The integrated density of states in strong magnetic fields

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Abstract

We consider three-dimensional Schrödinger operators with constant magnetic fields and ergodic electric potentials. We study the strong-magnetic-field asymptotic behaviour of the integrated density of states, distinguishing between the asymptotics far from the Landau levels, and the asymptotics near a given Landau level.

Keywords: Schrödinger operators, magnetic fields, ergodic potentials, integrated density of states.

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1 Introduction and main results

Random magnetic Schrödinger operators received much attention during the last few decades both in mathematics and physics literature (see for example the monograph [23] or the survey article [20] and the references cited there). In this paper we consider the 3D Schrödinger operator with constant magnetic field of scalar intensity $b > 0$, and ergodic electric potential, and investigate the asymptotic behavior as $b \to \infty$ of its integrated density of states (IDS).

The paper is a continuation of [12] and [19] where we considered electric potentials which are almost surely uniformly bounded and continuous, and studied only the asymptotics of the IDS near a given Landau level. In the present paper, we apply an approach based on the Pastur-Shubin representation of the IDS (see e.g. [20, Section 2]) and the Helffer-Sjöstrand representation of a smooth compactly supported function of a self-adjoint operator (see e.g. [6, Chapter 8]). A similar approach has been systematically used by F. Klopp and other authors (see e.g. [13], [14], [15], [5]) both in the cases of vanishing and non-vanishing magnetic fields. This approach allowed us to extend the results of [12] and [19] to a considerably larger class of potentials, and to obtain the leading asymptotic term as $b \to \infty$ of the IDS not only near a given Landau level, but also far from the Landau levels.

Let us pass to the precise formulation of our results. Denote by $H_{0,b} := (-i\nabla - A)^2 - b$ the unperturbed 3D Schrödinger operator with constant magnetic field $B = \text{curl } A = (0, 0, b)$, $b > 0$, generated by the magnetic potential $A := \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$. The self-adjoint operator

$H_{0,b}$ is defined originally on $C_0^\infty(\mathbb{R}^3)$, and then is closed in $L^2(\mathbb{R}^3)$. It is well-known that the spectrum of $H_{0,b}$ is purely absolutely continuous and coincides with $[0, \infty)$ (see e.g. [1]). Moreover, the Landau levels $2bq$, $q \in \mathbb{Z}_+$, play the role of thresholds in the spectrum of $H_{0,b}$. Further, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let

$$\Omega \times \mathbb{R}^3 \ni (\omega, x) \mapsto V_\omega(x) \in \mathbb{R}$$

be a function measurable with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}(\mathbb{R}^3)$ where $\mathcal{B}(\mathbb{R}^3)$ is the σ-algebra of the Borel sets in $\mathbb{R}^3$. We assume that V_ω is $\mathbb{G}^3$-ergodic with $\mathbb{G} = \mathbb{R}$ or $\mathbb{G} = \mathbb{Z}$, i.e. that there exists an ergodic group of measure preserving automorphisms $\mathcal{T}_{\mathbf{k}} : \Omega \to \Omega$, $\mathbf{k} \in \mathbb{G}^3$, such that $V_\omega(\mathbf{x} + \mathbf{k}) = V_{\mathcal{T}_{\mathbf{k}}\omega}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$ and $\omega \in \Omega$. We recall that a group G of measure preserving automorphisms of Ω is called ergodic if the G-invariance of a set $\mathcal{A} \in \mathcal{F}$ implies either $\mathbb{P}(\mathcal{A}) = 0$ or $\mathbb{P}(\mathcal{A}) = 1$. Also, we suppose that V_ω is $\mathbb{G}$-ergodic with $\mathbb{G} = \mathbb{R}$ or $\mathbb{G} = \mathbb{Z}$ in the direction of the magnetic field, i.e. that the subgroup $\{\mathcal{T}_{\mathbf{k}} | \mathbf{k} = (0, 0, k), \quad k \in \mathbb{G}\}$ is ergodic. Finally, we assume that V_ω satisfies

$$\mathbb{E}\left(\int_{\mathcal{C}} V_\omega(\mathbf{x})^4 d\mathbf{x}\right) < \infty \tag{1.1}$$

where $\mathbb{E}$ denotes the mathematical expectation, and $\mathcal{C} := \left(-\frac{1}{2}, \frac{1}{2}\right)^3$. Note that if V_ω is $\mathbb{R}^3$-ergodic, then

$$\mathbb{E}\left(\int_{\mathcal{C}} V_\omega(\mathbf{x})^4 d\mathbf{x}\right) = \mathbb{E}\left(V_\omega(\mathbf{0})^4\right). \tag{1.2}$$

Examples of random potentials satisfying the above assumptions can be found in [8], [23], and [12]. Introduce the operator $H_{V,b} := H_{0,b} + V = H_{0,b} + V_\omega$ which is almost surely essentially self-adjoint[1] on $C_0^\infty(\mathbb{R}^3)$ (see [8]). We define the IDS $\varrho_{V,b}$ associated with $H_{V,b}$ by the Shubin-Pastur formula

$$\varrho_{V,b}(E) := \mathbb{E}\left(\mathrm{Tr}\left(\chi_{\mathcal{C}} \chi_{(-\infty,E)}(H_{V,b}) \chi_{\mathcal{C}}\right)\right), \quad E \in \mathbb{R}, \tag{1.3}$$

where $\chi_{\mathcal{C}}$ is the multiplier by the characteristic function of $\mathcal{C}$, and $\chi_{(-\infty,E)}(H_{V,b})$ is the spectral projection of the operator $H_{V,b}$ corresponding to the interval $(-\infty, E)$. The correctness of this definition of the IDS, and its equivalence to the traditional definition involving a thermodynamical limit are discussed in [8] and [7]. Moreover, various properties of $\varrho_{V,b}$ have been studied in [9]. We define the free IDS $\varrho_{0,b}$ associated with the unperturbed operator $H_{0,b}$ in a completely analogous way. The explicit form of $\varrho_{0,b}$ is well-known:

$$\varrho_{0,b}(E) = \frac{b}{2\pi^2} \sum_{q=0}^{\infty} (E - 2bq)_+^{1/2}, \quad E \in \mathbb{R}. \tag{1.4}$$

[1] Whenever there is no risk of confusion, we will use the same notation for a given essentially self-adjoint operator, and its self-adjoint closure.

The aim of the article is to study the asymptotic behaviour as $b \to \infty$ of the quantities

$$\varrho_{V,b}(\mathcal{E}b + \lambda_2) - \varrho_{V,b}(\mathcal{E}b + \lambda_1),$$

the parameters $\mathcal{E} \in [0, \infty)$, λ_1, λ_2, $\lambda_1 \le \lambda_2$, being fixed. As discussed in [12], it is reasonable to distinguish two asymptotic regimes: asymptotics near a given Landau level which corresponds to $\mathcal{E} \in 2\mathbb{Z}_+$, and asymptotics far from the Landau levels which corresponds to $\mathcal{E} \in (0, \infty) \setminus 2\mathbb{Z}_+$. This distinction is justified by the fact that (1.4) implies

$$\lim_{b\to\infty} b^{-1} \left(\varrho_{0,b}(\mathcal{E}b + \lambda_2) - \varrho_{0,b}(\mathcal{E}b + \lambda_1)\right) = \frac{1}{2\pi^2} \left((\lambda_2)_+^{1/2} - (\lambda_1)_+^{1/2}\right) \tag{1.5}$$

if $\mathcal{E} \in 2\mathbb{Z}_+$, and

$$\lim_{b\to\infty} b^{-1/2} \left(\varrho_{0,b}(\mathcal{E}b + \lambda_2) - \varrho_{0,b}(\mathcal{E}b + \lambda_1)\right) = \frac{\lambda_2 - \lambda_1}{4\pi^2} \sum_{q=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2q)^{-1/2} \tag{1.6}$$

if $\mathcal{E} \in (0, \infty) \setminus 2\mathbb{Z}_+$, i.e. in the case $V = 0$ the main asymptotic term of the IDS near a given Landau level is of order b, while its main asymptotic term far from the Landau levels is or order $b^{1/2}$.

In order to formulate our results concerning the asymptotics of the IDS $\varrho_{V,b}$ near a given Landau level, we need some additional notations. For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we denote by $X_\perp = (x_1, x_2)$ the variables on the plane perpendicular to the magnetic field. Fix $X_\perp \in \mathbb{R}^2$. Due to the our assumption about the ergodicity of V in direction of the magnetic field, the function $\mathbb{R} \ni x_3 \mapsto V_\omega(X_\perp, x_3) \in \mathbb{R}$ is ergodic, and the operator

$$h_V(X_\perp) := -\frac{d^2}{dx_3^2} + V(X_\perp, x_3) \tag{1.7}$$

is almost surely essentially self-adjoint on $C_0^\infty(\mathbb{R})$ (see [18, Chapter III]). Denote by $\rho_V(\lambda; X_\perp)$, $\lambda \in \mathbb{R}$, the IDS for the operator $h_V(X_\perp)$. By [12], if V is $\mathbb{R}$-ergodic (respectively, $\mathbb{Z}$-ergodic) in the direction of the magnetic field, then the IDS $\rho_V(\lambda; X_\perp)$ is independent of $X_\perp \in \mathbb{R}^2$ (respectively, $\rho_V(\lambda; X_\perp)$ is $\mathbb{Z}^2$-periodic with respect to $X_\perp$). Set

$$k_V(\lambda) := \int_{(-\frac{1}{2},\frac{1}{2})^2} \rho_V(\lambda; X_\perp) dX_\perp.$$

Evidently, in the case of $\mathbb{R}$-ergodicity we have $k_V(\lambda) = \rho(\lambda; 0)$. Moreover, since the operator $h_V(X_\perp)$ is an ordinary differential operator, it is easy to check that the function $\mathbb{R} \ni \lambda \mapsto k_V(\lambda) \in \mathbb{R}$ is continuous (see [18, Chapter III]).

