A phase-space study of the quantum Loschmidt Echo in the semiclassical limit

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Abstract

The notion of Loschmidt echo (also called "quantum fidelity") has been introduced in order to study the (in)-stability of the quantum dynamics under perturbations of the Hamiltonian. It has been extensively studied in the past few years in the physics literature, in connection with the problems of "quantum chaos", quantum computation and decoherence.

In this paper, we study this quantity semiclassically (as $\hbar \to 0$), taking as reference quantum states the usual coherent states. The latter are known to be well adapted to a semiclassical analysis, in particular with respect to semiclassical estimates of their time evolution. For times not larger than the so-called "Ehrenfest time" $C|\log \hbar|$, we are able to estimate semiclassically the Loschmidt Echo as a function of t (time), $\hbar$ (Planck constant), and δ (the size of the perturbation). The way two classical trajectories merging from the same point in classical phase-space, fly apart or come close together along the evolutions governed by the perturbed and unperturbed Hamiltonians play a major role in this estimate.

We also give estimates of the "return probability" (again on reference states being the coherent states) by the same method, as a function of t and $\hbar$.

1 Introduction

The semiclassical time behaviour of quantum wavepackets has been the subject of intense interest in the last decades, in particular in situations where there is some hyperbolicity in the corresponding classical dynamics (Lyapunov exponents) [8], [15], [28]. Moreover the response of a quantum system to an external perturbation when the size δ of the perturbation increases can manifest intriguing properties such as recurrences or decay in time of the so-called Loschmidt Echo (or "quantum fidelity") [6], [7]. By Loschmidt Echo we mean the following:
starting from a quantum Hamiltonian $\hat{H}$ whose classical counterpart has a chaotic dynamics, and adding to it a " perturbation" $\hat{H}_\delta = \hat{H} + \delta\hat{V}$, then we compare the evolutions in time $U(t) := e^{-it\hat{H}/\hbar}$, $U_\delta(t) := e^{-it\hat{H}_\delta/\hbar}$ of initial quantum wavepackets φ sufficiently well localized around some point z in phase-space; more precisely the **overlap** between the two evolutions, or rather its square absolute value, is:

$$F_{\hbar,\delta}(t) := |\langle U_\delta(t)\varphi \ , \ U(t)\varphi\rangle|^2$$

For example for quantum dynamics in Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, d being the space dimension, φ can be chosen as the usual *coherent states*, since they are the quantum wavepackets "as most localized as possible" in phase-space $\mathbb{R}^{2d}$.

Since for $\delta = 0$, we obviously have $F_{\hbar,0}(t) \equiv 1$, and for any δ , $F_{\hbar,\delta}(0) = 1$, the type of decay in t of $F_{\hbar,\delta}(t)$ so to say *measures the (in)fidelity of the quantum evolution with respect to a perturbation of size* δ for generic initial wavepackets φ.

The notion of Loschmidt Echo seems to have been first introduced by Peres ([22]), in the following spirit: since the sensitivity to initial data which characterizes classical chaos has no quantum counterpart because of unitarity of the quantum evolution, at least the "sensitivity to perturbations" of the Hamiltonian could replace it as a characterization of chaoticity in the "quantum world".

A big amount of recent work appeared on the subject, studying in an essentially heuristic way the decay in time of $F_{\hbar,\delta}(t)$ as t increases from zero to infinity; some of them also study this point in relationship with the important question of *decoherence*. (See [1], [4], [12-14], [18], [22-26], [30-32]).

In this "jungle" of sometimes contradictory results, it is hard to see the various arguments involved, in particular the precise behaviour of $F_{\hbar,\delta}(t)$ as δ (the size of the perturbation), t (the time), and of course $\hbar$ (the Planck constant) are varied, in particular in which sense and order the various limts $\delta \to 0, \hbar \to 0, t \to \infty$ are taken.
Also an important point to consider is how $F_{\hbar,\delta}(t)$ depends on the location of the phase-space point z around which the initial wavepacket φ is peaked (since classical chaoticity distinguishes various zones in phase-space with "more or less regularity properties").

