a_j(\cdot, I^0)/\alpha! \in C^{\widetilde{l}-sj-|\alpha|}(\mathbb{T}^{n-1})$ and the corresponding map

$$C^l(\Gamma, \mathbb{R}) \ni K \to a_{j,\alpha} \in C^{\widetilde{l}-sj-|\alpha|}(\mathbb{T}^{n-1})$$

is continuous. We denote also by $\widetilde{R}_N(\mathbb{T}^{n-1}\times D;\mathcal{B};\lambda)$ a residual class of symbols

$$\begin{cases} r(\varphi,I,\lambda) \;=\; \sum_{j=0}^{N-1} r_j(\varphi,I)\lambda^{-j}\,, \\ r_j(\varphi,I) = \sum_{|\alpha|=N-j}(I-I^0)^\alpha r_{j,\alpha}(\varphi,I) \end{cases} \tag{3.22}$$

where $C^l(\Gamma,\mathbb{R})\ni K\to r_{j,\alpha}\in C_0^{2n}(\mathbb{T}^{n-1}\times D)$ is continuous in the sense that the support of $r_{j,\alpha}$ is contained in a fixed compact set in $\mathbb{T}^{n-1}\times D$ independent of K and the map $K\to r_{j,\alpha}\in C^{2n}(\mathbb{T}^{n-1}\times D)$ is continuous in $C^l(\Gamma,\mathbb{R})$. Note that the class $\widetilde{S}_{\widetilde{l},s,N}/\widetilde{R}_N$ does not depend on of ψ. The choice of the residual class is motivated by the proof of Proposition 3.3 below.

Denote by $\mathcal{L}_\omega$ the operator defined by $\mathcal{L}_\omega a(\varphi)=a(\varphi-2\pi\omega)-a(\varphi)$.

Proposition 3.1 *Fix $l\geq(M-1)([\tau]+n)+2n+2$ and suppose that K belongs to a bounded subset $\mathcal{B}$ of $C^l(\Gamma,\mathbb{R})$. Then there exists a λ-PDO $A(\lambda)$ of order 0 acting on $C^\infty(\mathbb{T}^{n-1},\mathbb{L})$ and a λ-FIO $W^0(\lambda)$ of the form (3.20) such that*

$$W_1(\lambda)A(\lambda) \;=\; A(\lambda)W^0(\lambda)+R^0(\lambda)\;+\;O_{\mathcal{B}}(|\lambda|^{-M})\,,$$

the full symbols of $A(\lambda)$ and of $W^0(\lambda)$ are

$$\sigma(A)(\varphi,I,\lambda)=a_0(I)+\lambda^{-1}a^0(\varphi,I,\lambda)\;,\quad \sigma(W^0)(\varphi,I,\lambda)=p_0(I)+\lambda^{-1}p^0(I,\lambda)\,,$$

with $a_0,p_0\in C_0^\infty(D)$, $a_0(I)=p_0(I)=1$ in a neighborhood D^0 of I^0, and

$$\begin{aligned} p^0 &\in \widetilde{S}_{l,[\tau]+n,M-1}(D;\mathcal{B};\lambda)\,,\\ a^0 &\in \widetilde{S}_{l-[\tau]-n,[\tau]+n,M-1}(\mathbb{T}^{n-1}\times D;\mathcal{B};\lambda)\,. \end{aligned} \tag{3.23}$$

Moreover, R^0 is a λ-FIOs of the form (3.20) with symbol

$$\begin{aligned} &\sigma(R^0)(\varphi,I,\lambda)=r_0(\varphi,I)+\lambda^{-1}r^0(\varphi,I,\lambda)\,,\\ &r^0=\textstyle\sum_{j=0}^{M-2} r_j^0\lambda^{-j}\in\widetilde{R}_{M-1}(\mathbb{T}^{n-1}\times D;\mathcal{B},\lambda)\,, \end{aligned} \tag{3.24}$$

$r_0=0$ in $\mathbb{T}^{n-1}\times D^0$ and

$$p_{0,0}^0=\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{T}^{n-1}} w_0^0(\varphi,I^0)d\varphi\,.$$

Proof. Given $f\in C^N(\mathbb{T}^{n-1}\times D)$ we denote by T_Nf its Taylor polynomial with respect to I at $I=I^0$, i.e.

$$T_Nf(\varphi,I)=\sum_{k=0}^{N}(I-I^0)^\alpha\, f_\alpha(\varphi)\,,$$

where $f_\alpha(\varphi)=\partial_I^\alpha f(\varphi,I^0)/\alpha!$ are the corresponding Taylor coefficients. We need the following

Lemma 3.2 *Let $A(\lambda)$ and $W^0(\lambda)$ have symbols $a_0(I)+\lambda^{-1}a^0(\varphi,I,\lambda)$ and $p_0(I)+\lambda^{-1}p^0(I,\lambda)$ respectively, where $a_0(I)=p_0(I)=1$ in a neighborhood D^0 of I^0, and a^0 and p^0 satisfy (3.23) with $l\geq(M-1)([\tau]+n)+2n+2$. Set*

$$R(\lambda):=W_1(\lambda)A(\lambda)-A(\lambda)W^0(\lambda).$$

Then

$$R(\lambda) = \lambda^{-1}R_1(\lambda) + R^0(\lambda) + O_{\mathcal{B}}(|\lambda|^{-M}),$$

where $R_1(\lambda)$ and $R^0(\lambda)$ are λ-FIOs of order 0 of the form (3.20), the symbbol

$$R_1(\varphi, I, \lambda) = \sum_{j=0}^{M-2} R_{1j}(\varphi, I)\lambda^{-j}$$

of $R_1(\lambda)$ belongs to $\widetilde{S}_{l,[\tau]+n,M-1}(\mathbb{T}^{n-1}\times D; \mathcal{B}, \lambda)$ and the symbol of $R^0(\lambda)$ satisfies (3.24). Moreover, for $0 \le j \le M-2$ we have

$$R_{1j}(\varphi, I) = \frac{1}{i}\mathcal{L}_\omega\, a_j^0(\varphi, I) \,+\, T_{M-j-2}w_j^0(\varphi, I) \,-\, p_j^0(I) \,+\, h_j^0(\varphi, I)\,, \tag{3.25}$$

$h_0^0 = 0$, and $h_j^0 = f_j^0 - g_j^0$, for $1 \le j \le M-2$, where the Taylor coefficient $f_{j,\alpha}^0(\varphi)$, $|\alpha| \le M-j-2$, of f_j^0 at $I = I^0$ is a linear combination of

$$\begin{cases} \partial_\varphi^\beta a_{s,\gamma}(\varphi - 2\pi\omega) & : \ 0 \le s \le j-1,\ |\beta+\gamma| \le 2(j-s) + |\alpha|\,, \\ w_{r,\delta}^0(\varphi)\partial_\varphi^\beta a_{s,\gamma}^0(\varphi - 2\pi\omega) & : \ 0 \le r+s \le j-1,\ |\beta+\gamma+\delta| \le 2(j-r-s-1) + |\alpha|\,, \end{cases} \tag{3.26}$$

while the Taylor coefficients $g_{j,\alpha}^0(\varphi)$, $|\alpha| \le M-j-2$, of g_j^0 at $I = I^0$ is a linear combination of

$$p_{k,\beta}^0\, a_{j-k-1,\gamma}^0(\varphi) : \quad 0 \le k \le j-1\,,\ \beta + \gamma = \alpha\,. \tag{3.27}$$

The proof of the lemma is given in the Appendix.

Recall that for each $|\alpha| \le l - 2j$ the map

$$C^l(\Gamma, \mathbb{R}) \ni K \ \to\ w_{j,\alpha}^0 \in C^{l-2j-|\alpha|}(\mathbb{T}^{n-1}) \tag{3.28}$$

is continuous.

