

# Localization for Random Unitary Operators

Eman Hamza

University of Alabama at Birmingham  
Department of Mathematics CH 452  
Birmingham, Al 35294-1170  
U.S.A.

Alain Joye

Institut Fourier  
Université de Grenoble, BP 74  
38402 Saint-Martin d'Hères  
France

Günter Stolz\*

University of Alabama at Birmingham  
Department of Mathematics CH 452  
Birmingham, Al 35294-1170  
U.S.A.

## Abstract

We consider unitary analogs of 1-dimensional Anderson models on  $l^2(\mathbb{Z})$  defined by the product  $U_\omega = D_\omega S$  where  $S$  is a deterministic unitary and  $D_\omega$  is a diagonal matrix of i.i.d. random phases. The operator  $S$  is an absolutely continuous band matrix which depends on a parameter controlling the size of its off-diagonal elements. We prove that the spectrum of  $U_\omega$  is pure point almost surely for all values of the parameter of  $S$ . We provide similar results for unitary operators defined on  $l^2(\mathbb{N})$  together with an application to orthogonal polynomials on the unit circle. We get almost sure localization for polynomials characterized by Verblunski coefficients of constant modulus and correlated random phases.

## 1 Introduction

Unitary operators displaying a band structure with respect to a distinguished basis of  $l^2(\mathbb{N})$  or  $l^2(\mathbb{Z})$  appear in different contexts. For example, such operators describe the quantum dynamics of certain models in solid state physics, see e.g. [BB], [BHJ] and references therein. Unitary band matrices also appear naturally in the study of orthogonal polynomials on the unit circle with respect to a measure, as was recognized recently in [CMV]. For a detailed account on orthogonal polynomials on the unit circle, see [S1] (which is briefly surveyed in [S3]). In both situations, the spectral properties of these unitary infinite matrices play an important role.

The spectral analysis of a certain set of deterministic and random unitary operators with band structure is undertaken in [BHJ] and [J1], [J2]. The random cases studied in the

---

\*partially supported through US-NSF grant DMS-0245210

first two papers concern a set of matrices on  $l^2(\mathbb{Z})$  which are (up to unitary equivalence) of the following form:  $U_\omega = D_\omega S$  where  $S$  is a deterministic unitary and  $D_\omega$  is a diagonal matrix of random phases,  $\text{diag}\{e^{-i\theta_k^\omega}\}$ . The operator  $S$  is an absolutely continuous band matrix which depends on a parameter  $t \in ]0, 1[$  which controls the size of its off-diagonal elements, see Section 2 below.

When the phases are i.i.d random variables, typical results obtained for discrete one-dimensional random Schrödinger operators are shown in [BHJ] and [J1] to hold in the unitary setting as well. For instance, the availability of a transfer matrix formalism to express generalized eigenvectors allows to introduce a Lyapunov exponent, to prove a unitary version of the Ishii-Pastur Theorem, and get absence of absolutely continuous spectrum [BHJ]. A density of states can be introduced and a Thouless formula is proven in [J1]. Since these operators can be naturally considered as unitary analogs of the self-adjoint Anderson model, generalizations to  $d$ -dimensions are introduced in [J2]. Their localization properties are studied by means of an adaptation to the unitary setup of the fractional moment method due to Aizenman and Molchanov [AM]. In [J2] it is shown for arbitrary dimension that localization takes place if the common distribution of the phases is absolutely continuous and the parameters of the  $d$ -dimensional deterministic unitary  $S$  are such that  $S$  is close to the identity. This is the unitary analog of the familiar large disorder regime under which localization holds for the  $d$ -dimensional Anderson model. When applied to the one-dimensional case, this yields localization only if the parameter  $t$  is sufficiently small.

One of the goals of the present paper is to prove localization for all values of the parameter  $t$ , thereby completing the analogy with the self-adjoint one dimensional Anderson model, where localization holds without an additional disorder assumption. We also complete the picture by considering products of the same sort on  $l^2(\mathbb{N})$ , for which we prove localization as well.