The aim of the present paper is to start a rigorous approach of the question of semiclassical estimate of $F_{\hbar,\delta}(t)$, in terms of classical characteristics of the (perturbed and unperturbed classical flows), for initial wavepackets $\varphi = \varphi_z$ being the coherent state at phase-space point z. These estimates are non-perturbative, and are carefully calculated in terms of parameters $(z, \delta, t, \hbar)$. The main tools we have used and developed in this respect are
1) semiclassical coherent states propagation estimates ([8])
2) a beautiful formula inspired by B. Mehlig and M. Wilkinson ([21]) about the Weyl symbol of a metaplectic operator, and thus of its expectation value in coherent states as a simple Gaussian phase-space integral (see [9] where we have completed the proof of Mehlig-Wilkinson, and treated in particular the case where the monodromy operator has eigenvalue 1).
Note that very recently, J. Bolte and T. Schwaibold have independently obtained a similar result about semiclassical estimates of the Quantum Fidelity ([2]).

The plan of this paper is as follows. In section 2 we give some preliminaries about the Echo for suitable quantum observables, and give the semiclassics of it. In Section 3, we consider the (integrable) $d = 1$ case, and consider the "return probability" in the semiclassical limit. We give a sketchy presentation of a beautiful result on "quantum revivals" (see [29], [19]). In Section 4 we consider the general d-dimensional case and give a semiclassical calculus of the "return probability" and of the **quantum fidelity**, with precise error estimates.

2 Preliminaries

Let us consider the Hamiltonian $\hat{H}_\delta = \hat{H}_0 + \delta\hat{V}$, depending on a real parameter δ. $U_\delta(t)$ is the time evolution unitary operator in Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$

$$U_\delta(t) = \exp\left(-\frac{it}{\hbar}\hat{H}_\delta\right)$$

The quantum echo is the unitary operator defined by

$$E_\delta^{(q)}(t) = U_0(-t)U_\delta(t) \tag{2.1}$$

and the quantum fidelity is defined, for a state ψ_0, $\|\psi_0\| = 1$, by

$$f_\delta^{(q)}(t) = \left|\langle\psi_0|E_\delta^{(q)}(t)\psi_0\rangle\right|^2 \tag{2.2}$$

This was introduced first in classical mechanics by Loschmidt (in discussions with Boltzmann) then adapted in quantum mechanics by Peres [22].
Let us denote by $\Phi_\delta(t)$ the classical flow defined in the phase space $\mathbb{R}^{2d}$ by the classical Hamiltonian H_δ. That means $z_{\delta,t} := \Phi_\delta(t, z_0)$ is the solution of the differential equation $\dot{z}_t = J\nabla H_\delta(z_t)$, $z_{t=0} = z_0$. So that the classical echo is

$$E_\delta^{(cl)}(t, z_0) = \Phi_0\left(-t, \Phi_\delta(t, z_0)\right). \tag{2.3}$$

We can see easily that in the semiclassical limit, $\hbar \to 0$, the quantum echo converges to the classical echo. In more mathematical terms, the quantum echo is a $\hbar$ - Fourier Integral Operator whose canonical relation is the classical echo. This is a consequence of the semiclassical Egorov theorem as we shall see now, at least when the reference quantum state is a "coherent state". Let us recall the definition of a coherent state:
Given $\varphi_0(x) := (\pi\hbar)^{-d/4}\exp(-x^2/2\hbar)$, we define, for $z := (q, p) \in \mathbb{R}^{2d}$:

$$\varphi_z := \hat{T}(z)\varphi_0$$

where

$$\hat{T}(z) := \exp\left(\frac{i(p.\hat{Q} - q.\hat{P})}{\hbar}\right)$$

Proposition 2.1 *Let be A a classical observable C^∞-smooth, with compact support for simplicity. Let be φ_z the coherent state living at z. Then we have:*

$$\lim_{\hbar\to 0}\langle E_\delta^{(q)}(t)\varphi_{z_0}|\hat{A}E_\delta^{(q)}(t)\varphi_{z_0}\rangle = A(E_\delta^{(cl)}(t, z_0)) \tag{2.4}$$

A more comprehensible formulation is to define the fidelity for observables by the equality $f^{(q)}_{\delta,A}(t) = |C^{(q)}(\hat{A}_\delta(t), \hat{A}_0(t))|^2$ where $C^{(q)}(\hat{B}, \hat{A})$ is the quantum correlation for a pair (A, B) of observables:

$$C^{(q)}(\hat{B}, \hat{A}) = \mathrm{Tr}\left(\hat{B}\hat{A}\right),$$

and $\hat{A}_\delta(t) = U_\delta(t)\hat{A}U_0(-t)$. Then the semiclassical Egorov theorem gives:

