High energy asymptotics and trace formulae for the perturbed harmonic oscillator

Alexander Pushnitski* and Ian Sorrell†

8 May 2005

Abstract

A one-dimensional quantum harmonic oscillator perturbed by a smooth compactly supported potential is considered. For the corresponding eigenvalues λ_n, a complete asymptotic expansion for large n is obtained, and the coefficients of this expansion are expressed in terms of the heat invariants. A sequence of trace formulas is obtained, expressing regularised sums of integer powers of eigenvalues λ_n in terms of the heat invariants.

Keywords: Harmonic oscillator, trace formulae, heat invariants, eigenvalue asymptotics.

1 Introduction and main results

1. Local heat invariants. In order to state our main results, we need to recall the notion of *local heat invariants.* Let $v \in C^\infty(\mathbb{R})$ be any real valued function such that v and all derivatives of v are uniformly bounded on $\mathbb{R}$. For the self-adjoint operator $h = -\frac{d^2}{dx^2} + v$ in $L^2(\mathbb{R}, dx)$, consider its heat kernel $e^{-th}(x, y)$, $t > 0$, $x, y \in \mathbb{R}$, i.e. the integral kernel of the operator e^{-th}. For any $x \in \mathbb{R}$, one has the asymptotic expansion

$$e^{-th}(x,x) \sim \frac{1}{\sqrt{4\pi t}} \sum_{j=0}^{\infty} t^j a_j[v(x)], \quad t \to +0, \tag{1.1}$$

where $a_j[v(x)]$ are polynomials in v and derivatives of v, known as the local heat invariants of h. Explicit formula for $a_j[v(x)]$ is available:

$$a_j[v(x)] = \sum_{k=0}^{j-1} \frac{(-1)^j \Gamma(j+\frac{1}{2})}{4^k k!(k+j)!(j-k)!\,\Gamma(k+\frac{3}{2})} (-\tfrac{d^2}{dy^2} + v(y))^{k+j} (|x-y|^{2k}) \,|_{y=x} \,. \tag{1.2}$$

*Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, U.K. email: a.b.pushnitski@lboro.ac.uk

†Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, U.K. email: i.sorrell@lboro.ac.uk

Formula (1.2) was derived in [7] on the basis of the results of [9, 10]; see also references in [9] to earlier works on this subject. From (1.2) or otherwise, one obtains

$$
\begin{aligned}
&a_0[v(x)] = 1, \quad a_1[v(x)] = -v(x), \quad a_2[v(x)] = \frac{1}{2}v^2(x) - \frac{1}{6}v''(x),\\
&a_3[v(x)] = -\tfrac{1}{6}v^3 + \tfrac{1}{6}vv'' + \tfrac{1}{12}v'^2 - \tfrac{1}{60}v^{(4)},\\
&a_4[v(x)] = \tfrac{1}{24}v^4 + \tfrac{1}{30}v'v''' + \tfrac{1}{60}vv^{(4)} + \tfrac{1}{40}(v'')^2 - \tfrac{1}{840}v^{(6)} - \tfrac{1}{12}v''v^2 - \tfrac{1}{12}v(v')^2.
\end{aligned}
$$

2. Perturbed harmonic oscillator. Consider the self-adjoint operators

$$
H_0 = -\frac{d^2}{dx^2} + x^2 \text{ and } H = -\frac{d^2}{dx^2} + x^2 + q(x) \text{ in } L^2(\mathbb{R}, dx), \text{ where } q \in C_0^\infty(\mathbb{R}).
$$

These operators can be defined as the closures of the symmetric operators, defined on $C_0^\infty(\mathbb{R})$ by the same differential expressions. Denote by $\lambda_n^0 = 2n-1$, $n = 1, 2, \dots$ the eigenvalues of H_0 and by $\lambda_1 < \lambda_2 < \cdots$ the eigenvalues of H. The aims of this paper are (i) to describe the asymptotic expansion of λ_n as $n \to \infty$, including explicit formulas for the coefficients of this expansion in terms of the local heat invariants; (ii) to derive a series of identities (trace formulas) which relate regularized sums of the type $\sum_{n=1}^\infty \lambda_n$, $\sum_{n=1}^\infty \lambda_n^2$, etc. to some explicit integrals involving heat invariants.

Our results are modelled on the Gel'fand-Levitan-Dikiĭ trace formulae for the Sturm-Liouville operator (see [5, 6, 2, 3] or [4]) and in part motivated by the recent advances in calculation of the heat invariants [9, 10].

First, as a preliminary result, we establish the asymptotic expansion

$$
\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \sim \frac{1}{\sqrt{4\pi t}} \sum_{j=1}^\infty t^j \int_{\mathbb{R}} (a_j[x^2 + q(x)] - a_j[x^2])dx, \quad t \to +0, \tag{1.3}
$$

where a_j are the local heat invariants. In formula (1.3) (as elsewhere in this paper) $q \in C_0^\infty(\mathbb{R})$ and thus all the integrals in the r.h.s. converge. On the formal level, (1.3) follows by subtracting (1.1) with $v(x) = x^2$ from (1.1) with $v(x) = x^2 + q(x)$ and integrating over x. A rigourous justification of this formal procedure is not difficult and will be given in Section 3.

3. High energy asymptotics. Suppose that q is given. Due to the explicit formula (1.2), we can regard the integrals appearing in the r.h.s. of (1.3) as known quantities. Below we describe the asymptotics of eigenvalues λ_n in terms of these integrals. Here is our main result:

Theorem 1.1. *(i) One has the asymptotic expansion*

$$
\lambda_n \sim \lambda_n^0 + \sum_{j=1}^\infty \frac{c_j}{(\lambda_n^0)^{j/2}}, \quad n \to \infty, \tag{1.4}
$$

with some coefficients $c_j \in \mathbb{R}$.

(ii) The coefficients c_j in (1.4) *can be calculated in the following way. Consider the formal asymptotic expansion*

$$\lambda_n^0 \sim \lambda_n + \sum_{j=1}^{\infty} \frac{b_j}{(\lambda_n)^{j-\frac{1}{2}}}, \quad n \to \infty, \tag{1.5}$$

with the coefficients

$$b_j = (\sqrt{\pi}\Gamma(\tfrac{3}{2} - j))^{-1} \int_{\mathbb{R}} (a_j[x^2 + q(x)] - a_j[x^2])dx. \tag{1.6}$$

Then inverting the formal asymptotic series (1.5) *gives* (1.4).

