Estimates of Initial Conditions of Parabolic Equations and Inequalities Via the Lateral Cauchy Data

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Abstract

A parabolic equation and, more generally, parabolic inequality is considered in the cylinder $Q_T = \Omega \times (0,T)$, where $\Omega \subset R^n$ is a bounded domain. Cauchy data, i.e., both Dirichlet and Neumann data are given at the lateral surface $S_T = \partial\Omega \times (0,T)$. Logarithmic stability estimates are obtained for the unknown initial condition at $\{t = 0\}$. These estimates enable one to establish convergence rate of a numerical method for the inverse problem of the determination of that initial condition.

1 Introduction

Let $\Omega \subset R^n$ be a bounded domain with the piecewise smooth boundary $\partial\Omega$ and $T = const. > 0$. Denote $Q_T = \Omega \times (0,T)$. Let $L = L(x,t,D)$ be an elliptic operator of the second order in Q_T,

$$Lu := L(x,t,D)u = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{ij} + \sum_{i,j=1}^{n} b_j(x,t)u_j + b_0(x,t)u,$$

where $u_j = \partial u/\partial x_j$. Here functions

$$a^{ij} \in C^1\left(\overline{Q}_T\right), b_j, b_0 \in B\left(\overline{Q}_T\right),$$

where $B\left(\overline{Q}_T\right)$ is the space of functions bounded in $\overline{Q}_T$. Naturally, we assume the existence of a positive number σ such that

$$\sigma\left|\xi\right|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j,\ \forall (x,t) \in \overline{Q}_T, \forall\xi \in R^n. \tag{1.1}$$

Let the function $u\in H^{2,1}\left(\overline{Q}_T\right)$ be a solution of the parabolic equation

$$u_t = Lu + f(x,t) \text{ in } Q_T, \tag{1.2}$$

where the function $f\in L_2\left(Q_T\right)$ with the unknown initial condition

$$u(x,0) = g(x). \tag{1.3}$$

Consider the Dirichlet and Neumann boundary data for this function at the cylindrical surface $S_T = \partial\Omega\times(0,T)$,

$$u\mid_{S_T} = h_1(x,t), \quad \frac{\partial u}{\partial n}\mid_{S_T} = h_2(x,t). \tag{1.4}$$

We study the topic of stability of the following

Inverse Problem. *Suppose that the initial condition g in (1.3) is unknown, but functions h_1 and h_2 in (1.4) are known. Determine the function $g(x)$.*

This is an inverse problem of the determination of the initial condition in the parabolic equation using lateral measurements. Applications are in such diffusion and heat conduction processes in which one is required to determine the initial state using boundary time dependent measurements. Uniqueness of this problem is well known and, therefore is not discussed here, although it follows from the stability result of Theorem 1. We shall also consider a more general

Stability Problem. *Suppose that the function $u\in H^{2,1}\left(Q_T\right)$ satisfies the parabolic inequality*

$$\left|u_t - L_0 u\right| \leq A\left[\left|\nabla u\right| + |u| + |f|\right], \text{ a.e. in } Q_T, \tag{1.5}$$

where $A = const. > 0$, $\nabla u = (u_1,...,u_n)$ and $L_0(x,t,D)$ is the principal part of the operator $L(x,t,D)$,

$$L_0 u := L_0(x,t,D)u = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{ij}.$$

Suppose that the initial condition $g(x)$ in (1.3) is unknown. Estimate the function $g(x)$ via functions h_1, h_2 and f.

The main results of this paper are Theorems 1 and 2.

Theorem 1. *Assume that above conditions imposed on the coefficients of the operator $L(x,t,D)$ are fulfilled. Denote $F = (h_1, h_2, f)$ and*

$$\|F\| = \left[\|h_1\|^2_{H^1(S_T)} + \|h_2\|^2_{L_2(S_T)} + \|f\|^2_{L_2(Q_T)}\right]^{1/2}. \tag{1.6}$$

Suppose that $\|F\|\leq B$, where B is a positive number. Assume also that in (1.3) the function $g\in H^1(\Omega)$. Then there exists a positive constant C such that for every number $\beta\in(0,2)$ there exists a number $\varepsilon_0\in(0,1)$ such that for any function $u\in H^{2,1}\left(Q_T\right)$ satisfying (1.3)-(1.5) the following stability estimate holds

$$\|g\|^2_{L_2(\Omega)} \leq \frac{C}{\beta\ln\left[\frac{B}{\varepsilon_0\|F\|}\right]}\cdot\left\|\left|\nabla g\right|\right\|^2_{L_2(\Omega)} + C\left(\frac{B}{\varepsilon_0}\right)^{\beta}\|F\|^{2-\beta}. \tag{1.7}$$

The constant C depends only on the domain Ω, the number T, $C^1\left(\overline{Q}_T\right)-$ norms of coefficients a^{ij}, the number σ in (1.1) and the number A in (1.5). The number ε_0 depends on these parameters, as well as on the parameter β.

To establish convergence rate for our numerical method (section 4), we need a more general

Theorem 2. *Assume that the function $u\in H^{2,1}(Q_T)$ satisfies boundary conditions (1.4) and the integral inequality*

$$\int\limits_{Q_T}(u_t-Lu)^2\,dxdt\le K^2, \tag{1.8}$$

where $K=const.>0$ and $L=L(x,t,D)$ is the above elliptic operator. Let $f(x,t)=const.=K$ and the notation (1.6) holds. Suppose that $\|F\|\le B$, where B is a positive number. Assume also that in (1.3) the function $g\in H^1(\Omega)$. Then there exists a positive constant C_1 such that for every number $\beta\in(0,2)$ there exists a number $\varepsilon_1\in(0,1)$ such that the following stability estimates hold

$$\|g\|^2_{L_2(\Omega)}\le\frac{C_1}{\beta\ln\left[\frac{B}{\varepsilon_1\|F\|}\right]}\cdot|||\nabla g|||^2_{L_2(\Omega)}+C_1\left(\frac{B}{\varepsilon_1}\right)^{\beta}\|F\|^{2-\beta}, \tag{1.9}$$

$$\|u\|^2_{H^{1,0}(Q_T)}\le\frac{C_1}{\beta\ln\left[\frac{B}{\varepsilon_1\|F\|}\right]}\cdot|||\nabla g|||^2_{L_2(\Omega)}+C_1\left(\frac{B}{\varepsilon_1}\right)^{\beta}\|F\|^{2-\beta}. \tag{1.10}$$

The constant C_1 depends only on the domain Ω, the number T, $C^1\left(\overline{Q}_T\right)-$ norms of coefficients a^{ij}, the number σ in (1.1) and $B\left(\overline{Q}_T\right)-$norms of coefficients $b_j(x,t)$ $(j=0,...,n)$ of the operator L. The number ε_1 depends on these parameters, as well as on the parameter β.

It follows from (1.7), (1.9) and (1.10) that if $\|F\|\to0$, then the first term in the right hand sides of these inequalities approaches zero with a "logarithmic speed", and the second one as a power. If the first term would be absent, we would obtain the Hölder stability. But because of the presence of this term, (1.7), (1.9) and (1.10) are logarithmic stability estimates. Throughout the paper we assume that conditions of Theorem 1 are satisfied, unless stated otherwise in proofs of theorems 2, 4, and 5. Notations of Theorem 1 are kept below. Also, throughout the paper C and C_1 denote different positive constants depending on parameters listed in Theorem 1 and Theorem 2 respectively.

Remarks. 1. To prove these theorems, we use Carleman estimates. It is well known that constants appearing in such estimates can be explicitly estimated via $C^1\left(\overline{Q}_T\right)-$ norms of coefficients a^{ij}, the number σ in (1.1) and numbers A and K. This means that one can derive explicit estimates for constants C and ε_0. However, we are not doing this here for brevity.

2. Estimates (1.7), (1.9) and (1.10) are the so-called conditional stability estimates, see, e.g., the book of Lavrent'ev, Romanov and Shishatskii [12] for the definition of conditional stability estimates. This is because the stronger norm $|||\nabla g|||_{L_2(\Omega)}$ is involved. Conditional stability estimates are typical ones for ill-posed problems such as, e.g., two problems formulated above. One of basic facts of the theory of ill-posed problems, which follows from the fundamental Tikhonov theorem [16] (the one about the continuity of the inverse operator on a compact set) is that a conditional stability estimate for an ill-posed problem enables one to obtain *a priori* estimate of the difference between the approximate and the exact solutions of this problem, provided that the exact solution belongs to *a priori* chosen compact set, see, e.g., (2.6) in §1 of Chapter 2 of [12]. This is quite helpful for establishing convergence rate of a corresponding numerical method, see, e.g., Theorem 5 in section 4.

3. A stability estimate for an ill-posed problem usually relies on the assumption that norms of certain input data are sufficiently small. For this reason, it is often assumed so without considering the case when those norms are not small, see, e.g., Chapter 4 in [12]. In order to avoid such a "smallness" assumption, we introduce constants B, ε_0 and ε_1. In the course of our proofs we first somewhat "normalize" the data making their respective norms less than ε_0 (or ε_1) and obtain a stability estimate this way. Next, we return to the original data. Also, see Chapter 2 of the book of Klibanov and Timonov [7] for a systematic use of this approach. While constants like B, ε_0 and ε_1 do not appear in well-posed problems (at least, in linear problems), their appearance is quite natural in ill-posed problems.