We are going to find the Taylor coefficients $p_{j,\alpha}^0 \in \mathbb{C}$ and

$$a_{j,\alpha} \in C^{l-(j+1)([\tau]+n)-|\alpha|}(\mathbb{T}^{n-1})\,,\ 0 \le j \le M-2\,,\ |\alpha| \le M-j-2\,,$$

so that $R_{1j} = 0$. Moreover, we shall prove by recurrence that the maps

$$K \mapsto p_{j,\alpha}^0 \in \mathbb{C}\,,\ K \mapsto a_{j,\alpha} \in C^{l-(j+1)([\tau]+n)-|\alpha|}(\mathbb{T}^{n-1}) \tag{3.29}$$

are continuous with respect to $K \in C^l(\Gamma, \mathbb{R})$. For $j = 0$ we have $h_0 = 0$, and we put

$$p_{0,\alpha}^0 \ = \ \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} w_{0,\alpha}^0(\varphi)\, d\varphi\,, \quad |\alpha| \le N-2\,.$$

Setting $u = a_{0,\alpha}$ and $v = p_{0,\alpha}^0 - w_{j,\alpha}^0$ we obtain from (3.25) equations of the form

$$\frac{1}{i}\,\mathcal{L}_\omega\, u(\varphi) \ = \ v(\varphi)\,, \quad \int_{\mathbb{T}^{n-1}} v(\varphi)\, d\varphi = 0\,. \tag{3.30}$$

We are going to solve (3.30). Suppose that $v \in C^m(\mathbb{T}^{n-1})$ for some $m \geq [\tau]+n$. Comparing the corresponding Fourier coefficients u_k and v_k, $0 \neq k \in \mathbb{Z}^{n-1}$, we get

$$u_k \;=\; \frac{i}{1-\exp(2\pi i\langle k,\omega\rangle)}\, v_k\,,\ \ k\neq 0\,,$$

and set $u_0 = 0$. Summing up and using the Diophantine condition (1.3) we get the function u. In this way we obtain an unique solution $u \in C^{m-[\tau]-n}(\mathbb{T}^{n-1})$ of (3.30) normalized by $\int_{\mathbb{T}^{n-1}} u(\varphi)\, d\varphi = 0$. Moreover,

$$\|u\|_{C^{m-[\tau]-n}} \leq C\|v\|_{C^m}\,,$$

hence, the linear map $v \mapsto u \in C^{m-[\tau]-n}(\mathbb{T}^{n-1})$ is continuous in $v \in C^m(\mathbb{T}^{n-1})$. In this way using (3.28) for $j=0$ and $|\alpha| \leq N-2$ we obtain $p^0_{0,\alpha} \in \mathbb{C}$ and $a_{0,\alpha} \in C^{l-([\tau]+n)-|\alpha|}(\mathbb{T}^{n-1})$ and we prove that the corresponding maps (3.29) are continuous. Moreover,

$$p^0_0(I^0) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} w^0_0(\varphi, I^0)d\varphi\,.$$

Fix $1 \leq j \leq M-2$ and suppose that the inductive assumption holds for all indices $k \leq j-1$. Then the maps

$$K \mapsto h_{j,\alpha} \in C^{l-j([\tau]+n)-|\alpha|}(\mathbb{T}^{n-1})\,,\ \ |\alpha| \leq M-j-2\,,$$

are continuous with respect to $K \in C^l(\Gamma,\mathbb{R})$ in view of (3.26) and (3.27). We set as above

$$p^0_{j,\alpha} \;=\; \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} (w^0_{j,\alpha}(\varphi) - h_{j,\alpha}(\varphi))\, d\varphi\,.$$

Obviously it depends continuously on $K \in C^l(\Gamma,\mathbb{R})$. Setting $u = a_{j,\alpha}$ and $v = p^0_{j,\alpha} - w^0_{j,\alpha} + h_{j,\alpha}$, $|\alpha| \leq M-j-2$, we solve (3.30) and prove as above that the maps (3.29) are continuous. In this way we obtain symbols p^0 and a^0 satisfying (3.23) and such that $R_{1j} = 0$ for $1 \leq j \leq M-2$. Now Lemma 3.2 implies that $R(\lambda) = R^0(\lambda) + O_{\mathcal{B}}(|\lambda|^{-M})$, where $R^0(\lambda)$ satisfies (3.24). □

We are going to write p^0_0 in an invariant form. For $j=0$ we have

$$p^0_0(I^0) \;=\; ic \,+\, 2i\sum_{j=0}^{m-1} \int_{\mathbb{T}^{n-1}} \frac{K\circ\pi_\Gamma}{\sin\theta}(B^j\chi(\varphi,I^0))\, d\varphi\,,$$

where c is independent of K. Denote by $d\mu_j$ the measure on $\Lambda_j = B^j(\Lambda) = B^j\left(\chi\mathbb{T}^{n-1}\right)$, $0 \leq j \leq m$, defined by $d\mu_j = (\chi^{-1}B^{-j})^*(d\varphi)$. It is easy to see that the latter is a Leray form on Λ_j. Indeed, setting $\Omega_j = (\chi^{-1}B^{-j})^*(dI_1 \wedge \cdots \wedge dI_{n-1})$ we obtain that $d\mu_j$ is the measure on Λ_j associated with the volume form $\imath_j^* V_j$, where $(n-1)!V_j \wedge \Omega_j = \omega_0^{n-1}$ in U_j, $\imath_j : \Lambda_j \to T^*\Gamma$ is the embedding map, and ω_0 is the symplectic two-form on $T^*\Gamma$. Moreover, $B^*(d\mu_{j+1}) = d\mu_j$ for any $0 \leq j \leq m-1$, and since P^0 acts on $\chi^{-1}(\Lambda_0)$ as a rotation by $2\pi\omega$, we get $d\mu_m = P^*(d\mu_0) = d\mu_0$, and we set $d\mu = d\mu_0$. This implies

$$p^0_0(I^0) \;=\; ic \,+\, 2i\frac{(2\pi)^{n-1}}{\mathrm{vol}\,(\Lambda)} \sum_{j=0}^{m-1} \int_\Lambda \frac{K\circ\pi_\Gamma}{\sin\theta} \circ B^j\, d\mu\,.$$

Consider the λ-FIOs $W^0(\lambda)$ and $R_1(\lambda)$ given by (3.20) with phase function Φ, and amplitudes $p_0 + \lambda^{-1}p^0$, $p^0(I) = \sum_{j=0}^{M-2} p^0_j(I)\lambda^{-j}$, and $r = r_0 + \lambda^{-1}r^0$, $r_0(\varphi,I) = \sum_{j=0}^{M-2} r^0_j(\varphi,I)\lambda^{-j}$, respectively, which are uniformly compactly supported with respect to I in D. We consider an almost analytic extensions of order $3M$ of the phase function Φ in $I = \xi + i\eta$ given by

$$\Phi(x, \xi + i\eta) = \sum_{|\alpha| \leq 3M} \partial_\xi^\alpha \Phi(x, \xi)(i\eta)^\alpha (\alpha!)^{-1} .$$

It is easy to see that $\overline{\partial}_I \Phi(x, \xi + i\eta) = O(|\eta|^{3M})$. In the same way we construct an almost analytic extension of order M of the function ψ, which was used to define the class $\widetilde{S}_{l,s,N}$. We have $\psi(\xi + i\eta) = 1$ in a complex neighborhood of I^0 and $\psi(\xi + i\eta) = 0$ for $\xi \notin D$.

Proposition 3.3 *We have*

$$W^0(\lambda)e_k(\varphi) \;=\; e^{i\lambda\Phi(\varphi,(k+\vartheta_0/4)/\lambda)}(p_0 + \lambda^{-1}p^0)((k + \vartheta_0/4)/\lambda, \lambda)e_k(\varphi) \;+\; O_{\mathcal{B}}(|\lambda|^{-M}) , \tag{3.31}$$

and

$$R(\lambda)e_k(\varphi) \;=\; O_{\mathcal{B}}(|\lambda|^{-M} + |I^0 - (k + \vartheta/4)/\lambda|^M) , \tag{3.32}$$

for any $\varphi \in \mathbb{T}^{n-1}$, $\lambda \in \mathcal{D}$, and $k \in \mathbb{Z}^{n-1}$, such that $|k| \leq C|\lambda|$ and $C \gg 1$.

Proof. We obtain as above

$$\begin{aligned} \widetilde{W^0(\lambda)e_k}(x) \;&=\; \widetilde{e_k}(x)\, e^{i\lambda\Phi(x,\xi_k)} \\ &\times \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{i\lambda\langle x-y+\Phi_0(x,\xi_k,\eta_k),\eta_k\rangle}\, (p_0 + \lambda^{-1}p^0)(I, \lambda)\, dI\, dy , \end{aligned}$$

where $\Phi_0(x, \xi, \eta) = \int_0^1 \nabla_\xi \Phi(x, \xi + \tau\eta) d\tau$, $\xi_k = (k+\vartheta_0/4)/\lambda$ and $\eta_k = I - (k+\vartheta_0/4)/\lambda$. Deforming the contour of integration we obtain

$$\begin{aligned} W^0(\lambda)e_k(\varphi) \;&=\; e_k(x)\, e^{i\lambda\Phi(\varphi,(k+\vartheta_0/4)/\lambda)} \\ &\times \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle u,v\rangle}\, (p_0 + \lambda^{-1}p^0)(v + (k + \vartheta_0/4)/\lambda, \lambda)\, du\, dv \;+\; O_{\mathcal{B}}(|\lambda|^{-M}) , \end{aligned}$$

which implies (3.31).

