SMILANSKY'S MODEL OF IRREVERSIBLE QUANTUM GRAPHS, II: THE POINT SPECTRUM

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Abstract. In the model suggested by Smilansky [6] one studies an operator describing the interaction between a quantum graph and a system of K one-dimensional oscillators attached at different points of the graph. This paper is a continuation of [3] in which we started an investigation of the case $K > 1$. For the sake of simplicity we consider $K = 2$, but our argument applies to the general situation. In this second part of the paper we apply the variational approach to the study of the point spectrum.

1. Introduction

In Smilansky's model of irreversible quantum graphs, the interaction between a quantum graph and a finite system of one-dimensional harmonic oscillators attached at various vertices of the graph is studied. The paper [6] may be consulted for the physical background and motivation, and [5] for a survey of recent work on quantum graphs. Our concern here is the spectral analysis of the self-adjoint operator which generates the dynamical system, and it suffices to have a precise description of the analytic problem. This paper continues the study in [3] where a detailed description of the problem may be found and a survey of earlier results in the literature given. As in [3], we consider the case of two oscillators attached to the graph constituted by $\mathbb{R}$ at vertices ± 1. This special case retains the main features of the general case without obscuring the argument with technical complications.

On a formal level, the problem is described by the differential expression

$$\mathcal{A}U = -U''_{x^2} + \frac{\nu_+^2}{2}(-U''_{q_+^2} + q_+^2 U) + \frac{\nu_-^2}{2}(-U''_{q_-^2} + q_-^2 U) \tag{1.1}$$

for $x \in \mathbb{R}$, $q_\pm \in \mathbb{R}$, together with the following 'transmission', or 'matching' conditions across the planes $x = \pm 1$ in $\mathbb{R}^3$:

$$\begin{aligned} U'_x(1+, q_+, q_-) - U'_x(1-, q_+, q_-) &= \alpha_+ q_+ U(0, q_+, q_-), \\ U'_x(-1+, q_+, q_-) - U'_x(-1-, q_+, q_-) &= \alpha_- q_- U(0, q_+, q_-). \end{aligned} \tag{1.2}$$

Date: 18th May, 2005.

1991 *Mathematics Subject Classification.* 81Q10, 35P20.

Key words and phrases. Quantum graphs, point spectrum.

The parameters $\alpha_\pm$ are real and can be assumed to be non-negative since, for instance, replacing α_+ by $-\alpha_+$ corresponds to replacing q_+ by $-q_+$ and this has no effect on the problem to be investigated. The parameters $\nu_\pm$ are fixed positive numbers throughout. To shorten our notation, we set $\boldsymbol{\alpha} = (\alpha_+, \alpha_-)$ and $\boldsymbol{\nu} = (\nu_+, \nu_-)$.

Let $\chi_n, n \in \mathbb{N}_0$, be the normalized Hermite functions in $L^2(\mathbb{R})$. The sequence $\{\chi_n\}_{n\in\mathbb{N}_0}$ is then an orthonormal basis in $L^2(\mathbb{R})$ and any $U \in L^2(\mathbb{R}^3)$ can be written as

$$U(x, q_+, q_-) = \sum_{m,n\in\mathbb{N}_0} u_{m,n}(x)\chi_m(q_+)\chi_n(q_-)$$

for some $u_{m,n} \in L^2(\mathbb{R})$. We write $U \sim \{u_{m,n}\}$ to indicate this representation. The mapping $U \mapsto \{u_{m,n}\}$ is an isometry of $\mathfrak{H} = L^2(\mathbb{R}^3)$ onto the Hilbert space $\ell^2(\mathbb{N}_0^2; L^2(\mathbb{R}))$. For $U \sim \{u_{m,n}\}$ we have $\mathcal{A}U \sim \{L_{m,n}u_{m,n}\}$, where

$$(L_{m,n}u)(x) = -u''(x) + r_{m,n}u(x), \qquad x \neq \pm 1; \tag{1.3}$$

$$r_{m,n} = \nu_+^2(m + 1/2) + \nu_-^2(n + 1/2), \qquad m, n \in \mathbb{N}_0. \tag{1.4}$$

The number

$$r_{0,0} = (\nu_+^2 + \nu_-^2)/2$$

plays a special role since it appears in the formulations of all our basic results.

The conditions (1.2) at $x = \pm 1$ become

$$\begin{aligned} \sum_{m,n\in\mathbb{N}_0} [u'_{m,n}](1)\chi_m(q_+)\chi_n(q_-) &= \sum_{m,n\in\mathbb{N}_0} \alpha_+ q_+ \chi_m(q_+)\chi_n(q_-), \\ \sum_{m,n\in\mathbb{N}_0} [u'_{m,n}](-1)\chi_m(q_+)\chi_n(q_-) &= \sum_{m,n\in\mathbb{N}_0} \alpha_- q_- \chi_m(q_+)\chi_n(q_-), \end{aligned} \tag{1.5}$$

where we have used the notation

$$[u'](a) := u'(a + 0) - u'(a - 0).$$

On using the recurrence relation

$$\sqrt{k+1}\chi_{k+1}(q) - \sqrt{2}q\chi_k(q) + \sqrt{k}\chi_{k-1}(q) = 0, \quad q \in \mathbb{R},$$

the matching conditions (1.5) reduce to

$$\begin{aligned} [u'_{m,n}](1) &= \frac{\alpha_+}{\sqrt{2}} \left(\sqrt{m+1}u_{m+1,n}(1) + \sqrt{m}u_{m-1,n}(1)\right); \\ [u'_{m,n}](-1) &= \frac{\alpha_-}{\sqrt{2}} \left(\sqrt{n+1}u_{m,n+1}(-1) + \sqrt{n}u_{m,n-1}(-1)\right). \end{aligned} \tag{1.6}$$

The operator realization of (1.1) and (1.2) in the Hilbert space $\mathfrak{H}$, which we denote by $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ can now be defined. Its domain $\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is given by

Definition 1.1. *An element $U \sim \{u_{m,n}\}$ lies in $\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ if and only if*
1. $u_{m,n} \in H^1(\mathbb{R})$ for all m, n;
2. for all m, n, the restriction of $u_{m,n}$ to each interval $(-\infty, -1), (-1, 1), (1, \infty)$, lies in H^2 and moreover,

$$\sum_{m,n} \int_{\mathbb{R}} |L_{m,n} u_{m,n}|^2 dx < \infty;$$

3. the conditions (1.6) are satisfied.

Along with the set $\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, we define its subset

$$\mathcal{D}^{\bullet}_{\boldsymbol{\alpha},\boldsymbol{\nu}} = \{U \in \mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}} : U \sim \{u_{m,n}\} \text{ finite}\},$$

where by *finite* we mean that the sequence has only a finite number of non-zero components.

The operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ in $\mathfrak{H}$ is defined on the domain $\mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ by

$$\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}} U \sim \{L_{m,n} u_{m,n}\} \text{ for } U \sim \{u_{m,n}\} \in \mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$$

where $L_{m,n}$ is given by (1.3). We denote the restriction of $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ to $\mathcal{D}^{\bullet}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ by $\mathbf{A}^{\bullet}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$.

The following statement is proved in [3], Theorem 2.3.

Theorem 1.2. *The operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is self-adjoint for all $\alpha_\pm \geq 0$, and is the closure of $\mathbf{A}^{\bullet}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$.*

Our main goal here, as well as in the preceding paper [3], is to study the spectrum of the operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ for different values of the parameters $\alpha_\pm$. Informally, the mains results of both papers can be summarized as follows: *the spectral properties of a K-oscillator system can be described in terms of the corresponding properties of K appropriate one-oscillator systems.* To obtain these one-oscillator systems, one divides the original graph into K pieces in such a way that each part contains only one point at which an oscillator is attached, and these points should not belong to the new boundary appearing as a result of the division. On this new boundary we put an additional boundary condition, for instance the Dirichlet condition. For our case ($\Gamma = \mathbb{R}$ and the oscillators attached at ± 1), it is most natural to take $x = 0$ as the point of division. Let us denote the corresponding operators by $\mathbf{A}_{\mathbb{R}_\pm;\alpha_\pm;\nu_\pm}$; see [3], section 2.4 for details.