The idea of proofs of Theorems 1 and 2 is to combine two Carleman estimates. The first one is for the backwards parabolic equation/inequality, i.e., for the case when the data are given at $\{t = t_0\}$, where $t_0 \in (0, T)$, and one wants to estimate the solution $u(x, t)$ for $t \in (0, t_0)$. And the second one is for the parabolic inequality (1.5) with the lateral Cauchy data. Both these estimates can be found in the book [12], and the second one can also be found in the book [7]. We note that, unlike (1.10), the second Carleman estimate does not estimate the function $u(x, t)$ in the entire cylinder Q_T via the lateral boundary data (1.4) and the function f. Instead, it estimates $u(x, t)$ only in a subdomain $G \subset Q_T$ bounded by the lateral side S_T and the level surface of the Carleman Weight Function $e^{2\mu\varphi}$, see section 3 for the definition of the function φ. The domain G_ω defined in (3.4) is a typical example of such a subdomain G. The single known estimate of the solution of the parabolic equation in the entire cylinder Q_T via the lateral Cauchy data (1.4) is one of Fursikov and Imanuvilov [4]. However, this is a weighted estimate, and the weight function of [4] vanishes at $\{t = 0, T\}$. Hence, Theorems 1 and 2 do not follow from [4].

The first Carleman estimate mentioned above traditionally enables one to obtain both the Hölder stability and uniqueness for backwards parabolic equations and inequalities in a sub-cylinder $Q(\tau, t_0) = \Omega \times (\tau, t_0)$ via $\|u(x, t_0)\|_{L_2(\Omega)}$, where $t_0 \in (0, T]$, assuming that either Dirichlet or Neumann zero boundary condition is given at the lateral surface S_T, see [12], as well as Lees and Protter [13]. However, estimates of [12] and [13] break down when $\tau \to 0^+$. Thus, an estimate of $\|u(x, 0)\|_{L_2(\Omega)}$ in the backwards parabolic problem is a more delicate matter. We obtain this estimate in Theorems 3 and 4 (section 2), which are new results. The main new observation enabling us to estimate $\|u(x, 0)\|_{L_2(\Omega)}$ is that a certain boundary integral over $\{t = 0\}$ occuring in the first Carleman estimate is non-negative, see the third term in the right hand side of (2.6).

It was shown in Exercise 3.1.2 of the book of Isakov [5] that in the backwards parabolic problem, the logarithmic convexity method leads to a logarithmic stability estimate of $\|u(x,0)\|_{L_2(\Omega)}$ via $\|u(x,t_0)\|_{L_2(\Omega)}$, $D_0 = \sup_{t\in(0,T)} \|u(x,t)\|_{L_2(\Omega)}$ and

$D_1 = \sup_{t\in(0,T)} \|u_t(x,t)\|_{L_2(\Omega)}$, see, e.g., books of Ames and Straugan [1], Isakov [5] and Payne [15] for this method. Note that in our Theorems 3 and 4 numbers D_0 and D_1 are not involved. The logarithmic convexity method can be applied only for the case of the equation $u_t = \widetilde{L}(x,D)u$, where $\widetilde{L}(x,D)$ is a self-adjoint elliptic operator with t-independent coefficients. Although there is a certain extension of this method on some inequalities including parabolic ones (see pp. 42-47 in [5]), but it does not include the case when $|\nabla u|$ is involved in the right hand side of such an inequality (compare with (1.5)), and it also needs some additional assumptions about t-dependencies of coefficients of the operator L_0, see example 3.1.8 in [5]. However, we consider a quite general case of the parabolic inequality (1.5), in which the principal part $L_0(x,t,D)$ of the elliptic operator is non self-adjoint, and we do not impose extra conditions on $t-$dependencies of coefficients. Finally, another important observation is that our proof of convergence of the numerical method (section 4) would not work if Theorem 1 would be valid only for the equation (1.2), rather than for the inequality (1.5). Actually, we need for this proof a more general result of Theorem 2.

The author is aware about three previously published similar results. Isakov and Kindermann [6] have proven an analog of the estimate (1.7) for the function $v(y,0)$, where the function $v(y,t)$ satisfies the equation $v_t = v_{yy}$, $y \in \mathbb{R}$,$t > 0$. The lateral Cauchy data $v(0,t)$ and $v_y(0,t)$ were used. Their proof is using the analyticity of the function $v(y,t)$ with respect to t. Note that the analyticity is not guaranteed in our case. Xu and Yamamoto [17] have proven an analog of Theorem 1 for the heat equation $u_t = \Delta u$ assuming the zero Dirichlet boundary condition $u \mid_{S_T} = 0$ and that the function $u(x,t)$ is known for $(x,t) \in \omega \times (0,T)$,where $\omega \subset \Omega$ is a subdomian. They have used a combination of the Carleman estimate of [4] with the logarithmic convexity method. It is assumed in [17] that $u \in C^{2,1}\left(\overline{Q}_T\right)$. In terms of Sobolev spaces, the proof of [17] is valid if $u \in H^{4,2}\left(Q_T\right)$, since it actually requires that $u_t(x,0) \in L_2(\Omega)$. This is because the logarithmic convexity method implies that in [17] the positive constant $C_4 \leq \widetilde{C}_4 \|u_t(x,0)\|_{L_2(\Omega)}$, where the positive constant $\widetilde{C}_4$ depends on the domain Ω. Note that we use a more relaxed smoothness condition $u \in H^{2,1}\left(Q_T\right)$. Also, Yamamoto and Zou [18] have extended the result of [17] to the case when the function $u(x,t)$ is known for $(x,t) \in \omega \times (\delta,T)$, where $\delta = const. \in (0,T)$. Because $\delta \neq 0$, *a pirori* upper estimate for $\|g\|_{H^{2\varepsilon}(\Omega)}$ with an $\varepsilon > 0$ is imposed in [18]. Also, $\left\{\ln\left(\|F\|^{-1}\right)\right\}^{-\kappa}$, where $\kappa \in (0,1)$, stands in [18] instead of our $\left\{\ln\left(\|F\|^{-1} \cdot B/\varepsilon_0\right)\right\}^{-1}$. The technique of [18] is similar with one of [17].

Our numerical method for the above Inverse Problem is a version of the quasi-reversibility method (QRM) of Lattes and Lions [11]. In the parabolic case, convergence of the QRM was proven in [11] only for the case when the function $g(x)$ is given and the lateral data (1.4) are given at a part $\Gamma \times (0,T)$ of the surface S_T, where $\Gamma \subset \partial\Omega$. One of goals of section 4 is to rigorously explain the robustness of previously published numerical results of the QRM for the parabolic case, see the book of Danilaev [3] and papers of Klibanov and Danilaev [8] and Tadi, Klibanov and Cai [16] for these numerical results. Computational studies of the QRM for the elliptic case were conducted in Bourgeois [2] and Klibanov and Santosa [9], and for the hyperbolic case in Klibanov and Rakesh [10]. All these numerical studies have consistently demonstrated a quite good robustness of the QRM.

It was shown in [7] that the QRM is a particular case of the Tikhonov regularization functional, and, therefore smoothness conditions imposed on the solution in [11] can be significantly relaxed. In addition, it was also shown in [7] how the convergence rates of the QRM for different equations are connected with both Carleman estimates and stability estimates. However, convergence of the QRM for the parabolic case was established in [7] only in the above indicated subdomain G of the cylinder Q_T, which is not completely satisfactory for the above Inverse Problem of the determination of the initial condition. Unlike this, we establish here the logarithmic convergence rate in the entire cylinder Q_T.

In section 2 obtain the logarithmic stability estimate of the initial condition $u(x,0)$ in the backwards parabolic inequalities (1.5) and (1.8). These results are used in section 3, where we prove Theorems 1 and 2. In section 4 we formulate a numerical method for the above Inverse Problem and establish its convergence rate.

2 An enhanced stability estimate for the backwards parabolic inequality

Although lemmata 1 and 2 of this section are analogs of lemmata 1 and 2 of §2 of Chapter 4 of [12] (and of similar results of [13]), but we need detailed proofs of these results here, because we need to obtain an estimate of $\|u(x,0)\|_{L_2(\Omega)}$, which is a new result (theorems 3 and 4). Compared with Lemma 3 of §2 of Chapter 4 of [12], the main new element of Lemma 3 of this section is the positive third term in the right hand side of (2.6). This term enables us to estimate $\|g\|_{L_2(\Omega)}$, see (2.11).