Remark 1.2. 1. Theorem 1.1 gives an algorithm of computing the 'unknown' coefficients c_j in the expansion (1.4) in terms of the 'known' integrals (1.6). The algorithm is given in the form of inverting an asymptotic series, which is a well defined algebraic procedure.

In order to compute a coefficient c_j, one needs to know finitely many coefficients b_j. For example,

$$c_1 = -b_1, \quad c_2 = 0, \quad c_3 = -b_2, \quad c_4 = -\frac{1}{2}b_1^2, \quad c_5 = -b_3, \quad c_6 = -2b_1b_2, \quad c_7 = \frac{1}{8}b_1^3 - b_4.$$

2. The fact that only half-integer (and not whole integer) negative powers of λ_n are present in the r.h.s. of (1.5) is equivalent to a series of identities for the coefficients c_j. For example, the first three identities of this type are

$$c_2 = 0, \quad c_1^2 + 2c_4 = 0, \quad c_6 + c_2^2 + 2c_1c_3 = 0.$$

3. From Theorem 1.1(ii) we obtain, in particular,

$$c_1 = \frac{1}{\pi}\int_{\mathbb{R}} q(x)dx, \quad c_2 = 0, \quad c_3 = \frac{1}{\pi}\int_{\mathbb{R}} q(x)x^2dx + \frac{1}{2\pi}\int q^2(x)dx,$$

$$c_4 = -\frac{1}{2}c_1^2, \quad c_5 = \frac{1}{16\pi}\int_{-\infty}^{\infty}(q^3(x) + 3q^2(x)x^2 + 3q(x)x^4 + \frac{1}{2}(q'(x))^2 + 2q(x))dx.$$

4. Trace formulas. As a by-product of our construction, we also obtain trace formulas for the eigenvalues λ_n and λ_n^0. This result is a direct analogue of the trace formulas for the Sturm-Liouville problem due to [5, 2, 3, 6] and our proof follows the reasoning of [3]. Let us introduce the Zeta functions

$$Z(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}, \quad Z_0(s) = \sum_{n=1}^{\infty} (\lambda_n^0)^{-s}, \quad \operatorname{Re} s > 1. \tag{1.7}$$

If $\lambda_n < 0$ for some n, then λ_n^{-s} should be understood as $|\lambda_n|^{-s}e^{-i\pi s}$. If $\lambda_n = 0$ for some n, then the corresponding term in the sum $\sum_{n=1}^{\infty} \lambda_n^{-s}$ is omitted.

Due to the explicit formula $\lambda_n^0 = 2n - 1$, we have $Z_0(s) = (1-2^{-s})\zeta(s)$, where $\zeta(s)$ is the Riemann Zeta function. By the properties of ζ, we conclude that $Z_0(s)$ has a meromorphic continuation into the whole complex plane with the only pole at $s = 1$, and this pole is simple. The real zeros of Z_0 are at $s = -2n$, $n = 0, 1, 2, \dots$.

Theorem 1.3. *The function $Z(s)$ admits meromorphic continuation into the whole complex plane. Its poles are simple and located at $s = 1$ and at $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. We have the identities:*

$$Z(-k) = Z_0(-k), \quad k \in \mathbb{N}. \tag{1.8}$$

As in [2, 3], formula (1.8) can be combined with the asymptotic expansion (1.4) to obtain regularised trace identities as follows. Exponentiating the asymptotics (1.4), we obtain for any $\operatorname{Re} s > 1$:

$$\lambda_n^{-s} \sim \sum_{j=0}^{\infty} d_j(s)(\lambda_n^0)^{-s-(j/2)}, \quad n \to \infty, \tag{1.9}$$

where $d_j(s)$ are explicit polynomials in s and c_j. For example,

$$d_0(s) = 1, \quad d_1(s) = d_2(s) = 0, \quad d_3(s) = -sc_1, \quad d_4(s) = -sc_2, \quad d_5(s) = -sc_3,$$
$$d_6(s) = -sc_4 + \frac{s(s+1)}{2}c_1^2, \quad d_7(s) = -sc_5 + s(s+1)c_1c_2.$$

Using this notation, we have for any $k \in \mathbb{N}$:

$$Z(s) = \sum_{n=0}^{\infty}\{\lambda_n^{-s} - \sum_{j=0}^{2k+2} d_j(s)(\lambda_n^0)^{-s-(j/2)}\} + \sum_{j=0}^{2k+2} d_j(s)Z_0(s + (j/2)), \quad \operatorname{Re} s > 1. \tag{1.10}$$

Now both sides of (1.10) can be meromorphically continued into the half-plane $\operatorname{Re} s > -k - \frac{1}{2}$. By Theorem 1.3, the l.h.s. of (1.10) is analytic at $s = -k$. By (1.9), the same applies to the first term in the r.h.s. of (1.10). Thus, the second term in the r.h.s. of (1.10) is also analytic at $s = -k$. As $Z_0(s)$ has a pole at $s = 1$ (and no other poles), it follows that $d_{2k+2}(-k) = 0$. Thus, we obtain

$$Z(-k) = \sum_{n=0}^{\infty}\{\lambda_n^{k} - \sum_{j=0}^{2k+1} d_j(-k)(\lambda_n^0)^{k-(j/2)}\} + \sum_{j=0}^{2k+1} d_j(-k)Z_0(-k + (j/2)).$$

Combined with (1.8), this yields a series of formulas

$$\sum_{n=0}^{\infty}\{\lambda_n^{k} - \sum_{j=0}^{2k+1} d_j(-k)(\lambda_n^0)^{k-(j/2)}\} + \sum_{j=1}^{2k+1} d_j(-k)Z_0(-k + (j/2)) = 0, \quad k \in \mathbb{N}. \tag{1.11}$$

In particular, for $k = 1, 2, 3$ we obtain (taking into account that $Z_0(0) = 0$)

$$\sum_{n=0}^{\infty}(\lambda_n - \lambda_n^0 - \frac{c_1}{\sqrt{\lambda_n^0}}) + c_1 Z_0(-\tfrac{1}{2}) = 0; \tag{1.12}$$

$$\sum_{n=0}^{\infty}(\lambda_n^2 - (\lambda_n^0)^2 - 2c_1\sqrt{\lambda_n^0} - \frac{2c_3}{\sqrt{\lambda_n^0}}) + 2c_1 Z_0(-\tfrac{1}{2}) + 2c_3 Z_0(\tfrac{1}{2}) = 0; \tag{1.13}$$

$$\sum_{n=0}^{\infty}\left(\lambda_n^3 - (\lambda_n^0)^3 - 3c_1(\lambda_n^0)^{3/2} - 3c_3(\lambda_n^0)^{1/2} - 3(c_4 + c_1^2) - 3c_5(\lambda_n^0)^{-1/2}\right)$$
$$+3c_1 Z_0(-\tfrac{3}{2}) + 3c_3 Z_0(-\tfrac{1}{2}) + 3c_5 Z_0(\tfrac{1}{2}) = 0.$$

Formulas (1.12), (1.13) (in a slightly different form) were obtained earlier in [8].