The following theorem is proved in [3], Theorem 2.6.

Theorem 1.3. *Let*

$$\mu_\pm := \frac{\nu_\pm \sqrt{2}}{\alpha_\pm}.$$

1. If $\mu_\pm > 1$*, then* $\sigma_{a.c.}(\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = [r_{0,0},\infty) = [(\nu_+^2+\nu_-^2)/2,\infty)$*.*

2. Let $\mu_+ = 1$ *and* $\mu_- > 1$*, or* $\mu_- = 1$ *and* $\mu_+ > 1$*. Then*

$$\sigma_{a.c.}(\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = [\nu_-^2/2,\infty) \qquad \text{or} \qquad \sigma_{a.c.}(\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = [\nu_+^2/2,\infty)$$

respectively.

3. Let $\mu_+ = \mu_- = 1$*, then* $\sigma_{a.c.}(\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = [0,\infty)$*.*

In all the cases $1-3$ *the multiplicity function* $\mathfrak{m}_{a.c.}(\lambda;\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}})$*, is finite for all* $\lambda \in \sigma_{a.c.}(\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}})$ *and is given by*

$$\begin{aligned}\mathfrak{m}_{a.c.}(\lambda;\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = &\sum_{n\in\mathbb{N}_0}\mathfrak{m}_{a.c.}(\lambda-\nu_-^2(n+1/2);\mathbf{A}_{\mathbb{R}_+;\alpha_+;\nu_+}) \\ &+\sum_{m\in\mathbb{N}_0}\mathfrak{m}_{a.c.}(\lambda-\nu_+^2(m+1/2);\mathbf{A}_{\mathbb{R}_-;\alpha_-;\nu_-}).\end{aligned} \tag{1.7}$$

4. Let $max(\mu_+,\mu_-) < 1$*. Then*

$$\sigma_{a.c.}(\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = \mathbb{R}, \qquad \mathfrak{m}_{a.c.}(\lambda;\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) \equiv \infty.$$

In the present paper we are concerned with the point spectrum below the threshold $r_{0,0}$ in the case that μ_+ and μ_- are both greater than 1. Below $N_-(\lambda;\mathbf{T})$, where λ is a real number, stands for the number of eigenvalues (counting multiplicities) of a self-adjoint operator $\mathbf{T}$, lying on the half-line $(-\infty,\lambda)$, provided that this part of the spectrum is discrete. We also set $N_+(\lambda;\mathbf{T}) = N_-(-\lambda;-\mathbf{T})$.

On the qualitative level, the main result of this paper can be described as follows.

For any $\mu_+,\mu_- < 1$ *the number* $N_-\left(r_{0,0};\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}\right)$ *is finite and asymptotically*

$$\begin{aligned}N_-\left(r_{0,0};\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}\right) \sim N_-(\nu_+^2/2;\mathbf{A}_{\mathbb{R}_+;\alpha_+;\nu_+}) + N_-(\nu_-^2/2;\mathbf{A}_{\mathbb{R}_-;\alpha_-;\nu_-}),\\ r_{0,0} = (\nu_+^2+\nu_-^2)/2, \qquad \mu_\pm \downarrow 1.\end{aligned} \tag{1.8}$$

In order to give the precise formulation, we need to describe the behaviour of the terms on the right-hand side of (1.8), and to explain what we mean when speaking about the asymptotics in two parameters. To achieve the first goal, we present a result which is a special case of Theorem 3.1 in [9], see also (3.10) in [8]. Let $\Gamma = [a,b]$ (with the standard change if $a = -\infty$ or $b = \infty$) be a finite or infinite interval and $o \in \operatorname{Int}\Gamma$. Consider the operator $\mathbf{A}_{\Gamma;\alpha;\nu}$ in $L^2(\Gamma\times\mathbb{R})$, defined by the differential expression

$$\mathcal{A}U = -U''_{x^2} + \frac{\nu^2}{2}(-U''_{q^2} + q^2U)$$

and the matching condition

$$U'_x(o+,q) - U'_x(o-,q) = \alpha q U(o,q),$$

cf (1.1) and (1.2). If $\Gamma \neq \mathbb{R}$, the Dirichlet or the Neumann boundary condition is posed on $\partial\Gamma \times \mathbb{R}$. We do not reflect the type of this condition in our notation. If $\Gamma = \mathbb{R}$, we drop the index Γ in the notation of the operator.

Proposition 1.4. *For any $\alpha \in (0, \nu\sqrt{2})$ the spectrum of the operator $\mathbf{A}_{\alpha,\nu}$ below the point $\nu^2/2$ is non-empty and finite, and the following asymptotic formula is satisfied:*

$$N_-(\nu^2/2; \mathbf{A}_{\Gamma;\alpha;\nu}) \sim \frac{1}{4\sqrt{2(\mu-1)}}, \qquad \mu := \frac{\nu\sqrt{2}}{\alpha} \downarrow 1. \tag{1.9}$$

It was assumed in [8] and [9] that $\nu = 1$, the general case reduces to this special case by scaling.

Our next theorem, together with the subsequent explanation of uniformity of the asymptotics, gives the precise meaning to (1.8). In the formulation of its second part an arbitrary positive function $\psi(t)$ on $(0,1)$ which is $o(t^{-1/4})$ as $t \to 0$ is involved. We also define the set

$$\Omega_\Psi := \{(x,y) : \Psi(x) \leq y \leq 1, \quad \Psi(y) \leq x \leq 1\}, \quad \Psi(t) = e^{-\psi(t)}. \tag{1.10}$$

Note that the co-ordinate axes are tangents of infinite order to Ω_Ψ at the origin.

Theorem 1.5. 1. *If $\mu_\pm := \sqrt{2}\nu_\pm/\alpha_\pm > 1$, then $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is bounded below and its spectrum in $(-\infty, r_{0,0})$ is non-empty and finite.*

2. *Let Ψ be chosen as in (1.10). Then, uniformly for $(1-\mu_+^{-1}, 1-\mu_-^{-1}) \in \Omega_\Psi$,*

$$N_-\left(r_{0,0}; \mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}\right) \sim \frac{1}{4\sqrt{2(\mu_+-1)}} + \frac{1}{4\sqrt{2(\mu_--1)}}, \qquad \mu_\pm \downarrow 1.$$

Now, let us explain what we mean by 'uniform asymptotics'. It means that on the domain $(1-\mu_+^{-1}, 1-\mu_-^{-1}) \in \Omega_\Psi$ there exists a bounded function $\Phi(\mu_+,\mu_-)$, such that $\Phi(\mu_+,\mu_-) \to 0$ as $\mu_\pm \to 1$ and

$$\begin{aligned} &\left|N_-\left(r_{0,0}; \mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}\right) - \frac{1}{4\sqrt{2}}((\mu_+-1)^{-1/2} + (\mu_--1)^{-1/2})\right| \\ &\leq \Phi(\mu_+,\mu_-)\left((\mu_+-1)^{-1/2} + (\mu_--1)^{-1/2}\right). \end{aligned}$$

The technical ideas which lead to this result were explained in the introduction to [3]. Here we only note that for $\mu_\pm \geq 1$ the operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is bounded below (see Theorem 2.1), which makes it possible to apply the variational approach. In contrast to the operator domain of the operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, its quadratic form domain for $\mu_\pm > 1$ does not depend on the parameters $\alpha_\pm$. This significantly simplifies the analysis. In particular, we do not need to divide the graph into two parts, as we did in [3]; cf. (2.9) and Theorem 2.8 there. In our proof of Theorem 1.5 we will be dealing with the operators $\mathbf{A}_{\alpha_\pm;\nu_\pm}$ (i.e., the corresponding graph is $\Gamma = \mathbb{R}$) rather than with $\mathbf{A}_{\mathbb{R}_\pm \alpha_\pm;\nu_\pm}$ as in (1.8). According to Proposition 1.4, the passage from $\mathbb{R}_\pm$ to $\mathbb{R}$ does not affect the asymptotic behaviour of the function N_- for these operators.