Lemma 1. *Let k be a positive constant. Then there exists a number $\lambda_0 > 1$ depending only on the number σ in (1.1) and $C^1\left(\overline{Q}_T\right)$-norms of functions a^{ij} such that for all $\lambda \geq \lambda_0$ and for all functions $v \in C^{2,1}\left(\overline{Q}_{t_0}\right)$ the following estimate is valid in Q_{t_0}*

$$(v_t - L_0 v)\, v\,(k+t_0-t)^{-2\lambda} \geq \frac{\sigma}{2}|\nabla v|^2 (k+t_0-t)^{-2\lambda} - C\lambda v^2 (k+t_0-t)^{-2\lambda}$$

$$+\sum_{i=1}^{n}\left(-\sum_{j=1}^{n} a^{ij} v_j v\,(k+t_0-t)^{-2\lambda}\right)_i + \left(\frac{v^2}{2}(k+t_0-t)^{-2\lambda}\right)_t.$$

Proof. We have

$$(v_t - L_0 v)\, v\,(k+t_0-t)^{-2\lambda} = v_t v\,(k+t_0-t)^{-2\lambda} - \sum_{i,j=1}^{n} a^{ij} v_{ij} v\,(k+t_0-t)^{-2\lambda}$$

$$= \left(\frac{v^2}{2}(k+t_0-t)^{-2\lambda}\right)_t - \lambda v^2 (k+t_0-t)^{-2\lambda-1} \qquad (2.1)$$

$$+\sum_{i=1}^{n}\left(-\sum_{j=1}^{n}a^{ij}v_jv\left(k+t_0-t\right)^{-2\lambda}\right)_i$$

$$+\sum_{i,j=1}^{n}a^{ij}v_iv_j\left(k+t_0-t\right)^{-2\lambda}-\sum_{i,j=1}^{n}a_i^{ij}v_jv\left(k+t_0-t\right)^{-2\lambda}.$$

Estimate from below the last two terms in the right hand side of (2.1),

$$\sum_{i,j=1}^{n}a^{ij}v_iv_j\left(k+t_0-t\right)^{-2\lambda}-\sum_{i,j=1}^{n}a_i^{ij}v_jv\left(k+t_0-t\right)^{-2\lambda}$$

$$\geq\sigma\left|\nabla v\right|^2\left(k+t_0-t\right)^{-2\lambda}-C\left|\nabla v\right|\left|v\right|\left(k+t_0-t\right)^{-2\lambda} \tag{2.2}$$

$$\geq\left(\sigma-\frac{C\varepsilon}{2}\right)\left|\nabla v\right|^2\left(k+t_0-t\right)^{-2\lambda}-\frac{C}{2\varepsilon}v^2\left(k+t_0-t\right)^{-2\lambda},\forall\varepsilon>0.$$

We have used here the Cauchy-Schwarz inequality "with ε", i.e.,

$ab\geq-\varepsilon a^2/2-b^2/2\varepsilon,\forall a,b\in\mathbb{R},\forall\varepsilon>0$. Let $\varepsilon=\sigma/C$. Given this ε, choose $\lambda_0>1$ such that $\lambda_0>C/\varepsilon=C^2/\sigma$. Then substituting the last line of (2.2) in (2.1), we obtain

$$\left(v_t-L_0v\right)v\left(k+t_0-t\right)^{-2\lambda}\geq\frac{\sigma}{2}\left|\nabla v\right|^2\left(k+t_0-t\right)^{-2\lambda}-\frac{3\lambda}{2}v^2\left(k+t_0-t\right)^{-2\lambda}$$

$$+\sum_{i=1}^{n}\left(-\sum_{j=1}^{n}a^{ij}v_jv\left(k+t_0-t\right)^{-2\lambda}\right)_i+\left(\frac{v^2}{2}\left(k+t_0-t\right)^{-2\lambda}\right)_t,\forall\lambda\geq\lambda_0.$$

□

Lemma 2. *Let k be a positive constant and $t_0\in(0,T)$. Then for every $\lambda>0$ and for all functions $v\in C^2\left(\overline{Q}_{t_0}\right)$ the following estimate is valid in Q_{t_0}*

$$\left(v_t-L_0v\right)^2\left(k+t_0-t\right)^{-2\lambda}\geq-C\left|\nabla v\right|^2\left(k+t_0-t\right)^{-2\lambda}+\lambda v^2\left(k+t_0-t\right)^{-2\lambda-2}$$

$$+\sum_{i=1}^{n}\left(-2\sum_{j=1}^{n}a^{ij}v_jv_t\left(k+t_0-t\right)^{-2\lambda}\right)_i$$

$$+\left(-\lambda v^2\cdot\left(k+t_0-t\right)^{-2\lambda-1}+\sum_{i,j=1}^{n}a^{ij}v_iv_j\left(k+t_0-t\right)^{-2\lambda}\right)_t.$$

Proof. Denote $w=v\left(k+t_0-t\right)^{-\lambda}$. Then $v=w\left(k+t_0-t\right)^{\lambda}$. Hence,

$$v_t=\left[w_t-\lambda\left(k+t_0-t\right)^{-1}w\right]\left(k+t_0-t\right)^{\lambda},\quad v_i=w_i\left(k+t_0-t\right)^{\lambda}.$$

Hence,

$$\left(v_t-L_0v\right)^2\left(k+t_0-t\right)^{-2\lambda}=\left(w_t-\lambda\left(k+t_0-t\right)^{-1}w-\sum_{i,j=1}^{n}a^{ij}w_{ij}\right)^2$$

$$\geq w_t^2 - 2\lambda w_t w\left(k+t_0-t\right)^{-1} - 2\sum_{i,j=1}^{n} a^{ij} w_{ij} w_t$$

$$= w_t^2 + \left(-\lambda w^2\left(k+t_0-t\right)^{-1}\right)_t + \lambda w^2\left(k+t_0-t\right)^{-2}$$

$$+\sum_{i=1}^{n}\left(-2\sum_{j=1}^{n} a^{ij} w_j w_t\right)_i + 2\sum_{i,j=1}^{n} a^{ij} w_j w_{ti} + 2\sum_{i,j=1}^{n} a_i^{ij} w_j w_t.$$

Hence,

$$\left(v_t - L_0 v\right)^2\left(k+t_0-t\right)^{-2\lambda}$$

$$\geq w_t^2 + \lambda w^2\left(k+t_0-t\right)^{-2} - C\left|\nabla w\right|\left|w_t\right|$$

$$+\left(-\lambda w^2\left(k+t_0-t\right)^{-1} + \sum_{i,j=1}^{n} a^{ij} w_i w_j\right)_t \tag{2.3}$$

$$+\sum_{i=1}^{n}\left(-2\sum_{j=1}^{n} a^{ij} w_j w_t\right)_i - \sum_{i,j=1}^{n} a_t^{ij} w_i w_j.$$

By the Cauchy-Schwarz inequality $-C\left|\nabla w\right|\left|w_t\right| \geq -w_t^2/2 - 2C^2\left|\nabla w\right|^2$. Hence, with a new constant C

$$w_t^2 + \lambda w^2\left(k+t_0-t\right)^{-2} - 2C\left|\nabla w\right|\left|w_t\right|$$

$$\geq \frac{1}{2} w_t^2 - C\left|\nabla w\right|^2 + \lambda w^2\left(k+t_0-t\right)^{-2}$$

$$\geq -C\left|\nabla w\right|^2 + \lambda w^2\left(k+t_0-t\right)^{-2}.$$

This and (2.3) lead to

$$\left(v_t - L_0 v\right)^2\left(k+t_0-t\right)^{-2\lambda} \geq -C\left|\nabla w\right|^2 + \lambda w^2\left(k+t_0-t\right)^{-2}$$

$$+\sum_{i=1}^{n}\left(-2\sum_{j=1}^{n} a^{ij} w_j w_t\right)_i + \left(-\lambda w^2\left(k+t_0-t\right)^{-1} + \sum_{i,j=1}^{n} a^{ij} w_i w_j\right)_t.$$

Replacing here w with $v = w\left(k+t_0-t\right)^{\lambda}$, we obtain the target inequality of this lemma. $\square$

Lemma 3. *Choose numbers $t_0 \in (0,T)$ and $k > 0$ so small that $k + t_0 < 1$ and*

$$\left(k+t_0\right)^{-2} > 8C\left\{\min\left[\frac{1}{\sqrt{6}A}, \frac{\sigma}{4\left(C+6A^2\right)}\right]\right\}^{-1}. \tag{2.4}$$

Choose a number θ such that

$$\frac{1}{2}\min\left[\frac{1}{\sqrt{6}A}, \frac{\sigma}{4\left(C+6A^2\right)}\right] \leq \theta \leq \min\left[\frac{1}{\sqrt{6}A}, \frac{\sigma}{4\left(C+6A^2\right)}\right]. \tag{2.5}$$

Then there exists a constant $\lambda_1 \geq \lambda_0 > 1$ depending only on the number σ in (1.1), $C^1\left(\overline{Q}_T\right)$-norms of functions a^{ij} and the constant A such that if a function $v \in H^{2,1}\left(Q_{t_0}\right)$ satisfies *the parabolic inequality (1.5) in Q_T, then for all $\lambda \geq \lambda_1$ the following estimate is valid in Q_{t_0}*

$$6\theta A^2 \int\limits_{Q_{t_0}} f^2 (k+t_0-t)^{-2\lambda} dxdt$$

$$\geq \frac{\sigma}{4} \int\limits_{Q_{t_0}} |\nabla v|^2 (k+t_0-t)^{-2\lambda} dxdt + \frac{\lambda\theta}{2} \int\limits_{Q_{t_0}} v^2 (k+t_0-t)^{-2\lambda} dxdt$$

$$+\lambda \left(\theta - \frac{k+t_0}{2\lambda}\right) (k+t_0)^{-2\lambda-1} \int\limits_{\Omega} v^2(x,0)dx \tag{2.6}$$

$$-\theta (k+t_0)^{-2\lambda} \sum_{i,j=1}^{n} \int\limits_{\Omega} \left(a^{ij} v_i v_j\right)(x,0)dx$$

$$-\lambda \left(\theta - \frac{k}{2\lambda}\right) k^{-2\lambda-1} \int\limits_{\Omega} v^2(x,t_0)dx + \theta k^{-2\lambda} \sum_{i,j=1}^{n} \int\limits_{\Omega} \left(a^{ij} v_i v_j\right)(x,t_0)dx$$

$$-\sum_{i=1}^{n} \int\limits_{S_{t_0}} \left(\sum_{j=1}^{n} (2\theta v_t + v) a^{ij} v_j (k+t_0-t)^{-2\lambda} \right) \cos(n,x_i)\, dS,$$

where n in $\cos(n,x_i)$ is the outward normal vector at S_T. The constant λ_1 is independent on a specific choice of positive numbers t_0 and k, as long as $t_0+k<1$ and the inequality (2.4) holds.