Theorem 1.1. *Assume that the random potential $V : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is measurable with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}(\mathbb{R}^3)$, and that (1.1) holds. Moreover, suppose that V is $\mathbb{R}^3$-ergodic or $\mathbb{Z}^3$-ergodic, and is $\mathbb{R}$-ergodic or $\mathbb{Z}$-ergodic in the direction of the magnetic*

field.
i) Let $\mathcal{E} \in (0,\infty) \setminus 2\mathbb{Z}_+$*, and* $\lambda_1, \lambda_2 \in \mathbb{R}$*,* $\lambda_1 < \lambda_2$*. Then we have*

$$\lim_{b\to\infty} b^{-1/2} \left(\varrho_{V,b}(\mathcal{E}b + \lambda_2) - \varrho_{V,b}(\mathcal{E}b + \lambda_1)\right) = \frac{\lambda_2 - \lambda_1}{4\pi^2} \sum_{q=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2q)^{-1/2}. \tag{1.8}$$

ii) Let $\mathcal{E} \in 2\mathbb{Z}_+$*, and* $\lambda_1, \lambda_2 \in \mathbb{R}$*,* $\lambda_1 < \lambda_2$*. Then we have*

$$\lim_{b\to\infty} b^{-1} \left(\varrho_{V,b}(\mathcal{E}b + \lambda_2) - \varrho_{V,b}(\mathcal{E}b + \lambda_1)\right) = \frac{1}{2\pi} \left(k_V(\lambda_2) - k_V(\lambda_1)\right). \tag{1.9}$$

Let us discuss briefly our results.

- Relation (1.8) (see also (1.6)) implies that far from the Landau levels the main asymptotic term of the IDS is independent of the potential V.

- It is easy to check that $k_0(\lambda) = \frac{1}{\pi}\lambda_+^{1/2}$, $\lambda \in \mathbb{R}$. Therefore, (1.5) is a trivial special case of (1.9).

- Under the additional assumption that V is almost surely bounded and continuous, relation (1.9) was proved in [12] in the case of the first Landau level, and in [19] in the case of the higher Landau levels. Here we cancel this restrictive assumption, and give a new simpler proof of (1.9) which is coherent with that of (1.8).
Results related to (1.9) in the case of the first Landau level can be found also in [23, Section 4.3].

- The results of Theorem 1.1 are typical for the 3D case, or more generally for the case of a non-full-rank magnetic field. In the 2D case the asymptotic behaviour of the IDS for the Schrödinger operator with strong constant magnetic field and random electric potentials has been considered by numerous authors (see e.g, [2], [3], [17], [22], [23, Subsection 3.2.2]). These results illustrate the situation where the constant magnetic field is of full rank. In this respect we would also mention the recent work [21] where the author studies various asymptotic properties (in particular, the high field asymptotics) of the IDS for the Pauli operator with random magnetic field in the case of even-dimensional underlying Euclidean spaces.

The article is organized as follows. Section 2 contains some auxiliary results and preliminary estimates. The proof of Theorem 1.1 can be found in Section 3.

2 Preliminary estimates

Introduce the Landau Hamiltonian

$$\mathcal{H}(b) := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \tag{2.1}$$

i.e. the two-dimensional Schrödinger operator with constant scalar magnetic field $b > 0$, essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. It is well-known that $\sigma(\mathcal{H}(b)) = \cup_{q=0}^\infty \{2bq\}$, and each eigenvalue $2bq$, $q \in \mathbb{Z}_+$, has infinite multiplicity (see e.g. [1]).
For $X_\perp, X_\perp' \in \mathbb{R}^2$ denote by $\mathcal{P}_{q,b}(X_\perp, X_\perp')$ the integral kernel of the orthogonal projection $p_q(b)$ onto the subspace Ker $(\mathcal{H}(b) - 2bq)$, $q \in \mathbb{Z}_+$. It is well-known that

$$\mathcal{P}_{q,b}(X_\perp, X_\perp') = \frac{b}{2\pi} L_q\left(\frac{b|X_\perp - X_\perp'|^2}{2}\right) \exp\left(-\frac{b}{4}(|X_\perp - X_\perp'|^2 + 2i(x_1x_2' - x_1'x_2))\right) \tag{2.2}$$

(see [16]) where $L_q(t) := \sum_{k=0}^q \binom{q}{k} \frac{(-t)^k}{k!}$, $t \in \mathbb{R}$, $q \in \mathbb{Z}_+$, are the Laguerre polynomials. Note that

$$\mathcal{P}_{q,b}(X_\perp, X_\perp) = \frac{b}{2\pi} \tag{2.3}$$

for each $q \in \mathbb{Z}_+$ and $X_\perp \in \mathbb{R}^2$. Set $P_q := p_q \otimes I_\parallel$, $q \in \mathbb{Z}_+$, where $I_\parallel$ denotes the identity operator in $L^2(\mathbb{R})$; thus P_q are orthogonal projections acting in $L^2(\mathbb{R}^3)$. For $\mathcal{E} \in (0, \infty) \setminus 2\mathbb{Z}_+$ set

$$P_-(\mathcal{E}) := \sum_{q=0}^{[\mathcal{E}/2]} P_q, \quad P_+(\mathcal{E}) = I - P_-(\mathcal{E}), \tag{2.4}$$

where I denotes the identity operator in $L^2(\mathbb{R}^3)$. For $\mathcal{E} \in 2\mathbb{Z}_+$ set

$$P_-(\mathcal{E}) := \begin{cases} \sum_{q=0}^{\mathcal{E}/2-1} P_q & \text{if} \quad \mathcal{E} \neq 0, \\ 0 \quad \text{if} \quad \mathcal{E} = 0, \end{cases} \qquad P_+(\mathcal{E}) := I - P_{\mathcal{E}/2} - P_-(\mathcal{E}). \tag{2.5}$$

Pick $l > 0$. Denote by $\chi_{\perp,l}$ the characteristic function of the disk

$$\Lambda_{\perp,l} := \left\{X_\perp \in \mathbb{R}^2 \,|\, |X_\perp| < l\right\},$$

and by $\chi_\parallel$ the characteristic function of the interval $\Lambda_\parallel := \left(-\frac{1}{2}, \frac{1}{2}\right)$. Thus $\chi_l(\mathbf{x}) = \chi_{\perp,l}(X_\perp)\chi_\parallel(x_3)$, $\mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3$, is the characteristic function of the cylindrical set

$$\Lambda_l := \Lambda_{\perp,l} \times \Lambda_\parallel. \tag{2.6}$$

Let $\zeta \in \mathbb{C} \setminus [0, \infty)$. Denote by $\sqrt{\zeta}$ the branch of the square root satisfying $\operatorname{Im} \sqrt{\zeta} > 0$. We have

$$\operatorname{Im} \sqrt{\zeta} = \left(\frac{|\zeta| - \operatorname{Re} \zeta}{2}\right)^{1/2}. \tag{2.7}$$

For $\zeta \in \mathbb{C} \setminus [0, \infty)$ set

$$R_0(\zeta) := (H_{0,b} - \zeta)^{-1}, \quad r_0(\zeta) := (h_0 - \zeta)^{-1}, \tag{2.8}$$

where $h_0 = -d^2/dx_3^2$ (see (1.7)).

Lemma 2.1. *Let $q \in 2\mathbb{Z}_+$, $\zeta \in \mathbb{C} \setminus [0, \infty)$. Then we have*

$$\|\chi_{\perp,l} p_q\|_{\mathrm{HS}}^2 = \mathrm{Tr}_{L^2(\mathbb{R}^2)}\left(\chi_{\perp,l} p_q \chi_{\perp,l}\right) = \|\chi_{\perp,l} p_q \chi_{\perp,l}\|_{\mathrm{Tr}} = \frac{bl^2}{2}, \tag{2.9}$$

$$\|\chi_{\|} r_0(\zeta)\|_{\mathrm{HS}}^2 = \frac{1}{4|\zeta| \operatorname{Im} \sqrt{\zeta}}, \tag{2.10}$$

$$\|\chi_l P_q R_0(\zeta)\|_{\mathrm{HS}}^2 = \frac{bl^2}{8|\zeta - 2bq| \operatorname{Im} \sqrt{\zeta - 2bq}}, \tag{2.11}$$

where $\|.\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm, and $\|.\|_{\mathrm{Tr}}$ denotes the trace-class norm.

Proof. Since

$$\|\chi_{\perp,l} p_q\|_{\mathrm{HS}}^2 = \mathrm{Tr}_{L^2(\mathbb{R}^2)}\left(p_q \chi_{\perp,l} p_q\right) = \|\chi_{\perp,l} p_q \chi_{\perp,l}\|_{\mathrm{Tr}} = \int_{\Lambda_{\perp,l}} \mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x})\, d\mathbf{x},$$

(2.9) follows immediately from (2.3). Similarly, (2.10) follows from the explicit expression for the integral kernel $\mathcal{R}_0(x_3, x_3'; \zeta)$ of the operator $r_0(\zeta)$, $\zeta \in \mathbb{C} \setminus [0, \infty)$, namely

$$\mathcal{R}_0(x_3, x_3'; \zeta) = \frac{i}{2\sqrt{\zeta}} \exp\left(i\sqrt{\zeta}|x_3 - x_3'|\right). \tag{2.12}$$

Finally, since $\chi_l P_q R_0(\zeta) = \chi_{\perp,l} p_q \otimes \chi_{\|} r_0(\zeta - 2bq)$, we find that (2.11) follows from (2.9) and (2.10). $\square$

Corollary 2.1. *Let $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$ with $x \in J$, where $J \subset \mathbb{R}$ is a compact interval.*
i) Let $\mathcal{E} \in (0, \infty)$, $q \in 2\mathbb{Z}_+$, $q < \mathcal{E}/2$. Then the estimate

$$\|\chi_l P_q R_0(\mathcal{E}b + z)\|_{\mathrm{HS}}^2 \leq c_1 b^{1/2} |y|^{-1} l^2 \tag{2.13}$$

holds with $c_1 = c_1(\mathcal{E}; b) := \frac{\sqrt{2}}{4(\mathcal{E} - 2q)^{1/2}}$ provided that $b \geq \frac{2 \max_{x \in J}(-x)}{\mathcal{E} - 2q}$.
ii) Let $\mathcal{E} \in 2\mathbb{Z}_+$. Then we have

$$\|\chi_l P_{\mathcal{E}/2} R_0(\mathcal{E}b + z)\|_{\mathrm{HS}}^2 \leq \frac{1}{4} b |y|^{-3/2} l^2. \tag{2.14}$$

iii) Let $\mathcal{E} \in (0, \infty)$, $q \in 2\mathbb{Z}_+$, $q > \mathcal{E}/2$. Then the estimate

$$\|\chi_l P_q R_0(\mathcal{E}b + z)\|_{\mathrm{HS}}^2 \leq c_2 b^{-1/2} l^2 \tag{2.15}$$

holds with $c_2 = c_2(\mathcal{E}; b) := \frac{\sqrt{2}}{8(2q - \mathcal{E})^{3/2}}$ provided that $b \geq \frac{2 \max_{x \in J} x}{2q - \mathcal{E}}$.