We mostly use the same notation as in [3]. However, in this paper we have to take special care in order to distinguish between the operators which correspond to the one-oscillator and to the two-oscillator cases. We always denote the first as $\mathbf{A}_{\alpha,\nu}$ and the second as $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, with the boldface α, ν in the indices. Besides, we almost never drop the index ν in the notation.

2. Variational description of $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ for $\mu_\pm > 1$

2.1. **The quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$.** If $U \sim \{u_{m,n}\} \in \mathcal{D}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}[U] := (\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}} U, U)$ is given by

$$\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}[U] = \mathbf{a}[U] + \alpha_+ \mathbf{b}_+[U] + \alpha_- \mathbf{b}_-[U], \tag{2.1}$$

where, in the notation (1.4),

$$\mathbf{a}[U] = \sum_{m,n\in\mathbb{N}_0} \int_{\mathbb{R}} \left(|u'_{m,n}(x)|^2 + r_{m,n}|u_{m,n}|^2\right) dx, \tag{2.2}$$

$$\mathbf{b}_+[U] = \operatorname{Re} \sum_{m,n\in\mathbb{N}_{\nvdash}} \sqrt{2m}\, u_{m,n}(1)\overline{u_{m-1,n}(1)}, \tag{2.3}$$

$$\mathbf{b}_-[U] = \operatorname{Re} \sum_{m,n\in\mathbb{N}_0} \sqrt{2n}\, u_{m,n}(-1)\overline{u_{m,n-1}(-1)}. \tag{2.4}$$

In (2.3) and (2.4) we took by default that $u_{-1,n} \equiv 0$ and $u_{m,-1} \equiv 0$ for all $m, n \in \mathbb{N}$.

The quadratic form $\mathbf{a}$ (which is the same as $\mathbf{a}_{\mathbf{0},\boldsymbol{\nu}}$) is positive definite in $\mathfrak{H}$. Completing the set $\mathcal{D}_{\mathbf{0},\boldsymbol{\nu}}$ with respect to the 'energy metric' $\mathbf{a}[U]$, we obtain a Hilbert space which we denote by $\mathbf{d}$.

Let us define H^1_γ, where $\gamma > 0$ is a real parameter, to be the Sobolev space $H^1(\mathbb{R})$ with the scalar product

$$(u_1, u_2)_\gamma = \int_{\mathbb{R}} \left(u_1'(x)\overline{u_2'(x)} + \gamma^2 u_1(x)\overline{u_2(x)}\right) dx \tag{2.5}$$

and the corresponding norm $\|u\|_\gamma$. The space $\mathbf{d}$ can be naturally identified with the orthogonal sum of the spaces $H^1_{\sqrt{r_{m,n}}}$. The topology in $\mathbf{d}$ does not depend on the values of $\nu_\pm$.

Our next goal is to prove the following

Theorem 2.1. *Let $\mu_\pm \geq 1$. Then the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is bounded below. If $\mu_\pm > 1$, then $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is closed on $\mathbf{d}$ and the corresponding self-adjoint operator in $\mathfrak{H}$ coincides with $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$.*

For the proof we need some auxiliary material. Let $\mathcal{F}_\gamma$ be the two-dimensional space of functions $v \in H^1_\gamma$ which for $x \neq \pm 1$ satisfy the equation

$$-v'' + \gamma^2 v = 0.$$

Evidently, each function $v \in H^1_\gamma$ is uniquely determined by its values at the points ± 1. The space $\mathcal{F}_\gamma$ was discussed in [3], sec. 3.1. In particular, it was shown there that for any $v \in \mathcal{F}_\gamma$ one has

$$[v'](p) = -\frac{2\gamma}{1 - e^{-4\gamma}} \left(v(p) - e^{-2\gamma} v(-p) \right), \qquad p = \pm 1. \tag{2.6}$$

It follows from (2.6) that the mapping $v \mapsto ([v'](1), [v'](-1))$ maps $\mathcal{F}_\gamma$ onto $\mathbb{C}^2$.

Denote by Π_γ the operator of ortogonal projection (in the scalar product (2.5)) of the space H^1_γ onto $\mathcal{F}_\gamma$.

Lemma 2.2. *For any $u \in H^1_\gamma$ its projection $\Pi_\gamma u$ is the function $v \in \mathcal{F}_\gamma$, defined by the conditions*

$$v(\pm 1) = u(\pm 1). \tag{2.7}$$

Proof. Let $v, w \in \mathcal{F}_\gamma$. We have

$$\begin{aligned}(u - v, w)_\gamma &= \left(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} \right) \left((u' - v')\overline{w'} + \gamma^2 (u - v)\overline{w} \right) dx \\ &= \left(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} \right) (u - v)(\overline{-w'' + \gamma^2 w}) dx \\ &- (u(1) - v(1))\overline{[w'](1)} - (u(-1) - v(-1))\overline{[w'](-1)}.\end{aligned}$$

The integrand in the second line vanishes and we get

$$(u - v, w)_\gamma = -(u(1) - v(1))\overline{[w'](1)} - (u(-1) - v(-1))\overline{[w'](-1)}.$$

By (2.6), the set of all possible pairs $([w'](1), [w'](-1))$ covers the whole of $\mathbb{C}^2$ which implies the result. $\square$

Lemma 2.3. *For all $u \in H^1_\gamma$,*

$$2\gamma(|u(-1)|^2 + |u(1)|^2) \leq (1 + e^{-2\gamma}) \int_{\mathbb{R}} \left(|u'|^2 + \gamma^2 |u|^2 \right) dx. \tag{2.8}$$

The constant is optimal. The equality in (2.8) is attained on the one-dimensional subspace in H^1_γ formed by the functions $v \in \mathcal{F}_\gamma$ such that $v(1) = v(-1)$.

Proof. Given a function $u \in H^1_\gamma$, take $v = \Pi_\gamma u$. Then

$$\|u-v\|^2_\gamma = \|u\|^2_\gamma - (v,u)_\gamma = \|u\|^2_\gamma - \int_{\mathbb{R}} \left(v'\overline{u'} + \gamma^2 v\overline{u}\right) dx.$$

Integrating by parts as in Lemma 2.2 and denoting $u(1) = A,\ u(-1) = B$, we get

$$\|u-v\|^2_\gamma = \|u\|^2_\gamma + \overline{A}[v'](1) + \overline{B}[v'](-1).$$

On using (2.6) and (2.7), we find from here:

$$\begin{aligned} 0 \le \|u-v\|^2_\gamma &= \|u\|^2_\gamma - \frac{2\gamma}{1-e^{-4\gamma}}\left(\overline{A}(A-e^{-2\gamma}B) + \overline{B}(B-e^{-2\gamma}A)\right) \\ &= \|u\|^2_\gamma - \frac{2\gamma}{1+e^{-2\gamma}}(|A|^2+|B|^2) - \frac{2\gamma e^{-2\gamma}}{1-e^{-4\gamma}}|A-B|^2, \end{aligned}$$

whence the Lemma. □

2.2. **Proof of Theorem 2.1.** We obtain from (2.3):

$$\mathbf{b}_+[U] \le \frac{1}{2}\sum_{m,n\in\mathbb{N}_0}\left(\sqrt{2m}+\sqrt{2(m+1)}\right)|u_{m,n}(1)|^2 \le \sum_{m\in\mathbb{N}_0,n\in\mathbb{N}}\sqrt{2m+1}|u_{m,n}(1)|^2$$

and similarly

$$\mathbf{b}_-[U] \le \sum_{m\in\mathbb{N},,n\in\mathbb{N}_0}\sqrt{2n+1}|u_{m,n}(-1)|^2.$$