Proof. Assume first that the function $v \in C^2\left(\overline{Q}_{t_0}\right)$. Multiply both sides of the inequality of Lemma 2 by θ, sum up with the inequality of Lemma 1 and integrate over Q_{t_0}. Noting that $(k+t_0-t)^{-2} \geq (k+t_0)^{-2}$ for $t \in (0,t_0)$, we obtain for all functions $v \in C^2\left(\overline{Q}_{t_0}\right)$

$$\int\limits_{Q_{t_0}} \left[\theta (v_t - L_0 v)^2 + (v_t - L_0 v) v\right] (k+t_0-t)^{-2\lambda} dxdt$$

$$\geq \left(\frac{\sigma}{2} - C\theta\right) \int\limits_{Q_{t_0}} |\nabla v|^2 (k+t_0-t)^{-2\lambda} dxdt$$

$$+\lambda\theta \left[(k+t_0)^{-2} - \frac{C}{\theta}\right] \int\limits_{Q_{t_0}} v^2 (k+t_0-t)^{-2\lambda} dxdt$$

$$+\lambda \left(\theta - \frac{k+t_0}{2\lambda}\right) (k+t_0)^{-2\lambda-1} \int\limits_{\Omega} v^2(x,0)dx \tag{2.7}$$

$$-\theta (k+t_0)^{-2\lambda} \sum_{i,j=1}^{n} \int\limits_{\Omega} \left(a^{ij} v_i v_j\right)(x,0)dx$$

$$-\lambda\left(\theta-\frac{k}{2\lambda}\right)k^{-2\lambda-1}\int\limits_{\Omega}v^2(x,t_0)dx+\theta k^{-2\lambda}\sum_{i,j=1}^{n}\int\limits_{\Omega}\left(a^{ij}v_iv_j\right)(x,t_0)dx$$

$$-\sum_{i=1}^{n}\int\limits_{S_{t_0}}\left(\sum_{j=1}^{n}\left(2\theta v_t+v\right)a^{ij}v_j\left(k+t_0-t\right)^{-2\lambda}\right)\cos\left(n,x_i\right)dS,\ \forall\lambda>\lambda_0,\forall\theta>0.$$

Since (2.7) is valid for all functions $v\in C^2\left(\overline{Q}_{t_0}\right)$ and the set $C^2\left(\overline{Q}_{t_0}\right)$ is dense in the space $H^{2,1}\left(Q_{t_0}\right)$, then (2.7) is also valid for all functions $v\in H^{2,1}\left(Q_{t_0}\right)$.

Suppose now that the function $v\in H^{2,1}\left(Q_{t_0}\right)$ in (2.7) satisfies the inequality (1.5). Using (1.5), (2.5) and the Cauchy-Schwarz inequality, we obtain that

$$\left[\theta\left(v_t-L_0v\right)^2+\left(v_t-L_0v\right)v\right]\left(k+t_0-t\right)^{-2\lambda}$$

$$\leq\left[2\theta\left(v_t-L_0v\right)^2+\frac{1}{2\theta}v^2\right]\left(k+t_0-t\right)^{-2\lambda}$$

$$\leq\left[6\theta A^2\left|\nabla v\right|^2+\left(\frac{1}{2\theta}+6\theta A^2\right)v^2+6\theta A^2f^2\right]\left(k+t_0-t\right)^{-2\lambda}$$

$$\leq\left[6\theta A^2\left|\nabla v\right|^2+\frac{2}{\theta}v^2+6\theta A^2f^2\right]\left(k+t_0-t\right)^{-2\lambda},\ \text{a.e. in } Q_{t_0}.$$

Integrating this inequality over Q_{t_0} and substituting then in (2.7), we obtain

$$6\theta A^2\int\limits_{Q_{t_0}}f^2\left(k+t_0-t\right)^{-2\lambda}dxdt\geq\left[\frac{\sigma}{2}-\theta\left(C+6A^2\right)\right]\int\limits_{Q_{t_0}}\left|\nabla v\right|^2\left(k+t_0-t\right)^{-2\lambda}dxdt$$

$$+\lambda\theta\left[\left(k+t_0\right)^{-2}-\frac{C}{\theta}-\frac{2}{\lambda\theta^2}\right]\int\limits_{Q_{t_0}}v^2\left(k+t_0-t\right)^{-2\lambda}dxdt$$

$$+\lambda\left(\theta-\frac{k+t_0}{2\lambda}\right)\left(k+t_0\right)^{-2\lambda-1}\int\limits_{\Omega}v^2(x,0)dx \tag{2.8}$$

$$-\theta\left(k+t_0\right)^{-2\lambda}\sum_{i,j=1}^{n}\int\limits_{\Omega}\left(a^{ij}v_iv_j\right)(x,0)dx$$

$$-\lambda\left(\theta-\frac{k}{2\lambda}\right)k^{-2\lambda-1}\int\limits_{\Omega}v^2(x,t_0)dx+\theta k^{-2\lambda}\sum_{i,j=1}^{n}\int\limits_{\Omega}\left(a^{ij}v_iv_j\right)(x,t_0)dx$$

$$-\sum_{i=1}^{n}\int\limits_{S_{t_0}}\left(\sum_{j=1}^{n}\left(2\theta v_t+v\right)a^{ij}v_j\left(k+t_0-t\right)^{-2\lambda}\right)\cos\left(n,x_i\right)dS.$$

Choose the number λ_1 such that $\lambda_1 \geq 2/(C\theta)$. Because of (2.5), it is sufficient to set

$$\lambda_1 \geq \frac{4}{C} \min \left[\frac{1}{\sqrt{6}A}, \frac{\sigma}{4(C+6A^2)} \right]^{-1}.$$

Since $k+t_0 \in (0,1)$,hen (2.4) implies that in (2.8)

$$(k+t_0)^{-2} - \frac{C}{\theta} - \frac{2}{\lambda\theta^2} > \frac{(k+t_0)^{-2}}{2} > \frac{1}{2}, \ \forall \lambda \geq \lambda_1 \tag{2.9a}$$

Also, by (2.4) and (2.5)

$$\frac{\sigma}{2} - \theta\left(C+6A^2\right) \geq \frac{\sigma}{2} - \frac{\sigma}{4} = \frac{\sigma}{4}. \tag{2.9b}$$

Estimates (2.8) and (2.9a,b) imply (2.6). □

Theorem 3. *Let the function $u \in H^{2,1}(Q_T)$ satisfies the parabolic inequality (1.5) and boundary conditions (1.4). Consider the vector function $W=(u,h_1,h_2,f)$. For $r \in (0,T)$ denote*

$$\|W\|_r = \left[\|u(x,r)\|_{L_2(\Omega)}^2 + \|h_1\|_{H^1(S_r)}^2 + \|h_2\|_{L_2(S_r)}^2 + \|f\|_{L_2(Q_r)}^2 \right]^{1/2}$$

Then there exist constants $C>0$ and $\bar{t} \in (0,T)$ such that for every $\beta \in (0,2)$ there exists a constant $\delta_0 \in (0,1)$ such that if $t_0 \in \left[\bar{t}/2, \bar{t}\right]$ and $\|W\|_{t_0} \leq B$, then the following logarithmic stability estimate is valid

$$\|g\|_{L_2(\Omega)}^2 \leq \frac{C}{\beta \ln \left[\frac{B}{\delta_0 \|W\|_{t_0}}\right]} \||\nabla g|\|_{L_2(\Omega)}^2 + C\left(\frac{B}{\delta_0}\right)^{\beta} \|W\|_{t_0}^{2-\beta},$$

where $g(x)=u(x,0)$ and the constant B is a given upper estimate of $\|W\|_{t_0}$. Constants C and $\bar{t}$ depend only on $C^1\left(\overline{Q}_T\right)$-norms of coefficients a^{ij} and numbers σ, T and A. The constant δ_0 depends on the same parameters, as well as on β. Neither of these numbers depends on t_0, as long as $t_0 \in \left[\bar{t}/2, \bar{t}\right]$.