2 Proof of Theorem 1.1(ii)

The proof of part (i) of Theorem 1.1 is fairly standard and is based on the asymptotic theory of solutions to ODEs and on various explicit formulas for parabolic cylinder functions (which give the solutions to the ODE corresponding to $q = 0$). We give this proof in Sections 5-6. The proof of part (ii) of Theorem 1.1 is the core of our construction and is presented in this section. The proof is based on the following

Lemma 2.1. *Let $\lambda_n^0 = 2n - 1$, $n \in \mathbb{N}$, and let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of real numbers such that $\lambda_n = \lambda_n^0 + O(1)$ as $n \to \infty$. Suppose that an asymptotic expansion*

$$\lambda_n^0 \sim \lambda_n + \sum_{j=1}^{\infty} p_j \lambda_n^{-\alpha_j} + \sum_{j=1}^{\infty} q_j \lambda_n^{-j}, \quad n \to \infty, \tag{2.1}$$

holds true, where $0 \leqslant \alpha_1 < \alpha_2 < \cdots$ are some non-integer exponents and $\{p_j\} \subset \mathbb{R}$, $\{q_j\} \subset \mathbb{R}$. Then one has the asymptotic expansion

$$\sum_{n=1}^{\infty} e^{-t\lambda_n} \sim \frac{1}{2t} + \sum_{j} \frac{p_j}{2} \Gamma(1 - \alpha_j) t^{\alpha_j} + \frac{1}{2} \log t \sum_{j=1}^{\infty} q_j \frac{(-1)^j}{(j-1)!} t^j + \sum_{k=1}^{\infty} r_k t^k \tag{2.2}$$

as $t \to +0$, with some coefficients $\{r_k\} \subset \mathbb{R}$.

Proof of Theorem 1.1(ii). Given Lemma 2.1 and part (i) of Theorem 1.1, the proof of Theorem 1.1(ii) is immediate. Indeed, inverting the asymptotic expansion (1.4) yields the expansion of the form

$$\lambda_n^0 \sim \lambda_n + \sum_{j=1}^{\infty} b_j \lambda_n^{\frac{1}{2} - j} + \sum_{j=1}^{\infty} \widetilde{b}_j \lambda_n^{-j}, \quad n \to \infty$$

with some real coefficients $\{b_j\}$, $\{\widetilde{b}_j\}$. Now using Lemma 2.1 and the explicit formula $\sum_{n=1}^{\infty} e^{-t\lambda_n^0} = (2 \sinh t)^{-1}$, we obtain the asymptotic expansion

$$\sum_{n=1}^{\infty} (e^{-t\lambda_n} - e^{-t\lambda_n^0}) \sim \frac{1}{\sqrt{t}} \sum_{j=1}^{\infty} \frac{b_j}{2} \Gamma(\tfrac{3}{2} - j) t^j + \frac{1}{2} \log t \sum_{j=1}^{\infty} \widetilde{b}_j \frac{(-1)^j}{(j-1)!} t^j + \sum_{k=1}^{\infty} \widetilde{r}_k t^k$$

with some real coefficients $\{\widetilde{r}_k\}$. Comparing this to (1.3), we see that all coefficients $\widetilde{b}_i$ vanish and the coefficients b_j are related to the heat invariants by formulas (1.6). This completes the proof of Theorem 1.1(ii). ∎

In the rest of this section, we prove Lemma 2.1. Broadly speaking, this Lemma can be regarded as a discrete analogue of the following version of Watson's Lemma:

Lemma 2.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a locally bounded measurable function, such that $\psi(\lambda) = 0$ for all λ near $-\infty$. Suppose that ψ has the following asymptotic expansion*

$$\psi(\lambda) = \sum_j p_j \lambda^{-\alpha_j} + \sum_j q_j \lambda^{-\beta_j} + O(\lambda^{-M}) \quad \lambda \to \infty, \tag{2.3}$$

where $\{\alpha_j\} \subset \mathbb{R} \setminus \mathbb{N}$, $\{\beta_j\} \subset \mathbb{N}$, $\{p_j\} \subset \mathbb{R}$, $\{q_j\} \subset \mathbb{R}$ are finite sets and $M > \max(\{\alpha_j\} \cup \{\beta_j\})$, $M \in (0,\infty) \setminus \mathbb{N}$. Then the following asymptotic formula for the Laplace transform of ψ holds true for $t \to +0$:

$$\int_{-\infty}^{\infty} e^{-t\lambda}\psi(\lambda)d\lambda \sim \sum_i p_i \Gamma(1-\alpha_i)t^{\alpha_i - 1} + (\log t)\sum_j q_j \frac{(-1)^{\beta_j}}{(\beta_j - 1)!} t^{\beta_j - 1} + \sum_{0 \leqslant k < M-1} r_k t^k + O(t^{M-1}) \tag{2.4}$$

with some coefficients $\{r_k\}$.

The proof can be performed, for example, by explicit computation, checking that each term in the asymptotics (2.3) gives the desired contribution to (2.4).

Proof of Lemma 2.1. 1. Let

$$N(\lambda) = \sharp\{n \mid \lambda_n < \lambda\}, \quad N_0(\lambda) = \sharp\{n \mid \lambda_n^0 < \lambda\}.$$

The main idea of the proof is to approximate $N(\lambda)$ by $N_0(\psi(\lambda))$, where ψ is a function with the asymptotic expansion (2.1). We construct ψ in terms of its inverse as follows.