Given a number $k \ge -r_{0,0}$, denote

$$\gamma_{m,n}(k) = \sqrt{r_{m,n}+k}. \tag{2.9}$$

The conditions $\mu_+, \mu_- \ge 1$ imply

$$\max(\alpha_+\sqrt{2m+1}, \alpha_-\sqrt{2n+1}) \le 2\gamma_{m,n}(0).$$

Hence,

$$\begin{aligned} &\alpha_+\sqrt{2m+1}|u_{m,n}(1)|^2 + \alpha_-\sqrt{2n+1}|u_{m,n}(-1)|^2 \\ &\le 2\gamma_{m,n}(0)\left(|u_{m,n}(1)|^2 + |u_{m,n}(-1)|^2\right). \end{aligned}$$

Applying Lemma 2.3 with $\gamma = \gamma_{m,n}(k)$ and k a positive constant to be chosen later, we obtain

$$\alpha_+\sqrt{2m+1}|u_{m,n}(1)|^2 + \alpha_-\sqrt{2n+1}|u_{m,n}(-1)|^2 \le C(m,n,k)\|u_{m,n}\|^2_{H^1_{\gamma_{m,n}(k)}} \tag{2.10}$$

where

$$C(m,n,k) = \frac{\gamma_{m,n}(0)}{\gamma_{m,n}(k)}(1+e^{-2\gamma_{m,n}(k)}).$$

Now we show that

$$C(m,n,k) \le 1, \qquad \forall m,n \in \mathbb{N}_0, \tag{2.11}$$

provided that k is large enough. To this end, consider the function

$$f_k(t) = (1 - kt^{-2})^{1/2}\left(1 + e^{-2t}\right), \qquad t \geq k^{1/2},\ k > 0,$$

then

$$C(m,n,k) = f_k(\gamma_{m,n}(k)).$$

Note that $f_k(k^{1/2}) = 0$ and $f_k(t) \to 1$ as $t \to \infty$. Hence, (2.11) will be proven if we show that $f_k'(t) \geq 0$ for all t.

We have

$$\begin{aligned} f'(t) &= \frac{k(1+e^{-2t})}{t^3(1-kt^{-2})^{1/2}} - 2(1-kt^{-2})^{1/2}e^{-2t} \\ &= \frac{(1+e^{-2t}+2te^{-2t})k - 2t^3e^{-2t}}{t^3(1-kt^{-2})^{1/2}} \geq \frac{k - 2t^3e^{-2t}}{t^3(1-kt^{-2})^{1/2}}, \end{aligned}$$

and the desired result follows for $k \geq 27e^{-3}/4 = \max\left(2t^3e^{-2t}\right)$.

On taking k such that (2.11) is satisfied, we derive from (2.10):

$$|\alpha_+\mathbf{b}_+[U] + \alpha_-\mathbf{b}_-[U]| \leq \sum_{m,n\in\mathbb{N}_0} \|u_{m,n}\|^2_{H^1_{\gamma_{m,n}(k)}} = \mathbf{a}[U] + k\|U\|^2_{\mathfrak{H}}.$$

So, the boundedness below of $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ for all $\mu_\pm \geq 1$ is established. The closedness of this quadratic form for all $\mu_\pm > 1$ easily follows from here, cf. [1]. Since the operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ has a unique self-adjoint realization, it necessarily coincides with the operator associated with the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$.

3. The spectrum of $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ below $r_{0,0}$.

We next prove that the spectrum below $r_{0,0}$ is finite and non-empty, and in the process, give an alternative proof of part 1 of Theorem 1.5. Our argument is similar to the one in [9] where the one-oscillator case was studied.

3.1. **Finiteness.** For some $L \in \mathbb{N}$, let us consider the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, see (2.1), on the set

$$\mathbf{d}^{(L)} = \left\{U \sim \{u_{m,n}\} : u_{m,n}(\pm 1) = 0,\ m+n \leq L\right\}. \tag{3.1}$$

For the operator $\mathbf{A}^{(L)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, associated with the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}} \restriction \mathbf{d}^{(L)}$, the subspace

$$\mathfrak{H}^{(L)} = \left\{U \sim \{u_{m,n}\} : u_{m,n} \equiv 0,\ m+n > L\right\}$$

is invariant, and the part $\mathbf{A}^{(L,-)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ of $\mathbf{A}^{(L)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ in $\mathfrak{H}^{(L)}$ decomposes in the orthogonal sum:

$$\mathbf{A}^{(L,-)}_{\boldsymbol{\alpha},\boldsymbol{\nu}} = \sum_{m+n\leq L}{}^{\oplus} \left(\mathbf{A} + r_{m,n}\right), \tag{3.2}$$

where $\mathbf{A} = -\frac{d^2}{dx^2}$ with domain $H^2(\mathbb{R})$. Since

$$\sigma(\mathbf{A}) = \sigma_{a.c.}(\mathbf{A}) = [0, \infty),$$

it follows that

$$\sigma(\mathbf{A}^{(L,-)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = \sigma_{a.c.}(\mathbf{A}^{(L,-)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = [r_{0,0}, \infty). \tag{3.3}$$

An explicit expression for the multiplicity function $\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}^{(L,-)}_{\boldsymbol{\alpha},\boldsymbol{\nu}})$ immediately follows from (3.2), but this is omitted.

On repeating the argument in section 2.2 with $k = 0$, we have that for $U \in \mathbf{d}$, $U \perp \mathfrak{H}^{(L)}$

$$\begin{aligned} \mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}[U] \ &\geq \sum_{m+n>L} \big(1 - \max\{\mu_+^{-1}, \mu_-^{-1}\}(1 + e^{-2\gamma_{m,n}(0)})\big) \\ &\times \int_{\mathbb{R}} \big(|u'_{m,n}|^2 + r_{m,n}|u_{m,n}|^2\big)\, dx. \end{aligned}$$

Let $\mathbf{A}^{(L,+)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ stand for the part of $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ in the subspace $(\mathfrak{H}^{(L)})^\perp$. It follows from the above inequality that for any $\lambda_0 > 0$ it is possible to choose L sufficiently large, to ensure that

$$(\mathbf{A}^{(L,+)}_{\boldsymbol{\alpha},\boldsymbol{\nu}} U, U) \geq \lambda_0 \|U\|^2. \tag{3.4}$$

Hence, in view of (3.3),

$$\sigma(\mathbf{A}^{(L)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = [r_{0,0}, \infty), \tag{3.5}$$

$$\sigma_{a.c.}(\mathbf{A}^{(L)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) \supseteq [r_{0,0}, \lambda_0). \tag{3.6}$$

The passage from the operator $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ to $\mathbf{A}^{(L)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ corresponds to the passage from the quadratic form domain $\mathbf{d}$ to its subspace $\mathbf{d}^{(L)}$ of finite co-dimension. In its turn, this corresponds to a finite rank perturbation of the resolvent. Such perturbations do not affect the absolutely continuous spectrum and its multiplicity. Hence,

$$\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}^{(L)}_{\boldsymbol{\alpha},\boldsymbol{\nu}}), \qquad \lambda \in [r_{0,0}, \infty).$$

This immediately leads to (1.7) for $r_{0,0} \leq \lambda < \lambda_0$ and therefore, for all $\lambda \geq r_{0,0}$.

Besides, the number of eigenvalues of $\mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ which may appear below $r_{0,0}$ under such a perturbation, does not exceed the rank of the perturbation and hence, is finite.