Another motivation comes from the application of such results to orthogonal polynomials on the unit circle (OPUC), with respect to a measure  $d\mu$ . These polynomials are characterized by a sequence of complex numbers  $\{\alpha_k\}_{k \in \mathbb{N}}$ , such that  $|\alpha_k| < 1$ ,  $k \in \mathbb{N}$ , called the Verblunski coefficients. These coefficients allow to construct a unitary infinite matrix on  $l^2(\mathbb{N})$ , the so-called CMV matrix [CMV], which is the equivalent in the OPUC setting of the Jacobi matrix for orthogonal polynomials on the real line. This matrix represents multiplication by  $z$ ,  $z \in S^1$ , on  $L^2(S^1, d\mu)$  so that its spectral measure is  $d\mu$ , see [S1]. When the Verblunski coefficients  $\{\alpha_k(\omega)\}_{k \in \mathbb{N}}$  are random, the fine structure of the corresponding random measure  $d\mu_\omega$  is of interest. As mentioned in [J1], when  $|\alpha_k(\omega)| = r$  for all  $k \in \mathbb{N}$ , and only the phases of  $\alpha_k(\omega)$  are random, the CMV matrix is unitarily equivalent to the product  $D_\omega S$  on  $l^2(\mathbb{N})$  which, modulo boundary conditions at site 0, is of the form considered on  $l^2(\mathbb{Z})$  above. The point now is that the random phases of the coefficients  $\alpha_k(\omega)$  are correlated if the phases of the diagonal matrix  $D_\omega$  are independent.

Therefore we get as a corollary of our general analysis that localization takes place for random OPUC with certain types of correlated Verblunski coefficients. Let us mention here that previous localization results for CMV-matrices provided in [GT], [T], [S1], [S2], and [Su] essentially consider independent Verblunski coefficients.

## 2 The Model and Main Result

We introduce in this section the set of infinite random unitary matrices on  $l^2(\mathbb{Z})$  we will be interested in. We focus on the properties related to the localization result we are to prove.

For further details and generalizations, we refer the reader to [BHJ], [J1] and [J2].

Let us denote by  $|k\rangle$  the unit vector at site  $k \in \mathbb{Z}$ , so that  $\{|k\rangle\}_{k \in \mathbb{Z}}$  forms an orthonormal basis of  $l^2(\mathbb{Z})$ . We introduce a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  is identified with  $\{\mathbb{T}^{\mathbb{Z}}\}$ ,  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$  being the torus, and  $\mathbb{P} = \otimes_{k \in \mathbb{Z}} \mathbb{P}_k$ , where  $\mathbb{P}_k = \nu$  for any  $k \in \mathbb{Z}$  and  $\nu$  is a fixed probability measure on  $\mathbb{T}$ , and  $\mathcal{F}$  the  $\sigma$ -algebra generated by the cylinders. We introduce a set of random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  by

$$\theta_k : \Omega \rightarrow \mathbb{T}, \quad \text{s.t.} \quad \theta_k^\omega = \omega_k, \quad k \in \mathbb{Z}. \quad (2.1)$$

These random variables  $\{\theta_k\}_{k \in \mathbb{Z}}$  are thus i.i.d. on  $\mathbb{T}$ .

We consider unitary operators of the form

$$U_\omega = D_\omega S, \quad \text{with} \quad D_\omega = \text{diag} \{e^{-i\theta_k^\omega}\} \quad (2.2)$$

and

$$S = \begin{pmatrix} \ddots & & & & & & & & \\ & rt & -t^2 & & & & & & \\ & r^2 & -rt & & & & & & \\ & rt & r^2 & rt & -t^2 & & & & \\ & -t^2 & -tr & r^2 & -rt & & & & \\ & & & rt & r^2 & & & & \\ & & & -t^2 & -tr & \ddots & & & \end{pmatrix}, \quad (2.3)$$

where the translation along the diagonal is fixed by  $\langle 2k-2|S|2k\rangle = -t^2$ ,  $k \in \mathbb{Z}$ . The real parameters  $t$  and  $r$  are linked by  $r^2 + t^2 = 1$  to ensure unitarity. Due to unitary equivalence it suffices to consider  $0 \leq t, r \leq 1$ . Thus  $S$  is determined by  $t$ . We shall sometimes write  $S(t)$  to emphasize this dependence. The spectrum of  $S(t)$  is purely absolutely continuous and is given by

$$\sigma(S(t)) = \Sigma(t) = \{e^{\pm i \arccos(1-t^2(1+\cos(y)))}, y \in \mathbb{T}\}. \quad (2.4)$$