Proposition 2.2 *With the above notations we have:*

$$\lim_{\hbar\to 0}(2\pi\hbar)^{2d} f^{(q)}_{\delta,A}(t) = f^{(cl)}_{\delta,A}(t) \tag{2.5}$$

where

$$f^{(cl)}_{\delta,A}(t) = \left|\int_{\mathbb{R}^{2d}} A(\Phi_\delta(t,z))A(\Phi_0(t,z))dz\right|^2$$

Remark 2.3 *An important question is to control the time of validity of the semiclassical approximation. Rigorous mathematical results are far from numerical and theoretical expected physical results. Without assumptions on classical flows this time is the Ehrenfest time (of order* $\log(\hbar^{-1})$*). If we assume that* Φ_0 *is completely integrable and* $\delta = \mathrm{O}(\hbar)$ *then this time is of order* $O(\hbar^{-1/3+\varepsilon})$*, for every* $\varepsilon > 0$*.(for details see [3]).*

3 Revivals for 1-D systems

In this section we give a flavor of results due to Robinett and Leichtle-Averbukh-Schleich ([29],[19] and more references therein contained) and show how to put them in a more rigorous mathematical framework. Let us consider a classical 1-D Hamiltonian H. One assumes H smooth, confining, with one well. Let e_n be an orthonormal basis of eigenstates, with eigenvalues E_n, $n \in \mathbb{N}$.
Let $\psi_0 = \sum_{n\in\mathbb{N}} c_n e_n$ an initial normalized state, and $\psi_t = U(t)\psi_0$. Then the autocorrelation fonction is :

$$a(t) := \langle\psi_0|\psi_t\rangle = \sum_{n\in\mathbb{N}} |c_n|^2 \mathrm{e}^{-\frac{it}{\hbar}E_n} \tag{3.6}$$

and the return probability is defined by:

$$\rho(t) = |a(t)|^2 \tag{3.7}$$

Let us remark here that a is an almost periodic function (in the sense of H. Bohr) in time t on $\mathbb{R}$ so it was remarked a long time ago that the return Theorem of Poincaré is true in quantum mechanics.
For 1-D systems much more accurate results are available because for these systems the spectrum can be computed with error $O(\hbar^\infty)$. Recall here this result.
We take the presentation from the paper by Helffer-Robert ([16]) and we refer to this paper for more details (see also the thesis of Bily for a proof with coherent states).
Let us consider a non critical energy interval $[E^-, E^+]$. The action integral is $J(E) = \int_{H(z)\leq E} dz$ and the period along the energy curve $H^{-1}(E)$ is $T_E = J'(E)$, $E \in [E^-, E^+]$ (the one well assumption means that $H^{-1}(E)$ has only one component).

Let us denote $F^\pm = J(E^\pm)$. We can determine the eigenvalues of $\hat{H}$ in $[E^-, E^+]$ by the following Bohr-Sommerfeld rule : there exist $F \mapsto b(F, \hbar)$ and C^∞ functions b_j defined on $[F^-, F^+]$ such that $b(F, \hbar) = \sum_{j\in\mathbb{N}} b_j(F)\hbar^j + O(\hbar^\infty)$ and the spectrum E_n of $\hat{H}$ is given by

$$E_n = b((n + \frac{1}{2})\hbar, \hbar) + O(\hbar^\infty), \text{ for } n \text{ such that } (n + \frac{1}{2})\hbar \in [F^-, F^+] \tag{3.8}$$

$$\text{where } b_0(F) = J^{-1}(F), \text{ and } b_1 = 0. \tag{3.9}$$

In a recent paper Colin de Verdière gives a method to compute explicitly the other corrections [5] .
Let us now choose an initial wave packet tightly spread around $E_{\bar{n}} \equiv J^{-1}((\bar{n} + \frac{1}{2})\hbar)$. For that we take $c_n = K_{\tau,\hbar}\chi\left(\frac{E_n - E_{\bar{n}}}{\tau}\right)$, where χ has a bounded support and $K_{\tau,\hbar}$ is defined such that $\sum_{n\in\mathbb{N}} |c_n|^2 = 1$.
Up to a small error it is possible to change the definitions of χ and $K_{\tau,\hbar}$ such that $c_n = K_{\tau,\hbar}\chi\left(\frac{n-\bar{n}}{\sigma}\right)$, with $\sigma = \frac{\tau}{\hbar}$. $K_{\tau,\hbar}$ is of order $\hbar\tau^{-1} = \sigma^{-1}$. Pratically, we shall choose $\tau = \hbar^\theta$ ($\sigma = \hbar^{\theta-1}$) with $0 < \theta < 1$ but it is more suggestive to keep the notation τ or σ.
For each $\hbar$ let us now fix an integer $\bar{n}$ (depending on $\hbar$) such that $(\bar{n}+\frac{1}{2})\hbar \in [F^-, F^+]$ and let us apply the Taylor formula :