3.2. **Non-emptiness of σ_p.** To prove that the spectrum below $r_{0,0}$ is non-empty, and hence complete the proof of Theorem 1.5, we apply the argument used to prove the analogous result in [7], Theorem 6.2. It is sufficient to find a function $U \in \mathbf{d}$ which is such that

$$\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}[U] < r_{0,0}\|U\|^2_{\mathfrak{H}}. \tag{3.7}$$

Choose $U \sim \{u_{m,n}\}$ as follows. We take

$$u_{0,0}(x) = -\varepsilon^{-1/2}\min(1, e^{-(\varepsilon|x|-1)}),$$

with $\varepsilon \in (0,1)$ to be chosen later. Note that

$$\int_{\mathbb{R}} |u'_{0,0}|^2 dx = 1, \qquad u_{0,0}(\pm 1) = -\varepsilon^{-1/2}.$$

We also take $u_{1,0}(x) = e^{-|x-1|}$, $u_{0,1}(x) = e^{-|x+1|}$, then $u_{1,0}(1) = u_{0,1}(-1) = 1$ and

$$\int_{\mathbb{R}} |u'_{1,0}|^2 dx = \int_{\mathbb{R}} |u'_{0,1}|^2 dx = \int_{\mathbb{R}} |u_{1,0}|^2 dx = \int_{\mathbb{R}} |u_{0,1}|^2 dx = 1.$$

We take all the other components $u_{m,n}$ to be zero. For such U we have

$$\begin{gathered}\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}[U] - r_{0,0}\|U\|^2_{\mathfrak{H}} \\ = \textstyle\int_{\mathbb{R}} \left(|u'_{0,0}|^2 + |u'_{1,0}|^2 + |u'_{0,1}|^2 + \nu_+^2|u_{1,0}|^2 + \nu_-^2|u_{0,1}|^2\right) dx \\ +\sqrt{2}\alpha_+ u_{1,0}(1)u_{0,0}(1) + \sqrt{2}\alpha_- u_{0,1}(-1)u_{0,0}(-1) \\ = 3 + \nu_+^2 + \nu_-^2 - \varepsilon^{-1/2}\sqrt{2}(\alpha_+ + \alpha_-).\end{gathered}$$

On choosing ε sufficiently small we obtain a function U which satisfies (3.7). This completes the proof of part 1 of Theorem 2.1.

4. Asymptotics: reduction to a problem in ℓ^2

4.1. **Removing the component $u_{0,0}$.** In what follows it is convenient for us to consider the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, defined in (2.1), for the elements $U \sim \{u_{m,n}\} \in \mathbf{d}$ subject to the additional conditions

$$u_{0,0}(1) = u_{0,0}(-1) = 0. \tag{4.1}$$

For any $\alpha_\pm < \nu_\pm\sqrt{2}$ the quadratic form $\mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}}$, restricted to this domain, generates in $\mathfrak{H}$ a self-adjoint operator, for which the subspace

$$\mathfrak{H}_{0,0} = \{U \sim \{u_{0,0}, 0, 0, \dots\}\}$$

is invariant. The part of this operator in $\mathfrak{H}_{0,0}$ is $-u''_{0,0} + r_{0,0}u_{0,0}$ under the conditions (4.1) and it has no spectrum below $r_{0,0}$. Removing this subspace yields the Hilbert space

$$\mathfrak{H}^\circ = \{U \sim \{u_{m,n}\} : u_{0,0} \equiv 0\}$$

and the quadratic form $\mathbf{a}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}} = \mathbf{a}_{\boldsymbol{\alpha},\boldsymbol{\nu}} \restriction \mathbf{d}^\circ$.

Below we denote

$$\gamma_{m,n} = \gamma_{m,n}(-r_{0,0}) = \sqrt{\nu_+^2 m + \nu_-^2 n},$$

cf (2.9). We shall consider $\mathbf{d}^\circ$ as a Hilbert space with the norm given by

$$\|U\|^2_{\mathbf{d}^\circ} = \mathbf{a}^\circ[U] - r_{0,0}\|U\|^2_{\mathfrak{H}^\circ} = \sum_{m+n>0} \int_{\mathbb{R}} \left(|u'_{m,n}|^2 + \gamma^2_{m,n}|u_{m,n}|^2\right) dx \tag{4.2}$$

and the corresponding scalar product $(.,.)_{\mathbf{d}^\circ}$. The norm $\|U\|_{\mathbf{d}^\circ}$ and the "energy norm" $\sqrt{\mathbf{a}[U]}$ are equivalent on $\mathbf{d}^\circ$. On the whole of $\mathbf{d}$ this is not true. This explains, why the passage from $\mathbf{d}$ to $\mathbf{d}^\circ$ is useful.

Let $\mathbf{A}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ stand for the self-adjoint operator in $\mathfrak{H}^\circ$, associated with the quadratic form $\mathbf{a}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}}$. It follows from the variational argument that

$$0 \le N_-(r_{0,0}; \mathbf{A}_{\boldsymbol{\alpha},\boldsymbol{\nu}}) - N_-(r_{0,0}; \mathbf{A}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}}) \le 2, \qquad \forall \mu_\pm > 1. \tag{4.3}$$

Therefore, both counting functions have the same asymptotic behaviour as $\mu_\pm \downarrow 1$.

According to the variational principle,

$$N_-(r_{0,0}; \mathbf{A}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}}) = \min_{E\in\mathcal{E}} \operatorname{codim} E \tag{4.4}$$

where $\mathcal{E}$ is the set of all subspaces $E \subset \mathbf{d}^\circ$ such that

$$\mathbf{a}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}}[U] \ge r_{0,0}\|U\|^2_{\mathfrak{H}^\circ}, \qquad \forall U \in E.$$

The latter inequality can be re-written as

$$\|U\|^2_{\mathbf{d}^\circ} + \alpha_+\mathbf{b}_+[U] + \alpha_-\mathbf{b}_-[U] \ge 0, \qquad \forall U \in E. \tag{4.5}$$

4.2. **Shrinking the space.** Our next goal is to show that it is enough to take the maximum in (4.4) over the set of subspaces $E \subset \mathcal{F}$ where

$$\mathcal{F} = \sum_{m+n>0}{}^{\oplus} \mathcal{F}_{\gamma_{m,n}}$$

(recall that the two-dimensional spaces $\mathcal{F}_\gamma$ were defined in section 2.1). Indeed, in the variational description of the non-zero spectrum of a self-adjoint operator $\mathbf{T}$ one can always consider only the subspaces orthogonal to $\ker \mathbf{T}$. Let us apply this remark to the operator $\mathbf{B}$ in $\mathbf{d}^\circ$, generated by the right-hand side in (4.5). It follows from Lemma 2.2 that the orthogonal complement to $\mathcal{F}$ in $\mathbf{d}^\circ$ is given by

$$\mathcal{F}^\perp = \{U \sim \{u_{m,n}\} : u_{m,n}(1) = u_{m,n}(-1) = 0.\}$$

Therefore, $\mathcal{F}^\perp \subset \ker \mathbf{B}$, which yields the desired result; see [8], proof of Theorem 3.1, or [9], proof of Theorem 10.1, for further details.

Now we construct a convenient orthogonal basis in $\mathcal{F}$. It is enough to choose a basis in each component $\mathcal{F}_{\gamma_{m,n}}$. To simplify notation, in calculations below we drop the indices m, n.