To prove (3.32) we write $\widetilde{R^0(\lambda)e_k}(x)$ as an oscillatory integral as above, and then we change the contour of integration with respect to y by

$$y \to v = y - x - \Phi_0(x, (k + \vartheta_0/4)/\lambda, I - (k + \vartheta_0/4)/\lambda) .$$

This implies

$$\begin{aligned} R^0(\lambda)e_k(\varphi) \;&=\; e_k(\varphi)\, e^{i\lambda\Phi(\varphi,(k+\vartheta_0/4)/\lambda)} \\ &\times \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v,I-(k+\vartheta_0/4)/\lambda\rangle}\, (r_0 + \lambda^{-1}r^0)(\varphi, I, \lambda)\, dI\, dv \end{aligned}$$

modulo $O_{\mathcal{B}}(|\lambda|^{-M})$. We write now r^0 in the form (3.22). Integrating $N-j-1$ times by parts with respect to v in the corresponding oscillating integral with amplitude $r^0_{j,\alpha}(\varphi, I)(I-I^0)^\alpha$, $|\alpha| = M-j-1$, we replace $(I-I^0)^\alpha$ by $((k+\vartheta_0/4)/\lambda) - I^0)^\alpha$. Hence,

$$\begin{aligned} R^0(\lambda)e_k(\varphi) &= e_k(\varphi)e^{i\lambda\Phi(\varphi,(k+\vartheta_0/4)/\lambda)} \\ &\times \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v, I-(k+\vartheta_0/4)/\lambda\rangle} f_k(\varphi, I, \lambda)\, dI\, dv \,+\, O_{\mathcal{B}}(|\lambda|^{-M})\,, \end{aligned}$$

where

$$\begin{aligned} f_k(\varphi, I, \lambda) &= |(k+\vartheta_0/4)/\lambda) - I^0|^{2M} r_0(\varphi, I)|I-I^0|^{-2M} \\ &+ \sum_{j=0}^{M-2} \sum_{|\alpha|=M-j-1} \lambda^{-j}\, ((k+\vartheta_0/4)/\lambda) - I^0)^\alpha\, r^0_{j,\alpha}(\varphi, I)\,. \end{aligned}$$

Since $r^0_{j,\alpha} \in C^{2n}(\mathbb{T}^{n-1} \times D)$ is continuous with respect to $K \in \mathcal{B}$ and $\mathcal{B}$ is bounded in C^l, integrating n times by parts with respect to I in the last integral we gain $O_{\mathcal{B}}((1+|\lambda v|)^{-n})$, and we obtain (3.32). □

3.4 *Construction of quasimodes.*

The index set $\mathcal{M}$ of the quasimode $\mathcal{Q}$ we are going to construct is defined as follows. We say that the pair $q = (k, \ell) \in \mathbb{Z}^{n-1} \times \mathbb{Z}$ belongs to $\mathcal{M}$ if there exists $\mu^0_q > 0$ such that the following quantization conditions hold:

$$\mu^0_q(I^0, L(I^0)) \;=\; (k + \vartheta_0/4, 2\pi\ell - \pi\vartheta/4) \,+\, O(1)\,, \tag{3.33}$$

as $|q| = |k| + |\ell| \to \infty$. We have $(I^0, L(I^0)) \neq (0,0)$ in view of (3.18), hence, there is $C > 0$ such that $\mu^0_q \geq C|q|$. Note that (3.33) still holds if we replace μ^0_q by

$$\lambda \in B(\mu^0_q) := \{\lambda \in \mathbb{C} \,:\, |\lambda - \mu^0_q| \leq C_0\}\,,$$

where $C_0 \gg 1$ is fixed, and the estimate $O(1)$ in (3.33) remains uniform with respect to $q \in \mathcal{M}$ and $\lambda \in B(\mu^0_q)$. Using (3.31) for $q \in \mathcal{M}$ and $\lambda \in B(\mu^0_q)$ we obtain

$$W_0(\lambda)e_k \;=\; Z_q(\lambda)\, e_k \,+\, O_{\mathcal{B}}(|\lambda|^{-M})e_k\,,$$

where

$$\begin{aligned} Z_q(\lambda) &= e^{i\lambda L((k+\vartheta_0/4)/\lambda) + i\pi\vartheta/4} \left(1 + \lambda^{-1}p^0((k+\pi\vartheta_0/4)/\lambda, \lambda)\right) \\ &= \exp\left[i\lambda L((k+\vartheta_0/4)/\lambda) + i\pi\vartheta/4 + \operatorname{Log}\left(1 + \lambda^{-1}p^0((k+\vartheta_0/4)/\lambda, \lambda)\right)\right]\,, \end{aligned}$$

where $\operatorname{Log} z = \ln|z| + i\arg z,\; -\pi < \arg z < \pi$. On the other hand, (3.32) and (3.33) imply

$$R(\lambda)e_k \;=\; O_{\mathcal{B}}(|\lambda|^{-M})\, e_k\,.$$

Hence,

$$W_1(\lambda)A(\lambda)e_k \;=\; \left(e^{i\pi\vartheta/4} Z_q(\lambda) + O_{\mathcal{B}}(|\lambda|^{-M})\right) e_k\,. \tag{3.34}$$

We are going to solve the equation

$$e^{i\pi\vartheta/4} Z_q(\lambda) \;=\; 1\,, \quad \lambda\in B_1(\mu_q^0)\,,$$

modulo $O_{\mathcal B}(|\lambda|^{-M})$. To this end we are looking for a perturbation $\lambda=\mu_q$ of μ_q^0 such that

$$\mu_q L((k+\vartheta_0/4)/\mu_q) + \pi\vartheta/4$$

$$+\frac{1}{i}\,\mathrm{Log}\,\left(1+\mu_q^{-1}p^0((k+\vartheta_0/4)/\mu_q,\mu_q)\right) \;=\; 2\pi\ell + O_{\mathcal B}(|\mu_q|^{-M})\,.$$

Introduce a small parameter $\varepsilon_q=(\mu_q^0)^{-1}$. We are looking for

$$\mu_q \;=\; \mu_q^0 + c_{q,0} + c_{q,1}\varepsilon_q + \cdots c_{q,M-1}\varepsilon_q^{M-1}\;, \quad \zeta_q \;=\; I^0 + b_{q,0}\varepsilon_q + \cdots b_{q,M-1}\varepsilon_q^{M} + b_{q,M}\varepsilon_q^{M+1}$$

such that

$$\begin{cases} \mu_q\zeta_q &=& k+\vartheta_0/4 \\ \mu_q L(\zeta_q) &=& 2\pi\ell - \pi\vartheta/4 - \frac{1}{i}\,\mathrm{Log}\,(1+\mu_q^{-1}p^0(\zeta_q,\mu_q)) + O_{\mathcal B}(\varepsilon_q^M)\,. \end{cases}$$

Recall that

$$p^0(\zeta_q,\mu_q) = p_0^0(\zeta_q) + \cdots + p_{M-2}^0(\zeta_q)\mu_q^{-M+2}\,, \quad p_m^0(\zeta_q) = \sum_{|\alpha|\le M-m-2} p_{m,\alpha}^0(\zeta_q - I^0)^\alpha\,.$$

Then

$$\mathrm{Log}\,\left(1+\mu_q^{-1}p^0(\zeta_q,\mu_q)\right) = \sum_{j=1}^{M-1} u_{q,j}\varepsilon_q^j + O_{\mathcal B}(\varepsilon_q^M)\,,$$

where $u_{q,j}$ are polynomials of $c_{q,m}$ and $b_{q,m}$, $0\le m\le j-2$, the coefficients of which polynomials of $p_{m,\alpha}^0$, $m+|\alpha|\le j-1$. Moreover, $u_{q,1}=-p_{0,0}^0$. Using the Taylor expansion of $L(I)$ at I^0 up to order M as well as (3.33) we obtain for $0\le j\le M-1$ the following linear system

$$\begin{cases} b_{q,j} + c_{q,j}I^0 &=& W_{q,j} \\ L(I^0)c_{q,j} + 2\pi\langle\omega, b_{q,j}\rangle &=& V_{q,j}\,, \end{cases}$$

where $V_{q,j}$ and $W_{q,j}$ are polynomials of $c_{q,m}$ and $b_{q,m}$, $0\le m<j$, the coefficients of which are polynomials of $p_{m,\alpha}^0$, $m+|\alpha|<j$. It is easy to see that the corresponding determinant is

$$L(I^0) - 2\pi\langle I^0,\omega\rangle = (2\pi)^{1-n}\int_\Lambda A(\varrho)\,d\mu > 0\,,$$

in view of (3.18), and we obtain an unique solution $(c_{q,j}, b_{q,j})$, $0\le j\le M-1$. More precisely,

$$c_{q,j} = (L(I^0) - 2\pi\langle I^0,\omega\rangle)^{-1}(V_{q,j} - 2\pi\langle\omega, W_{q,j}\rangle)\,,$$

and $b_{q,j} = W_{q,j} - c_{q,j}I^0$. We choose $b_{q,M}$ so that $\mu_q\zeta_q = k+\vartheta_0/4$.