The formal inversion of the expansion (2.1) has the form

$$\lambda_n \sim \lambda_n^0 + \sum_{j=1}^{\infty} s_j (\lambda_n^0)^{-\eta_j}, \quad n \to \infty, \tag{2.5}$$

where $0 \leqslant \eta_1 < \eta_2 < \dots$ and $\{s_j\} \subset \mathbb{R}$. Fix some sufficiently large $M \in (0,\infty) \setminus \mathbb{N}$; we have

$$\lambda_n = \lambda_n^0 + \sum_{\eta_j < M} s_j (\lambda_n^0)^{-\eta_j} + O((\lambda_n^0)^{-M}), \quad n \to \infty. \tag{2.6}$$

Let $\phi \in C^\infty(\mathbb{R})$ be such that

(i) $\phi(\lambda) \geqslant 0$ for all $\lambda \in \mathbb{R}$ and $\phi(\lambda) = 0$ for all $\lambda \leqslant 1$;

(ii) $\phi(\lambda)$ is strictly increasing for $\lambda > 1$;

(iii) $\phi(\lambda) = \lambda + \sum_{\eta_j < M} s_j \lambda^{-\eta_j}$ for all sufficiently large $\lambda > 0$.

Let $\psi \in C^\infty(0,\infty)$ be such that $\phi(\psi(\lambda)) = \lambda \ \forall \lambda > 0$. Finally, for $\lambda > 0$ let us write $N_0(\lambda) = \frac{1}{2}\lambda + \omega(\lambda)$, where $\omega(\lambda)$ is a 2-periodic function.

With this notation we have:

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-t\lambda_n} &= \int_{-\infty}^{\infty} e^{-t\lambda} dN(\lambda) = t\int_{-\infty}^{\infty} e^{-t\lambda} N(\lambda) d\lambda \\ &= t\int_{-\infty}^{\infty} e^{-t\lambda}\left(N(\lambda) - N_0(\psi(\lambda))\right)d\lambda + \frac{1}{2}t\int_0^{\infty} e^{-t\lambda}\psi(\lambda)d\lambda + t\int_0^{\infty} e^{-t\lambda}\omega(\psi(\lambda))d\lambda \\ &=: F_1(t) + F_2(t) + F_3(t).\end{aligned}$$

Below we consider separately the integrals $F_1(t)$, $F_2(t)$ and $F_3(t)$.

2. Consider $F_2(t)$. By the construction of ψ, we have the asymptotics

$$\psi(\lambda) = \lambda + \sum_{\alpha_j < M} p_j \lambda^{-\alpha_j} + \sum_{j<M} q_j \lambda^{-j} + O(\lambda^{-M}) \quad \lambda \to \infty$$

with the same exponents and coefficients as in (2.1). By Lemma 2.2, we obtain:

$$F_2(t) \sim \frac{1}{2t} + \sum_{\alpha_j<M} \frac{p_j}{2}\Gamma(1-\alpha_j)t^{\alpha_j} + \frac{1}{2}(\log t)\sum_{j<M} q_j \frac{(-1)^j}{(j-1)!}t^j + \sum_{0<k<M} r_k t^k + O(t^M), \quad t \to +0. \tag{2.7}$$

3. Consider $F_1(t)$. By the construction of ϕ, we have $\lambda_n = \phi(\lambda_n^0) + O(n^{-M})$, $n \to \infty$, and so

$$F_1(t) = \sum_{n=1}^{\infty}(e^{-t\lambda_n} - e^{-t\phi(\lambda_n^0)}) = \sum_{n=1}^{\infty} e^{-t\lambda_n}(1 - e^{tO(n^{-M})}).$$

It follows that $F_1(t)$ has at least $[M]-1$ continuous derivatives in t on $[0,\infty)$ and therefore, by the Taylor formula,

$$F_1(t) = \sum_{0\leqslant k<[M]-1} F_1^{(k)}(0)t^k + o(t^{[M]-1}), \quad t \to +0. \tag{2.8}$$

4. Let us prove that F_3 has continuous derivatives in $t \in [0,\infty)$ of any order, and so

$$F_3(t) \sim \sum_{k=0}^{\infty} F_3^{(k)}(0)t^k, \quad t \to +0. \tag{2.9}$$

Fix $N \in \mathbb{N}$. Integrating by parts N times, we obtain

$$\begin{aligned} F_3(t) = t\int_0^\infty e^{-t\lambda}\omega(\psi(\lambda))d\lambda &= t\int_0^\infty e^{-t\phi(\mu)}\phi'(\mu)\omega(\mu)d\mu \\ &= -\int_0^\infty (e^{-t\phi(\mu)})'\omega(\mu)d\mu = (-1)^{N+1}\int_0^\infty (e^{-t\phi(\mu)})^{(N+1)}\omega_N(\mu)d\mu, \end{aligned}$$

where ω_N is a periodic function. Using the property (iii) of ϕ, we obtain

$$(e^{-t\phi(\mu)})^{(N+1)} = e^{-t\phi(\mu)}\{(-t)^{N+1}(\phi'(\mu))^{N+1} + \sum_{l=0}^{N} t^l O(\lambda^{l-N-2-\eta_1})\}.$$

It follows that F_3 has at least $N-1$ continuous derivatives on $[0,\infty)$. As $N \in \mathbb{N}$ can be taken arbitrary large, this proves the statement.

5. Combining (2.7) – (2.9), and using the fact that M can be taken arbitrary large, we get the desired statement. ∎

3 Proof of the asymptotic expansion (1.3)

1. Asymptotic expansion (1.1)**.** First let us prove that for any bounded *from below* function $v \in C^\infty(\mathbb{R})$, the asymptotic expansion (1.1), (1.2) holds true locally uniformly in $x \in \mathbb{R}$. The expansion (1.1) as such is well known, but all treatments of this expansion in the literature that we are aware of, assume boundedness of v, whereas here we have to deal with potentials of the type $x^2 + v(x)$. Below is a simple argument which shows that the boundedness from above condition can be lifted. Let us fix any $R > 0$ and prove that (1.1) holds true uniformly in $x \in [-R, R]$.