The functions $u^{\pm}(x) = e^{-\gamma|x\mp 1|}$ form a linear basis in $\mathcal{F}_\gamma$. We have

$$\|u^{\pm}\|_\gamma^2 = 2\gamma, \qquad (u^+, u^-)_\gamma = 2\gamma e^{-2\gamma}$$

where the norm and the scalar product are taken in H^1_γ, see (2.5). Let now

$$v^+ := \frac{u^+ + \varkappa u^-}{\|u^+ + \varkappa u^-\|_\gamma}, \quad v^- := \frac{u^- + \varkappa u^+}{\|u^- + \varkappa u^+\|_\gamma}, \tag{4.6}$$

for a constant $\varkappa$. These are normalized and are orthogonal in H^1_γ if and only if $\varkappa^2 + 2e^{2\gamma}\varkappa + 1 = 0$. We choose the root

$$\varkappa = -e^{2\gamma} + \sqrt{e^{4\gamma} - 1} = -\frac{1}{2}e^{-2\gamma}(1 + O(e^{-4\gamma})); \tag{4.7}$$

then

$$\rho^2 := \|u^{\pm} + \varkappa u^{\mp}\|_\gamma^2 = 2\gamma(1 + \varkappa^2 + 2\varkappa e^{-2\gamma}) = 2\gamma(1 + O(e^{-4\gamma})). \tag{4.8}$$

Also, using the equation for $\varkappa$, we find

$$v^+(1) = v^-(-1) = \widehat{\rho}^{\,-1}, \qquad v^+(-1) = v^-(1) = -\varkappa\widehat{\rho}^{\,-1}$$

where

$$\widehat{\rho} = \rho(1 - e^{-4\gamma})^{-1/2}.$$

Below we indicate the dependence of γ, $\varkappa$ and ρ on m, n. In particular, we write $v^{\pm}_{m,n}$. Note that by (4.7), (4.8) we have

$$\varkappa_{m,n} = -\frac{e^{-2\gamma_{m,n}}}{2}\left(1 + O(e^{-4\gamma_{m,n}})\right), \qquad \rho_{m,n} = \sqrt{2\gamma_{m,n}}\left(1 + O(e^{-4\gamma_{m,n}})\right). \tag{4.9}$$

Let $U \sim \{C^+_{m,n}v^+_{m,n} + C^-_{m,n}v^-_{m,n}\} \in \mathcal{F}$, then the mapping

$$U \mapsto \mathcal{C} = \{C^+_{m,n}, C^-_{m,n}\}$$

is an isometry of $\mathcal{F}$ onto the Hilbert space $\mathcal{G} = \ell^2(\mathbb{N}_0^2 \setminus \{(0,0)\})$. We denote by $\mathcal{G}^{\pm}$ the subspaces in $\mathcal{G}$, formed by the elements

$$\mathcal{C}^+ = \{C^+_{m,n}, 0\}, \qquad \mathcal{C}^- = \{0, C^-_{m,n}\}$$

respectively. On $\mathbf{d}^\circ$ the quadratic forms $\mathbf{b}_\pm$ become

$$\mathbf{b}_+[U] = \mathbf{b}'_+[\mathcal{C}]$$
$$= \sum_{m+n>0} \frac{\sqrt{2m}}{\widehat{\rho}_{m,n}\widehat{\rho}_{m-1,n}} \operatorname{Re}[(C^+_{m,n} - \varkappa_{m,n}C^-_{m,n})\overline{(C^+_{m-1,n} - \varkappa_{m-1,n}C^-_{m-1,n})}],$$
$$\mathbf{b}_-[U] = \mathbf{b}'_-[\mathcal{C}]$$
$$= \sum_{m+n>0} \frac{\sqrt{2n}}{\widehat{\rho}_{m,n}\widehat{\rho}_{m,n-1}} \operatorname{Re}[(C^-_{m,n} - \varkappa_{m,n}C^+_{m,n})\overline{(C^-_{m,n-1} - \varkappa_{m,n-1}C^+_{m,n-1})}],$$

and the quadratic form $\mathbf{a}^\circ_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ becomes

$$\mathbf{a}'_{\boldsymbol{\alpha},\boldsymbol{\nu}}[\mathcal{C}] = \|\mathcal{C}\|^2_{\mathcal{G}} + \alpha_+\mathbf{b}'_+[\mathcal{C}] + \alpha_-\mathbf{b}'_-[\mathcal{C}].$$

Denote by $\mathbf{B}'_\pm$ the operators in $\mathcal{G}$ associated with the quadratic forms $\mathbf{b}'_\pm$; then the operator associated with $\mathbf{a}'_{\boldsymbol{\alpha},\boldsymbol{\nu}}$ is $\mathbf{I} + \alpha_+\mathbf{B}'_+ + \alpha_-\mathbf{B}'_-$.

It follows from this construction and (4.4), (4.5) that

$$N_-(r_{0,0}; \mathbf{A}^\circ_{\boldsymbol{\alpha};\boldsymbol{\nu}}) = N_-(0; \mathbf{I} + \alpha_+\mathbf{B}'_+ + \alpha_-\mathbf{B}'_-) = N_+(1; -\alpha_+\mathbf{B}'_+ - \alpha_-\mathbf{B}'_-). \tag{4.10}$$

Consider now the case when one of the parameters $\alpha_\pm$ is equal to zero. Below we denote

$$\boldsymbol{\alpha}_+ = (\alpha_+, 0), \qquad \boldsymbol{\alpha}_- = (0, \alpha_-).$$

For $\boldsymbol{\alpha} = \boldsymbol{\alpha}_\pm$ the equality (4.10) can be re-written in the standard form of the Birman – Schwinger principle:

$$N_-(r_{0,0}; \mathbf{A}^\circ_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}}) = N_+(\alpha_+^{-1}; -\mathbf{B}'_+), \qquad N_-(r_{0,0}; \mathbf{A}^\circ_{\boldsymbol{\alpha}_-;\boldsymbol{\nu}}) = N_+(\alpha_-^{-1}; -\mathbf{B}'_-). \tag{4.11}$$

4.3. **Structure of the operators $\mathbf{B}'_\pm$.** Denote by $\mathbf{b}''_\pm$ the leading terms in the expressions for $\mathbf{b}'_\pm$, i.e.

$$\mathbf{b}''_+[\mathcal{C}] = \mathbf{b}''_+[\mathcal{C}^+] = \sum_{m+n>0} \frac{\sqrt{2m}}{\widehat{\rho}_{m,n}\widehat{\rho}_{m-1,n}} \operatorname{Re}(C^+_{m,n}\overline{C^+_{m-1,n}}),$$
$$\mathbf{b}''_-[\mathcal{C}] = \mathbf{b}''_-[\mathcal{C}^-] = \sum_{m+n>0} \frac{\sqrt{2n}}{\widehat{\rho}_{m,n}\widehat{\rho}_{m,n-1}} \operatorname{Re}(C^-_{m,n}\overline{C^-_{m,n-1}}).$$

Let $\mathbf{B}''_\pm$ stand for the corresponding operators in $\mathcal{G}^\pm$.

Now we are in a position to explain the scheme of our further analysis. It is natural to expect that the number $N_-(0; \mathbf{I} + \alpha_+\mathbf{B}'_+ + \alpha_-\mathbf{B}'_-)$, is close to $N_-(0; (\mathbf{I}_+ + \alpha_+\mathbf{B}''_+) \oplus (\mathbf{I}_- + \alpha_-\mathbf{B}''_-))$. Indeed, consider the operator

$$\begin{aligned} \mathbf{X}_{\boldsymbol{\alpha}} &:= (\mathbf{I} + \alpha_+\mathbf{B}'_+ + \alpha_-\mathbf{B}'_-) - (\mathbf{I}_+ + \alpha_+\mathbf{B}''_+) \oplus (\mathbf{I}_- + \alpha_-\mathbf{B}''_-) \\ &= \alpha_+(\mathbf{B}'_+ - (\mathbf{B}''_+ \oplus 0)) + \alpha_-(\mathbf{B}'_- - (0 \oplus \mathbf{B}''_-)), \end{aligned} \tag{4.12}$$

then

$$(\mathbf{X}_{\boldsymbol{\alpha}}\mathcal{C},\mathcal{C})_{\mathcal{G}} = \alpha_+(\mathbf{b}'_+[\mathcal{C}] - \mathbf{b}''_+[\mathcal{C}^+]) + \alpha_-(\mathbf{b}'_-[\mathcal{C}] - \mathbf{b}''_-[\mathcal{C}^-]).$$

This quadratic form is expressed by a sum of terms with exponentially decaying coefficients, and adding this sum cannot affect the asymptotic behaviour of the function N_-. Further, the behaviour of N_- for the operator involving $\mathbf{B}''_\pm$ is easy to understand, due to its special structure.