Proof. Estimates (2.13) – (2.15) follow immediately from (2.7) and (2.11). $\square$

Let T be an operator symmetric on $C_0^\infty(\mathbb{R}^3)$ such that the operator $H_T := H_0 + T$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. For $z \in \mathbb{C} \setminus \mathbb{R}$ denote by

$$R_T(z) := (H_T - z)^{-1} \tag{2.16}$$

the resolvent of the self-adjoint operator H_T. In what follows we will often use the resolvent equations

$$R_0(z) - R_T(z) = R_T(z)TR_0(z) = R_0(z)TR_T(z) \tag{2.17}$$

where $R_0(z)$ is the free resolvent introduced in (2.8), and $R_T(z)TR_0(z)$ (respectively, $R_0(z)TR_T(z)$) should be understood as the (bounded) closure of the operator defined originally on the domain $(H_0 - z)C_0^\infty(\mathbb{R}^3)$ (respectively, $(H_T - z)C_0^\infty(\mathbb{R}^3)$) which is dense in $L^2(\mathbb{R}^3)$.

Corollary 2.2. *Let $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$ with $x \in J$, where $J \subset \mathbb{R}$ is a compact interval. Suppose in addition that V is an $\mathbb{R}^3$-ergodic potential measurable with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}(\mathbb{R}^3)$, and satisfying $\mathbb{E}(V(\mathbf{0})^2) < \infty$.*
i) Let $\mathcal{E} \in (0, \infty)$, $q \in 2\mathbb{Z}_+$, $q < \mathcal{E}/2$. Then we have

$$\mathbb{E}\left(\|\chi_l P_q R_V(\mathcal{E}b + z)\|_{\mathrm{HS}}^2\right) \leq \frac{2c_1 b^{1/2}}{|y|} l^2 \left(1 + \frac{1}{y^2}\mathbb{E}(V(\mathbf{0})^2)\right) \tag{2.18}$$

provided that $b \geq \frac{2\max_{x\in J}(-x)}{(\mathcal{E}-2q)}$.
ii) Let $\mathcal{E} \in 2\mathbb{Z}_+$. Then we have

$$\mathbb{E}\left(\|\chi_l P_{\mathcal{E}/2} R_V(\mathcal{E}b + z)\|_{\mathrm{HS}}^2\right) \leq \frac{bl^2}{2|y|^{3/2}} \left(1 + \frac{1}{y^2}\mathbb{E}(V(\mathbf{0})^2)\right). \tag{2.19}$$

Proof. For both cases, *i)* and *ii)*, the resolvent equation and the almost sure estimate $\|R_V(\mathcal{E}b + z)\| \leq \frac{1}{|y|}$, imply

$$\mathbb{E}\left(\|\chi_l P_q R_V(\mathcal{E}b + z)\|_{\mathrm{HS}}^2\right) \leq$$

$$2\mathbb{E}\left(\|\chi_l P_q R_0(\mathcal{E}b + z)\|_{\mathrm{HS}}^2 + y^{-2}\|\chi_l P_q R_0(\mathcal{E}b + z)V\|_{\mathrm{HS}}^2\right). \tag{2.20}$$

Further, by the $\mathbb{R}^3$-ergodicity

$$\mathbb{E}\left(\|\chi_l P_q R_0(\mathcal{E}b + z)V\|_{\mathrm{HS}}^2\right) =$$

$$\mathbb{E}\left(\int_{\Lambda_l}\int_{\mathbb{R}^3} |V(X_\perp', x_3')|^2 |\mathcal{P}_{q,b}(X_\perp, X_\perp')|^2 |\mathcal{R}_0(x_3, x_3'; \zeta)|^2 dX_\perp' dx_3' dX_\perp dx_3\right) =$$

$$\mathbb{E}(V(\mathbf{0})^2)\int_{\Lambda_l}\int_{\mathbb{R}^3} |\mathcal{P}_{q,b}(X_\perp, X_\perp')|^2 |\mathcal{R}_0(x_3, x_3'; \zeta)|^2 dX_\perp' dx_3' dX_\perp dx_3 =$$

$$\mathbb{E}(V(\mathbf{0})^2)\|\chi_l P_q R_0(\mathcal{E}b + z)\|_{\mathrm{HS}}^2. \tag{2.21}$$

Combining (2.20)-(2.21) for $q < \mathcal{E}/2$ or $q = \mathcal{E}/2$, with (2.13) or (2.14), we get (2.18) or (2.19). $\square$

Corollary 2.3. *Assume the hypotheses of Corollary 2.2. Let $\mathcal{E} \in [0,\infty)$. Then there exists $b_0(J,\mathcal{E}) > 0$, and $c_j(\mathcal{E}) > 0$, $j = 3,4$, such that $b \geq b_0$ implies*

$$\mathbb{E}(\|\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b+z)\|_{\mathrm{HS}}^2) \leq c_3 \frac{b^{1/2} l^2}{|y|} \left(1 + \frac{1}{y^2}\mathbb{E}(V(\mathbf{0})^2)\right), \tag{2.22}$$

$$\mathbb{E}(\|\chi_l P_+(\mathcal{E}) R_V(\mathcal{E}b+z)\|_{\mathrm{HS}}^2) \leq c_4 \frac{l^2}{b^{1/2}} \left(1 + \frac{1}{y^2}\mathbb{E}(V(\mathbf{0})^2)\right). \tag{2.23}$$

Moreover,

$$\mathbb{E}(\|\chi_l R_V(\mathcal{E}b+z)\|_{\mathrm{HS}}^2) \leq 2b^{1/2} l^2 (c_3|y|^{-1} + c_4 b^{-1}) \left(1 + \frac{1}{y^2}\mathbb{E}(V(\mathbf{0})^2)\right), \tag{2.24}$$

if $\mathcal{E} \notin 2\mathbb{Z}_+$, and

$$\mathbb{E}(\|\chi_l R_V(\mathcal{E}b+z)\|_{\mathrm{HS}}^2) \leq 4bl^2 \left(\frac{1}{2}|y|^{-3/2} + c_3 b^{-1/2}|y|^{-1} + c_4 b^{-3/2}\right) \left(1 + \frac{1}{y^2}\mathbb{E}(V(\mathbf{0})^2)\right), \tag{2.25}$$

if $\mathcal{E} \in 2\mathbb{Z}_+$.

Proof. By analogy with (2.20) - (2.21) we have

$$\mathbb{E}\left(\|\chi_l P_\pm R_V(\mathcal{E}b+z)\|_{\mathrm{HS}}^2\right) \leq 2\|\chi_l P_\pm R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2 \left(1 + y^{-2}\mathbb{E}(V(\mathbf{0})^2)\right). \tag{2.26}$$

Let us check (2.22). Let at first $\mathcal{E} \notin 2\mathbb{Z}_+$. Since

$$\|\chi_l P_- R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2 = \sum_{q=0}^{[\mathcal{E}/2]} \|\chi_l P_q R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2,$$

we find that estimate (2.22) with $c_3 := 2\sum_{q=0}^{[\mathcal{E}/2]} c_1(\mathcal{E};q)$ follows from (2.26) and (2.13), if $b \geq \frac{2\max_{x\in J}(-x)}{(\mathcal{E}-2[\mathcal{E}/2])}$. Let now $\mathcal{E} \in 2\mathbb{Z}_+$. If $\mathcal{E} = 0$, (2.22) is trivial. Assume $\mathcal{E} > 0$. We have

$$\|\chi_l P_-(\mathcal{E}) R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2 = \sum_{q=0}^{\mathcal{E}/2-1} \|\chi_l P_q R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2.$$

Thus we find that (2.22) holds with $c_3 := 2\sum_{q=0}^{\mathcal{E}/2-1} c_1(\mathcal{E};q)$ if $b \geq 2\max_{x\in J}(-x)$. Similarly, estimates (2.26) and (2.15) combined with

$$\|\chi_l P_+ R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2 = \sum_{q=[\mathcal{E}/2]+1}^{\infty} \|\chi_l P_q R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2$$

imply that (2.23) holds with $c_4 := 2\sum_{q=[\mathcal{E}/2]+1}^{\infty} c_2(\mathcal{E};q)$ if $b \geq \frac{2\max_{x\in J} x}{2([\mathcal{E}/2]+1)-\mathcal{E}}$.
Finally, (2.24) or, respectively, (2.25), follows from (2.22), (2.23), and the inequality,

$$\|\chi_l R_V(\mathcal{E}b+z)\|_{\rm HS}^2 \leq 2(\|\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b+z)\|_{\rm HS}^2 + \|\chi_l P_+(\mathcal{E}) R_V(\mathcal{E}b+z)\|_{\rm HS}^2)$$

or, respectively, from (2.19), (2.22), (2.23), and

$$\|\chi_l R_V(\mathcal{E}b+z)\|_{\rm HS}^2 \leq$$

$$4(\|\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b+z)\|_{\rm HS}^2 + \|\chi_l P_{\mathcal{E}/2} R_V(\mathcal{E}b+z)\|_{\rm HS}^2) + \|\chi_l P_+(\mathcal{E}) R_V(\mathcal{E}b+z)\|_{\rm HS}^2.$$

□

Assume that V is an $\mathbb{R}^3$-ergodic potential satisfying the assumptions of Theorem 1.1. For $x_3 \in \mathbb{R}$ and $\omega \in \Omega$ put

$$v(x_3) = v_\omega(x_3) := V_\omega(0, x_3). \tag{2.27}$$

Set $h_v := h_V(0)$ (see (1.7)). For $z \in \mathbb{C}\setminus\mathbb{R}$ and ω in a full-probability subset of Ω denote by $r_v(z) := (h_v - z)^{-1}$ the resolvent of the self-adjoint operator h_v.