Proof. Choose a number $\bar{t} \in (0,T) \cap (0,1)$ such that

$$\left(\frac{2}{3\bar{t}}\right)^2 > 8C\left\{\min\left[\frac{1}{\sqrt{6}A}, \frac{\sigma}{4(C+6A^2)}\right]\right\}^{-1}, \tag{2.10}$$

where C is the constant of Lemma 3. Let $k := \bar{t}/2$. Because of (2.10), (2.4) is satisfied for every $t_0 \in \left[\bar{t}/2, \bar{t}\right]$. Choose an arbitrary $t_0 \in \left[\bar{t}/2, \bar{t}\right]$. Since

$$\left(\frac{\bar{t}+2t_0}{2}\right)^{-2\lambda} = (k+t_0)^{-2\lambda} \leq (k+t_0-t)^{-2\lambda} \leq \left(\frac{2}{\bar{t}}\right)^{2\lambda}, \forall t \in [0,t_0],$$

then (2.6) implies that

$$\left(\frac{\bar{t}+2t_0}{2}\right)^{-2\lambda} \left[\frac{\sigma}{4}\||\nabla u|\|_{L_2(Q_{t_0})}^2 + \frac{\lambda\theta}{2}\|u\|_{L_2(Q_{t_0})}^2 + \lambda\left(\theta - \frac{\bar{t}+2t_0}{4\lambda}\right)\frac{2}{\bar{t}+2t_0}\|g\|_{L_2(\Omega)}^2\right]$$

$$\leq C\left(\frac{2}{\bar{t}}\right)^{2\lambda}\lambda\|W\|_{Q_{t_0}}^2+C\left(\frac{\bar{t}+2t_0}{2}\right)^{-2\lambda-1}\||\nabla g|\|_{L_2(\Omega)}^2,\ \forall\lambda\geq\lambda_1, \tag{2.11}$$

where λ_1 and θ are the same as in Lemma 3. Choose $\lambda_2 \geq \lambda_1$ depending on $C^1\left(\overline{Q}_T\right)$-norms of coefficients a^{ij} and numbers σ, T and A such that

$$\frac{\bar{t}}{\lambda_2}\leq\frac{1}{4}\min\left[\frac{1}{\sqrt{6}A},\frac{\sigma}{4\left(C+6A^2\right)}\right].$$

Since $t_0 \in \left[\bar{t}/2, \bar{t}\right]$, then (2.4) and (2.5) imply that

$$\theta-\frac{\bar{t}+2t_0}{2\lambda}\geq\frac{\theta}{2}\geq\frac{1}{4}\min\left[\frac{1}{\sqrt{6}A},\frac{\sigma}{4\left(C+6A^2\right)}\right],\forall\lambda\geq\lambda_2.$$

Multiplying both sides of (2.11) by $\lambda^{-1}\cdot\left[\left(\bar{t}+2t_0\right)/2\right]^{2\lambda}$ and keeping in mind that $\left(\bar{t}+2t_0\right)/\bar{t}\leq 3$, we obtain

$$\|g\|_{L_2(\Omega)}^2+\|u\|_{L_2\left(Q_{t_0}\right)}^2\leq C\cdot 3^{2\lambda}\|W\|_{t_0}^2+\frac{C}{\lambda}\||\nabla g|\|_{L_2(\Omega)}^2,\forall\lambda\geq\lambda_2. \tag{2.12}$$

Denote

$$\widetilde{u}=\frac{\delta_0}{B}u,\ \widetilde{g}=\frac{\delta_0}{B}g,\ \widetilde{W}=\frac{\delta_0}{B}W, \tag{2.13}$$

where the positive number δ_0 will be chosen later. Then (2.12) becomes

$$\|\widetilde{g}\|_{L_2(\Omega)}^2+\|\widetilde{u}\|_{L_2\left(Q_{t_0}\right)}^2\leq C\cdot 3^{2\lambda}\left\|\widetilde{W}\right\|_{t_0}^2+\frac{C}{\lambda}\||\nabla\widetilde{g}|\|_{L_2(\Omega)}^2,\forall\lambda\geq\lambda_1. \tag{2.14}$$

Note that by (2.13)

$$\left\|\widetilde{W}\right\|_{t_0}\leq\delta_0. \tag{2.15}$$

Choose an arbitrary constant $\beta \in (0,2)$. Choose λ such that

$$3^{2\lambda}\left\|\widetilde{W}\right\|_{t_0}^2=\left\|\widetilde{W}\right\|_{t_0}^{2-\beta}.$$

Hence,

$$\lambda=\frac{\beta}{\ln 9}\cdot\ln\left(\frac{1}{\left\|\widetilde{W}\right\|_{t_0}}\right). \tag{2.16}$$

Since we should have $\lambda \geq \lambda_2$, then (2.15) and (2.16) lead to the following choice for δ_0

$$0<\delta_0\leq\exp\left[-\frac{\lambda_2\ln 9}{\beta}\right].$$

Replacing in (2.14) the vector $\left(\widetilde{u},\widetilde{g},\widetilde{W}\right)$ with the vector (u,g,W), we obtain the target estimate of this theorem. $\square$

To prove Theorem 2, we also need to prove

Theorem 4. *Let the function $u\in H^{2,1}(Q_T)$ satisfies the parabolic inequality (1.8) and boundary conditions (1.4). Let*

$$\max_{0\leq j\leq n}\|b_j\|_{B\left(\overline{Q}_T\right)}\leq A_1,$$

where $A=const.>0$. Denote $f=const.\equiv K$. Consider the vector function $W=(u,h_1,h_2,f)$ and denote $r\in(0,T)$

$$\|W\|_r=\left[\|u(x,r)\|_{L_2(\Omega)}^2+\|h_1\|_{H^1(S_r)}^2+\|h_2\|_{L_2(S_r)}^2+K^2\right]^{1/2}$$

Then there exist constants $C_1>0$ and $\overline{t}\in(0,T)$ such that for every $\beta\in(0,2)$ there exists a constant $\delta_1\in(0,1)$ such that if $t_0\in\left[\overline{t}/2,\overline{t}\right]$ and $\|W\|_{t_0}\leq B$, then the following logarithmic stability estimate is valid

$$\|g\|_{L_2(\Omega)}^2\leq\frac{C_1}{\beta\ln\left[\frac{B}{\delta_1\|W\|_{t_0}}\right]}\,\||\nabla g|\|_{L_2(\Omega)}^2+C_1\left(\frac{B}{\delta_1}\right)^{\beta}\|W\|_{t_0}^{2-\beta},$$

where $g(x)=u(x,0)$ and the constant B is a known upper estimate of $\|W\|_{t_0}$. Constants C_1 and $\overline{t}$ depend only on $C^1\left(\overline{Q}_T\right)$-norms of coefficients a^{ij} the numbers σ,T,K and A_1. The constant δ_1 depends on the same parameters, as well as on β. Neither of these numbers depends on t_0, as long as $t_0\in\left[\overline{t}/2,\overline{t}\right]$.

Proof. Let $\overline{t}\in(0,T)\cap(0,1)$ and $k=\overline{t}/2$. We will specify the number $\overline{t}$ later. Choose and arbitrary $t_0\in\left[\overline{t}/2,\overline{t}\right]$. We have for all $\lambda>0$

$$\int\limits_{Q_T}(u_t-Lu)^2\,dxdt\geq\int\limits_{Q_{t_1}}(u_t-Lu)^2(k+t_0-t)^{-2\lambda}(k+t_0-t)^{2\lambda}\,dxdt$$

$$\geq k^{2\lambda}\int\limits_{Q_{t_1}}(u_t-Lu)^2(k+t_0-t)^{-2\lambda}\,dxdt.$$

Hence, by (1.8)

$$\int\limits_{Q_{t_0}}(u_t-Lu)^2(k+t_0-t)^{-2\lambda}\,dxdt\leq\left(\frac{2}{\overline{t}}\right)^{2\lambda}C_1\|f\|_{L_2(Q_T)}^2. \tag{2.17}$$

Since

$$(u_t-Lu)^2\geq\frac{1}{2}(u_t-L_0u)^2-3\left[(L-L_0)\,u\right]^2\geq\frac{1}{2}(u_t-L_0u)^2-C_1\left(|\nabla u|^2+u^2\right),$$

then we obtain from (2.17)

$$\int\limits_{Q_{t_0}} (u_t - L_0 u)^2 (k + t_0 - t)^{-2\lambda} \, dxdt \tag{2.18}$$

$$\leq C_1 \int\limits_{Q_{t_0}} \left(|\nabla u|^2 + u^2\right) (k + t_0 - t)^{-2\lambda} \, dxdt + \left(\frac{2}{\overline{t}}\right)^{2\lambda} C_1 \|f\|^2_{L_2(Q_T)}.$$

Since (2.7) holds for all functions $v \in H^{2,1}(Q_{t_0})$, for all $\lambda > \lambda_0 > 1$ and for all $\theta > 0$, then (2.7) also holds for the function u. Hence, it follows from the proof of Lemma 3 that one can choose numbers $\theta_1, \theta_2 > 0, \theta_1 < \theta_2$, as well as numbers $\overline{t} \in (0, T) \cap (0, 1)$ and $\widehat{\lambda}_1 \geq \lambda_0$, all depending only on C_1 and σ such that $\forall \theta \in [\theta_1, \theta_2], \forall \lambda \geq \widehat{\lambda}_1, \forall t_0 \in \left[\overline{t}/2, \overline{t}\right]$ the inequality (2.6) is valid with the replacement $(6\theta A^2, v) \to (C_1, u)$. Thus, the rest of the proof is the same as in Theorem 3. □