Let $\tilde v \in C_0^\infty(\mathbb{R})$ be such that $\tilde v(x) = v(x)$ for all $|x| \leqslant 4R$, and let $\tilde h = -\frac{d^2}{dx^2} + \tilde v$. The heat kernel expansion for C_0^∞-potentials is certainly well known (see e.g. [7] and references to earlier work therein), and so we have

$$e^{-t\tilde h}(x,x) \sim \frac{1}{\sqrt{4\pi t}} \sum_{j=0}^{\infty} t^j a_j[v(x)], \quad t \to +0, \quad |x| \leqslant R,$$

uniformly in $x \in [-R, R]$. Thus, it suffices to prove the estimate

$$\sup_{|x|\leqslant R} |e^{-t\tilde h}(x,x) - e^{-th}(x,x)| = O(e^{-c/t}), \quad t \to +0, \quad c > 0. \tag{3.1}$$

Let χ_R be the characteristic function of $(-R, R)$ in $\mathbb{R}$, and let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| \leqslant 2R$ and $\phi(x) = 0$ for $|x| \geqslant 3R$. Denoting $D = \frac{d}{dx}$, we obtain

$$\begin{aligned}
\chi_R(e^{-th} - e^{-t\tilde h})\chi_R &= \chi_R(e^{-th}\phi - \phi e^{-t\tilde h})\chi_R = -\int_0^t ds\, \chi_R e^{-(t-s)h}(h\phi - \phi\tilde h)e^{-s\tilde h}\chi_R \\
&= \int_0^t ds\, \chi_R e^{-(t-s)h}\phi'' e^{-s\tilde h}\chi_R + \int_0^t ds\, \chi_R e^{-(t-s)h}\phi' D e^{-s\tilde h}\chi_R \\
&= \int_0^t ds\, \chi_R e^{-(t-s)h}\phi'' e^{-s\tilde h}\chi_R + \int_0^t ds\, \chi_R e^{-(t-s)h}\phi' D e^{sD^2}\chi_R \\
&\qquad - \int_0^t ds \int_0^s ds_1\, \chi_R e^{-(t-s)h}\phi' D e^{(s-s_1)D^2}\tilde v e^{-s_1\tilde h}\chi_R.
\end{aligned}$$

From here, using the explicit formula for the heat kernel $e^{t\frac{d^2}{dx^2}}$ and the well known estimate

$$|e^{-th}(x,y)| \leqslant \frac{1}{\sqrt{4\pi t}} \exp(-\tfrac{(x-y)^2}{4t} - t\inf_{\mathbb{R}} v), \quad x, y \in \mathbb{R}, \tag{3.2}$$

we obtain (3.1).

2. Asymptotic expansion (1.3)**.** Now we are ready to prove the asymptotic expansion (1.3). Let $R > 0$ be sufficiently large so that $\operatorname{supp} q \subset (-R, R)$. Let χ_{2R} be the characteristic function of $(-2R, 2R)$ and let $\tilde\chi_{2R} = 1 - \chi_{2R}$. By the previous step of the proof, it suffices to prove that

$$\operatorname{Tr}(\tilde\chi_{2R}(e^{-tH} - e^{-tH_0})\tilde\chi_{2R}) = O(e^{-c/t}), \quad t \to +0, \quad c > 0. \tag{3.3}$$

By (3.2), we obtain

$$\|\tilde{\chi}_{2R} e^{-tH} \chi_R\|_{S_2} = O(e^{-c/t}), \quad t \to +0, \quad c > 0;$$
$$\|\chi_R e^{-tH_0} \tilde{\chi}_{2R}\|_{S_2} = O(e^{-c/t}), \quad t \to +0, \quad c > 0.$$

From these estimates and the formula

$$\tilde{\chi}_{2R}(e^{-tH} - e^{-tH_0})\tilde{\chi}_{2R} = -\int_0^t \tilde{\chi}_{2R} e^{-(t-s)H} \chi_R \, q \, \chi_R e^{-sH_0} \tilde{\chi}_{2R} \, ds$$

we get the required result (3.3).

4 Proof of Theorem 1.3

We follow the arguments of [3]. First let us assume that $\lambda_n \neq 0$ for all n. Fix $k \in \mathbb{N}$ and consider formula (1.10). The second term in the r.h.s. is meromorphic in $\mathbb{C}$ with possible poles at $s = 1 - \frac{j}{2}$, $j = 0, 1, 2, \dots, 2k + 2$. The first term in the r.h.s. of (1.10) admits analytic continuation into the half-plane $\operatorname{Re} s > -k - \frac{1}{2}$. As k can be taken arbitrary large, it follows that Z admits a meromorphic continuation into the whole complex plane, all poles of Z are simple and located at the points $s = 1 - \frac{j}{2}$, $j = 0, 1, 2, \dots$.

Next, from the formula

$$e^{-t\lambda_n} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (t\lambda_n)^{-s} \Gamma(s) ds, \quad \gamma > 0, \quad (t\lambda_n) \in \mathbb{R} \setminus \{0\},$$

we get

$$\sum_{n=1}^{\infty} (e^{-t\lambda_n} - e^{-t\lambda_n^0}) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (Z(s) - Z_0(s)) t^{-s} \Gamma(s) ds, \quad t > 0, \quad \gamma > 0.$$

By a standard argument involving shifting the contour of integration to the left, the last formula yields the following asymptotic expansion as $t \to +0$:

$$\sum_{n=1}^{\infty} (e^{-t\lambda_n} - e^{-t\lambda_n^0}) \sim \sum_{j=0}^{\infty} \operatorname*{Res}_{s=1-(j/2)} \big((Z(s) - Z_0(s)) t^{-s} \Gamma(s)\big), \quad t \to +0. \tag{4.1}$$

First note that $Z(s) - Z_0(s)$ does not have poles at any of the points $s = 0, -1, -2, \dots$. Indeed, if $Z(s) - Z_0(s)$ did have a pole at $s = -n$ say, then $(Z(s) - Z_0(s))\Gamma(s)$ would have a double pole there and then the expansion (4.1) would involve a term $Ct^n \log t$. But by (1.3), no logarithmic terms actually occur in the asymptotic expansion.

Next, by (1.3), there are no integer powers of t in the asymptotic expansion, which by the same argument leads to the conclusion that $Z(-k) - Z_0(-k) = 0$ for all $k = 0, 1, 2, \dots$.