Lemma 2.2. *Let $z \in \mathbb{C}\setminus\mathbb{R}$. Assume that V is an $\mathbb{R}^3$-ergodic potential satisfying the assumptions of the Theorem 1.1. Then almost surely the resolvent $r_v(z)$, $z \in \mathbb{C}\setminus\mathbb{R}$, admits an integral kernel $\mathcal{R}_v(x,x';z)$, x, $x' \in \mathbb{R}$, such that the function $\mathbb{R} \ni x \mapsto \int_{\mathbb{R}} |\mathcal{R}_v(x,x';z)|^2 dx' \in [0,\infty)$ is well-defined and continuous.*

Proof. Fix $\omega \in \Omega$ such that $v \in L^2_{\rm loc}(\mathbb{R})$, and the operator h_v is essentially self-adjoint on $C_0^\infty(\mathbb{R})$. By (1.1) - (1.2) the set of such ω has probability one. Since $v \in L^2_{\rm loc}(\mathbb{R})$, every solution u of the differential equation

$$-u'' + vu = zu, \quad z \in \mathbb{C}, \tag{2.28}$$

is absolutely continuous. Suppose now that $z \in \mathbb{C}\setminus\mathbb{R}$ and denote by u_- (respectively, u_+) a non-trivial solution of (2.28) whose restriction on any interval $(-\infty, a)$ (respectively, (a,∞)), is square-integrable. The existence of such solutions is well-known (see e.g. [4, Lemma III.1.2]). Obviously, u_- and u_+ are linearly independent. Let $w := u_+'(x)u_-(x) - u_-'(x)u_+(x)$, $x \in \mathbb{R}$, be the Wronskian of the solutions u_+ and u_-. It is well-known that $w \neq 0$ is independent of $x \in \mathbb{R}$, and we have

$$w\mathcal{R}_v(x,x';z) = \begin{cases} u_-(x)u_+(x') & \text{if} \quad x \leq x', \\ u_+(x)u_-(x') & \text{if} \quad x' \leq x. \end{cases}$$

Therefore,

$$\int_{\mathbb{R}} |\mathcal{R}_v(x,x';z)|^2 \, dx' =$$

$$|w|^{-2}\left(|u_+(x)|^2 \int_{-\infty}^x |u_-(x')|^2 dx' + |u_-(x)|^2 \int_x^\infty |u_+(x')|^2 dx'\right)$$

which proves the claim of the lemma. □

Lemma 2.3. *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Suppose that V is an $\mathbb{R}^3$-ergodic potential satisfying the assumptions of the Theorem 1.1. Then*

$$\mathbb{E}\left(\left(\int_{\mathbb{R}} |\mathcal{R}_v(0,x';z)|^2 dx'\right)^2\right) \leq \frac{1}{2(|z|\mathrm{Im}\sqrt{z})^2}\left(1 + \frac{1}{y^4}\mathbb{E}\left(V(\mathbf{0})^4\right)\right). \tag{2.29}$$

Proof. Let $\mathcal{J} \in C_0^\infty(\mathbb{R})$, $\mathcal{J} \geq 0$, and $\int_{\mathbb{R}} \mathcal{J}(x)dx = 1$. Set $\mathcal{J}_\epsilon(x) := \epsilon^{-1}\mathcal{J}(x/\epsilon)$, $x \in \mathbb{R}$, $\epsilon > 0$, and

$$Q_\epsilon = Q_\epsilon(z) :=$$

$$\mathbb{E}\left(\|\mathcal{J}_\epsilon^{1/2} r_v(z)\|_{\mathrm{HS}}^4\right) = \mathbb{E}\left(\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{J}_\epsilon(x)|\mathcal{R}_v(x,x';z)|^2 dx dx'\right)^2\right), \ \epsilon > 0, \ z \in \mathbb{C}\setminus\mathbb{R}.$$

Since almost surely the operator $r_v(z)$ is self-adjoint and $v \in L^2_{\mathrm{loc}}(\mathbb{R})$, by analogy with (2.17) we can justify almost surely that the resolvent equation

$$r_v(z) = r_0(z) - r_0(z) v r_v(z)$$

with $r_0(z) = (h_0 - z)^{-1}$ (see (2.8)). Similarly, almost surely we have $\|r_v(z)\| \leq 1/|y|$. Therefore,

$$Q_\epsilon \leq 8\mathbb{E}\left(\|\mathcal{J}_\epsilon^{1/2} r_0(z)\|_{\mathrm{HS}}^4 + y^{-4}\|\mathcal{J}_\epsilon^{1/2} r_0(z) v\|_{\mathrm{HS}}^4\right). \tag{2.30}$$

By the Cauchy-Schwarz inequality and the $\mathbb{R}^3$-ergodicity of V, we have

$$\mathbb{E}\left(\|\mathcal{J}_\epsilon^{1/2} r_0(z) v\|_{\mathrm{HS}}^4\right) \leq \mathbb{E}\left(V(\mathbf{0})^4\right)\|\mathcal{J}_\epsilon^{1/2} r_0(z)\|_{\mathrm{HS}}^4. \tag{2.31}$$

On the other hand,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{J}_\epsilon(x)|\mathcal{R}_0(x,x';z)|^2 dx dx' =$$

$$\frac{1}{4|z|}\int_{\mathbb{R}} \mathcal{J}_\epsilon(x)dx \int_{\mathbb{R}} \exp\left(-2\mathrm{Im}\sqrt{z}|s|\right)ds = \frac{1}{4|z|\mathrm{Im}\sqrt{z}}. \tag{2.32}$$

The combination of (2.30), (2.31), and (2.32) yields

$$Q_\epsilon \leq \frac{1}{2(|z|\mathrm{Im}\sqrt{z})^2}\left(1 + \frac{1}{y^4}\mathbb{E}\left(V(\mathbf{0})^4\right)\right), \quad \epsilon > 0. \tag{2.33}$$

Further, by Lemma 2.2, the function $x \ni \mathbb{R} \mapsto \int_{\mathbb{R}} |\mathcal{R}_v(x,x';z)|^2 dx'$ is almost surely continuous. Therefore, almost surely

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{J}_\epsilon(x)|\mathcal{R}_v(x,x';z)|^2 dx dx' = \int_{\mathbb{R}} |\mathcal{R}_v(0,x';z)|^2 dx'. \tag{2.34}$$

Next, almost surely we have

$$\|\mathcal{J}_\epsilon^{1/2} r_v(z)\|_{\mathrm{HS}}^4 \leq 8\left(\|\mathcal{J}_\epsilon^{1/2} r_0(z)\|_{\mathrm{HS}}^4 + \frac{1}{y^4}\|\mathcal{J}_\epsilon^{1/2} r_0(z) v\|_{\mathrm{HS}}^4\right). \tag{2.35}$$

Finally, it follows from (2.12) that the estimates

$$\|\mathcal{J}_\epsilon^{1/2} r_0(z) v\|_{\rm HS}^4 \leq \frac{1}{16|z|^2} \left(\int_{\mathbb{R}} \mathcal{J}(x) e^{2\epsilon {\rm Im}\,\sqrt{z}|x|} dx \int_{\mathbb{R}} e^{-2{\rm Im}\,\sqrt{z}|x'|} v(x')^2 dx' \right)^2 \leq$$

$$c_5 \left(\int_{\mathbb{R}} v(x')^2 e^{-2{\rm Im}\,\sqrt{z}|x'|} dx' \right)^2 \leq c_5 \int_{\mathbb{R}} e^{-2{\rm Im}\,\sqrt{z}|x'|} dx' \int_{\mathbb{R}} v(x')^4 e^{-2{\rm Im}\,\sqrt{z}|x'|} dx' \qquad (2.36)$$

hold with $c_5 = c_5(z) > 0$ provided that ϵ is small enough. The r.h.s of (2.36) is in $L^1(\Omega; d\mathbb{P})$ since $\mathbb{E}(V(\mathbf{0})^4) < \infty$, and is independent of ϵ. By (2.35), this is also true for $\|\mathcal{J}_\epsilon^{1/2} r_v(z)\|_{\rm HS}^4$. Bearing in mind this fact and limiting relation (2.34), we apply the dominated convergence theorem, and get

$$\lim_{\epsilon \to 0} Q_\epsilon = \lim_{\epsilon \to 0} \mathbb{E}\left(\|\mathcal{J}_\epsilon^{1/2} r_v(z)\|_{\rm HS}^4 \right) = \mathbb{E}\left(\left(\int_{\mathbb{R}} |\mathcal{R}_v(0, x'; z)|^2 dx' \right)^2 \right)$$

which, combined with (2.33), proves the claim of the lemma. □

3 Proof of Theorem 1.1

3.1. Throughout the section we assume the hypotheses of Theorem 1.1. Also, until Subsection 3.8 we suppose that V is $\mathbb{R}^3$-ergodic and $\mathbb{R}$-ergodic in the direction of the magnetic field.
Since $\mathbb{R} \ni \lambda \mapsto \varrho_{V,b}(\mathcal{E}b + \lambda)$ is a non-decreasing function, (1.8) and (1.9) can be interpreted as measure-convergence relations. Since the measures related to the r.h.s. of (1.8) and (1.9) have no atoms, we conclude that (1.8) is equivalent to the validity of the relation

$$\lim_{b \to \infty} b^{-1/2} \int_{\mathbb{R}} \varphi(\lambda) d\varrho_{V,b}(\mathcal{E}b + \lambda) = \frac{1}{4\pi^2} \sum_{q=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2q)^{-1/2} \int_{\mathbb{R}} \varphi(\lambda) d\lambda, \quad \mathcal{E} \in (0, \infty) \setminus 2\mathbb{Z}_+, \quad (3.1)$$

for any $\varphi \in C_0^\infty(\mathbb{R})$, while (1.9) is equivalent to the validity of the relation

$$\lim_{b \to \infty} b^{-1} \int_{\mathbb{R}} \varphi(\lambda) d\varrho_{V,b}(\mathcal{E}b + \lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda) dk_V(\lambda), \quad \mathcal{E} \in 2\mathbb{Z}_+. \qquad (3.2)$$

for any $\varphi \in C_0^\infty(\mathbb{R})$ (see e.g. [11]).

3.2. The assumption that V is $\mathbb{R}^3$-ergodic implies

Lemma 3.1. [8, Corollary 3.3] *Assume that V is an $\mathbb{R}^3$-ergodic potential satisfying the hypotheses of Theorem 1.1. Let $\Lambda \subset \mathbb{R}^3$ be a Lebesgue-measurable set of positive finite Lebesgue measure $|\Lambda|$. Then we have*

$$\int_{\mathbb{R}} \psi(\lambda) d\varrho_{V,b}(\lambda) = \frac{1}{|\Lambda|} \mathbb{E}\left({\rm Tr}\left(\chi_\Lambda \psi(H_{V,b}) \chi_\Lambda \right) \right), \quad \forall \psi \in C_0^\infty(\mathbb{R}), \qquad (3.3)$$

where χ_Λ is the characteristic function of Λ.