$$E_n - E_{\bar{n}} = \hbar b_0'(n-\bar{n}) + \frac{\hbar^2}{2}b_0''(n-\bar{n})^2 + \frac{\hbar^3}{6}b_0'''(n-\bar{n})^3 + \hbar^2 b_2'(n-\bar{n})\hbar + O(\tau^4) \tag{3.10}$$

where the derivatives of b_j in F are computed at $\bar{F} = (\bar{n} + \frac{1}{2})\hbar$.

Let us denote by $a_i(t)$ the approximation for $a(t)$ obtained by plugging in (3.6) the i-first terms of the Taylor expansion of $\frac{E_n - E_{\bar{n}}}{\hbar}$ resulting from (3.10), denoted by $\kappa_i(n)$ $(1 \leq i \leq 3)$. So we get the following preliminary result:

Proposition 3.1 *There exists θ as above, such that we have*

$$|a(t)|^2 = |a_i(t)|^2 + \mathrm{O}(|t|\hbar^{-1}\tau^{i+1}) \tag{3.11}$$

In particular we see that for any $\varepsilon > 0$, $|a_i(t)|^2$ is a semiclassical approximation for $|a(t)|^2$ valid for large time, $|t|$ less than $\hbar^{1+\varepsilon}\tau^{-1-i}$, with a reminder term $\mathrm{O}(\hbar^\varepsilon)$.

From this proposition we can give a rough idea about the collapses and revivals phenomenon for the return probability $\rho(t)$.
Let us remark first that $\kappa_1(n) = b_0'(n - \bar{n})$ so $|a_1(t)|^2$ is periodic with period

$$T_{cl} = \frac{2\pi}{b_0'}$$

(classical period along the orbit of energy $E_{\bar{n}}$). So the return probability $\rho(t)$ is close to 1 for $t = NT_{cl}$ as far as $|t|$ is less than $\tau^{-2}\hbar^{1+\varepsilon}$.

For larger times, we have to consider $\kappa_2(n) = b_0'(n - \bar{n}) + \frac{\hbar}{2}b_0''(n - \bar{n})^2$ and a second time scale dependent on $\hbar$, the revival time defined as

$$T_{rev} = \frac{4\pi}{\hbar b_0''}$$

We shall see now that for large time intervals

$$J_\hbar = [T_{rev}\sigma^{-\delta-2}, T_{rev}\sigma^{-1-\delta'}],$$

where $\delta > 0$, $\delta' > 0$ are any small fixed real numbers, we have

$$\lim_{\hbar \to 0, t \in J_\hbar} \rho(t) = 0 \tag{3.12}$$

It is simpler to prove the collapse property (3.12) for a Gaussian cut-off, $\chi(x) = \mathrm{e}^{-x^2/4}$. Even if it is not the real problem we want to consider here, it is enough to give the intuition of the collapse phenomenon. The real problem with a smooth χ

with compact support is more technical to check.
The trick here is to apply the Poisson formula in the time variable to

$$a_2(t) = K_{\tau,\hbar} \sum_{m\in\mathbb{Z}} \exp\left(-\frac{m^2}{2\sigma^2} + 2i\pi t \frac{m^2}{T_{rev}}\right) \exp\left(2i\pi t \frac{m}{T_{cl}}\right) \tag{3.13}$$

So, applying the classical formula for the Fourier transform of a Gaussian we get

$$a_2(t) = K_{\tau,\hbar} \sqrt{\frac{2\pi}{\gamma_{t,\hbar}}} \sum_{\ell\in\mathbb{Z}} \exp\left(-4\pi^2 \frac{(\ell - \frac{t}{T_{cl}})^2}{2\gamma_{t,\hbar}}\right) \tag{3.14}$$

where $\gamma_{t,\hbar} = \left(\frac{1}{\sigma^2} - \frac{4i\pi t}{T_{rev}}\right)$.
We have

$$\gamma_{t,\hbar} = \gamma_{0,\hbar}\left(1 - \frac{4i\pi t}{T_{rev}}\sigma^2\right)$$

and each Gaussian term in the sum in (3.14) has width δ_t, given by

$$\delta_t = \left(\Re({\gamma_{t,\hbar}}^{-1})\right)^{-1/2} = \left(\frac{1}{\sigma^2} + 16\pi^2 \frac{t^2\sigma^2}{{T_{rev}}^2}\right)^{1/2}$$

From formula (3.14), we can see that a sufficient condition for t to be a collapse time for $\rho(t)$ is that $|\frac{\gamma_{0,\hbar}}{\gamma_{t,\hbar}}|$ and δ_t tend to 0 with $\hbar$. So we get (3.12).