Finally, consider the case when one of the eigenvalues of H vanishes: $\lambda_m = 0$. Then the preceding arguments should be repeated for the sequence $\{\lambda_n\}$, $n \in \mathbb{N} \setminus \{m\}$. This leads to the same set of results, apart from the formula $Z(0) = 0$; this should be replaced by $Z(0) = -1$. ∎

5 Proof of Theorem 1.1(i)

Let us define two solutions $\psi^0_\pm = \psi^0_\pm(x,\lambda)$ of the equation $-\psi'' + x^2\psi = \lambda\psi$ by

$$\psi^0_+(x,\lambda) = U(-\frac{\lambda}{2}, x\sqrt{2}), \quad \psi^0_-(x,\lambda) = U(-\frac{\lambda}{2}, -x\sqrt{2}),$$

where U is the parabolic cylinder function (see [1, §19.3]). For any $x \in \mathbb{R}$, the solutions $\psi^0_\pm(x,\lambda)$ are entire functions of λ. For any $\lambda \in \mathbb{C}$, the solutions $\psi^0_\pm(x,\lambda)$ have the asymptotics

$$\psi^0_+(x,\lambda) = \psi^0_-(-x,\lambda) = (x\sqrt{2})^{(\lambda-1)/2} e^{-x^2/2}(1 + o(1)), \quad x \to +\infty,$$

and the Wronskian $w_0(\lambda) = W(\psi^0_-, \psi^0_+) = (\psi^0_-)'_x \psi^0_+ - \psi^0_-(\psi^0_+)'_x$ is given by

$$w_0(\lambda) = \frac{2\sqrt{\pi}}{\Gamma(\frac{1-\lambda}{2})} = \frac{2}{\sqrt{\pi}}\Gamma(\tfrac{1+\lambda}{2})\cos(\tfrac{\pi\lambda}{2}). \tag{5.1}$$

At the eigenvalues $\lambda^0_n = 2n - 1$, the Wronskian $w_0(\lambda)$ vanishes and we have

$$\psi^0_+(x,\lambda^0_n) = (-1)^{n+1}\psi^0_-(x,\lambda^0_n) = 2^{-(n-1)/2} e^{-x^2/2} H_{n-1}(x), \tag{5.2}$$

where H_n is the n'th Hermite polynomial.

Next, let $\psi_\pm = \psi_\pm(x,\lambda)$ be the solutions of the equation $-\psi'' + (x^2 + q(x))\psi = \lambda\psi$, normalised by

$$\begin{aligned} \psi_+(x,\lambda) &= \psi^0_+(x,\lambda), \quad x > \sup\operatorname{supp} q, \\ \psi_-(x,\lambda) &= \psi^0_-(x,\lambda), \quad x < \inf\operatorname{supp} q. \end{aligned}$$

The eigenvalues λ_n coincide with the zeros of the Wronskian $w(\lambda) = W(\psi_-, \psi_+)$. In Section 6 we prove the following Lemma, which describes the asymptotics of $w(\lambda)$ as $\operatorname{Re}\lambda \to +\infty$. Let Ω be the half-strip

$$\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geqslant 0, |\operatorname{Im}\lambda| \leqslant 1\},$$

for $\lambda \in \Omega$ let us denote by $\sqrt{\lambda}$ the principal branch of the square root, so that $\operatorname{Re}\sqrt{\lambda} \geqslant 0$.

Lemma 5.1. *The Wronskian $w(\lambda)$ is analytic in $\lambda \in \Omega$. The following asymptotic expansion holds true:*

$$w(\lambda) \sim \frac{2}{\sqrt{\pi}}\Gamma(\tfrac{1+\lambda}{2})\left(\cos(\tfrac{\pi\lambda}{2})\sum_{j=0}^{\infty}\frac{Q_j}{(\sqrt{\lambda})^j} + \sin(\tfrac{\pi\lambda}{2})\sum_{j=0}^{\infty}\frac{P_j}{(\sqrt{\lambda})^j}\right), \tag{5.3}$$

as $|\lambda| \to \infty$, $\lambda \in \Omega$. Here $Q_j, P_j \in \mathbb{C}$ are some coefficients, $Q_0 = 1$, $P_0 = 0$.

Given Lemma 5.1, we can prove Theorem 1.1(i) as follows. Fix any sufficiently small $\varepsilon > 0$, denote $B_{n,\varepsilon} = \{z \mid |z - \lambda^0_n| \leqslant \varepsilon\}$, and let $\Gamma_{n,\varepsilon}$ be the contour $\partial B_{n,\varepsilon}$ oriented anti-clockwise. By Rouche's Theorem combined with a simple continuity argument, we obtain that $\lambda_n \in B_{n,\varepsilon}$ for all sufficiently large n. Next, the zeros of w in the half-strip Ω coincide with the zeros of

$$\widetilde{w}(\lambda) = \frac{\sqrt{\pi}w(\lambda)}{2\Gamma(\frac{1+\lambda}{2})}.$$

It follows that for all sufficiently large n we have

$$\lambda_n = \frac{1}{2\pi i}\int_{\Gamma_{n,\varepsilon}} \lambda \frac{\widetilde{w}'(\lambda)}{\widetilde{w}(\lambda)} d\lambda. \tag{5.4}$$

By analyticity of w, the asymptotic expansion (5.3) can be differentiated. Thus, we obtain the following asymptotic expansion for $\lambda \in \Gamma_{n,\varepsilon}$, $n \to \infty$:

$$\frac{\widetilde{w}'(\lambda)}{\widetilde{w}(\lambda)} = -\frac{\pi}{2}\tan\frac{\pi\lambda}{2} + \frac{1}{\sqrt{\lambda}}g_0(\lambda) + \frac{1}{\lambda\sqrt{\lambda}}g_1(\lambda)\tan\frac{\pi\lambda}{2} + \sum_{m=2}^{\infty}(\tan\frac{\pi\lambda}{2})^m(\sqrt{\lambda})^{-m+1}g_m(\lambda), \tag{5.5}$$

where the functions $g_m(\lambda)$ are analytic in $\lambda \in \Omega$ and have the asymptotic expansions

$$g_m(\lambda) \sim \sum_{k=0}^{\infty} c_{mk}(\sqrt{\lambda})^{-k}, \quad |\lambda| \to \infty, \quad \lambda \in \Omega. \tag{5.6}$$

Substituting the expansions (5.5) and (5.6) into (5.4) and computing the integrals of the type $\int_{\Gamma_{n,\varepsilon}}(\tan\frac{\pi\lambda}{2})^m\lambda^j d\lambda$, we arrive at the expansion (1.4). ∎