In what follows we pick an arbitrary $l > 0$ and choose Λ in (3.3) as the cylindrical set Λ_l defined in (2.6) (hence $|\Lambda| = \pi l^2$). In consistence with our previous notations we will write χ_l instead of χ_{Λ_l}. Thus (3.3) implies

$$\int_{\mathbb{R}} \varphi(\lambda) d\varrho_{V,b}(\lambda + \mathcal{E}b) = \int_{\mathbb{R}} \varphi(\lambda - \mathcal{E}b) d\varrho_{V,b}(\lambda) = \frac{1}{\pi l^2} \mathbb{E}\left(\mathrm{Tr}\left(\chi_l \varphi(H_{V,b} - \mathcal{E}b)\chi_l\right)\right). \quad (3.4)$$

3.3. Further, we introduce a representation of Tr $(\chi_l \varphi(H_{V,b} - \mathcal{E}b)\chi_l)$ by the Helffer-Sjöstrand formula. Let $\varphi \in C_0^\infty(\mathbb{R})$, and $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^2)$ be a quasi-analytic extension of φ. Note that

$$\left|\frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x, y)\right| = O(|y|^N), \quad y \to 0, \quad (3.5)$$

for each $N \in \mathbb{Z}_+$, and

$$\tilde{\varphi}(x, 0) = \varphi(x), \quad x \in \mathbb{R}. \quad (3.6)$$

Lemma 3.2. [6, Theorem 8.1] *Let L be a self-adjoint operator. Let $\varphi \in C_0^\infty(\mathbb{R})$. Then for each $m \in \mathbb{Z}_+$ we have*

$$\varphi(L) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x, y)(z - i)^m (L - i)^{-m} (L - z)^{-1} dxdy, \quad (3.7)$$

the integral being understood in the Riemann-Bochner sense.

Remark: The Helffer-Sjöstrand formula (3.7) is written in [6, Theorem 8.1] only for $m = 0$, but this special case easily implies the formula for arbitrary $m \in \mathbb{Z}_+$ (see e.g. [13], [14]).

For $\varphi \in C_0^\infty(\mathbb{R})$ and $\mathcal{E} \in [0, \infty)$ set

$$\Phi(b, \mathcal{E}) := \frac{1}{\pi^2 l^2} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x, y)\ (z - i)\mathbb{E}\left(\mathrm{Tr}\ (\chi_l R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z)\chi_l)\right) dxdy, \quad (3.8)$$

the notation R_V being introduced in (2.16). Applying the Cauchy-Schwarz inequality

$$|\mathbb{E}\left(\mathrm{Tr}\ (\chi_l R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z)\chi_l)\right)| \leq$$

$$\left(\mathbb{E}(\|\chi_l R_V(\mathcal{E}b + i)\|_{\mathrm{HS}}^2)\right)^{1/2} \left(\mathbb{E}(\|\chi_l R_V(\mathcal{E}b + z)\|_{\mathrm{HS}}^2)\right)^{1/2},$$

estimates (2.24)-(2.25), assumption (1.1), and (3.5), we find that the quantity $\Phi(b, \mathcal{E})$ is well-defined. Using (3.4), (3.7) with $L = H_{V,b} - \mathcal{E}b$ and $m = 1$, we get the following

Corollary 3.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$. Then under the assumptions of the Theorem 1.1 we have*

$$\int_{\mathbb{R}} \varphi(\lambda - \mathcal{E}b)\, d\varrho_{V,b}(\lambda) = \Phi(b, \mathcal{E}). \quad (3.9)$$

3.4. For $\varphi \in C_0^\infty(\mathbb{R})$ and $\mathcal{E} \in [0,\infty)$ set

$$\Phi_1(b,\mathcal{E})) := \left\{ \begin{array}{l} \frac{1}{\pi^2 l^2} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x,y)\, \mathbb{E}\left(\mathrm{Tr}\, \left(\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b+z) P_-(\mathcal{E})\chi_l\right)\right) dxdy \text{ if } \mathcal{E} \notin 2\mathbb{Z}_+, \\ \frac{1}{\pi^2 l^2} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x,y)\, \mathbb{E}\left(\mathrm{Tr}\, \left(\chi_l P_{\mathcal{E}/2} R_V(\mathcal{E}b+z) P_{\mathcal{E}/2}\chi_l\right)\right) dxdy \text{ if } \mathcal{E} \in 2\mathbb{Z}_+, \end{array} \right.$$

the orthogonal projections P_q, $q \in \mathbb{Z}_+$, and P_- being introduced in (2.4) - (2.5). In order to check that $\Phi_1(b,\mathcal{E})$ is well-defined, set

$$P := \left\{ \begin{array}{lll} P_-(\mathcal{E}) & \text{if} & \mathcal{E} \notin 2\mathbb{Z}_+, \\ P_{\mathcal{E}/2} & \text{if} & \mathcal{E} \in 2\mathbb{Z}_+, \end{array} \right.$$

and note that resolvent equation almost surely implies

$$\chi_l P R_V(\mathcal{E}b+z) P \chi_l = \chi_l P R_0(\mathcal{E}b+z) P \chi_l - \chi_l P R_0(\mathcal{E}b+z) V R_V(\mathcal{E}b+z) P \chi_l.$$

Therefore,

$$|\mathbb{E}\left(\mathrm{Tr}\left(\chi_l P R_V(\mathcal{E}b+z) P\chi_l\right)\right)| \leq \|\chi_l P R_0(-1)^{1/2}\|_{\mathrm{HS}}^2 \|(H_{0,b}+1) R_0(\mathcal{E}b+z)\| +$$

$$\left(\mathbb{E}\left(\|\chi_l P R_0(\mathcal{E}b+z) V\|_{\mathrm{HS}}^2\right)\right)^{1/2} \left(\mathbb{E}\left(\|R_V(\mathcal{E}b+z) P\chi_l\|_{\mathrm{HS}}^2\right)\right)^{1/2}. \tag{3.10}$$

Since $\|\chi_l P R_0(-1)^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}\left(\chi_l P R_0(-1)\right)$ and $\chi_l P_q R_0(-1) = \chi_{\perp,l} p_q \otimes \chi_{\|} r_0(-2bq-1)$, $q \in 2\mathbb{Z}_+$, we easily find that

$$\|\chi_l P R_0(-1)^{1/2}\|_{\mathrm{HS}}^2 = \frac{bl^2}{4} \left\{ \begin{array}{lll} \sum_{q=0}^{[\mathcal{E}/2]} (2qb+1)^{-1/2} & \text{if} & \mathcal{E} \notin 2\mathbb{Z}_+, \\ (\mathcal{E}b+1)^{-1/2} & \text{if} & \mathcal{E} \in 2\mathbb{Z}_+, \end{array} \right. \tag{3.11}$$

taking into account (2.9) and (2.12). Moreover,

$$\|(H_{0,b}+1) R_0(\mathcal{E}b+z)\| = \sup_{\lambda \in [0,\infty)} \frac{\lambda+1}{\sqrt{(\lambda - \mathcal{E}b - x)^2 + y^2}} \leq 1 + \frac{(\mathcal{E}b+x+1)_+}{|y|}. \tag{3.12}$$

Finally, by analogy with (2.21) we have

$$\mathbb{E}\left(\|\chi_l P R_0(\mathcal{E}b+z) V\|_{\mathrm{HS}}^2\right) = \mathbb{E}(V(\mathbf{0})^2) \mathbb{E}\left(\|\chi_l P R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^2\right)$$

which combined with (2.20) or (2.26) yields

$$\mathbb{E}\left(\|\chi_l P R_0(\mathcal{E}b+z) V\|_{\mathrm{HS}}^2\right) \mathbb{E}\left(\|R_V(\mathcal{E}b+z) P\chi_l\|_{\mathrm{HS}}^2\right) \leq$$

$$2\|\chi_l P R_0(\mathcal{E}b+z)\|_{\mathrm{HS}}^4 \mathbb{E}(V(\mathbf{0})^2) \left(1 + y^{-2}\mathbb{E}(V(\mathbf{0})^2)\right). \tag{3.13}$$

Putting together (3.10) - (3.13), (2.13) or (2.14), and (3.5), we conclude that $\Phi_1(b,\mathcal{E})$ is well-defined.

Proposition 3.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$. Then under the assumptions of Theorem 1.1 we have*

$$\Phi(b) = \Phi_1(b, \mathcal{E}) + O(1), \quad b \to \infty, \tag{3.14}$$

if $\mathcal{E} \notin 2\mathbb{Z}_+$, and

$$\Phi(b) = \Phi_1(b, \mathcal{E}) + O(b^{3/4}), \quad b \to \infty, \tag{3.15}$$

if $\mathcal{E} \in 2\mathbb{Z}_+$.