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 1

Denote $y = (x_2, ..., x_n)$. Without loss of generality we can assume that

$$\overline{\Omega} \subset \left\{ x_1 > x_{10} = const. > 0, x_1 + |y|^2 < \frac{1}{2} \right\}. \tag{3.1}$$

Consider the function $\psi(x,t)$ defined as

$$\psi(x,t) = x_1 + |y|^2 + \frac{\left(t - \overline{t}\right)^2}{b^2}, \tag{3.2}$$

where the number $\overline{t} \in (0, T)$ was chosen in the proof of Theorem 3 (see (2.10)). By (2.10) we can assume without loss of generality that $\overline{t} < T/2$. Let $s = \max_{\overline{\Omega}} \left(x_1 + |y|^2\right)$. We choose the number b such that

$$\frac{3}{4} \cdot \overline{t} \left(\frac{1}{2} - s\right)^{-1/2} < b < \overline{t} \left(\frac{1}{2} - s\right)^{-1/2}. \tag{3.3}$$

For $\omega \in (0, 1/2)$ denote

$$G_\omega = \left\{ (x,t) : x \in \Omega, \psi(x,t) < \frac{1}{2} - \omega. \right\} \tag{3.4}$$

It follows from (3.1)-(3.4) that

$$\text{if } \ 0 < \omega_1 < \omega_2 < 1/2, \text{ then } \ G_{\omega_2} \subset G_{\omega_1} \subset G_0 \subset Q_T. \tag{3.5}$$

Also, by (3.3) one can choose a number $\omega_0 \in (0, 1/6)$ so small that

$$\overline{\Omega} \times \left[\frac{\overline{t}}{2}, \overline{t}\right] \subset G_{3\omega_0} \subset G_0 \subset Q_T. \tag{3.6}$$

The boundary ∂G_ω of the domain G_ω consists of two parts,

$$\partial G_\omega = \partial_1 G_\omega \cup \partial_2 G_\omega, \tag{3.7}$$

where

$$\partial_1 G_\omega = \overline{G}_\omega \cap S_T,\ \partial_2 G_\omega = \overline{G}_\omega \cap \left\{\psi(x,t) = \frac{1}{2} - \omega\right\}. \tag{3.8}$$

For a positive parameter ν denote

$$\varphi(x,t) = [\psi(x,t)]^{-\nu}.$$

It follows from Lemma 3 of §1 of Chapter 4 of [12] that the following pointwise Carleman estimate is valid for any function $w \in C^{2.1}\left(\overline{G}_0\right)$

$$(w_t - L_0 w)^2 \exp(2\mu\varphi) \geq M\left[\mu\nu\left|\nabla w\right|^2 + \mu^3\nu^4\psi^{-2\nu-2}w^2\right]\exp(2\mu\varphi)$$

$$+\nabla\cdot U + V_t, \quad \forall\,(x,t) \in G_0, \forall\nu \geq \nu_0, \forall\mu \geq \lambda_2,$$

where constants $\nu_0, \lambda_2 > 1$ and M depend only on the domain Ω, the number b in (3.2) and (3.3) and $C^1\left(\overline{G}_0\right)$-norms of coefficients a^{ij}. The vector function (U, V) satisfies the following estimate

$$|(U,V)| \leq M\mu^3\nu^3\psi^{-2\nu-2}\left(\left|\nabla w\right|^2 + w^2\right)\exp(2\mu\varphi).$$

Setting here $\nu = \nu_0$, we obtain with a different constant M depending on the same parameters

$$(w_t - L_0 w)^2 \exp(2\mu\varphi) \geq M\left(\mu\left|\nabla w\right|^2 + \mu^3 w^2\right)\exp(2\mu\varphi) + \nabla\cdot U + V_t, \tag{3.9}$$

$$\forall\,(x,t) \in G_0, \forall\mu \geq \lambda_2, \forall w \in C^{2.1}\left(\overline{G}_0\right),$$

$$|(U,V)| \leq M\mu^3\left(\left|\nabla w\right|^2 + w^2\right)\exp(2\mu\varphi). \tag{3.10}$$

Choose a function $\chi(x,t) \in C^2\left(\overline{G}_0\right)$ such that

$$\chi(x,t) = \left\{\begin{array}{c} 1,\ \text{for } (x,t) \in G_{2\omega_0}, \\ 0,\ \text{for } (x,t) \in G_0 \diagdown G_{\omega_0}, \\ \text{between 0 and 1 for all other } (x,t) \in G_0. \end{array}\right\}. \tag{3.11}$$

Denote $p(x,t) = u(x,t)\chi(x,t)$, where the function $u(x,t)$ satisfies conditions (1.4), (1.5). Hence, the function $p \in H^{2,1}(G_0)$ and (1.4), (1.5), (3.7), (3.8) and (3.11) lead to

$$|p_t - L_0 p| \leq A\left[|\nabla p| + |p| + |f|\right] + \widetilde{A}\,(1-\chi)\left[|\nabla u| + |u|\right],\ \text{a.e. in } G_0, \tag{3.12}$$

$$p\mid_{\partial_1 G_0}=h_1\chi,\quad \frac{\partial p}{\partial n}\mid_{\partial_1 G_0}=h_2\chi+\frac{\partial\chi}{\partial n}\cdot h_1, \tag{3.13}$$

$$p\mid_{\partial_2 G_0}=\nabla p\mid_{\partial_2 G_0}=0, \tag{3.14}$$

where the positive constant $\widetilde{A}\geq A$ depends on the constant A and the number ω_0.

Integrate the inequality (3.9) over G_0 using (3.10) and the Gauss' formula. Since in (3.10) $w\in C^{2,1}\left(\overline{G}_0\right)$ is an arbitrary function and the set $C^{2,1}\left(\overline{G}_0\right)$ is dense in $H^{2,1}\left(G_0\right)$, then we can pass to the limit as $w\rightarrow p\in H^{2,1}\left(G_0\right)$. Hence, (3.13) and (3.14) imply that

$$\int\limits_{G_0}\left(p_t-L_0p\right)^2\exp\left(2\mu\varphi\right)dxdt\geq C\mu\int\limits_{G_0}\left[\left|\nabla p\right|^2+\mu^2p^2\right]\exp\left(2\mu\varphi\right)dxdt \tag{3.15}$$

$$-C\mu^3\exp(2\mu c)\left[\|h_1\|^2_{H^1(S_T)}+\|h_2\|^2_{L_2(S_T)}\right],\forall\mu\geq\lambda_2.$$

where

$$c=\max_{\overline{Q}_T}\varphi(x,t).$$

Since by (3.11) and (3.12)

$$\int\limits_{G_0}\left(p_t-L_0p\right)^2\exp\left(2\mu\varphi\right)dxdt\leq 6A^2\int\limits_{G_0}\left[\left|\nabla p\right|^2+\left|p\right|^2+\left|f\right|^2\right]\exp\left(2\mu\varphi\right)dxdt$$

$$+4A_1^2\int\limits_{G_0}(1-\chi)\left[\left|\nabla u\right|^2+u^2\right]\exp\left(2\mu\varphi\right)dxdt,$$

and by (3.15) $1-\chi=0$ in $G_{2\omega_0}$, then (3.11) implies that for all $\mu\geq\lambda_2$

$$4A_1^2\int\limits_{G_0\diagdown G_{2\omega_0}}\left[\left|\nabla u\right|^2+u^2\right]\exp\left(2\mu\varphi\right)dxdt+6A^2\int\limits_{G_0}\left[\left|\nabla p\right|^2+\left|p\right|^2+\left|f\right|^2\right]\exp\left(2\mu\varphi\right)dxdt$$

$$\geq M\mu\int\limits_{G_0}\left[\left|\nabla p\right|^2+\mu^2p^2\right]\exp\left(2\mu\varphi\right)dxdt \tag{3.16}$$

$$-M\mu^3\exp(2\mu c)\left[\|h_1\|^2_{H^1(S_T)}+\|h_2\|^2_{L_2(S_T)}\right].$$

Let $\lambda_1>1$ be the number which was chosen in the proof of Theorem 3. Choose $\lambda_3\geq\max\left(\lambda_1,\lambda_2\right)$ such that

$$6A^2\leq\frac{M\mu}{2},\ \forall\mu\geq\lambda_3$$

and

$$\mu^3\exp(2\mu c)<\exp(3\mu c),\forall\mu\geq\lambda_3. \tag{3.17}$$

Hence, by (3.16)

$$4A_1^2 \int\limits_{G_0 \diagdown G_{2\omega_0}} \left[|\nabla u|^2 + u^2 \right] \exp(2\mu\varphi)\, dxdt + M\mu^3 \exp(2\mu c) \left[\|h_1\|_{H^1(S_T)}^2 + \|h_2\|_{L_2(S_T)}^2 \right] \tag{3.18}$$

$$\geq \frac{M\mu}{2} \int\limits_{G_0} \left[|\nabla p|^2 + \mu^2 p^2 \right] \exp(2\mu\varphi)\, dxdt.$$