We have

$$W_{q,0} = k+\vartheta_0/4 - \mu_q^0 I^0 = O(1)\,,\; V_{q,0} = 2\pi\ell - \pi\vartheta/4 - \mu_q^0 L(I^0) = O(1)\,,\; q\in\mathcal M\,,$$

in view of (3.33). Hence, $b_{q,0}$ and $c_{q,0}$ are uniformly bounded and they do not depend on K. By recurrence we prove that $b_{q,j}$ and $c_{q,j}$ are continuous with respect to K and uniformly bounded with respect to $q \in \mathcal{M}$ and $K \in \mathcal{B}$. For $j = 1$ we obtain $W_{q,1} = -c_{q,0} b_{q,0}$ and $V_{q,1} = -2\pi\langle \omega, b_{q,0}\rangle - \frac{1}{2}\langle \nabla^2 L(I^0) b_{q,0}, b_{q,0}\rangle + \frac{1}{i} p^0_{0,0}$, and we get

$$c_{q,1} \, = \, c'_{q,1} \, + \, \frac{2(2\pi)^{n-1}}{\int_\Lambda A(\varrho) d\mu} \sum_{j=0}^{m-1} \int_{\mathbb{T}^{n-1}} \frac{K \circ \pi_\Gamma}{\sin\theta} \circ B^j \, d\mu \, ,$$

where $c'_{q,1}$ does not depend on K.

For each $q = (k, \ell) \in \mathcal{M}$ we set

$$v^0_q \; := \; T(\mu_q) A(\mu_q) e_k \quad \text{and} \quad u^0_q \; := \; G(\mu_q) v^0_q \; = \; G(\mu_q) T(\mu_q) A(\mu_q) e_k \, .$$

Then using (3.34), we obtain

$$(W(\mu_q) - \operatorname{Id}) v^0_q = O_{\mathcal{B}}(|\lambda|^{-M}) \, v^0_q \, , \tag{3.35}$$

and we get

$$\left| \begin{array}{rcl} \left(\Delta \, - \, \mu_q^2\right) u^0_q & = & O_{\mathcal{B}}(|\mu_q|^{-M}) \, u^0_q \text{ in } X \, , \\ \mathcal{N} u^0_q|_\Gamma & = & O_{\mathcal{B}}(|\mu_q|^{-M}) \, u^0_q \end{array} \right.$$

Lemma 3.4 *There is $C > 0$ such that*

$$C^{-1}(1 + |\mu_q|)^{-1} \leq \|u^0_q\|_{L^2(X)} \leq C$$

for any $q \in \mathcal{M}$.

Proof. Since $T(\lambda)$, $A(\lambda)$ and $G(\lambda)$ are uniformly bounded in the corresponding L^2 norms, we obtain

$$\forall q \in \mathcal{M} \, , \quad \|u^0_q\|_{L^2(X)} \; \leq C \, ,$$

where $C > 0$ is a constant. We have

$$\|u^0_q|_\Gamma\|_{L^2(\Gamma)} \; \leq C \|u^0_q\|_{H^1(X)} \tag{3.36}$$

for some $C > 0$ and any $q \in \mathcal{M}$, where $H^1(X)$ is the corresponding Sobolev space. We are going to show that

$$\|u^0_q\|_{H^1(X)} \leq C(1 + |\mu_q|) \|u^0_q\|_{L^2(X)} + O(|\mu_q|^{-1}) \|u^0_q|_\Gamma\|_{L^2(\Gamma)} \, , \quad q \in \mathcal{M} \, . \tag{3.37}$$

Let $\chi_1 \in C^\infty_0(X)$ have its support in the interior of X and $\chi_2 = 1 - \chi_1$. Denote by $\Psi(\lambda)$ a λ-PDO with $\mathrm{WF}'(\Psi)$ contained in the interior of T^*X and such that

$$\mathrm{WF}'(\Psi - \operatorname{Id}) \cap \{(x, \xi) \in T^*X \, : \, h(x, \xi) < 2 \, , \, x \in \operatorname{supp}(\chi_1)\} = \emptyset \, .$$

Then for any first order differential operator V in X the operator $\lambda^{-1} V \Psi(\lambda) : L^2(X) \to L^2(X)$ is uniformly bounded and we have

$$\|\chi_1 G(\lambda) v\|_{H^1(X)} \leq C(1 + |\lambda|) \|G(\lambda) v\|_{L^2(X)} + O(|\lambda|^{-1}) \|v\|_{L^2(\Gamma)} \, ,$$

$\lambda \in \mathcal{D}$, $v \in L^2(X)$. Near the boundary we choose local coordinates so that $X = \{x_1 \geq 0\}$ and suppose that $0 \leq x_1 \leq \varepsilon$ and $\varepsilon \ll 1$ on the support of χ_2. Now we write $H_j(\lambda)$ in these local coordinates with a phase function $\phi(x,\xi') + \langle y', \xi'\rangle$, $\xi' = (\xi_2, \ldots, \xi_n)$, $y' = (y_2, \ldots, y_n)$, where $\phi(0, x', \xi') = \langle x', \xi'\rangle$ and with a C^∞ compactly supported amplitude $a(x, \xi', \lambda)$ of order 0. Then $\chi_2(\partial/\partial x_k)H_j(\lambda)u = \lambda\chi_2 B_k(\lambda)H_j(\lambda)u + O(|\lambda|^{-1})u$, where B_k stands for a continuous family of λ-PDOs of order 0 on the boundary $x_1 \mapsto B_k(x_1, x', D')$. This implies

$$\|\chi_2 G(\lambda)v\|_{H^1(X)} \leq C(1 + |\lambda|)\|G(\lambda)v\|_{L^2(X)} + O(|\lambda|^{-1})\|v\|_{L^2(\Gamma)}\,,$$

$\lambda \in \mathcal{D}$, $v \in L^2(X)$, and we obtain (3.37).

Since $i^*_\Gamma G(\lambda) = \psi(\lambda) + \widetilde{W}(\lambda)\psi(\lambda) + O_{\mathcal{B}}(|\lambda|^{-M})$, using (3.35) we obtain

$$u^0_q|_\Gamma = i^*_\Gamma G(\mu_q)v^0_q = v^0_q + \widetilde{W}(\mu_q)v^0_q = v^0_q + Q^{-1}_m(\mu_q)W(\mu_q)v^0_q = 2v^0_q + O(|\mu_q|^{-1})v^0_q.$$

This estimate combined with (3.36) and (3.37) implies the lemma. □

Normalizing $u_q = u^0_q\|u^0_q\|^{-1}$ we obtain a quasimode (μ_q, u_q) of order $N = M - 1$. Next we show that μ_q can be chosen real-valued. Applying Green's formula we get

$$|\mu_q^2 - \overline{\mu_q}^2| \;\leq\; |\langle \mu_q^2 u_q, u_q\rangle \,-\, \langle u_q, \mu_q^2 u_q\rangle| \;=\; O_{\mathcal{B}}(|\mu_q|^{-N})\,,$$

which allows us to take μ_q in $\mathbb{R}$. Choosing $|q| \gg 1$ we can suppose that μ_q is positive. Notice that K should be in $C^k(\Gamma, \mathbb{R})$ with $k \geq (M-1)([\tau]+n) + 2n + 2 = N([\tau]+n) + 2n + 2$.

4 Spectral invariants for continuous deformations of the potential

Let V_t, $t \in [0,1]$, be a continuous family of C^ℓ real-valued potentials in X, $\ell \in \mathbb{N}$, which means that the map $[0,1] \ni t \mapsto V_t$ is continuous in $C^\ell(X, \mathbb{R})$. Denote by Δ_t the selfadjoint operators $\Delta + V_t$ in $L^2(X)$ with Dirichlet or Robin (Neumann) boundary conditions on Γ. We consider the corresponding spectral problem

$$\begin{cases} \Delta u \;+\; V_t u \;=\; \lambda u \quad \text{in X}\,, \\ \qquad\quad \mathcal{B}u \;=\; 0 \quad \text{in } \Gamma\,, \end{cases}$$

where $\mathcal{B}u = u|_\Gamma$ or $\mathcal{B}u = \frac{\partial u}{\partial \nu}|_\Gamma - K\,u|_\Gamma$, K being a smooth real valued function on Γ independent of t. As above we suppose that there exists a Kronecker torus Λ of $P = B^m$ satisfying (H_3) and we set

$$W_t(x,\xi) \;=\; \int_0^{T(x,\xi)} V_t\left(\pi_X(\exp(sX_g)(x,\xi^+))\right)\, ds\;, \quad (x,\xi) \in \Lambda\,,$$

where $T(x,\xi)$ is the return time function and $\pi_X : T^*X \to X$ is the natural projection. Set $\ell = ([2d]+1)([\tau]+n) + 2n + 2$, where τ is the exponent in the Diophantine condition.