So, our immediate task is to take care of the errors coming from the difference $\mathbf{b}'_\pm[\mathcal{C}] - \mathbf{b}''_\pm[\mathcal{C}]$. Each term in these quadratic forms involves at least one of the factors $\varkappa_{m,n}, \varkappa_{m-1,n}, \varkappa_{m,n-1}$. We have

$$\gamma_{m,n} = \sqrt{\nu_+^2 m + \nu_-^2 n} \geq \delta'\sqrt{m+n}, \qquad \delta' = \min(\nu_+,\nu_-).$$

Note also that by (4.9) the factors $\sqrt{2m}(\widehat{\rho}_{m,n}\widehat{\rho}_{m-1,n})^{-1}$ and $\sqrt{2n}(\widehat{\rho}_{m,n}\widehat{\rho}_{m,n-1})^{-1}$ appearing in the expressions for $\mathbf{b}'_\pm, \mathbf{b}''_\pm$ are bounded uniformly in m,n. Taking this into account, applying the Cauchy – Schwartz inequality, and using the asymptotic result (4.9) for $\varkappa_{m,n}$, we come to the inequality

$$|\mathbf{b}'_\pm[\mathcal{C}] - \mathbf{b}''_\pm[\mathcal{C}]| \leq c \sum_{m+n>0} e^{-2\delta\sqrt{m+n}}|C^\pm_{m,n}|^2,$$

with some $c<\infty$ and a positive $\delta<\delta'$. Now it follows from the variational principle that the consecutive eigenvalues of the operator $|\mathbf{X}_{\boldsymbol{\alpha}}|$ do not exceed the numbers $c\max(\alpha_+,\alpha_-)e^{-2\delta\sqrt{m+n}}$, repeated twice and then rearranged in decreasing order. Hence, given an $\varepsilon>0$, we derive an estimate, uniform in $\alpha_\pm \leq \nu_\pm\sqrt{2}$:

$$N_+(\varepsilon;|\mathbf{X}_{\boldsymbol{\alpha}}|) \leq \#\left\{(m,n)\in\mathbb{N}^2 : C_0 e^{-2\delta\sqrt{m+n}} > \varepsilon\right\} \leq R\log^4(K/\varepsilon), \tag{4.13}$$

with some $R,K>0$. Note that another way to obtain this inequality is based on the connection between the eigenvalues and the approximation numbers of a compact operator, see [2].

Now, let us consider the operator $\mathbf{A}_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}}$. Since $\alpha_-=0$, the variable q_- can be separated and the operator decomposes into the orthogonal sum (see (1.5) in [3])

$$\mathbf{A}_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}} = \sum_{n\in\mathbb{N}_0}^{\oplus}(\mathbf{A}_{\alpha_+;\nu_+} + \nu_-^2(n+1/2)).$$

This decomposition yields

$$N_-(r_{0,0};\mathbf{A}_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}}) = \sum_{n\in\mathbb{N}_0} N_-(\nu_+^2/2 - \nu_-^2 n;\mathbf{A}_{\alpha_+;\nu_+}). \tag{4.14}$$

For $\alpha_+ \leq \nu_+\sqrt{2}$ the operator $\mathbf{A}_{\alpha_+;\nu_+}$ is non-negative (see [7]), therefore the sum in (4.14) has only a finite number of non-zero terms. Besides, the terms corresponding to any $n>0$, are finite, since the essential spectrum of $\mathbf{A}_{\alpha_+,\nu_+}$

is $[\nu_+^2,\infty)$. Taking into account that by (4.3) the asymptotic behaviour as $\alpha_+\to\nu_+\sqrt{2}$ of the function $N_-(r_{0,0};.)$ for the operators $\mathbf{A}_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}}$ and $\mathbf{A}^\circ_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}}$ is the same, we conclude from (1.9) that

$$N_-(r_{0,0};\mathbf{A}^\circ_{\boldsymbol{\alpha}_+;\boldsymbol{\nu}})\sim N_-(\nu_+^2/2;\mathbf{A}_{\alpha_+;\nu_+})\sim\frac{1}{4\sqrt{2(\mu_+-1)}},\qquad \mu_+\downarrow 1.$$

From the last equality and (4.11) we derive that

$$N_-(0;\mathbf{I}+\alpha_+\mathbf{B}'_+)=N_+(\alpha_\pm^{-1};-\mathbf{B}'_+)\sim\frac{1}{4\sqrt{2(\mu_+-1)}},\qquad \mu_\pm\downarrow 1.$$

The analogous equality is valid for the operator $\mathbf{B}'_-$.

The same asymptotic formula holds for the operators $\mathbf{B}''_\pm$:

$$N_-(0;\mathbf{I}_\pm+\alpha_\pm\mathbf{B}''_\pm)\sim\frac{1}{4\sqrt{2(\mu_+-1)}},\qquad \mu_\pm\downarrow 1. \tag{4.15}$$

This follows (for the 'plus' sign, say) from the evident equality

$$N_-(0;\mathbf{I}_+ +\alpha_+\mathbf{B}''_+)=N_-(0;\mathbf{I}+\alpha_+\mathbf{B}''_+\oplus 0)$$

and from the estimate (4.13) for the case $\alpha_-=0$.

5. Proof of Theorem 1.5, part 2

The proof is based upon (4.10) and the equality

$$\mathbf{I}+\alpha_+\mathbf{B}'_+ +\alpha_-\mathbf{B}'_- =(\mathbf{I}_+ +\alpha_+\mathbf{B}''_+)\oplus(\mathbf{I}_- +\alpha_-\mathbf{B}''_-)+\mathbf{X}_{\boldsymbol{\alpha}}$$

where the last term is given by (4.12).

Set $\eta_\pm:=\mu_\pm-1=\frac{\nu_\pm\sqrt{2}}{\alpha_\pm}-1$ and $M=(4\sqrt{2})^{-1}$. Then (4.15) means that there exist two non-negative functions $\varphi_\pm(\mu_\pm)$, defined for $\mu_\pm>1$, vanishing as $\mu_\pm\to 1$ and such that

$$\left|N_-(0;\mathbf{I}_\pm+\alpha_\pm\mathbf{B}''_\pm)-M(\mu_\pm-1)^{-1/2}\right|\le\varphi_\pm(\mu_\pm)(\mu_\pm-1)^{-1/2}. \tag{5.1}$$

To determine the asymptotic behaviour of $N_-(0;\mathbf{I}+\alpha_+\mathbf{B}'_+ +\alpha_-\mathbf{B}'_-)$, we have to estimate the smallest co-dimension of subspaces in $\mathcal{G}$ on which

$$\|\mathcal{C}^+\|^2_{\mathcal{G}}+\|\mathcal{C}^-\|^2_{\mathcal{G}}+\alpha_+\mathbf{b}''[\mathcal{C}^+]+\alpha_-\mathbf{b}''[\mathcal{C}^-]+(\mathbf{X}_{\boldsymbol{\alpha}}\mathcal{C},\mathcal{C})_{\mathcal{G}}\ge 0 \tag{5.2}$$

for all $\mathcal{C}$. By (4.13), for any $\varepsilon>0$ there exists a subspace $\mathcal{K}(\varepsilon)\subset\mathcal{G}$ such that

$$\operatorname{codim}\mathcal{K}(\varepsilon)\le R\log^4(K/\varepsilon), \tag{5.3}$$

and for all $\mathcal{C}\in\mathcal{K}(\varepsilon)$

$$|(\mathbf{X}_{\boldsymbol{\alpha}}\mathcal{C},\mathcal{C})_{\mathcal{G}}|\le\varepsilon\|\mathcal{C}\|^2_{\mathcal{G}}=\varepsilon(|\mathcal{C}^+\|^2_{\mathcal{G}}+|\mathcal{C}^-\|^2_{\mathcal{G}}). \tag{5.4}$$