For example if $\tau = \hbar^{3/4}$ we have:

$$\rho(NT_{cl}) = 1 + O(\hbar^\varepsilon)$$

as long as $NT_{cl} < \hbar^{\varepsilon-1/2}$, and

$$\rho(t) = O(\hbar^\varepsilon)$$

provided $t \in J_{\hbar,\tau} = [\hbar^{-1/2-\delta}, \hbar^{-3/4-\delta'}]$. So we have a large collapse interval. But for larger time, $\rho(t)$ can again be close to 1. This is the revival phenomenon. For example we have a revival if the resonant condition : $\frac{T_{rev}}{T_{cl}} \in \mathbb{Z}$ is satisfied; in this case

$$\rho(T_{rev}) = 1 + O(\hbar^\varepsilon)$$

as follows easily from Proposition 3.1, and equ. (3.13) (recall that $T_{rev} = O(\hbar^{-1})$)

As it is shown in [29], it is also possible to observe fractional revivals.

Remark 3.2 *The above analysis could be extended to completely integrable systems in d degrees of freedom*

Remark 3.3 *In the next section, for d-multidimensional sytems, we shall start with a Gaussian coherent φ_z of classical energy $E = H(z)$. Let us consider χ as above and such that $\chi = 1$ in a small neighborhood of E. Then, modulo an error term* $\mathrm{O}(\hbar^\infty)$, *we have easily*

$$\langle \varphi_z | U(t)\varphi_z \rangle = \sum_{n\in\mathbb{N}} \chi\left(\frac{E_n - E}{\tau}\right) |\langle \varphi_z | e_n \rangle|^2 \mathrm{e}^{-\frac{it}{\hbar}E_n} \tag{3.15}$$

We get something similar to the definition of $a(t)$ but with coefficients c_n not necessary smooth in the variable n, so application of the Poisson formula seems difficult.

4 Fidelity on coherent states

Let us recall the time dependent propagation theorem for coherent states proved in [8] and revisited in [28].

Theorem 4.1 *Under the assumptions (4.17), there exists a family of polynomials $\{b_j(t,x)\}_{j\in\mathbb{N}}$ in d real variables $x = (x_1, \cdots, x_d)$, with time dependent coefficients, such that for all $\hbar \in]0,1]$, we have*

$$\left\| U(t)\varphi_z - \exp\left(\frac{i\gamma_t}{\hbar}\right) \hat{T}(z_t)\Lambda_\hbar \widehat{R_1}(F_t)\left(\sum_{0\le j\le N} \hbar^{j/2} b_j(t) g\right)\right\|_{L^2(\mathbb{R}^d)} \le C(N,t,\hbar)\hbar^{(N+1)/2} \tag{4.16}$$

such that for every $N \in \mathbb{N}$, and every $T < +\infty$ we have $\sup\limits_{0<\hbar\le 1, |t|\le T} C(N,t,z,\hbar) < +\infty$

with the following notations and assumptions :

1. H is a smooth Hamiltonian such that for every multiindex α there exist $C_\alpha > 0$ and $M_\alpha \in \mathbb{R}$ such that
$$|\partial_X^\alpha H(X)| \le C_\alpha (1 + |X|)^{M_\alpha}, \quad \text{for } X \in \mathbb{R}^{2d}. \tag{4.17}$$

2. $\hat{H}$ is self-adjoint on $L^2(\mathbb{R}^d)$.

3. $t \mapsto z_t$ is the classical path ($z_0 = z$) and F_t is the stability matrix along this path.

4. $$\gamma_t(z) = \frac{1}{2}\int_0^t z_s \cdot \nabla H(z_s) ds - tH(z)$$

5.
$$\Lambda_\hbar\psi(x) = \hbar^{-d/4}\psi\left(x\hbar^{-1/2}\right) \quad \text{and}$$
$$g(x) = \pi^{-d/4}\exp\left(-|x|^2/2\right). \tag{4.18}$$

6. $\widehat{R}_1(F)$ is the usual metaplectic representation (for $\hbar = 1$) associated to F (see [9]). In particular if
$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
is the 4 $d \times d$ block-matrix form of the symplectic matrix F, the action of $\widehat{R}_1(F)$ on the state g is given by:
$$\widehat{R}_1(F)g = \pi^{-d/4}(\det(A+iB))^{-1/2}\exp\left(\frac{i}{2}\Gamma x\cdot x\right)$$
with $\Gamma := (C+iD)(A+iB)^{-1}$

Let us denote by $\psi_{z,t}^{(N)}$ the approximation of $U(t)\varphi_z$ given by (4.16).
Let us recall some more accurate estimate obtained in [8] and [28].