Theorem 4.1 *Let Λ be a Kronecker torus of the billiard ball map with a Diophantine vector of rotation. Let V_t, $t \in [0,1]$, be a continuous family of real-valued potentials in $C^\ell(X, \mathbb{R})$ such that Δ_t satisfy the isospectral condition $(H_1) - (H_2)$. Then*

$$\forall\, t \in [0,1], \quad \sum_{j=0}^{m-1} \int_\Lambda \frac{W_t \circ \pi_\Gamma}{\sin\theta} \circ B^j\, d\mu \;=\; \sum_{j=0}^{m-1} \int_\Lambda \frac{W_0 \circ \pi_\Gamma}{\sin\theta} \circ B^j\, d\mu\,.$$

To prove the theorem we construct as in Theorem 2.2 a continuous family of quasimodes

$$(\mu_q(t), u_q(t))_{q\in\mathcal{M}}\ ,\ \mathcal{M}\subset\mathbb{Z}^n\,,$$

of Δ_t of order N such that

$$\mu_q(t) = \mu_q^0 + c_{q,0} + c_{q,1}(t)(\mu_q^0)^{-1} + \cdots + c_{q,N}(t)(\mu_q^0)^{-N}$$

where μ_q^0 and $c_{q,0}$ are independent of t, $\mu_q^0 \ge C|q|$, $C>0$, and $c_{q,j}(t)$ is continuous in $t\in[0,1]$. Moreover,

$$c_{q,1}(t) = c'_{q,1} + c''_1 \sum_{j=0}^{m-1} \int_\Lambda \frac{W_t\circ\pi_\Gamma}{\sin\theta}\circ B^j\, d\mu\,,$$

$c'_{q,1}$ is independent of t, and

$$c''_1(t) = 2(2\pi)^{n-1}\left(\int_\Lambda A(\varrho)\, d\mu\right)^{-1}\,.$$

To construct the quasimodes we consider for each $j=0,\dots,m-1$ the microlocal outgoing parametrix $\widetilde{H}_j : C^\infty(\Gamma)\to C^\infty(\widetilde{X})$, of the Dirichlet problem for $\Delta-\lambda^2-V$ which is defined as follows

$$\begin{cases} (\Delta-\lambda^2-V_t)\widetilde{H}_j(\lambda) = O_M(|\lambda|^{-N-1}) \text{ in } \widetilde{X}\,, \\ \mathrm{WF}'(\imath_\Gamma^*\, H_j(\lambda)) \subset U_j\cup U_{j+1}\,, \\ \mathrm{WF}'(\imath_\Gamma^*\,\widetilde{H}_j(\lambda) - \mathrm{Id})\cap \mathrm{WF}'(\psi_j(\lambda)) = \emptyset\,, \\ \mathrm{WF}'(\widetilde{H}_j(\lambda))\cap (U_j\times\pi_\Sigma^{-1}(U_j)) \subset U_j\times\pi_\Sigma^{+}(U_j)\,, \end{cases}$$

We are looking for $\widetilde{H}_j(\lambda)$ of the form $\widetilde{H}_j(\lambda) = H_j(\lambda)+\lambda^{-1}H_j^0(\lambda)$, where $H_j^0(\lambda)$ is a FIO of order 1/4 having the same canonical relation as $H_j(\lambda)$. It satisfies the equation

$$(\Delta-\lambda^2-V_t)H_j^0(\lambda) - V_tH_j(\lambda) = O_N(|\lambda|^{-N-1}) \text{ in } \widetilde{X}\,,$$

hence, its principal symbol $p_j^0(x,\xi)$ satisfies the equation $\{g, p_j^0\} = iV_t$. Taking into account the boundary values at U_j we get

$$p_j^0(\varrho, s) = i\int_0^s V_t(\exp(uX_g)(\varrho))\, du\ ,\quad \varrho\in U_j\,.$$

Then

$$\widetilde{G}_j(\lambda) := G_j(\lambda) + \lambda^{-1}G_j^0(\lambda)$$

is a λ-FIO the canonical relation of which is just the graph of the restriction of the billiard ball map $B: U_j\to U_{j+1}$. Moreover, the principal symbol of $G_j^0(\lambda)$ is equal to $p_j^0(\varrho, T_j(\varrho))$. Arguing as in Sect. 3 we complete the construction of the quasimodes.

5 Spectral rigidity for Liouville billiard tables

We recall from [13] the definition of Liouville billiard tables of dimension two. We consider two even functions $f \in C^\infty(\mathbb{R})$, $f(x+2\pi) = f(x)$, and $q \in C^\infty([-N,N])$, $N > 0$, such that

- $f > 0$ if $x \notin \pi\mathbb{Z}$, and $f(0) = f(\pi) = 0$, $f''(0) > 0$;
- $q < 0$ if $y \neq 0$, $q(0) = 0$ and $q''(0) < 0$;
- $f^{(2k)}(\pi l) = (-1)^k q^{(2k)}(0)$, $l = 0, 1$, for every natural $k \in \mathbb{N}$.

Consider the quadratic forms

$$\begin{aligned} dg^2 &= (f(x) - q(y))(dx^2 + dy^2) \\ dI^2 &= (f(x) - q(y))(q(y)dx^2 + f(x)dy^2) \end{aligned}$$

defined on the cylinder $C = \mathbb{T}^1 \times [-N, N]$.

The involution $\sigma_0 : (x,y) \mapsto (-x,-y)$ induces an involution of the cylinder C, that will be denoted by σ_0 as well. We identify the points m and $\sigma_0(m)$ on the cylinder and denote by $\widetilde{C} := C/\sigma_0$ the topological quotient space. Let $\sigma : C \to \widetilde{C}$ be the corresponding projection. A point $x \in C$ is called *singular* if $\sigma^{-1}(\sigma(x)) = x$, otherwise it is a *regular* point of σ. Obviously, the singular points are $F_1 = \sigma(0,0)$ and $F_1 = \sigma(1/2,0)$. It is shown in [13] that the quotient space $\widetilde{C}$ is homeomorphic to the unit disk $\mathbf{D}^2$ in $\mathbb{R}^2$ and that there exist an unique differential structure on C such that the projection $\sigma : C \to \widetilde{C}$ is a smooth map, σ is a local diffeomorphism in the regular points, and the push-forward $\sigma_* g$ gives a smooth Riemannian metric while $\sigma_* I$ is a smooth integral of the corresponding billiard flow on it. We denote by X the space $\widetilde{C}$ provided with that differentiable structure and call $(X, \sigma_* g)$ a Liouville billiard table. Any Liouville billiard table possesses the string property which means that any broken geodesic starting from the singular point F_1 (F_2) passes through F_2 (F_1) after the first reflection at the boundary and the sum of distances from any point of Γ to F_1 and F_2 is constant.

We impose the following additional conditions:

- the boundary Γ of X is locally geodesically convex which amounts to $q'(N) < 0$;
- $f(x) = f(\pi - x)$ for any x and f is strictly monotone on the interval $[0, \pi]$;

Liouville billiard tables satisfying the conditions above will be said to be *of classical type*. One of the consequences of the last condition is that there is a group $I(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on (X, g) by isometries. It is generated by the involutions σ_1 and σ_2 defined by $\sigma_1(x,y) = (x,-y)$ and $\sigma_2(x,y) = (\pi - x, y)$. We point out that in contrast to [13] we do not assume f and q to be analytic. Examples of Liouville billiard tables of classical type on surfaces of constant curvature and quadrics are provided in [13]. The only Liouville billiard table in $\mathbb{R}^2$ is the interior of the ellipse because of the string property.

Proof of Corollary 1.3. A first integral of B in $B^*\Gamma$ is the function $\mathcal{I}(x,\xi) = f(x) - \xi^2$ the regular values h of which belong to $(q(N), 0) \cup (0, f(\pi/2))$ (see [13], Lemma 4.1 and Proposition 4.2). Each regular level set L_h consists of two connected circles $\Lambda^\pm(h)$ which are invariant with respect to B for $h \in (q(N), 0)$ and to B^2 for $h \in (0, f(1/4))$. The Leray form on L_h is

$$\lambda_h = \begin{cases} \frac{dx}{\sqrt{f(x)-h}}, & \xi > 0\,, \\ -\frac{dx}{\sqrt{f(x)-h}}, & \xi < 0\,. \end{cases}$$

Given a continuous function G on Γ we consider the corresponding Radon transform assigning to each circle $\Lambda^{\pm}(h)$ the integral

$$R_G(\Lambda^{\pm}(h)) = \int_{\Lambda^{\pm}(h)} (G \circ \pi_\Gamma)\lambda_h \,.$$

We take the exponent in the Diophantine condition to be $\tau = 3/2$. Then $\ell = 3[2d] + 9$. Set $G_t(x) = K_t(x)/\sin\theta(x,h)$, $t = 1, 2$. Since $G_0,\, G_1 \in \mathrm{Symm}^{\ell}(\Gamma)$, using Theorem 1.1 we obtain that $R_{G_0}(\Lambda^{\pm}(h)) = R_{G_1}(\Lambda^{\pm}(h))$ for each regular value h such that the corresponding frequency ω is Diophantine with exponent $\tau = 3/2$. On the other hand, the set of all Diophantine numbers with a fixed exponent $\tau > 1$ is dense in $\mathbb{R}$ and by continuity we get it for any regular value. It is easy to see that

$$\sin\theta \;=\; \sqrt{\frac{h - q(N)}{f(x) - q(N)}}\,,$$

hence,

$$R_{G_t}(\Lambda^{\pm}(h)) = \pm\frac{1}{\sqrt{h - q(N)}} \int\limits_0^{2\pi} \frac{K_t(x)}{\sqrt{f(x) - h}} \sqrt{f(x) - q(N)}\, dx \;,\;\; h \in (q(N), 0) \cup (0, f(\pi/2))\,,$$

does not depend on $t \in [0, 1]$. Since K_t, $t = 0, 1$, are invariant with respect to the action of $I(X)$, this implies $K_0 \equiv K_1$ as in [13]. □

Spectral rigidity for higher dimensional Liouville billiard tables will be obtained in [14]. We point out that we do not need analyticity and the billiard tables we consider are supposed to be smooth only.