Note that

$$\exp(2\mu\varphi) \leq \exp\left[2\mu \left(\frac{1}{2} - 2\omega_0 \right)^{-\nu_0} \right] \quad \text{in } G_0 \diagdown G_{2\omega_0}. \tag{3.19}$$

Also,

$$\exp\left[2\mu \left(\frac{1}{2} - 2\omega_0 \right)^{-\nu_0} \right] < \exp(2\mu c). \tag{3.20}$$

The standard energy estimate for the parabolic equation, whose proof can be easily extended to the parabolic inequality (1.5) leads to

$$\|u\|_{L_2(Q_T)}^2 + \||\nabla u|\|_{L_2(Q_T)}^2 \leq C \left[\|g\|_{L_2(\Omega)}^2 + \|h_1\|_{L_2(S_T)}^2 + \|h_2\|_{L_2(S_T)}^2 \right].$$

Hence, (3.19) leads to

$$\int\limits_{G_0 \diagdown G_{2\omega_0}} \left[|\nabla u|^2 + u^2 \right] \exp(2\mu\varphi)\, dxdt$$

$$\leq \exp\left[2\mu \left(\frac{1}{2} - 2\omega_0 \right)^{-\nu_0} \right] \int\limits_{G_0 \diagdown G_{2\omega_0}} \left[|\nabla u|^2 + u^2 \right] dxdt \tag{3.21}$$

$$\leq C \left[\|g\|_{L_2(\Omega)}^2 + \|h_1\|_{L_2(S_T)}^2 + \|h_2\|_{L_2(S_T)}^2 + \|f\|_{L_2(Q_T)}^2 \right] \cdot \exp\left[2\mu \left(\frac{1}{2} - 2\omega_0 \right)^{-\nu_0} \right].$$

Recall that

$$\|F\|^2 = \|h_1\|_{H^1(S_T)}^2 + \|h_2\|_{L_2(S_T)}^2 + \|f\|_{L_2(Q_T)}^2.$$

Hence, (3.17)-(3.21) imply that

$$C \exp\left[2\mu \left(\frac{1}{2} - 2\omega_0 \right)^{-\nu_0} \right] \|g\|_{L_2(\Omega)}^2 + C \|F\|^2 \cdot \exp(3\mu c) \tag{3.22}$$

$$\geq \mu \int\limits_{G_0} \left[|\nabla p|^2 + \mu^2 p^2 \right] \exp(2\mu\varphi)\, dxdt, \forall \mu \geq \lambda_3.$$

Note that by (3.5) and (3.11) $u=p$ in $G_{3\omega_0}$. Also,

$$\exp(2\mu\varphi) \geq \exp\left[2\mu\left(\frac{1}{2}-3\omega_0\right)^{-\nu_0}\right] \text{ in } G_{3\omega_0}.$$

Hence, (3.5) and (3.22) imply that

$$C\exp\left[2\mu\left(\frac{1}{2}-2\omega_0\right)^{-\nu_0}\right]\|g\|^2_{L_2(\Omega)} + C\|F\|^2\exp(3\mu c)$$

$$\geq \mu \int\limits_{G_{3\omega_0}} \left[|\nabla p|^2 + \mu^2 p^2\right]\exp(2\mu\varphi)\,dxdt$$

$$\geq \exp\left[2\mu\left(\frac{1}{2}-3\omega_0\right)^{-\nu_0}\right]\int\limits_{G_{3\omega_0}}\left[|\nabla u|^2+u^2\right]dxdt,\ \forall\mu\geq\lambda_3.$$

Dividing both sides by

$$\exp\left[2\mu\left(\frac{1}{2}-3\omega_0\right)^{-\nu_0}\right],$$

we obtain

$$C\exp(-2\mu\rho)\|g\|^2_{L_2(\Omega)} + C\|F\|^2\exp(3\mu c) \geq \int\limits_{G_{3\omega_0}}\left[|\nabla u|^2+u^2\right]dxdt,\ \forall\mu\geq\lambda_3, \tag{3.23}$$

where

$$\rho = \left(\frac{1}{2}-3\omega_0\right)^{-\nu_0} - \left(\frac{1}{2}-2\omega_0\right)^{-\nu_0} > 0.$$

By (3.6) and the mean value theorem there exists a number $t_0 \in \left(\overline{t}/2, \overline{t}\right)$ such that

$$\int\limits_\Omega u^2(x,t_0)dx \leq \frac{2}{\overline{t}}\int\limits_{G_{3\omega_0}} u^2 dxdt.$$

This and (3.23) lead to

$$\int\limits_\Omega u^2(x,t_0)dx \leq C\exp(-2\mu\rho)\|g\|^2_{L_2(\Omega)} + C\|F\|^2\exp(3\mu c), \forall\mu\geq\lambda_3. \tag{3.24}$$

We now recall the notation $\|W\|_{t_0}$ of Theorem 3. Since $\|F\| \geq \|W\|_{t_0}$, then substituting (3.24) in (2.12), we obtain for all $\lambda,\mu\geq\lambda_3$

$$\|g\|^2_{L_2(\Omega)} \leq C\cdot 3^{2\lambda}\exp(-2\mu\rho)\|g\|^2_{L_2(\Omega)} + C\cdot 3^{2\lambda}\exp(3\mu c)\|F\|^2 + \frac{C}{\lambda}\||\nabla g|\|^2_{L_2(\Omega)}, \tag{3.25}$$

Choose $\mu = \mu(\lambda)$ as follows

$$\mu = \mu(\lambda) = \frac{2\lambda}{\rho} \ln 3.$$

Then there exists a number $\lambda_4 \geq \lambda_3$ such that $\mu(\lambda) \geq \lambda_3, \forall \lambda \geq \lambda_4$. Furthermore with such a choice of μ we have

$$3^{2\lambda} \exp(-2\mu\rho) = 3^{-2\lambda}$$

and

$$3^{2\lambda} \exp(3\mu c) = \exp(\lambda\rho_1), \quad \rho_1 = \frac{6c}{\rho} + \ln 9.$$

Hence, (3.25) leads to

$$\|g\|^2_{L_2(\Omega)} \leq C \cdot 3^{-2\lambda} \|g\|^2_{L_2(\Omega)} + C \exp(\lambda\rho_1) \|F\|^2 + \frac{C}{\lambda} \||\nabla g|\|^2_{L_2(\Omega)}, \forall \lambda \geq \lambda_4.$$

Choose a number $\lambda_5 \geq \lambda_4$ such that $C \cdot 3^{-2\lambda} \leq 1/2$. Then

$$\|g\|^2_{L_2(\Omega)} \leq C \exp(\lambda\rho_1) \|F\|^2 + \frac{C}{\lambda} \||\nabla g|\|^2_{L_2(\Omega)}, \ \forall \lambda \geq \lambda_5. \tag{3.26}$$

Similarly with (2.13) denote

$$\widetilde{g} = \frac{\varepsilon_0}{B} g, \widetilde{F} = \frac{\varepsilon_0}{B} F, \tag{3.27}$$

where the number ε_0 will be chosen later. Then (3.26) holds for functions $\widetilde{g}$ and $\widetilde{F}$, i.e.,

$$\|\widetilde{g}\|^2_{L_2(\Omega)} \leq C \exp(\lambda\rho_1) \left\|\widetilde{F}\right\|^2 + \frac{C}{\lambda} \||\nabla \widetilde{g}|\|^2_{L_2(\Omega)}, \ \forall \lambda \geq \lambda_5. \tag{3.28}$$

Take an arbitrary $\beta \in (0, 2)$ and choose λ such that

$$\exp(\lambda\rho_1) \left\|\widetilde{F}\right\|^2 = \left\|\widetilde{F}\right\|^{2-\beta}.$$

Hence,

$$\lambda = \frac{\beta}{\rho_1} \ln\left(\frac{1}{\left\|\widetilde{F}\right\|}\right). \tag{3.29}$$

Since $\left\|\widetilde{F}\right\| \leq \varepsilon_0$ and we should have $\lambda \geq \lambda_5$, then (3.27) and (3.29) lead to the following requirement for ε_0

$$\varepsilon_0 \leq \exp\left(-\frac{\rho_1\lambda_5}{\beta}\right).$$

Thus, (3.27)-(3.29) imply (1.7). $\square$

3.2 Proof of Theorem 2

We keep notations of the proof of Theorem 1. By (1.8)

$$K^2 \geq \int\limits_{Q_T} (u_t - Lu)^2 \exp(2\mu\varphi) \exp(-2\mu\varphi)\, dxdt \geq e^{-2\mu c} \int\limits_{Q_T} (u_t - Lu)^2 \exp(2\mu\varphi)\, dxdt.$$

Since by (3.5) $G_0 \subset Q_T$, then this inequality leads to

$$\int\limits_{G_0} (u_t - Lu)^2 \exp(2\mu\varphi)\, dxdt \leq C_1 \|f\|^2_{L_2(Q_T)} e^{2\mu c}. \tag{3.30}$$