(i) Let be N fixed and $R > 0$ such that $|z_t| \le R$, $\forall t \in \mathbb{R}$. Then there exist $c_N > 0$, $k_R > 0$ such that
$$\hbar^{(N+1)/2}C(N,t,z,\hbar) \le c_N k_R\left(\sqrt{\hbar}|F_t|^3\right)^{N+1}(1+|t|)^{N+1} \tag{4.19}$$
In particular, in the generic case, we have a positive Lyapunov exponent γ such that $|F_t| \le \mathrm{e}^{\gamma|t|}$, so that the semiclassical approximation is valid for $|t| \le \frac{1-\varepsilon}{6\gamma}|\log\hbar|$.

In the integrable case we have $|F_t| \le c|t|$ and the semiclassical approximation is valid for $|t| \le \hbar^{-1/6+\varepsilon}$, for any $\varepsilon > 0$.

(ii) If H satisfies the following analyticity assumption in the set
$$\Omega_\rho = \{X \in \mathbb{C}^{2n}, |\Im X| < \rho\} \tag{4.20}$$
where $\Im X = (\Im X_1, \cdots, \Im X_{2d})$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{2d}$ for the Hermitean norm in $\mathbb{C}^{2d}$. So we assume there exist $\rho > 0$, $C > 0$, $\nu \ge 0$, such that H is holomorphic in Ω_ρ and for all $X \in \Omega_\rho$, we have
$$|H(X)| \le C\mathrm{e}^{\nu|X|}. \tag{4.21}$$

Then the N-dependent constant c_N in (4.19) can be estimated by

$$c_N \le C^{N+1}(N+1)^{\frac{N+1}{2}} \tag{4.22}$$

From this estimate we get an approximation for $U(t)\varphi_z$ modulo an exponentially small error (see also [15])

(iii) There exist $\tau > 0$, $a > 0$, $k > 0$ such that for $N = \{\frac{a}{\hbar}\}$ (the nearest integer to $\frac{a}{\hbar}$), we have

$$\|U(t)\varphi_z - \psi_{z,t}^{(N)}\| \le k\mathrm{e}^{-\frac{\tau}{\hbar}}, \ \forall \hbar \in]0,1]. \tag{4.23}$$

Now we apply the above estimates and the results already proven [9] concerning the action of metaplectic transformations on Gaussians. Our aim is to study the fidelity

$$f_{\delta,z}(t) = |\langle U_0(t)\varphi_z | U_\delta(t)\varphi_z\rangle|^2 \tag{4.24}$$

We shall add the index δ to keep track of the dependence on the Hamiltonian H_δ. z is fixed so we shall omit index z.

The assumptions on the family of Hamiltonians H_δ are always supposed to be satisfied uniformly in the parameter $\delta \in [0,1]$.
From now on we denote by $F_{\delta,t}$ the stability matrix for H_δ on the trajectory $z_{\delta,t}$. With the same notations as in Section 2 we thus have:

$$F_{\delta,t} = \frac{\partial}{\partial z}\Phi_\delta(t,z)$$

Let us denote

$$F_t = F_{0,t}^{-1} \cdot F_{\delta,t}$$

For any symplectic matrix F, we define
$\Gamma_F = (\mathbb{1}+iJ)K_F(\mathbb{1}-iJ)-\mathbb{1}$, $K_F := (\mathbb{1}+F)(2V)^{-1}$ and $V_F = \frac{1}{2}(\mathbb{1}+F+iJ(\mathbb{1}-F))$. V_F can be shown to be invertible, and moreover has determinant greater than or equal to 1 (see Lemma 4.4 below).