Appendix

We consider families of λ-PDOs with symbols of finite smoothness which depend continuously on $K \in C^l(\Gamma)$. Given four positive integers l, $\widetilde{l}$, $N \geq 1$ and $m \geq 2$ such that $\widetilde{l} \geq mN + 2n$, and a bounded subset $\mathcal{B}$ of $C^l(\Gamma, \mathbb{R})$, we say that a family of operators Q depending on $K \in \mathcal{B}$ belongs to $\mathrm{PDO}_{\widetilde{l},m,N}(\Gamma; \mathcal{B}; \lambda)$ if in any local coordinates it can be written in the form $\mathrm{OP}_\lambda(q) + O_{\mathcal{B}}(|\lambda|^{-N})$, where the distribution kernel of $\mathrm{OP}_\lambda(q)$ is

$$\mathrm{OP}_\lambda(q)(x,y) := (\lambda/2\pi)^{n-1} \int e^{i\lambda\langle x-y,\xi\rangle} q(x,\xi,\lambda)\, d\xi\,, \tag{A.1}$$

with amplitude

$$q(x,\xi,\lambda) = \sum_{k=0}^{N-1} q_k(x,\xi)\lambda^{-k}\,, \tag{A.2}$$

and $q_k \in C_0^{\widetilde{l}-mk}(T^*R^{n-1})$, $0 \leq k \leq N-1$, depends continuously in $K \in C^l(\Gamma, \mathbb{R})$ in the sense that the support of q_k is contained in a fixed compact set independent of K and the map

$$C^l(\Gamma, \mathbb{R}) \ni K \;\to\; q_k \in C^{\widetilde{l}-mk}(T^*R^{n-1})$$

is continuous. Hereafter, $O_{\mathcal{B}}(|\lambda|^{-N}) : L^2(\Gamma) \to L^2(\Gamma)$ stands for a family of operators depending on $K \in \mathcal{B}$, the norm of which is uniformly bounded by $C_{\mathcal{B}}(1+|\lambda|)^{-N}$, and λ belongs to the complex strip $\mathcal{D}$. We denote the class of symbols q by $S_{\widetilde{l},m,N}(T^*\mathbb{R}^{n-1};\mathcal{B};\lambda)$. Using the L^2-continuity theorem, [8], Theorem 18.1.11$'$, it is easy to see that the operators of the class $\mathrm{PDO}_{\widetilde{l},m,N}(\Gamma;\mathcal{B};\lambda)$ are uniformly bounded in L^2 with respect to $K \in \mathcal{B}$ (it suffices $\widetilde{l} \geq mN+n$). Moreover, the class $\mathrm{PDO}_{\widetilde{l},m,N}(\Gamma;\mathcal{B};\lambda)$ is closed under multiplication and transposition and it does not depend on the choice of the local coordinates modulo $O_{\mathcal{B}}(|\lambda|^{-N})$ (see Remark A.1).

Consider now a λ-FIO A_λ acting on $C_0^\infty(\mathbb{R}^{n-1})$ with distribution kernel

$$K_{A_\lambda}(x,y) = (\lambda/2\pi)^{n-1} \int e^{i\lambda(\langle x-y,\xi\rangle+\psi(x,\xi))} q(x,\xi,\lambda)\, d\xi\,, \tag{A.3}$$

where $q_\lambda = q(\cdot,\cdot,\lambda) \in C_0^n(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, its support is contained in a fixed compact F for each λ, and $\sup_\lambda \|q_\lambda\|_{C^n} < \infty$. We suppose that the phase function $S(x,\xi) = \langle x,\xi\rangle + \psi(x,\xi)$ is C^∞ and non-degenerate in a neighborhood U of F, which amounts to $|\det \partial_x\partial_\xi S| \geq \delta > 0$ in U. Using a result of Boulkhemair [2], Corollary 1, we obtain

$$\|A_\lambda\|_{\mathcal{L}(L^2)} \;\leq\; C \sup_\lambda \|q_\lambda\|_{C^n}\,, \tag{A.4}$$

where $C = C(S,F) > 0$ does not depend on q_λ. Indeed, if $F \subset B_\varepsilon(\varrho^0) := \{\varrho : |\varrho-\varrho^0| < \varepsilon\} \subset U$, where $\varrho^0 \in F$ and $\varepsilon > 0$ is sufficiently small we can extend S to a globally defined smooth function $\widetilde{S}$ in $T^*\mathbb{R}^{n-1}$ which coincides with S in $B_\varepsilon(\varrho^0)$ and equals the Taylor polynomial of degree 2 of S at ϱ^0 outside $B_{2\varepsilon}(\varrho^0)$ and such that $|\det \partial_x\partial_\xi \widetilde{S}| \geq \delta/2$ in $T^*\mathbb{R}^{n-1}$. Then applying [2], Corollary 1, to the oscillatory integral with phase function $\widetilde{S}$ and amplitude q we obtain (A.4). In the general case we use a suitable partition of the unity of F.

We are going to estimate the following integral for suitable functions a and b

$$q_\lambda(z) = \lambda^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle y,\eta\rangle} a(z,y,\eta,\lambda) b(z,y,\eta,\lambda) dy d\eta\;, \quad z = (x,\xi) \in T^*\mathbb{R}^{n-1}\,,\ \lambda \in \mathcal{D}\,.$$

Lemma A. 1 *Suppose that $a_\lambda = a(\cdot,\lambda)$ and $b_\lambda = b(\cdot,\lambda)$, $\lambda \in \mathcal{D}$, are C^{2n}-smooth and uniformly compactly supported functions, i.e.* $\operatorname{supp} a_\lambda \subset F_1$, $\operatorname{supp} b_\lambda \subset F_2$, *for all λ, where F_1 and F_2 are compact. Then*

$$\sup_\lambda \|q_\lambda\|_{C^n} \;\leq\; C \sup_\lambda \|a_\lambda\|_{C^{2n}} \times \sup_\lambda \|b_\lambda\|_{C^{2n}}\,.$$

where $C = C(F_1,F_2) > 0$. In particular the FIO A_λ with amplitude $q_\lambda(x,\xi)$ satisfies (A.4).

Proof. We have

$$q_\lambda(z) = \lambda^{2n-2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \widehat{a}(z,\lambda\xi,\eta,\lambda) \widehat{b}(z,\lambda(\eta-\xi),\eta,\lambda) d\xi d\eta\,,$$

where $\widehat{a}(z,\lambda\xi,\eta,\lambda)$ stands for the partial Fourier transform $(y \to \lambda\xi)$ of $a(z,y,\eta,\lambda)$. Integrating n times by parts with respect to y we get

$$\|q_\lambda\|_{C^n} \;\leq\; C\|a_\lambda\|_{C^{2n}} \|b_\lambda\|_{C^{2n}} \lambda^{2n-2} \int_{\mathbb{R}^{2n-2}} (1+|\lambda||\xi|)^{-n} (1+|\lambda||\eta-\xi|)^{-n} d\xi d\eta\,,$$

which implies the lemma. □

The frequency set $\mathrm{WF}'(Q_\lambda)$ (modulo $O(|\lambda|^{-N})$) of a λ-PDO Q_λ with symbol q locally given by (A.2) is

$$\mathrm{WF}'\,(Q_\lambda) := \cup_{j=0}^{N-1}\,\mathrm{supp}\,(q_j)$$

in each local chard.

Using Lemma A.1 one can commute λ-PDOs in $\mathrm{PDO}_{\widetilde{l},s,N}(\Gamma,\mathcal{B};\lambda)$ with a classical λ-FIOs $G(\lambda)$ associated to a smooth canonical transformation $\kappa : T^*\Gamma \to T^*\Gamma$ and having a C_0^∞ amplitude in each local cart. More precisely, we have

Lemma A. 2 *Let $Q(\lambda) \in \mathrm{PDO}_{\widetilde{l},m,N}(\Gamma;\mathcal{B};\lambda)$, $\widetilde{l} \geq mM+2n$, and let $G(\lambda)$ be elliptic on $WF'(Q)$. Then there exists $Q'(\lambda) \in \mathrm{PDO}_{\widetilde{l},m,N}(\Gamma;\mathcal{B};\lambda)$ such that*

$$Q(\lambda)G(\lambda) - G(\lambda)Q'(\lambda) = O_{\mathcal{B}}(|\lambda|^{-N}) \ : \ L^2(\Gamma) \longrightarrow L^2(\Gamma) \tag{A.5}$$

and wise versa. The principal symbol of $Q'(\lambda)$ is given by the Egorov's theorem, $\sigma(Q') = \sigma(Q)\circ\kappa$.