6 Proof of Lemma 5.1

Let $x > \sup\operatorname{supp} q$; then

$$w(\lambda) = (\psi_-(x,\lambda))'_x\psi_+^0(x,\lambda) - \psi_-(x,\lambda)(\psi_+^0(x,\lambda))'_x. \tag{6.1}$$

We will use this formula and construct ψ_- in a standard way as a solution to the integral equation

$$\psi_-(x,\lambda) = \psi_-^0(x,\lambda) + \int_{-\infty}^{x} G_\lambda(x,y)q(y)\psi_-(y,\lambda)dy, \tag{6.2}$$

where the integral kernel $G_\lambda(x,y)$ is given by

$$G_\lambda(x,y) = -\frac{1}{w_0(\lambda)}(\psi_+^0(x,\lambda)\psi_-^0(y,\lambda) - \psi_-^0(x,\lambda)\psi_+^0(y,\lambda)). \tag{6.3}$$

The kernel $G_\lambda(x,y)$ is an entire function of λ due to the analyticity of $\psi_\pm^0(x,\lambda)$ and the relation (5.2).

Let $R > 0$ be sufficiently large so that $\operatorname{supp} q \subset (-R,R)$. Denote $\Delta = [-2R,2R]$; let $L_\lambda : C(\Delta) \to C(\Delta)$ be the Volterra type integral operator from (6.2),

$$L_\lambda : f(x) \mapsto \int_{-2R}^{x} G_\lambda(x,y)q(y)f(y)dy.$$

Then the solution of the integral equation (6.2) can be written as

$$\psi_- = \sum_{n=0}^{\infty} L_\lambda^n \psi_-^0,$$

and so for the Wronskian (6.1) we have the series representation

$$w(\lambda) = \sum_{n=0}^{\infty} W(L_\lambda^n \psi_-^0, \psi_+^0)(x), \quad x \in (R, 2R).$$

Lemma 6.1. *For any $n \in \mathbb{N}$ and any $x \in (R, 2R)$, the Wronskian $W(L_\lambda^n \psi_-^0, \psi_+^0)(x)$ is analytic in $\lambda \in \Omega$ and one has the estimate*

$$|W(L_\lambda^n \psi_-^0, \psi_+^0)(x)| \leqslant \frac{C(\lambda)^n}{n!} |\Gamma(\tfrac{1+\lambda}{2})|, \quad C(\lambda) = O(|\lambda|^{-1/2}), \quad |\lambda| \to \infty, \lambda \in \Omega. \tag{6.4}$$

The asymptotic expansion

$$W(L_\lambda^n \psi_-^0, \psi_+^0)(x) \sim \Gamma(\tfrac{1+\lambda}{2}) \left(\cos(\tfrac{\pi\lambda}{2}) \sum_{j=n}^{\infty} \frac{Q_j^{(n)}}{(\sqrt{\lambda})^j} + \sin(\tfrac{\pi\lambda}{2}) \sum_{j=n}^{\infty} \frac{P_j^{(n)}}{(\sqrt{\lambda})^j} \right), \tag{6.5}$$

with some coefficients $Q_j^{(n)}$, $P_j^{(n)}$ holds true as $|\lambda| \to \infty$, $\lambda \in \Omega$.

Clearly, Lemma 5.1 follows from Lemma 6.1.

Proof of Lemma 6.1: 1. It is convenient to introduce two linear combinations e_+ and e_- of of the solutions $\psi_\pm^0$:

$$e_+(x, \lambda) = \frac{\sqrt{\pi} 2^{(1-\lambda)/4}}{\cos(\frac{\pi\lambda}{2}) \Gamma(\frac{1+\lambda}{4})} \left(e^{-i\pi(\lambda+1)/4} \psi_+^0(x, \lambda) + e^{i\pi(\lambda+1)/4} \psi_-^0(x, \lambda) \right),$$

$e_-(x, \lambda) = e_+(-x, \lambda)$. The solutions $e_\pm(x, \lambda)$ are analytic in $\lambda \in \Omega$ (with removeable singularities at λ_n^0 — see (5.2)). These solutions are chosen so that they satisfy the following asymptotic expansions:

$$e_\pm(x, \lambda) \sim e^{\pm i\sqrt{\lambda} x} \left(1 + \sum_{j=1}^{\infty} \frac{R_j^\pm(x)}{(\sqrt{\lambda})^j} \right), \quad \lambda \to \infty, \quad \lambda \in \Omega, \tag{6.6}$$

$$(e_\pm(x, \lambda))'_x \sim e^{\pm i\sqrt{\lambda} x} \left(\pm i\sqrt{\lambda} x + \sum_{j=0}^{\infty} \frac{\widetilde{R}_j^\pm(x)}{(\sqrt{\lambda})^j} \right), \quad \lambda \to \infty, \quad \lambda \in \Omega, \tag{6.7}$$

where $R_j^\pm$, $\widetilde{R}_j^\pm$ are polynomials in x. The expansion (6.6) follows directly from the formulae 19.9.4, 19.9.5, 19.4.2 of [1], and (6.7) is obtained by application of the recurrence formulas [1, §19.6].

2. Let us first prove the bound (6.4). We have

$$W(L_\lambda^n\psi_-^0,\psi_+^0)(x) = (L_\lambda^n\psi_-^0(x,\lambda))'_x\psi_+^0(x,\lambda) - L_\lambda^n\psi_-^0(x,\lambda)(\psi_+^0(x,\lambda))'_x; \qquad (6.8)$$

let us obtain appropriate bounds for each term in the r.h.s. of (6.8). Expressing $\psi_\pm^0$ in terms of $e_\pm$,

$$\psi_\pm^0(x,\lambda) = \frac{1}{2\sqrt{\pi}i}2^{(\lambda-1)/4}\Gamma(\tfrac{1+\lambda}{4})\left(e^{i\pi(\lambda+1)/4}e_\mp(x,\lambda) - e^{-i\pi(\lambda+1)/4}e_\pm(x,\lambda)\right), \qquad (6.9)$$

and using (6.6), (6.7), we obtain

$$\|\psi_\pm^0(\cdot,\lambda)\|_{C(\Delta)} = O(|2^{\lambda/4}\Gamma(\tfrac{1+\lambda}{4})|), \quad |\lambda|\to\infty, \quad \lambda\in\Omega, \qquad (6.10)$$

$$\|(\psi_\pm^0(\cdot,\lambda))'_x\|_{C(\Delta)} = O(|\lambda^{1/2}2^{\lambda/4}\Gamma(\tfrac{1+\lambda}{4})|), \quad |\lambda|\to\infty, \quad \lambda\in\Omega. \qquad (6.11)$$