Theorem 4.2 *Let us assumme (4.17) satisfied. Then we have for the fidelity, the asymptotic formula as $\hbar \to 0$,*

$$f_\delta(t) = |\det(V_{F_t})|^{-1}\mathrm{e}^{\frac{2\Re\triangle_t}{\hbar}} + \mathrm{O}(\sqrt{\hbar}) \tag{4.25}$$

where

$$\triangle_t = \frac{1}{4}\Gamma_{F_t} F_{0,t}^{-1}(z_{0,t} - z_{\delta,t}) \cdot F_{0,t}^{-1}(z_{0,t} - z_{\delta,t})$$

Moreover we have

$$\Re\triangle_t \leq -(2 + 2|F_t|^2)^{-1}|F_{0,t}^{-1}(z_{0,t} - z_{\delta,t})|^2, \tag{4.26}$$

where $|F_t|$ *is the norm of the symplectic matrix* F_t *in the Euclidean space* $\mathbb{R}^{2d}$.
In particular $f_\delta(t) = 1 + \mathrm{O}(\sqrt{\hbar})$ *if and only if* $z_{\delta,t} = z_{0,t}$ *and* $F_t = F_{0,t}^{-1}F_{\delta,t}$ *is a unitary matrix.*

Moreover if the analyticity assumption (4.21) is satisfied then we have for the matrix elements of the echo operator, the following full asymptotic expansion

$$\left|\langle\varphi_z|E_\delta^{(q)}(t)\varphi_z\rangle\right| = \left(\sum_{0\leq j\leq N} \alpha_j(t)\hbar^{j/2}\right) \mathrm{e}^{\frac{\Re\triangle_t}{\hbar}} + \mathrm{O}(\mathrm{e}^{-\frac{c}{\hbar}}) \tag{4.27}$$

where $N = \{\frac{a}{\hbar}\}$ *(for some* $a > 0$ *and for some* $c > 0$*). There exists* $C > 0$ *such that, for all* $j \in \mathbb{N}$,

$$|\alpha_j(t)| \leq C^{j+1}(j+1)^{(j+1)/2}$$

and $\alpha_0(t) = |\det(V_t)|^{-1/2}$.

Proof

Let us first consider the first order approximation. Using the propagation theorem 4.1, we get

$$\langle\varphi_z|E_\delta^{(q)}(t)\varphi_z\rangle = \mathrm{e}^{\frac{i}{\hbar}\beta_t}\langle\varphi_{F_{0,t}^{-1}(z_{0,t}-z_{\delta,t})}|\hat{R}(F_{0,t}^{-1}F_{\delta,t})\varphi_0\rangle + \mathrm{O}(\sqrt{\hbar}),$$

where

$$\beta_t = \gamma_{\delta,t} - \gamma_{0,t} - \frac{1}{2}\sigma(z_{\delta,t}, z_{0,t})$$

We have established in [9] the following result for expectation values of a metaplectic operator in the coherent states:

Lemma 4.3 *The matrix elements of* $\hat{R}(F)$ *on coherent states* φ_z *are given by the following formula:*

$$\langle\varphi_{z+X}|\hat{R}(F)\varphi_z\rangle =$$

$$(\det V_t)^{-1/2}\exp\left\{-\frac{1}{\hbar}|z + \frac{X}{2}|^2 + \frac{i}{2\hbar}\sigma(X,z) + \frac{1}{\hbar}K_F(z + \frac{X - iJX}{2})\cdot(z + \frac{X - iJX}{2})\right\} \tag{4.28}$$

Moreover we have:

Lemma 4.4 *For any symplectic matrix F, $|\det V_F| \geq 1$, and $|\det V_F| = 1$ if and only if F is unitary.*

Proof of Lemma 4.4: Let

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be the 4-block decomposition of the $2d \times 2d$ symplectic matrix F. We have the following diagonalization property of the hermitian matrix iJ:

$$iJ = U \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} U^*$$

where U is the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ i\mathbb{1} & -i\mathbb{1} \end{pmatrix}$$

Thus we have:

$$V_F = \frac{1}{2} U \left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} U^* F U + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) U^*$$

and therefore

$$\det V_F = \det \frac{1}{2} \left\{ \begin{pmatrix} A + D + i(B - C) & A - D - i(B + C) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

$$= \det \frac{1}{2} \begin{pmatrix} A + D + i(B - C) & A - D - i(B + C) \\ 0 & 2 \end{pmatrix} = \det \frac{1}{2}(A + D + i(B - C))$$

We conclude that:

$$|\det V_F|^2 = \det \frac{1}{4}[\tilde{A} + \tilde{D} - i(\tilde{B} - \tilde{C})][A + D + i(B - C)] = \det[\mathbb{1} + L^* L]$$

with

$$L = \frac{1}{2}[A - D + i(B + C)]$$

where we have used the symplecticity of F, namely that

$$\tilde{A}C - \tilde{C}A = \tilde{D}B - \tilde{B}D = 0$$

$$\tilde{A}D - \tilde{D}A = \mathbb{1}$$

(Note that $\tilde{A}$ denotes the transpose of matrix A.) $\square$

End of Proof of Theorem 4.2: Putting $z = 0,\ X = F_{0,t}^{-1}(z_{0,t} - z_{\delta,t})$ in (4.28), we get (4.27).