Proof. We define $Q' = BQA$, where $\mathrm{WF}'(AB - I) \cap \mathrm{WF}'(Q) = \emptyset$. To prove that $Q'(\lambda) \in \mathrm{PDO}_{\widetilde{l},m,N}(\Gamma;\mathcal{B};\lambda)$, we choose local coordinates x in Γ and write the distribution kernel of $Q(\lambda)$ in the form (A.1) with symbol $q \in S_{\widetilde{l},m,N}(T^*\mathbb{R}^{n-1};\mathcal{B};\lambda)$. We can suppose that distribution kernel of $G(\lambda)$ is given by (A.3) with a smooth compactly supported amplitude a. More generally, we suppose that $a \in S_{\widetilde{l},m,N}(T^*\mathbb{R}^{n-1};\mathcal{B};\lambda)$. Then the distribution kernel of $Q(\lambda)G(\lambda)$ modulo $O_{\mathcal{B}}(|\lambda|^{-N})$ is given by the oscillatory integral (A.3) with amplitude

$$K_1(x,\xi,\lambda)$$

$$= \sum_{j=0}^{N-1}\sum_{r+s=j} \lambda^{-j}\left(\frac{\lambda}{2\pi}\right)^{n-1}\int_{\mathbb{R}^{2n-2}} e^{i\lambda(\langle x-z,\eta-\xi\rangle+\psi(z,\xi)-\psi(x,\xi))}\, q_r(x,\eta)a_s(z,\xi)\,d\eta dz\ .$$

Set

$$\psi_1(x,z,\xi) = \int_0^1 \nabla_x\psi(x+\tau z,\xi)d\tau\,.$$

Changing the variables we get

$$K_1(x,\xi,\lambda) \ = \ \sum_{j=0}^{N-1}\sum_{r+s=j} \lambda^{-j}\left(\frac{\lambda}{2\pi}\right)^{n-1}\int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle z,\eta\rangle}\, q_r(x,\eta+\xi+\psi_1(x,z,\xi))a_s(z+x,\xi)\,d\eta dz\,.$$

We develop q_r in Taylor polynomials with respect to η at $\eta = 0$ up to order $O(|\eta|^{N-j})$. On the other hand $\partial_\eta^\beta q_r \in C^{\tilde{l}-mr-|\beta|}(T^*\mathbb{R}^{n-1})$, and

$$\tilde{l} - mr - 2|\beta| \geq \tilde{l} - mr - 2(N-r) \geq \tilde{l} - mN \geq 2n \tag{A.6}$$

for $|\beta| \leq N - j \leq N - r$, and integrating β times by parts with respect to η we obtain

$$K_1(x,\xi,\lambda) = \sum_{j=0}^{N} F_j(x,\xi)\lambda^{-j}\,,$$

where

$$F_j(x,\xi) \;=\; \sum_{r+s+|\beta|=j} \frac{1}{\beta!}\left[D_z^\beta\left(\partial_\eta^\beta\, q_r(x,\eta+\xi+\psi_1(x,z,\xi)\, a_s(z+x,\xi)\right)\right]_{|z=0,\eta=0} \tag{A.7}$$

for $j \leq N-1$. Moreover, using (A.6) and Lemma A.1 we estimate

$$\|F_N\|_{C^n} \leq C \sum_{r+s=j} \sup_\lambda \|q_r(\cdot,\cdot,\lambda)\|_{C^{\tilde{l}-mr}} \times \sup_\lambda \|a_s(\cdot,\cdot,\lambda\|_{C^{\tilde{l}-ms}}\,.$$

In the same way, we write $G(\lambda)Q'(\lambda)$ modulo $O_{\mathcal{B}}(|\lambda|^{-N})$ as a λ-FIO with distribution kerlel (A.3) with amplitude given by the oscillatory integral

$$K_2(x,\xi,\lambda) = \sum_{j=0}^{N-1}\sum_{s+r=j}\left(\frac{\lambda}{2\pi}\right)^{n-1}\lambda^{-j}\int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle z,\eta\rangle}\, a_s(x,\eta+\xi)q'_r(z+x+\psi_2(x,\xi,\eta),\xi)d\eta dz\,,$$

where $\psi_2(x,\xi,\eta) = \int_0^1 \nabla_\xi\psi(x,\xi+\tau\eta)d\tau$. We get as above

$$K_2(x,\xi,\lambda) = \sum_{j=0}^{N} H_j(x,\xi)\lambda^{-j}$$

where

$$\|H_N\|_{C^n} \leq C \sum_{r+s=j} \sup_\lambda \|a_r(\cdot,\cdot,\lambda)\|_{C^{\tilde{l}-mr}} \times \sup_\lambda \|q'_s(\cdot,\cdot,\lambda\|_{C^{\tilde{l}-ms}}$$

and

$$H_j(x,\xi) \;=\; \sum_{r+s+|\beta|=j} \frac{1}{\beta!}\left[D_\eta^\beta\left(a_s(x,\eta+\xi)\partial_z^\beta\, q'_r(z+x+\psi_2(x,\xi,\eta),\xi)\right)\right]_{|\eta=0,z=0} \tag{A.8}$$

for $0 \leq j \leq N-1$. Note that $\psi_1(x,0,\xi) = \nabla_x\psi(x,\xi)$, $\psi_2(x,\xi,0) = \nabla_\xi\psi(x,\xi)$, and that locally graph $\kappa = \{(x,\xi+\nabla_x\psi(x,\xi), x+\nabla_\xi\psi(x,\xi),\xi)\}$. Since $G(\lambda)$ is elliptic on $\mathrm{WF}'(Q)$ we can assume that $a_0(x,\xi) \neq 0$ on the support of $(x,\xi) \to q_r(x,\xi+\nabla_x\psi(x,\xi))$ for any r, and we determine q'_j by recurrence from the equations $H_j(x,\xi) = F_j(x,\xi)$, $j = 0,\ldots,N-1$. It is easy to see by recurrence that $q'_j \in C^{\tilde{l}-mj}(T^*R^{n-1})$ is continuous with respect to $K \in C^l(\Gamma)$. □

Remark A. 1 *We have proved that if $Q(\lambda)$ is a family of λ-PDOs in $\mathbb{R}^{n-1}$ the distribution kernels of which have the form (A.1) with symbol $q \in S_{\tilde{l},m,N}(T^*\mathbb{R}^{n-1};\mathcal{B};\lambda)$ and if the distribution kernels of $G(\lambda)$ are given by (A.3) with amplitude $a \in S_{\tilde{l},m,N}(T^*\mathbb{R}^{n-1};\mathcal{B};\lambda)$, then $Q(\lambda)G(\lambda)$ and $G(\lambda)Q(\lambda)$ are λ-FIOs in $\mathbb{R}^{n-1}$ with distribution kernels (A.3) and amplitudes in $S_{\tilde{l},m,N}(T^*\mathbb{R}^{n-1};\mathcal{B};\lambda)$. By the same argument, the class $\mathrm{PDO}_{\tilde{l},m,N}(\Gamma;\mathcal{B};\lambda)$ is closed under multiplication and transposition and it does not depend on the choice of the local coordinates modulo $O_{\mathcal{B}}(|\lambda|^{-N})$.*

Proof of Lemma 3.2. First we write the operator $W_1(\lambda)A(\lambda)$ in the form (3.20) with amplitude given by the oscillatory integral

$$F(x,I,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{n-1}\int_{\mathbb{R}^{2n-2}} e^{i\lambda(\langle x-z,\xi-I\rangle+\Phi(x,\xi)-\Phi(x,I))}\, w(x,\xi,\lambda)a(z,I,\lambda)\, d\xi dz$$

modulo $O_{\mathcal{B}}(|\lambda|^{-N})$. Set $\Phi_0(x,I,\eta) = L_0(I,\eta) + H_0(x,I,\eta)$, where

$$L_0(I,\eta) = \int_0^1 \nabla_I L(I+\tau\eta)d\tau\ , \quad H_0(x,I,\eta) = \int_0^1 \nabla_I R(x,I+\tau\eta)d\tau\ .$$

Changing the variables and using (3.21) we obtain as above modulo $O_{\mathcal{B}}(|\lambda|^{-N})$

$$F(x,I,\lambda)\ =\ \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v,\eta\rangle} \left(c_0 + \sum_{j=0}^{M-2} \lambda^{-j-1} c_j^0\right)(x,I,v,\eta)\, d\eta dv\ ,$$

where $c_0(x,I,v,\eta) = w_0(x,I+\eta)a_0(v+x+\Phi_0(x,I,\eta),I)$, and

$$\begin{aligned} & c_j^0(x,I,v,\eta) = w_j^0(x,I+\eta)a_0(v+x+\Phi_0(x,I,\eta),I) \\ & + \psi(I)w_0(x,I+\eta) \sum_{|\alpha|\le M-j-2} a_{j,\alpha}^0(v+x+\Phi_0(x,I,\eta),I)\,(I-I^0)^\alpha \\ & + \sum_{r+s=j-1}\ \sum_{|\alpha|\le M-s-2} \psi(I)\, w_r^0(x,I+\eta)a_{s,\alpha}^0(v+x+\Phi_0(x,I,\eta),I)\,(I-I^0)^\alpha\ . \end{aligned}$$