Further,

$$u = \chi u + (1-\chi)\, u = p + (1-\chi)\, u.$$

Hence,

$$u_t - Lu = (p_t - Lp) + (1-\chi)(u_t - Lu) + Q(x,t),$$

where

$$|Q(x,t)| \leq C_1 (|\nabla\chi| + |\chi_t|)(|\nabla u| + |u|).$$

Hence, using the Cauchy-Schwarz inequality, (3.30) and the fact that by (3.11) $|\nabla\chi| + |\chi_t| = 0$ in $G_{2\omega_0}$, we obtain that

$$\int\limits_{G_0} (p_t - L_0 p)^2 \exp(2\mu\varphi)\, dxdt$$

$$\leq C_1 \int\limits_{G_0} \left[|\nabla p|^2 + |p|^2\right] \exp(2\mu\varphi)\, dxdt$$

$$+C_1 \int\limits_{G_0 \diagdown G_{2\omega_0}} \left[|\nabla u|^2 + u^2\right] \exp(2\mu\varphi)\, dxdt + C_1 \|f\|^2_{L_2(Q_T)} e^{2\mu c},\ \forall \mu \geq \lambda_2.$$

This and (3.15) lead to a direct analog of (3.16), i.e.,

$$C_1 \|f\|^2_{L_2(Q_T)} e^{2\mu c} + C_1 \int\limits_{G_0 \diagdown G_{2\omega_0}} \left[|\nabla u|^2 + u^2\right] \exp(2\mu\varphi)\, dxdt + C_1 \int\limits_{G_0} \left[|\nabla p|^2 + |p|^2\right] \exp(2\mu\varphi)\, dxdt$$

$$\geq \mu \int\limits_{G_0} \left[|\nabla p|^2 + \mu^2 p^2\right] \exp(2\mu\varphi)\, dxdt$$

$$-\mu^3 \exp(2\mu c) \left[\|h_1\|^2_{H^1(S_T)} + \|h_2\|^2_{L_2(S_T)}\right],\ \forall \mu \geq \lambda_2.$$

The rest of the proof is the same as the proof of Theorem 1 after (3.16). $\square$

4 Convergent Numerical Method

We want to find an approximate solution (u, g) of the problem (1.2)-(1.4), assuming, of course that the initial condition $g(x)$ is unknown. First, it is convenient to obtain zero boundary conditions in (1.4). Suppose that there exists a function $P \in H^{2,1}(Q_T)$ such that

$$P \mid_{S_T} = h_1, \ \frac{\partial P}{\partial n} \mid_{S_T} = h_2.$$

Denote

$$w = u - P, \quad \widetilde{f} = f - (P_t - LP), \quad \widetilde{g} = g - P(x,0).$$

Then

$$w_t - Lw = \widetilde{f} \ \text{ in } Q_T, \tag{4.1}$$

$$w \mid_{S_T} = \frac{\partial w}{\partial n} \mid_{S_T} = 0, \tag{4.2}$$

$$w(x,0) = \widetilde{g}(x). \tag{4.3}$$

Introduce a Sobolev space $H(Q_T)$ by

$$H(Q_T) = \left\{ y : \|y\|_H^2 := \|y\|_{H^{2,1}(Q_T)}^2 + \||\nabla y_t|\|_{L_2(Q_T)}^2 < \infty, y \mid_{S_T} = \frac{\partial y}{\partial n} \mid_{S_T} = 0 \right\}.$$

Let $\langle , \rangle$ be the scalar product in $H(Q_T)$. We minimize the Tikhonov functional, see, e.g., Tikhonov and Arsenin [16],

$$J_\alpha(w) = \left\| w_t - Lw - \widetilde{f} \right\|_{L_2(Q_T)}^2 + \alpha \|w\|_H^2, \tag{4.4}$$

where $\alpha > 0$ is the regularization parameter. Since the Frechét derivative of $J_\alpha(w)$ is zero at the minimizer w^α, then the minimizer w^α satisfies the following conditions

$$\int\limits_{Q_T} (w_t^\alpha - Lw^\alpha)(y_t - Ly)\, dxdt + \alpha \langle w_\alpha, y \rangle = \int\limits_{Q_T} \widetilde{f}(y_t - Ly)\, dxdt, \tag{4.5}$$

$$w^\alpha \in H(Q_T), \forall y \in H(Q_T). \tag{4.6}$$

Hence, the function $w^\alpha \in H(Q_T)$ is the weak solution of the problem (4.5), (4.6). The following result follows immediately from the Riesz' theorem

Lemma 4. *For every $\alpha > 0$ there exists unique solution $w^\alpha \in H(Q_T)$ of the boundary value problem (4.5), (4.6) and the following estimate holds*

$$\|w^\alpha\|_H^2 \leq \frac{C}{\alpha} \left\| \widetilde{f} \right\|_{L_2(Q_T)}^2.$$

To address a more difficult (than existence) question about convergence, we need to introduce error in the data and to use Theorem 1. Following the concept of Tikhonov for solutions of ill-posed problems [16], we assume that there exists an "ideal" exact data $\widetilde{f}^* \in L_2(Q_T)$ and the "ideal" exact solution $w^* \in H(Q_T)$ of the problem (4.1), (4.2) corresponding to this data (It follows from Theorem 1 that if such a solution exists, then it is unique). However, since actual data $\widetilde{f}$ is always given with an error, one cannot find that ideal solution w^*. Instead, one can only hope to find an approximation for this solution. Hence, we assume that

$$\left\| \widetilde{f} - \widetilde{f}^* \right\|_{L_2(Q_T)} \leq \delta < 1, \tag{4.7}$$

where δ is an upper estimate of the level of the error in the data. So, we want to figure out the choice of the regularization parameter $\alpha = \alpha(\delta)$ and to estimate the difference between the approximate solution $w^{\alpha(\delta)}$ and the exact one w^* as $\delta \to 0^+$. The following convergence result is valid.

Theorem 5. *Let functions w^* and $\widetilde{f}^*$ be those which were introduced above. Let in (4.4)-(4.6) $\alpha = \alpha(\delta) = \delta$ and the inequality (4.7) be fulfilled. Let $s_\delta = w^{\alpha(\delta)} - w^*$ be the difference between the regularized $w^{\alpha(\delta)}$ and the exact w^* solutions. Then there exists a positive constant C_1 such that for every number $\beta \in (0,2)$ there exists a number $\delta_2 \in (0,1)$ such that for all $\delta \in (0, \delta_2)$ the following estimates hold*

$$\left\| \left(w^{\alpha(\delta)} - w^* \right)(x,0) \right\|^2_{L_2(\Omega)} \leq C_1 \delta_2^{-\beta} \left[1 + \| w^* \|^2_H \right] \cdot \frac{1}{\beta \ln \left(\frac{1}{\delta_2 \delta} \right)}, \tag{4.8}$$

$$\| s_\delta \|^2_{H^{1,0}(Q_T)} \leq C_1 \delta_2^{-\beta} \left[1 + \| w^* \|^2_H \right] \cdot \frac{1}{\beta \ln \left(\frac{1}{\delta_2 \delta} \right)} \tag{4.9}$$

The constant C_1 depends only on the domain Ω, the number T, $C^1\left(\overline{Q}_T\right)-$ norms of coefficients a^{ij}, the number σ in (1.1) and $B\left(\overline{Q}_T\right)$-norms of coefficients at low order terms of the operator L. The number δ_2 depends on the same parameters, as well as on the parameter β.

Remark. Hence, estimates (4.8) and (4.9) tell one that if one *a priori* imposes a bound on $H(Q_T)$-norms of solutions, $\|w^*\|_H \leq M_1$, where $M_1 = const. > 0$, and sets a connection $\alpha(\delta) = \delta$ between the regularization parameter α and the upper estimate δ of the level of the error in the data, then the regularized solution $w^{\alpha(\delta)}$ converges to the exact one with the logarithmic speed, as long as $\delta \to 0^+$.

Proof of Theorem 5. Let $q = \widetilde{f} - \widetilde{f}^*$. Then by (4.7) $\|q\|_{L_2(Q_T)} \leq \delta$. Using (4.1), (4.2), (4.5) and (4.6), we obtain

$$\int\limits_{Q_T} (s_{\delta t} - L s_\delta)(y_t - Ly)\, dxdt + \alpha \langle s_\delta, y \rangle = \int\limits_{Q_T} q (y_t - Ly)\, dxdt + \alpha \langle w^*, y \rangle, \forall y \in H(Q_T).$$

Setting here $y := s$, $\alpha := \delta$ and applying the Cauchy-Schwarz inequality and (4.7), we obtain

$$\int\limits_{Q_T} (s_{\delta t} - L s_\delta)^2\, dxdt \leq \left(1 + \|w^*\|^2_H \right) \delta, \tag{4.10}$$

$$\|s_\delta\|_H^2 \leq 1 + \|w^*\|_H^2 . \tag{4.11}$$

To apply Theorem 2, set

$$K^2 := \left(1 + \|w^*\|_H^2\right)\delta, f := const. = K, F = (0,0,f), B = \sqrt{1 + \|w^*\|_H^2}.$$

It follows from the definition of the space $H(Q_T)$, the trace theorem and (4.11) that $\||\nabla s_\delta(x,0)|\|^2 \leq C_1\left(1 + \|w^*\|_H^2\right)$. Hence, Theorem 2 and (4.10) imply (4.8) and (4.9). $\square$

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