Now the estimate (4.26) easily follows from the following:

Lemma 4.5 *Let*

$$\gamma_F(X) = \frac{1}{4} X \cdot \Gamma_F X$$

Then for any $X \in \mathbb{R}^{2d}$ we have:

$$\Re(\gamma_F(X)) \leq -\frac{|X|^2}{2(1+s_F)}$$

where s_F is the largest value of $F\tilde{F}$ ($\tilde{F}$ being the transpose of the matrix F).

Proof: let us begin to assume that $\det(\mathbb{1} + F) \neq 0$. Then we have:

$$K_F = (\mathbb{1} + iN)^{-1} \text{ where } N = J(\mathbb{1} - F)(\mathbb{1} + F)^{-1} \text{ is real symmetric}$$

so we can compute

$$\Re(K_F) = (\mathbb{1} + N^2)^{-1} = K_F K_F^* \text{ and } \Im(K_F) = -N(\mathbb{1} + N^2)^{-1}$$

So we get:

$$\gamma_F(X) = \frac{1}{4}\left((\mathbb{1} + JN)K_F K_F^*(\mathbb{1} - NJ)X \cdot X - 2|X|^2\right)$$

By definition of K_F, we have:

$$(\mathbb{1} + JN)K_F = 2\left((\mathbb{1} + iJ)F^{-1} + \mathbb{1} - iJ\right)^{-1} := 2T_F$$

We have, using that F is symplectic

$$(T_F^*)^{-1} T_F^{-1} = 2(\tilde{F}^{-1}F^{-1} + \mathbb{1})$$

Hence we get:

$$T_F T_F^* - \frac{\mathbb{1}}{2} = \left(2(\tilde{F}^{-1}F^{-1} + \mathbb{1})\right)^{-1} - \frac{\mathbb{1}}{2} = -\frac{\mathbb{1}}{2(\mathbb{1} + \tilde{F}F)}$$

$$T_F T_F^* X \cdot X - \frac{|X|^2}{2} = -\frac{1}{2}(\mathbb{1} + \tilde{F}F)^{-1} X \cdot X \leq -\frac{1}{2(1+s_F)}|X|^2$$

and the conclusion follows for $\det(\mathbb{1} + F) \neq 0$, hence for every symplectic matrix F by continuity.

□

We could also give an asymptotic expansion modulo $\mathrm{O}(\hbar^{+\infty})$.
The exponentially small estimates, in the analytic case, are obtained using estimates stated in [28]. In both cases we could get estimates for large times, smaller that the Ehrenfest time.
We obtain a very similar result for the semiclassical behavior of the "return probability" in the coherent states:

Theorem 4.6 *Let us assume (4.17). Then we have for the return probability* $r(t,z) := |\langle U(t)\varphi_z|\varphi_z\rangle|$ *the asymptotic formula as* $\hbar \to 0$,

$$r(t,z) = |\det(V_t)|^{-1/2} \mathrm{e}^{\frac{\Re \triangle_t}{\hbar}} + \mathrm{O}(\sqrt{\hbar}) \tag{4.29}$$

where now

$$V_t = \frac{1}{2}(\mathbb{1} + F(t) + iJ(\mathbb{1} - F(t)))$$

$F(t)$ *being the stability matrix for the flow, and*

$$\triangle_t = \frac{1}{4}\Gamma_{F(t)}(z_t - z)\cdot(z_t - z)$$

and $\Gamma_{F(t)} = (\mathbb{1} + iJ)(\mathbb{1} + F(t))(2V_t)^{-1}(\mathbb{1} - iJ) - \mathbb{1}$
In particular if z *lies on a periodic orbit* γ *of the classical flow, with period* T_γ*, and if* $F(T_\gamma)$ *is unitary, we get:*

$$r(T_\gamma, z) = 1 + O(\hbar^{1/2})$$

namely we have almost "quantum revival" when $\hbar \to 0$.

Acknowledgements We thank Jens Bolte for communicating ref. [2] before publication.

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