For this and other properties of  $S$ , relations between  $U_\omega$  and the physical model alluded to in Section 1 or links with orthogonal polynomials, see [J1] and Section 6. Note that the band structure (2.3) is the simplest one a unitary operator can take without being trivial from the point of view of its spectrum, [BHJ].

**Remarks:**

- i) In this definition,  $S$  plays the role of the free Laplacian in the self-adjoint Anderson model. The band structure of  $S$  is inherited by  $U_\omega$ .
- ii) As  $t \rightarrow 0$ ,  $S(t)$  tends to the identity operator, whereas as  $t \rightarrow 1$ ,  $S(t)$  tends to a direct sum of shift operators. Accordingly, the spectrum of  $U_\omega$  is pure point if  $t = 0$  and purely absolutely continuous if  $t = 1$ , [BHJ].

The localization result of [J2], when restricted to the present one dimensional setting, says the following. If the i.i.d. phases  $\theta_k$  have an absolutely continuous distribution with bounded density, i.e.  $d\nu(\theta) = \tau(\theta)d\theta$  on  $\mathbb{T}$ , where  $0 \leq \tau(\theta) \in L^\infty(\mathbb{T})$ , then there exists  $t_0 > 0$  such that  $t < t_0$  implies that  $\sigma(U_\omega)$  is pure point almost surely. Thus  $t$  takes the role of a disorder parameter, small  $t$  corresponding to large disorder.

In the one-dimensional context, one expects to have localization as soon as  $t < 1$  and  $\nu$  has an absolutely continuous component. Indeed, we show the

**Theorem 2.1** *Let  $U_\omega$  be defined by (2.1), (2.2) and (2.3). If the distribution  $d\nu$  of the i.i.d. phases possesses a non-trivial absolutely continuous component and  $\text{supp } \nu$  has non-empty interior, then  $U_\omega$  is pure point almost surely, with exponentially decaying eigenfunctions.*

Here  $\text{supp } \nu$  refers to the topological support of  $\nu$ , i.e. the set of all  $\lambda$  such that  $\nu(\lambda - \varepsilon, \lambda + \varepsilon) > 0$  for all  $\varepsilon > 0$ . The requirement that the  $\text{supp } \nu$  contains an open set stems from the method used here to prove positivity of Lyapunov exponents, see Proposition 3.1 and its proof in Section 7 below. This assumption can be dropped, as follows from the work in preparation [HS]. For example, it will be shown there that the Lyapunov exponents are strictly positive if the support of  $\nu$  contains at least three points.

We need that  $\nu$  has an a.c. component due to the use of a spectral averaging argument in Section 4 below. One may expect that Theorem 2.1 carries over to singularly distributed  $\nu$ , as was proven for discrete and continuous one-dimensional Anderson models in [CKM] and [DSS], respectively. But this would require to develop a completely different and much more involved approach.

We further show in Section 5 that our main result is also true for unitary matrices with a similar structure defined on  $l^2(\mathbb{N})$ , see Theorem 5.1. This extension allows for an application of our analysis to random OPUC which is described in Section 6.

### 3 Properties of the Model

In order to prove Theorem 2.1, we need to collect some facts about the unitary operator  $U_\omega$  that are proven in [BHJ] and [J1].