Proof. First of all, note that $\int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x,y)\, dxdy = 0$. Further, the resolvent equation implies

$$R_V(\mathcal{E}b + z) - R_V(\mathcal{E}b + i) = (z - i) R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z).$$

Hence,

$$\Phi_1(b, \mathcal{E}) = \frac{1}{\pi^2 l^2} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x,y)\; (z-i) \mathbb{E} \left(\mathrm{Tr}\; (\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_-(\mathcal{E}) \chi_l)\right) dxdy$$

if $\mathcal{E} \notin 2\mathbb{Z}_+$, and

$$\Phi_1(b, \mathcal{E}) = \frac{1}{\pi^2 l^2} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x,y)\; (z-i) \mathbb{E} \left(\mathrm{Tr}\; \left(\chi_l P_{\mathcal{E}/2} R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_{\mathcal{E}/2} \chi_l\right)\right) dxdy$$

if $\mathcal{E} \in 2\mathbb{Z}_+$.
Assume $\mathcal{E} \notin 2\mathbb{Z}_+$. Then we have

$$\mathrm{Tr}\; (\chi_l R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) \chi_l) = \mathrm{Tr}\; (\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_-(\mathcal{E}) \chi_l) +$$

$$\mathrm{Tr}\; (\chi_l P_+(\mathcal{E}) R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_-(\mathcal{E}) \chi_l) + \mathrm{Tr}\; (\chi_l R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_+(\mathcal{E}) \chi_l) =$$

$$\mathrm{Tr}\; (\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_-(\mathcal{E}) \chi_l) + \mathrm{I} + \mathrm{II}.$$

Therefore,

$$\Phi(b) - \Phi_1(b, \mathcal{E}) = \frac{1}{\pi^2 l^2} \int_{\mathbb{R}^2} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(x,y)(z-i)\; \mathbb{E}(\mathrm{I} + \mathrm{II})\, dxdy. \tag{3.16}$$

Evidently,

$$|\mathbb{E}(\mathrm{I})| \leq \mathbb{E}\left(\|\chi_l P_+(\mathcal{E}) R_V(\mathcal{E}b + i)\|_{\mathrm{HS}}^2\right)^{1/2} \mathbb{E}\left(\|R_V(\mathcal{E}b + z) P_-(\mathcal{E}) \chi_l\|_{\mathrm{HS}}^2\right)^{1/2}, \tag{3.17}$$

$$|\mathbb{E}(\mathrm{II})| \leq \mathbb{E}\left(\|\chi_l R_V(\mathcal{E}b + i)\|_{\mathrm{HS}}^2\right)^{1/2} \mathbb{E}\left(\|R_V(\mathcal{E}b + z) P_+(\mathcal{E}) \chi_l\|_{\mathrm{HS}}^2\right)^{1/2}. \tag{3.18}$$

Combining (3.17) - (3.18) with Corollary 2.3, we find that (3.16) implies (3.14).
Assume now $\mathcal{E} \in 2\mathbb{Z}_+$. Then we have

$$\mathrm{Tr}\; (\chi_l R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) \chi_l) = \mathrm{Tr}\; \left(\chi_l P_{\mathcal{E}/2} R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_{\mathcal{E}/2} \chi_l\right) +$$

$$\mathrm{Tr}\; \left(\chi_l (P_-(\mathcal{E}) + P_+(\mathcal{E})) R_V(\mathcal{E}b + i) R_V(\mathcal{E}b + z) P_{\mathcal{E}/2} \chi_l\right) +$$

$$\text{Tr } (\chi_l R_V(\mathcal{E}b+i)R_V(\mathcal{E}b+z)(P_-(\mathcal{E})+P_+(\mathcal{E}))\chi_l) =$$
$$\text{Tr } (\chi_l P_-(\mathcal{E}) R_V(\mathcal{E}b+i)R_V(\mathcal{E}b+z)P_-(\mathcal{E})\chi_l) + \text{III} + \text{IV}.$$

Hence, in this case

$$\Phi(b,\mathcal{E}) - \Phi_1(b,\mathcal{E}) = \frac{1}{\pi^2 l^2}\int_{\mathbb{R}^2} \frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)(z-i)\,\mathbb{E}(\text{III}+\text{IV})\,dxdy.$$

Obviously,

$$|\mathbb{E}(\text{III})| \leq \Big(\big(\mathbb{E}(\|\chi_l P_+(\mathcal{E})R_V(\mathcal{E}b+i)\|^2_{\text{HS}})\big)^{1/2} + \big(\mathbb{E}(\|\chi_l P_-(\mathcal{E})R_V(\mathcal{E}b+i)\|^2_{\text{HS}})\big)^{1/2}\Big) \times \big(\mathbb{E}(\|R_V(\mathcal{E}b+z)P_{\mathcal{E}/2}\chi_l\|^2_{\text{HS}})\big)^{1/2}$$

and

$$|\mathbb{E}(\text{IV})| \leq \Big(\big(\mathbb{E}(\|R_V(\mathcal{E}b+z)P_+(\mathcal{E})\chi_l\|^2_{\text{HS}})\big)^{1/2} + \big(\mathbb{E}(\|R_V(\mathcal{E}b+z)P_-(\mathcal{E})\chi_l\|^2_{\text{HS}})\big)^{1/2}\Big) \times \big(\mathbb{E}(\|\chi_l R_V(\mathcal{E}b+i)\|^2_{\text{HS}})\big)^{1/2}.$$

Arguing as in the proof of (3.14), we obtain (3.15) with the help of Corollaries 2.2 – 2.3. □

3.5. In this subsection we establish several auxiliary results needed in the further steps of the proof of Theorem 1.1. For $\mathbf{x} \in \mathbb{R}^3$ set

$$C(\mathbf{x}) := \mathbb{E}\left((V(\mathbf{0}) - V(\mathbf{x}))^4\right). \tag{3.19}$$

Obviously,

$$\sup_{\mathbf{x}\in\mathbb{R}^3} |C(\mathbf{x})| \leq\; 8(\mathbb{E}(V(\mathbf{0})^4). \tag{3.20}$$

We will need the continuity of the function C proved in a somewhat more general context in the following

Lemma 3.3. *Let*

$$\Omega \times \mathbb{R}^d \ni (\omega, x) \mapsto \mathcal{V}_\omega(x) \in \mathbb{R}$$

be a function measurable with respect to the product σ-algebra $\mathcal{F}\times\mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the σ-algebra of the Borel sets in $\mathbb{R}^d$, $d \geq 1$. Assume that $\mathcal{V}_\omega$ is $\mathbb{R}^d$-ergodic, and for some $p \geq 1$ we have $\mathbb{E}(|V(0)|^p) < \infty$. Then

$$\lim_{x\to 0} \mathbb{E}(|\mathcal{V}_\omega(x) - \mathcal{V}_\omega(0)|^p) = 0.$$

Proof. It suffices to prove that for each sequence $\{x_j\}_{j\in\mathbb{N}}$, $x_j \in \mathbb{R}^d$, such that $\lim_{j\to\infty} x_j = 0$, we have

$$\lim_{j\to\infty} \mathbb{E}(|\mathcal{V}_\omega(x_j) - \mathcal{V}_\omega(0)|^p) = 0. \tag{3.21}$$

Note that for every Lebesgue-measurable set $\mathcal{O} \subset \mathbb{R}^d$ of positive finite Lebesgue measure $|\mathcal{O}|$ we have

$$|\mathcal{O}|\,\mathbb{E}(|\mathcal{V}_\omega(0)|^p) = \mathbb{E}\left(\int_{\mathcal{O}} |\mathcal{V}_\omega(y)|^p dy\right), \tag{3.22}$$

$$|\mathcal{O}|\,\mathbb{E}\left(|\mathcal{V}_\omega(x_j) - \mathcal{V}_\omega(0)|^p\right) = \mathbb{E}(\int_{\mathcal{O}} |\mathcal{V}_\omega(y + x_j) - \mathcal{V}_\omega(y)|^p dy), \quad j \in \mathbb{N}. \tag{3.23}$$

Set $\mathcal{C} := \left(-\frac{1}{2}, \frac{1}{2}\right)^d$, $\mathcal{D} := (-1,1)^d$. Evidently, if $|x_j|$ is small enough, $\mathcal{C} \pm x_j \subset \mathcal{D}$.
By (3.22) with $\mathcal{O} = \mathcal{D}$ there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$, and $\mathcal{V}_{\omega|\mathcal{D}} \in L^p(\mathcal{D})$ (and, hence, $\mathcal{V}_{\omega|\mathcal{C}} \in L^p(\mathcal{C})$) for all $\omega \in \Omega_0$. Pick $\omega \in \Omega_0$ and set $\mathcal{W} := \chi_{\mathcal{D}}\mathcal{V}_\omega$ where $\chi_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$. Then $\mathcal{W} \in L^p(\mathbb{R}^d)$, and we have

$$\int_{\mathbb{R}^d} |\mathcal{W}(y + x_j) - \mathcal{W}(y)|^p dy \to 0 \quad \text{as} \quad j \to \infty. \tag{3.24}$$

Set

$$f_j(\omega) = \begin{cases} \int_{\mathcal{C}} |\mathcal{V}_\omega(y + x_j) - \mathcal{V}_\omega(y)|^p dy & \text{if} \quad \omega \in \Omega_0, \\ 0 \quad \text{if} \quad \omega \in \Omega \setminus \Omega_0. \end{cases}$$

By (3.23) with $\mathcal{O} = \mathcal{C}$ we have

$$\mathbb{E}(|\mathcal{V}_\omega(x_j) - \mathcal{V}_\omega(0)|^p) = \int_\Omega f_j(\omega) d\mathbb{P}(\omega). \tag{3.25}$$

Let us now show that

$$\lim_{j\to\infty} f_j(\omega) = 0, \quad \forall \omega \in \Omega. \tag{3.26}$$

For $\omega \in \Omega \setminus \Omega_0$ this is trivial. If $\omega \in \Omega_0$ we have

$$\int_{\mathbb{R}^d} |\mathcal{W}(y + x_j) - \mathcal{W}(y)|^p dy \geq \int_{\mathcal{C}} |\mathcal{V}_\omega(y + x_j) - \mathcal{V}_\omega(y)|^p dy = f_j(\omega). \tag{3.27}$$

Since f_j is non-negative, (3.24) and (3.27) imply (3.26). Now set

$$\tilde{f}(\omega) = \begin{cases} \int_{\mathcal{D}} |\mathcal{V}_\omega(y)|^p dy & \text{if} \quad \omega \in \Omega_0, \\ 0 \quad \text{if} \quad \omega \in \Omega \setminus \Omega_0. \end{cases}$$

By (3.22) with $\mathcal{O} = \mathcal{D}$ we have $0 \leq \tilde{f} \in L^1(\Omega; d\mathbb{P})$. Moreover, there exists a constant $c = c(p)$ such that

$$f_j(\omega) \leq c(p)\tilde{f}(\omega), \quad \forall \omega \in \Omega. \tag{3.28}$$

Putting together (3.26) and (3.28), we find that the dominated convergence theorem implies $\lim_{j\to\infty} \int_\Omega f_j(\omega) d\mathbb{P}(\omega) = 0$ which combined with (3.25) yields (3.21). $\square$

For $l > 0$ set

$$D_l = \frac{1}{4\pi l^2} \int_{\Lambda_{\perp,l}} C(X_\perp, 0) dX_\perp. \tag{3.29}$$

Remark: By Lemma 3.3, we have $D_l \to 0$ as $l \to 0$.
For $l > 0$, $q \in \mathbb{Z}_+$, and $x_3 \in \mathbb{R}$, set

$$K_{q,b}(x_3) := \int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} |\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2 \left(v(x_3) - V(X'_\perp, x_3)\right)^2 dX_\perp dX'_\perp, \tag{3.30}$$

$\mathcal{P}_{q,b}$ being defined in (2.2), and v - in (2.27).