We develop $a_{j,\alpha}^0(v+x+\Phi_0,I)$ in Taylor polynomials with respect to v at $v=0$ up to order $O(|v|^{M-j-1-|\alpha|})$. Since $a^0\ \in \widetilde{S}_{l-[\tau]-n,[\tau]+n,M-1}(\mathbb{T}^{n-1}\times D;\mathcal{B};\lambda)$ and Φ_0 is a smooth function independent of K, we obtain $\partial_x^\beta a_{j,\alpha}^0 \in C^p$ for $|\alpha+\beta|\le M-j-1$, where

$$\begin{aligned} & p = l-(j+1)([\tau]+n) - |\alpha+\beta| \ge |\beta| + l - (j+1)([\tau]+n) - 2|\alpha+\beta| \\ & \ge |\beta| + l - (j+1)([\tau]+n-2) - 2M \ge |\beta| + l - (M-1)([\tau]+n) - 2 \ge |\beta| + 2n\,. \end{aligned} \tag{A.9}$$

In particular, $\partial_x^\beta a_{j,\alpha}^0 \in C^{|\beta|+2n}(\mathbb{T}^{n-1})$, $|\alpha+\beta|\le M-j-1$, $j\le M-2$, depends continuously on $K\in\mathcal{B}$. Integrating β times by parts with respect to η we gain $\lambda^{-|\beta|}$. Notice that all the derivatives of H_0 vanish for $(\eta,I)=(0,I^0)$, and we have $\partial_\eta^\gamma H_0(x,I,0) = O(|I-I^0|^M)$ for any γ. In this way we get

$$F(x,I,\lambda) = F_0(x,I) + \lambda^{-1}\sum_{j=0}^{M-2} F_j^0(x,I)\lambda^{-j} + \lambda^{-1}F^1(x,I,\lambda) + \lambda^{-M}F_M\ ,$$

where $F_0 = 1$ in $\mathbb{T}^{n-1}\times D^0$,

$$F_j^0(\varphi,I)\ =\ a_j^0(\varphi - \nabla L(I),I) + w_j^0(\varphi,I) + f_j^0(\varphi,I)\,,$$

$f_0^0 = 0$, and for $j\ge 1$ we have

$$\begin{aligned} f_j^0(\varphi,I)\ =\ & \sum_{s=0}^{j-1}\ \sum_{|\beta|=j-s}\ \sum_{|\gamma|\le M-j-2} \frac{1}{\beta!}\left[D_\eta^\beta \partial_x^\beta\, a_{s,\gamma}^0(\varphi - L_0(I,\eta))\right]_{|\eta=0} (I-I^0)^\gamma \\ + & \sum_{r+s+|\beta|=j-1}\ \sum_{|\gamma|\le M-j-2} \frac{1}{\beta!}\left[D_\eta^\beta\left(w_r^0(\varphi,I+\eta)\,\partial_x^\beta\, a_{s,\gamma}^0(\varphi - L_0(I,\eta))\right)\right]_{|\eta=0} (I-I^0)^\gamma\,. \end{aligned} \tag{A.10}$$

We have also $F^1 \in \widetilde{R}_{M-1}(\mathbb{T}^{n-1} \times D; \mathcal{B}, \lambda)$ in view of (A.9). Moreover, using (A.9) and Lemma A.1 we obtain

$$\|F_M\|_{C^n} \leq C \left(\sum_{j \leq M-2} \sup_\lambda \|w_j^0(\cdot, \cdot, \lambda)\|_{C^{l-2j}} \right) \left(\sum_{j+|\gamma| \leq M-2} \sup_\lambda \|a_{j,\gamma}^0(\cdot, \lambda\|_{C^{l-(j+1)([\tau]+n)-|\gamma|}} \right) ,$$

hence, the corresponding λ-FIO is uniformly bounded with respect to $K \in \mathcal{B}$ in L^2. In the same way we write $A(\lambda)W_0(\lambda)$ in the form (3.20) with amplitude $G(x, I, \lambda)$ given by the oscillatory integral

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} (p_0 + \lambda^{-1}p^0)(I, \lambda) \int_{\mathbb{R}^{2n-2}} e^{i\lambda(\langle x-z, \xi-I\rangle + \Phi(z,I) - \Phi(x,I))}\, a(x, \xi, \lambda) d\xi dz \, .$$

Changing the variables we obtain $G = a(p_0 + \lambda^{-1}p^0) + u$, where u is given by

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} (p_0 + \lambda^{-1}p^0(I, \lambda)) \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v, \eta\rangle} \left[a(x, \eta + I + H_1(x, v, I), \lambda) - a(x, \eta + I, \lambda)\right] d\eta dv \ ,$$

and $H_1(x, v, I) = \int_0^1 \nabla_x R(x + \tau v, I) d\tau$. Notice that H_1 and all its derivatives vanish at $I = I^0$. Then u satisfies (3.24) and we get

$$G(\varphi, I, \lambda) = G_0(\varphi, I) + \lambda^{-1} \sum_{j=0}^{M-2} G_j^0(\varphi, I)\lambda^{-j} + \lambda^{-1}G^1(\varphi, I, \lambda) + \lambda^{-M} F_M(\varphi, I, \lambda) \, ,$$

where $G_0 = 1$ in $\mathbb{T}^{n-1} \times D^0$, $G^1 \in \widetilde{R}_{M-1}(\mathbb{T}^{n-1} \times D; \mathcal{B}, \lambda)$, the λ-FIO corresponding to F_M is $O_{\mathcal{B}}(|\lambda|^{-M})$, and

$$G_j^0(\varphi, I) \ = \ a_j^0(\varphi, I) + p_j^0(I) + g_j^0(\varphi, I) \, .$$

Moreover, $g_0^0 = 0$ and for $j \geq 1$ we have

$$g_j^0(\varphi, I) = \sum_{k=0}^{j-1} a_k^0(\varphi, I) p_{j-k-1}^0(I) \, . \tag{A.11}$$

Taking into account (A.10) and (A.11) we obtain

$$R_1(\varphi, I, \lambda) = \sum_{j=0}^{M-2} T_{M-j-2}(F_j^0 - G_j^0)(\varphi, I)\, \lambda^{-j} \ \in \ \widetilde{S}_{l-[\tau]-n, [\tau]+n, M-1}(\mathbb{T}^{n-1} \times D; \mathcal{B}, \lambda)$$

and we denote by $R_1(\lambda)$ the corresponding FIO. Moreover, the symbol of the reminder term $R^0(\lambda)$ satisfies (3.24).

We are going to show that the coefficient $f_{j,\alpha}^0(\varphi)$ of $(I - I^0)^\alpha$ in the Taylor series of (A.10) at $I = I^0$ is a linear combination of functions given by (3.26). First note that $(\partial_\eta^k L_0)(I, 0) = (1 + |k|)^{-1} \partial_I^k \nabla_I L(I)$ for any $k \in \mathbb{N}^{n-1}$ and that $\nabla_I L(I^0) = 2\pi\omega$. Expand $\partial_I^k \nabla_I L(I)$, in Taylor series at $I = I^0$ up to order $O(|I - I^0|^M)$, $k \in \mathbb{N}^{n-1}$. Then use the Taylor expansions of

$$\partial_x^\beta a_{s,\gamma}^0 \left(\varphi - 2\pi\omega + \sum_{1 \leq |k| \leq M} L_k (I - I^0)^k \right) \tag{A.12}$$

at $\varphi - 2\pi\omega$ up to order $O(|I-I^0|^{|\alpha|-|\gamma|+1})$. Hence, the corresponding terms in the first sum of (A.10) are linear combinations of $\partial_x^{\beta+k} a^0_{s,\gamma}(\varphi - 2\pi\omega)$, where $0 \le s \le j-1$ and $|\beta| \le 2(j-s)$, $|k| + |\gamma| \le |\alpha|$. In the second sum of (A.10) write

$$D_I^{\beta'} w_r^0(\varphi, I) = \sum_{\beta' \le \delta, |\delta| \le M-r-1} w^0_{r,\delta}(\varphi)\,(I-I^0)^{\delta-\beta'}\delta!/(\delta-\beta')!\,,\ \beta' \le \beta\,,$$

and expand (A.12) in Taylor series up to order $O(|I-I^0|^{|\alpha|-|\gamma|-|\delta-\beta'|+1})$. Then the corresponding terms in the second sum are linear combinations of $w^0_{r,\delta}(\varphi)\partial_x^{\beta+k} a^0_{s,\gamma}(\varphi-2\pi\omega)$, where $0 \le r+s \le j-1$, $|\beta+\beta'| \le 2(j-s-r-1)$, and $k + |\delta-\beta'| + |\gamma| \le |\alpha|$ for some $\beta' \le \beta$, $\beta' \le \delta$, and we prove the assertion. In the same way we prove that $g^0_{j,\alpha}(\varphi)$ is a linear combination of functions in (3.27). □

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