Our definition (2.2) leads to ergodicity of the unitary operator  $U_\omega$ . Indeed, introducing the shift operator  $W$  on  $\Omega$  by

$$W(\omega)_k = \omega_{k+2}, \quad k \in \mathbb{Z}, \quad (3.5)$$

we get an ergodic set  $\{W^j\}_{j \in \mathbb{Z}}$  of translations. With the unitary operators  $V_j$  defined on the canonical basis of  $l^2(\mathbb{Z})$  by

$$V_j|k\rangle = |k - 2j\rangle, \quad \forall k \in \mathbb{Z}, \quad (3.6)$$

we observe that for any  $j \in \mathbb{Z}$

$$U_{W^j \omega} = V_j U_\omega V_j^*. \quad (3.7)$$

Therefore, our random operator  $U_\omega$  is an ergodic unitary operator. The general theory of ergodic operators, as for example presented in [CL], chapter V, for the self-adjoint case, carries over to the unitary setting. In particular, it follows that the spectrum of  $U_\omega$  is almost surely deterministic, i.e. there is a subset  $\Sigma$  of the unit circle such that  $\sigma(U_\omega) = \Sigma$  for almost every  $\omega$ . The same holds for the absolutely continuous, singular continuous and pure point parts of the spectrum: There are  $\Sigma_{ac}$ ,  $\Sigma_{sc}$  and  $\Sigma_{pp}$  such that almost surely  $\sigma_{ac}(U_\omega) = \Sigma_{ac}$ ,  $\sigma_{sc}(U_\omega) = \Sigma_{sc}$  and  $\sigma_{pp}(U_\omega) = \Sigma_{pp}$ . Moreover, as shown in [J1], we can characterize  $\Sigma$  in terms of the support of  $d\nu$  and of the spectrum  $\Sigma(t)$  of  $S(t)$ .

**Theorem 3.1** *Under the above hypotheses, the almost sure spectrum of  $U_\omega$  consists in the set*

$$\Sigma = \exp(i \text{supp } \nu) \Sigma(t) = \{e^{i\alpha} \Sigma(t) \mid \alpha \in \text{supp } \nu\}. \quad (3.8)$$

Let us proceed by recalling some facts concerning the generalized eigenvectors and the associated Lyapunov exponent. Following [BB], [BHJ] and [J1], we study the generalized eigenvectors of  $U_\omega$  by means of  $2 \times 2$  transfer matrices. This is possible due to the band structure of  $U_\omega$ . Our unitary operators differ from those considered in the works above by a unitary transform, so that the formulas differ a little.

Consider

$$\begin{aligned} U_\omega \psi &= e^{i\alpha} \psi, \\ \psi &= \sum_{k \in \mathbb{Z}} c_k |k\rangle, \quad c_k \in \mathbb{C}, \quad \alpha \in \mathbb{C}. \end{aligned} \quad (3.9)$$

This equation is equivalent to the relations for all  $k \in \mathbb{Z}$ ,

$$\begin{pmatrix} c_{2(k+1)-1} \\ c_{2(k+1)} \end{pmatrix} = T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)) \begin{pmatrix} c_{2k-1} \\ c_{2k} \end{pmatrix}, \quad (3.10)$$

where the function  $T : \mathbb{T}^2 \rightarrow M_2(\mathbb{C})$  is defined by

$$\begin{aligned} T(\theta, \eta)_{11} &= -e^{-i\eta} \\ T(\theta, \eta)_{12} &= \frac{r}{t} \left( e^{i(\theta-\eta)} - e^{-i\eta} \right) \\ T(\theta, \eta)_{21} &= \frac{r}{t} \left( 1 - e^{-i\eta} \right) \\ T(\theta, \eta)_{22} &= -\frac{1}{t^2} e^{i\theta} + \frac{r^2}{t^2} \left( (e^{i(\theta-\eta)} + 1) - e^{-i\eta} \right), \end{aligned} \quad (3.11)$$

and the phases by

$$\theta_k^\omega(\alpha) = \theta_k^\omega + \alpha. \quad (3.12)$$

Note that  $\det T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)) = e^{i(\theta_{2k}^\omega - \theta_{2k+1}^\omega)}$  is independent of  $\alpha$  and of modulus one.