Lemma 3.4. *Suppose that $\mathbb{E}\,(V(\mathbf{0})^4)) < \infty$. Then*

$$\limsup_{b\to\infty} \mathbb{E}\left(b^{-2} K_{q,b}(0)^2\right) \leq l^4 D_l. \tag{3.31}$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\mathbb{E}\left(b^{-2} K_{q,b}(0)^2\right) = b^{-2}\mathbb{E}\left(\int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} |\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2 \left(V(\mathbf{0}) - V(X'_\perp, 0)\right)^2 dX_\perp dX'_\perp\right)^2 \leq$$

$$b^{-2}\mathbb{E} \int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} |\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2 dX_\perp dX'_\perp \times$$

$$\int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} |\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2 (V(\mathbf{0}) - V(X'_\perp, 0))^4 dX_\perp dX'_\perp. \tag{3.32}$$

By (3.19) and the identity

$$\int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} |\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2 dX_\perp dX'_\perp = \|\chi_{\Lambda_{\perp,l}} p_q\|^2_{\mathrm{HS}} = \frac{bl^2}{2}$$

(see (2.9)), we get

$$\mathbb{E}\left(b^{-2} K_{q,b}(0)^2\right) \leq \frac{b^{-1} l^2}{2} \int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} |\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2 C(X'_\perp, 0) dX_\perp dX'_\perp. \tag{3.33}$$

Changing the variables $X_\perp = Y$, $X'_\perp = Y_\perp + b^{-1/2} Y'_\perp$, we find that

$$\int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} \left|\mathcal{P}_{q,b}(X_\perp, X'_\perp)\right|^2 C(X'_\perp, 0) dX_\perp dX'_\perp =$$

$$b \int_{\Lambda_{\perp,l}} \int_{\mathbb{R}^2} \left|\mathcal{P}_{q,1}(0, Y'_\perp)\right|^2 C(Y_\perp + b^{-1/2} Y'_\perp, 0) dY_\perp dY'_\perp.$$

Hence, by the dominated convergence theorem and (2.3),

$$\lim_{b\to\infty} b^{-1}\int_{\Lambda_{\perp,l}}\int_{\mathbb{R}^2}|\mathcal{P}_{q,b}(X_\perp,X'_\perp)|^2 C(X'_\perp,0)dX'_\perp =$$

$$\int_{\mathbb{R}^2}|\mathcal{P}_{q,1}(0,Y_\perp)|^2dY_\perp\int_{\Lambda_{\perp,l}} C(X_\perp,0)dX_\perp = 2l^2D_l. \tag{3.34}$$

The combination of (3.32) – (3.34) yields (3.31). □

3.6. For $\mathcal{E}\in[0,\infty)$, $q\in\mathbb{Z}_+$, and $\varphi\in C_0^\infty(\mathbb{R})$, set

$$\Phi_{2,q}(b,\mathcal{E}) = \frac{1}{\pi^2l^2}\int_{\mathbb{R}^2}\frac{\partial\tilde\varphi}{\partial\bar z}(x,y)\,\mathbb{E}\left(\mathrm{Tr}\left(\chi_lP_qR_{p_q\otimes v}(\mathcal{E}b+z)P_q\chi_l\right)\right)dxdy, \tag{3.35}$$

the notation R_T being introduced in (2.16), and v being defined in (2.27). Using (2.9), we immediately get

$$\Phi_{2,q}(b,\mathcal{E}) := \frac{b}{2\pi^2}\int_{\mathbb{R}^2}\frac{\partial\tilde\varphi}{\partial\bar z}(x,y)\,\mathbb{E}\left(\mathrm{Tr}_{L^2(\mathbb{R})}\left(\chi_\parallel r_v(\mathcal{E}b-2bq+z)\chi_\parallel\right)\right)dxdy. \tag{3.36}$$

As above $r_v(z) = (h_v-z)^{-1}$, and $h_v = h_V(0)$ (see (1.7)). It is easy to see that the quantities $\Phi_{2,q}(b,\mathcal{E})$ are well-defined. Put

$$\Phi_2(b,\mathcal{E}) := \begin{cases}\sum_{q=0}^{[\mathcal{E}/2]}\Phi_{2,q}(b,\mathcal{E}) & \text{if}\quad \mathcal{E}\notin 2\mathbb{Z}_+,\\ \Phi_{2,\mathcal{E}/2}(b,\mathcal{E}) & \text{if}\quad \mathcal{E}\in 2\mathbb{Z}_+.\end{cases}$$

Proposition 3.2. *Let $\varphi\in C_0^\infty(\mathbb{R})$, $\mathcal{E}\in[0,\infty)$. Assume that V satisfies the hypotheses of Theorem 1.1. Then we have*

$$\limsup_{b\to\infty} b^{-1/2}|\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E})| \le c_6D_l^{1/2} \tag{3.37}$$

if $\mathcal{E}\notin 2\mathbb{Z}_+$, and

$$\limsup_{b\to\infty} b^{-1}|\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E})| \le c_7D_l^{1/2} \tag{3.38}$$

if $\mathcal{E}\in 2\mathbb{Z}_+$, where c_6 and c_7 do not depend on l.

Proof. Assume at first $\mathcal{E}\notin\mathbb{Z}_+$. Then we have

$$\Phi_2(b,\mathcal{E}) = \frac{1}{\pi^2l^2}\int_{\mathbb{R}^2}\frac{\partial\tilde\varphi}{\partial\bar z}(x,y)\mathbb{E}\left(\mathrm{Tr}\left(\chi_lP_-(\mathcal{E})R_{T(\mathcal{E})}(\mathcal{E}b+z)P_-(\mathcal{E})\chi_l\right)\right)dxdy$$

where $T(\mathcal{E}) := \sum_{q=0}^{[\mathcal{E}/2]}p_q\otimes v$, and the resolvent equation justified by analogy with (2.16). Therefore,

$$\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E}) =$$

$$\frac{1}{\pi^2 l^2}\int_{\mathbb{R}^2}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)\mathbb{E}\left(\mathrm{Tr}\left(\chi_l P_-(\mathcal{E})R_{T(\mathcal{E})}(\mathcal{E}b+z)(T(\mathcal{E})-V)R_V(\mathcal{E}b+z)P_-(\mathcal{E})\chi_l\right)\right)dxdy =$$

$$\frac{1}{\pi^2 l^2}\sum_{q=0}^{[\mathcal{E}/2]}\int_{\mathbb{R}^2}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)\times$$

$$\mathbb{E}\left(\mathrm{Tr}\left(\chi_l P_q R_{p_q\otimes v}(\mathcal{E}b+z)(p_q\otimes v-P_qV)R_V(\mathcal{E}b+z)P_-(\mathcal{E})\chi_l\right)\right)dxdy. \qquad (3.39)$$

Applying the Cauchy-Schwarz inequality, we get

$$b^{-1/2}\left|\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E})\right|\leq$$

$$\frac{1}{\pi^2 l^2}\sum_{q=0}^{[\mathcal{E}/2]}\int_{\mathbb{R}^2}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)\right|\left(\mathbb{E}\left(b^{-1/2}\|\chi_l P_q R_{p_q\otimes v}(\mathcal{E}b+z)(p_q\otimes v-P_qV)\|_{\mathrm{HS}}^2\right)\right)^{1/2}\times$$

$$\left(\mathbb{E}\left(b^{-1/2}\|R_V(\mathcal{E}b+z)P_-(\mathcal{E})\chi_l\|_{\mathrm{HS}}^2\right)\right)^{1/2}dxdy. \qquad (3.40)$$

Note that we have $\chi_l P_q R_{p_q\otimes v}(\mathcal{E}b+z) = \chi_{\perp,l}\, p_q\otimes\chi_{\|} r_v(\mathcal{E}b-2bq+z)$. Recall that $\mathcal{R}_v(x_3,x_3';\zeta)$, $x_3,x_3'\in\mathbb{R}$, denotes the integral kernel of the operator $r_v(\zeta)$, $\zeta\in\mathbb{C}\setminus\mathbb{R}$. Using the ergodicity properties of V_ω, we find that for $q=0,\ldots,[\mathcal{E}/2]$, we have

$$\mathbb{E}\left(b^{-1/2}\|\chi_{\perp,l}\,p_q\otimes\chi_{\|}r_v(\mathcal{E}b-2bq+z)(p_q\otimes v-P_qV)\|_{\mathrm{HS}}^2\right)=$$

$$\mathbb{E}\left(b^{-1/2}\int_{-1/2}^{1/2}\int_{\mathbb{R}}\left|\mathcal{R}_v(x_3,x_3';\mathcal{E}b-2bq+z)\right|^2K_{q,b}(x_3')dx_3'dx_3\right)=$$

$$\mathbb{E}\left(b^{-1/2}K_{q,b}(0)\int_{\mathbb{R}}\left|\mathcal{R}_v(0,s;\mathcal{E}b-2bq+z)\right|^2ds\right)\leq$$

$$\leq\left(\mathbb{E}(b^{-2}K_{q,b}(0)^2)\right)^{1/2}\left(\mathbb{E}\left(b\left(\int_{\mathbb{R}}|\mathcal{R}_v(0,s;\mathcal{E}b-2bq+z)|^2ds\right)^2\right)\right)^{1/2} \qquad (3.41)$$

(see (3.30)) for the definition of $K_{q,b}(0)$). By Lemma 2.3,

$$\mathbb{E}\left(b\left(\int_{\mathbb{R}}|\mathcal{R}_v(0,s;\mathcal{E}b-2bq+z)|^2ds\right)^2\right)\leq$$

$$\frac{b}{2(|\mathcal{E}b-2bq+z|\mathrm{Im}\sqrt{\mathcal{E}b-2bq+z})^2}\left(1+\frac{1}{y^4}\mathbb{E}(V(\mathbf{0})^4)\right)\leq$$

$$\frac{1}{2(\mathcal{E}-2q)y^2}\left(1+\frac{1}{y^4}\mathbb{E}(V(\mathbf{0})^4)\right),$$

provided that $x=\mathrm{Re}\,z\in\mathrm{supp}\,\varphi$, and $b\geq\frac{2\max_{x\in\mathrm{supp}\,\varphi}(-x)}{\mathcal{E}-2q}$. Therefore, (3.41) entails

$$\mathbb{E}\left(b^{-1/2}\|\chi_l P_q R_{p_q\otimes v}(\mathcal{E}b+z)(p_q\otimes v-P_qV)\|_{\mathrm{HS}}^2\right)\leq$$

$$\left(\mathbb{E}\left(b^{-2}K_{q,b}(0)^2\right)\right)^{1/2}\left(\frac{1}{2(\mathcal{E}-2q)y^2}\left(1+\frac{1}{y^4}\mathbb{E}(V(\mathbf{0})^4)\right)\right)^{1/2}. \tag{3.42}$$