Choose $\varepsilon\in(0,1)$ to be such that $\alpha_\pm/(1-\varepsilon)<\sqrt{2}\nu_\pm$, or equivalently,

$$\eta_\pm>\varepsilon\mu_\pm. \tag{5.5}$$

Let $\mathcal{L}_\pm(\varepsilon)$ be subspaces of $\mathcal{G}_\pm$ of co-dimension $N_-(0;\mathbf{I}_\pm+\frac{\alpha_\pm}{1-\varepsilon}\mathbf{B}''_\pm)$ which are such that

$$\|\mathcal{C}^\pm\|^2_{\mathcal{G}}+\frac{\alpha_\pm}{1-\varepsilon}\mathbf{b}''_\pm[\mathcal{C}^\pm]\geq 0,\qquad \forall\mathcal{C}^\pm\in\mathcal{L}_\pm(\varepsilon).$$

Then, for $\mathcal{C}\in(\mathcal{L}_+(\varepsilon)\oplus\mathcal{L}_-(\varepsilon))\cap\mathcal{K}(\varepsilon)$ the inequality (5.2) is satisfied. It follows that

$$\begin{aligned}&F(\eta_+,\eta_-):=N_-(0;\mathbf{I}+\alpha_+\mathbf{B}'_++\alpha_-\mathbf{B}'_-)\\ &\leq N_-(0;\mathbf{I}_++\frac{\alpha_+}{1-\varepsilon}\mathbf{B}''_+)+N_-(0;\mathbf{I}_-+\frac{\alpha_-}{1-\varepsilon}\mathbf{B}''_-)+R\log^4(K/\varepsilon).\end{aligned}\tag{5.6}$$

By (5.1), this gives

$$\begin{aligned}&F(\eta_+,\eta_-)\leq\{M+\varphi_+((1-\varepsilon)\mu_+)\}(\eta_+-\varepsilon\mu_+)^{-1/2}\\ &+\{M+\varphi_-((1-\varepsilon)\mu_-)\}(\eta_--\varepsilon\mu_-)^{-1/2}+R\log^4(K/\varepsilon).\end{aligned}\tag{5.7}$$

The inequalities (5.5) guarantee that the estimate (5.1) with $\mu_\pm$ replaced by $(1-\varepsilon)\mu_\pm$ and, correspondingly, $\eta_\pm$ replaced by $\eta_\pm-\varepsilon\mu_\pm$ is still valid.

Now we choose ε, keeping in mind to optimize the right-hand side in (5.7). Let $\Psi(t)$ be a function described in (1.10). Since $\psi(t)=o(t^{-1/4})$, on choosing

$$\varepsilon=\varepsilon(\mu_+,\mu_-)=\frac{1}{2}\Psi\left(\min\{\frac{\eta_+}{\mu_+},\frac{\eta_-}{\mu_-}\}\right),$$

we find that the inequalities (5.5) are satisfied. For if $\eta_+/\mu_+\leq\eta_-/\mu_-$, then

$$\varepsilon<\Psi(\eta_+/\mu_+)\leq\Psi(\eta_-/\mu_-)\leq\eta_+/\mu_+\leq\eta_-/\mu_-.$$

Also, $\varepsilon=o(\eta_\pm)$ as $\eta_\pm\to 0$.

Introduce the function

$$\varphi(\mu_+,\mu_-)=\varphi_+((1-\varepsilon)\mu_+)\frac{\eta_+^{1/2}}{(\eta_+-\varepsilon\mu_+)^{1/2}}+\varphi_-((1-\varepsilon)\mu_-)\frac{\eta_-^{1/2}}{(\eta_--\varepsilon\mu_-)^{1/2}}.$$

It is well-defined for $(1-\mu_+^{-1},1-\mu_-^{-1})\in\Omega_\Psi$ and $\varphi(\mu_+,\mu_-)\to 0$ as $\mu_\pm\to 1$. The inequality (5.7) turns into

$$F(\eta_+,\eta_-)\leq M\eta_+^{-1/2}+M\eta_-^{-1/2}+\varphi(\mu_+,\mu_-)(\eta_+^{-1/2}+\eta_-^{-1/2})+R\log^4(K/\varepsilon).$$

By (1.10), the last term here is $o(\eta_\pm^{-1/2})$ and so

$$F(\eta_+,\eta_-)\leq M\eta_+^{-1/2}+M\eta_-^{-1/2}+\Phi(\mu_+,\mu_-)(\eta_+^{-1/2}+\eta_-^{-1/2})$$

where Φ is a bounded function, defined on the same domain as φ and having the same properties. The estimate is uniform for $(\eta_+/\mu_+,\eta_-/\mu_-)\in\Omega_\Psi$.

To obtain the lower estimate we again choose $\mathcal{K}(\varepsilon)$ as in (5.3). There is a subspace $\mathcal{L}(\varepsilon)$ of $\mathcal{G}$ of co-dimension $N_-(0;\mathbf{I}+\alpha_+\mathbf{B}'_++\alpha_-\mathbf{B}'_-)$ which is such that

$$\|\mathcal{C}\|^2_{\mathcal{G}}+\alpha_+\mathbf{b}'_+[\mathcal{C}]+\alpha_-\mathbf{b}'_-[\mathcal{C}]\geq 0,\qquad\forall\mathcal{C}\in\mathcal{K}(\varepsilon).$$

Then, for $\mathcal{C} \in \mathcal{G}(\varepsilon) \cap \mathcal{K}(\varepsilon)$,

$$\|\mathcal{C}\|_{\mathcal{G}}^2 + \alpha_+ \mathbf{b}''_+[\mathcal{C}] + \alpha_- \mathbf{b}''_-[\mathcal{C}] \geq \varepsilon \|\mathcal{C}\|_{\mathcal{G}}^2.$$

It follows that

$$\begin{aligned} &N_-(0; \mathbf{I}_+ + \frac{\alpha_+}{1+\varepsilon}\mathbf{B}''_+) + N_-(0; \mathbf{I}_- + \frac{\alpha_-}{1+\varepsilon}\mathbf{B}''_-) \\ &\leq N_-(0; \mathbf{I} + \alpha_+ \mathbf{B}'_+ + \alpha_- \mathbf{B}'_-) + R \log^4(K/\varepsilon), \end{aligned}$$

and so

$$F(\eta_+, \eta_-) \geq N_-(0; \mathbf{I}_+ + \frac{\alpha_+}{1+\varepsilon}\mathbf{B}''_+) + N_-(0; \mathbf{I}_- + \frac{\alpha_-}{1+\varepsilon}\mathbf{B}''_-) - R \log^4(K/\varepsilon).$$

The rest of the argument is the same as for the upper estimate. Actually it is easier, for if $\mu_\pm(\varepsilon) := \sqrt{2}\nu_\pm(1+\varepsilon)/\alpha_\pm$, then $(1-\mu_+^{-1}(\varepsilon), 1-\mu_-^{-1}(\varepsilon))$ automatically lies in Ω_Ψ if $(1-\mu_+^{-1}, 1-\mu_-^{-1})$ does, and $\mu_\pm(\varepsilon) \to 1$ as $\mu_\pm \to 1$.

All in all we have therefore shown that there exists a bounded function $\Phi(\mu_+, \mu_-)$ on Ω_Ψ which vanishes as $(\mu_+, \mu_-) \to (1,1)$ and such that, uniformly for $(\eta_+/\mu_+, \eta_-/\mu_-) \in \Omega_\Psi$,

$$\left| N_-(r_{0,0}; \mathbf{A}'_{\boldsymbol{\alpha},\boldsymbol{\nu}}) - M\eta_+^{-1/2} - M\eta_-^{-1/2} \right| \leq \Phi(\mu_+, \mu_-) \left(\eta_+^{-1/2} + \eta_-^{-1/2} \right).$$

The proof of Theorem 1.5 is therefore complete.

6. Acknowledgments

The work on the paper started in the Summer of 2004 when one of the authors (M.S.) was a guest of the School of Mathematics, Cardiff University. M.S. takes this opportunity to express his gratitude to the University for its hospitality and to the EPSRC for financial support under grant GR/T01556.

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