Next, expressing the kernel $G_\lambda(x,y)$ in terms of $e_\pm$,

$$G_\lambda(x,y) = \frac{1}{4i}\frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})}(e_+(x,\lambda)e_-(y,\lambda) - e_-(x,\lambda)e_+(y,\lambda)), \qquad (6.12)$$

and using the asymptotics (6.6), we obtain

$$\sup_{|x|\leqslant R}\sup_{|y|\leqslant R}|G_\lambda(x,y)| = O(|\lambda|^{-1/2}), \quad |\lambda|\to\infty, \quad \lambda\in\Omega.$$

Using this estimate and the fact that L_λ is a Volterra type operator, we obtain

$$\|L_\lambda^n\|_{C(\Delta)\to C(\Delta)} \leqslant \frac{C(\lambda)^n}{n!}, \quad C(\lambda) = O(|\lambda|^{-1/2}), \quad |\lambda|\to\infty, \quad \lambda\in\Omega. \qquad (6.13)$$

Finally, in order to estimate the term $(L_\lambda^n\psi_-^0)'_x$, let us introduce the operator $L'_\lambda : C(\Delta)\to C(\Delta)$ by

$$L'_\lambda : f(x)\mapsto \int_{-R}^x \frac{\partial G_\lambda(x,y)}{\partial x}q(y)f(y)dy.$$

Then $(L_\lambda^n\psi_-^0(x,\lambda))'_x = L'_\lambda L_\lambda^{n-1}\psi_-^0(x,\lambda)$. Using the asymptotics (6.6), (6.7), we obtain

$$\|L'_\lambda\|_{C(\Delta)\to C(\Delta)} = O(1), \quad |\lambda|\to\infty, \quad \lambda\in\Omega. \qquad (6.14)$$

Combining (6.8), (6.10)–(6.14), we obtain (6.4).

3. Let us prove the asymptotic expansion (6.5). Using (6.9), we obtain

$$W(L_\lambda^n\psi_-^0,\psi_+^0) = \frac{1}{4\pi}2^{(\lambda-1)/2}\Gamma(\tfrac{1+\lambda}{4})^2\big(-W(L_\lambda^n e_+,e_+) - W(L_\lambda^n e_-,e_-) + e^{-i\pi(1+\lambda)/2}W(L_\lambda^n e_-,e_+) + e^{i\pi(1+\lambda)/2}W(L_\lambda^n e_+,e_-)\big)$$

Denote

$$g_n^\pm(x,\lambda) = \frac{L_\lambda^n e_\pm(x,\lambda)}{e_\pm(x,\lambda)}; \qquad (6.15)$$

by (6.6), the denominator does not vanish for all sufficiently large λ. Using this notation, we obtain

$$W(L_\lambda^n\psi_-^0, \psi_+^0) = i\sqrt{\pi}\Gamma(\tfrac{1+\lambda}{2})(e^{i\pi(1+\lambda)/2}g_n^+ - e^{-i\pi(1+\lambda)/2}g_n^-) \\ + (g_n^+(x,\lambda))'_x O(|\Gamma(\tfrac{1+\lambda}{2})|) + (g_n^-(x,\lambda))'_x O(|\Gamma(\tfrac{1+\lambda}{2})|).$$

It suffices to show that $g_n^\pm$ have the asymptotic expansions

$$g_n^\pm(x,\lambda) \sim \sum_{j=n}^{\infty} \frac{S_j^\pm(x)}{(\sqrt{\lambda})^j}, \quad |\lambda| \to \infty, \quad \lambda \in \Omega \tag{6.16}$$

for some coefficients $S_j^\pm \in C(\mathbb{R})$, and that for any $x \in (R, 2R)$,

$$(g_n^\pm(x,\lambda))'_x = O(|\lambda|^{-\infty}), \quad |\lambda| \to \infty, \quad \lambda \in \Omega. \tag{6.17}$$

By the definition of $g_n^\pm$, we have

$$g_{n+1}^\pm(x,\lambda) = \int_{-R}^{x} \frac{G_\lambda(x,y)}{e_\pm(x,\lambda)} g_n(y,\lambda) e_\pm(y,\lambda) q(y) dy. \tag{6.18}$$

Using this formula, the expression (6.12) for $G_\lambda(x,y)$ and the asymptotics (6.6), the expansion (6.16) can be easily proven by induction. The asymptotics (6.17) follows by differentiation of (6.18).

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover, 1972.

[2] L. A. Dikiĭ, *On a formula of Gelfand-Levitan.* (Russian) Uspehi Matem. Nauk (N.S.) **8** no. 2(54) (1953), 119–123.

[3] L. A. Dikiĭ, *The zeta function of an ordinary differential equation on a finite interval.* (Russian) Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 187–200.

[4] L. A. Dikiĭ, *Trace formulas for Sturm-Liouville differential operators* (Russian) Uspehi Mat. Nauk (N.S.) **13** no. 3(81) (1958), 111–143. English translation in: Amer. Math. Soc. Transl. (2) **18** (1961), 81–115.

[5] I. M. Gelfand, B. M. Levitan, *On a simple identity for the characteristic values of a differential operator of the second order.* (Russian) Doklady Akad. Nauk SSSR (N.S.) **88** (1953), 593–596.

[6] I. M. Gelfand, *On identities for eigenvalues of a differential operator of second order.* (Russian) Uspehi Mat. Nauk (N.S.) **11** no. 1(67) (1956), 191–198.

[7] M. Hitrik, I. Polterovich, *Regularized traces and Taylor expansions for the heat semigroup*, J. London Math. Soc. (2) **68** (2003), no. 2, 402–418.

[8] G. V. Kozlov, V. A. Lyubishkin, *Regularized higher-order traces for a harmonic oscillator.* (Russian) Differentsial'nye Uravneniya *29* no. 1 (1993), 61–63, 180; translation in Differential Equations *29* (1993), no. 1, 51–53.

[9] I. Polterovich, *A commutator method for computation of heat invariants*, Indag. Math. (N.S.) **11** no. 1 (2000), 139–149.

[10] I. Polterovich, *Heat invariants of Riemannian manifolds*, Israel J. Math. **119** (2000), 239–252.