Combining (3.39), (3.40), (3.42), and (2.22), we get

$$b^{-1/2}\left|\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E})\right| \leq \max_{q\leq[\mathcal{E}/2]}\left(b^{-2}l^{-4}\mathbb{E}\left(K_{q,b}(0)^2\right)\right)^{1/4}\sum_{q=0}^{[\mathcal{E}/2]}\frac{1}{\sqrt{2}\pi^2(\mathcal{E}-2q)^{1/2}}\times$$

$$\int_{\mathbb{R}^2}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)\right|\left(1+\frac{1}{y^2}(\mathbb{E}(V(\mathbf{0})^4)^{1/2}\right)|y|^{-1}dxdy. \tag{3.43}$$

Putting together (3.43) and (3.31), we get (3.37).
Assume now $\mathcal{E}\in 2\mathbb{Z}_+$. By analogy with (3.40) we have

$$b^{-1}\left|\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E})\right| \leq$$

$$\frac{1}{\pi^2l^2}\int_{\mathbb{R}^2}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)\right|\left(b^{-1}\mathbb{E}\left(\|\chi_l P_{\mathcal{E}/2}R_{p_{\mathcal{E}/2}\otimes v}(\mathcal{E}b+z)(p_{\mathcal{E}/2}\otimes v-P_{\mathcal{E}/2}V)\|_{\mathrm{HS}}^2\right)\right)^{1/2}\times$$

$$\left(b^{-1}\mathbb{E}\left(\|R_V(\mathcal{E}b+z)P_{\mathcal{E}/2}\chi_l\|_{\mathrm{HS}}^2\right)\right)^{1/2}dxdy, \tag{3.44}$$

and, similarly to (3.41), we get

$$\mathbb{E}\left(b^{-1}\|\chi_l P_{\mathcal{E}/2}R_{p_{\mathcal{E}/2}\otimes v}(\mathcal{E}b+z)(p_{\mathcal{E}/2}\otimes v-P_{\mathcal{E}/2}V)\|_{\mathrm{HS}}^2\right)\leq$$

$$\left(\mathbb{E}\left(b^{-2}K_{\mathcal{E}/2,b}(0)^2\right)\right)^{1/2}\left(\mathbb{E}\left(\int_{\mathbb{R}}\left|\mathcal{R}_v(0,s;z)\right|^2ds\right)\right)^{1/2}. \tag{3.45}$$

Using again Lemma 2.3, we obtain

$$\mathbb{E}\left(\left(\int_{\mathbb{R}}|\mathcal{R}_v(0,s;z)|^2ds\right)^2\right)\leq$$

$$\frac{1}{2(|z|\mathrm{Im}\sqrt{z})^2}\left(1+\frac{1}{y^4}\mathbb{E}(V(\mathbf{0})^4\right)\leq\frac{2}{|y|^3}\left(1+\frac{1}{y^4}\mathbb{E}(V(\mathbf{0})^4\right). \tag{3.46}$$

Putting together (3.44) - (3.46) and (2.19), we get

$$b^{-1}\left|\Phi_1(b,\mathcal{E})-\Phi_2(b,\mathcal{E})\right|\leq$$

$$\left(l^{-4}b^{-2}\mathbb{E}\left(K_{\mathcal{E}/2,b}(0)^2\right)\right)^{1/4}\frac{1}{\sqrt{2}\pi^2}\int_{\mathbb{R}^2}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x,y)\right|\left(1+\frac{1}{y^2}(\mathbb{E}(V(\mathbf{0})^4)^{1/2}\right)|y|^{-3/2}dxdy$$

which combined with (3.31) yields (3.38).

□

3.7. Let $\mathcal{E} \in [0,\infty)$, $q \in \mathbb{Z}_+$, $\varphi \in C_0^\infty(\mathbb{R})$. The Helffer-Sjöstrand formula (see (3.7)) with $L = h_V - \mathcal{E}b$, and $m = 0$, and the Pastur-Shubin formula (see [20, Section 2]) imply

$$\Phi_{2,q}(b,\mathcal{E}) = \frac{b}{2\pi}\int_{\mathbb{R}} \varphi(\lambda - \mathcal{E}b + 2bq)d\rho_V(\lambda;0).$$

Therefore,

$$\Phi_2(b,\mathcal{E}) = \begin{cases} \frac{b}{2\pi}\sum_{q=0}^{[\mathcal{E}/2]}\int_{\mathbb{R}} \varphi(\lambda - \mathcal{E}b + 2bq)d\rho_V(\lambda;0) \text{ if } \mathcal{E} \notin 2\mathbb{Z}_+, \\ \frac{b}{2\pi}\int_{\mathbb{R}} \varphi(\lambda)d\rho_V(\lambda;0) \text{ if } \mathcal{E} \ \in 2\mathbb{Z}_+. \end{cases} \tag{3.47}$$

Assume at first $\mathcal{E} \notin \mathbb{Z}_+$. In order to complete the proof of (1.8) we need the following

Lemma 3.5. [18, Theorem 6.5] *Assume that V satisfies the assumptions of Theorem 1.1. Then we have*

$$\rho_V(E;0) = \frac{E^{1/2}}{\pi} - \frac{E^{-1/2}}{2\pi}\mathbb{E}(v(0)) + o(E^{-1}), \quad E \to \infty. \tag{3.48}$$

Applying Lemma 3.5, we easily get

$$\lim_{b\to\infty} b^{1/2}\int_{\mathbb{R}} \varphi(\lambda - \mathcal{E}b + 2bq)d\rho_V(\lambda;0) = \frac{1}{2\pi}(\mathcal{E}-2q)^{-1/2}\int_{\mathbb{R}}\varphi(\lambda)d\lambda, \quad q = 0,\ldots,[\mathcal{E}/2]. \tag{3.49}$$

Putting together (3.14), (3.37), (3.47), and (3.49), we obtain

$$\limsup_{b\to\infty}\left|b^{-1/2}\Phi(b,\mathcal{E}) - \frac{1}{4\pi^2}\sum_{q=0}^{[\mathcal{E}/2]}(\mathcal{E}-2q)^{-1/2}\int_{\mathbb{R}}\varphi(\lambda)d\lambda\right| \leq c_6 D_l^{1/2}. \tag{3.50}$$

Since $\lim_{l\downarrow 0} D_l = 0$, we conclude that in the case of $\mathbb{R}^3$-ergodic potentials V, relation (3.1) follows from (3.50).
Now assume $\mathcal{E} \in 2\mathbb{Z}_+$. Then (3.15), (3.38), and (3.47) directly entail

$$\limsup_{b\to\infty}\left|b^{-1}\Phi(b,\mathcal{E}) - \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(\lambda)d\rho_V(\lambda;0)\right| \leq c_7 D_l^{1/2}$$

yielding (3.2) in the case of $\mathbb{R}^3$-ergodic potentials V.

3.8. In this subsection we prove Theorem 1.1 in the case where $V = V_\omega$ is $\mathbb{Z}^3$-ergodic and $\mathbb{Z}$-ergodic in the direction of the magnetic field. To this end we will use the so-called suspension method ([10], [18]). Set

$$\tilde{V}_{\omega,\theta}(\mathbf{x}) = V_\omega(\mathbf{x}+\theta), \quad \omega \in \Omega, \quad \theta \in \left(-\frac{1}{2},\frac{1}{2}\right)^3, \quad \mathbf{x} \in \mathbb{R}^3.$$

Then the potential $\tilde{V}_{\omega,\theta}$ is $\mathbb{R}^3$-ergodic and $\mathbb{R}$-ergodic in the direction of the magnetic field on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ which is the product of the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ where $\Omega_0 := \left(-\frac{1}{2}, \frac{1}{2}\right)^3$, $\mathcal{F}_0$ is the σ-algebra of the Borel subsets of Ω_0, and $\mathbb{P}_0$ is the Lebesgue measure on Ω_0 (see [10] or [18, pp. 28-29]). Moreover, due to the unitary equivalence of the operators $H_{0,b} + \tilde{V}_{\omega,\theta}$ and $H_{0,b} + V_\omega$, we have

$$\varrho_{V,b}(\lambda) = \varrho_{\tilde{V},b}(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.51}$$

Now relation (1.8) for $\mathbb{Z}^3$-ergodic potentials follows from the same relation applied to the $\mathbb{R}^3$-potential $\tilde{V}$, and (3.51). On the other hand, (1.9) applied to the $\mathbb{R}^3$-potential $\tilde{V}$ and (3.51) imply

$$\lim_{b\to\infty} b^{-1} \left(\varrho_{V,b}(\mathcal{E}b + \lambda_2) - \varrho_{V,b}(\mathcal{E}b + \lambda_1)\right) = \frac{1}{2\pi} \left(k_{\tilde{V}}(\lambda_2) - k_{\tilde{V}}((\lambda_1)\right). \tag{3.52}$$

Using again the Pastur-Shubin formula for $\rho_{\tilde{V}}(\lambda; 0)$ i.e.

$$\rho_{\tilde{V}}(\lambda; 0) = \frac{1}{|\Lambda|} \tilde{\mathbb{E}} \left(\mathrm{Tr}\ \left(\chi_\Lambda \chi_{(-\infty,\lambda)}(h_{\tilde{V}}(0))\chi_\Lambda\right)\right),$$

it is easy to check that we have

$$k_{\tilde{V}}(\lambda) = \rho_{\tilde{V}}(\lambda; 0) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \rho_V(\lambda; X_\perp) dX_\perp = k_V(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.53}$$

Combining (3.52) and (3.53), we get (1.9) for $\mathbb{Z}^3$-ergodic potentials.

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