Introducing the notation

$$T(k, \omega) \equiv T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)), \quad (3.13)$$

we compute for any  $k \in \mathbb{N}$ , assuming  $(c_{-1}, c_0)$  known,

$$\begin{aligned} \begin{pmatrix} c_{2k-1} \\ c_{2k} \end{pmatrix} &= T(k-1, \omega) \cdots T(1, \omega) T(0, \omega) \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix} \equiv \Phi(k, \omega) \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix} \\ \begin{pmatrix} c_{-2k+1} \\ c_{-2k+2} \end{pmatrix} &= T(-k, \omega)^{-1} \cdots T(-2, \omega)^{-1} T(-1, \omega)^{-1} \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix} \equiv \Phi(-k, \omega) \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix}, \end{aligned} \quad (3.14)$$

with  $\Phi(0, \omega) = \mathbb{I}$ . The dynamical system defined that way is ergodic,

$$\begin{aligned} \Phi(k, \omega) &= T(0, W^{k-1}(\omega)) \cdots T(0, W(\omega)) T(0, \omega) \\ \Phi(-k, \omega) &= T(-1, W^{-k+1}(\omega)) \cdots T(-1, W^{-1}(\omega)) T(-1, \omega) \end{aligned} \quad (3.15)$$

and the determinant of the transfer matrices is of modulus one. As shown in [BHJ], it follows that for any  $\alpha \in \mathbb{C}$ , the Lyapunov exponent

$$\gamma_\omega(e^{i\alpha}) = \lim_{k \rightarrow \pm\infty} \frac{1}{|k|} \ln \|\Phi(k, \omega)\| \quad (3.16)$$

almost surely exists, has the same value for  $k \rightarrow \infty$  and  $k \rightarrow -\infty$ , and takes the deterministic value

$$\gamma(e^{i\alpha}) = \lim_{k \rightarrow \pm\infty} \frac{1}{|k|} \mathbb{E}(\ln \|\Phi(k, \omega)\|). \quad (3.17)$$

We can allow spectral parameters of the form  $e^{i\alpha} = z \in \mathbb{C} \setminus \{0\}$ , and get from classical arguments, see *e.g.* [CFKS], that  $\gamma$  is a subharmonic function of  $z$ . For other properties of  $\gamma(e^{i\alpha})$  and, in particular for its relations with the density of states, see [J1].

The link between the behaviour at infinity of the generalized eigenvectors and the spectrum of  $U_\omega$  is provided by Sh'nol's Theorem. This is a deterministic fact which, as proven in [BHJ], carries over to the unitary operators considered here (here  $E_\omega(\cdot)$  is the spectral resolution of  $U_\omega$ , which we consider to be supported on  $\mathbb{T}$ ):

**Theorem 3.2**  $\sigma(U_\omega)$  is the closure of the set

$$S_\omega = \{\alpha \in \mathbb{T}; U_\omega \phi = e^{i\alpha} \phi \text{ has a non-trivial polynomially bounded solution}\} \quad (3.18)$$

and  $E_\omega(\mathbb{T} \setminus S_\omega) = 0$ .

A version of the Ishii-Pastur theorem suited to unitary matrices with a band structure follows as a corollary to these arguments.

**Theorem 3.3** Let  $U_\omega$  be defined by (2.2), (2.1) and (2.3) and  $\gamma(e^{i\alpha})$  by (3.16). Then

$$\Sigma_{ac} \subseteq \overline{\{e^{i\alpha} \in S^1; \gamma(e^{i\alpha}) = 0\}}^{ess}. \quad (3.19)$$

The positivity of the Lyapunov exponent is assessed in [BHJ] by means of Fürstenberg's Theorem. The situation considered in [BHJ] actually concerns phases which are uniformly distributed on  $\mathbb{T}$  and this is therefore easily obtained. In the present case, the support of  $d\nu$  is more arbitrary and a more detailed investigation is necessary. The argument leading to the following result is presented in Section 7.

**Proposition 3.1** Assume the support  $\nu$  has non-empty interior. Then, the Lyapunov exponent  $\gamma(e^{i\alpha})$  associated with the ergodic linear dynamical system (3.14) is strictly positive for any  $\alpha \in \mathbb{T}$ .

As a direct corollary, we get that

$$\Sigma_{ac} = \emptyset. \quad (3.20)$$

## 4 Proof of Theorem 2.1:

We adapt an argument which, in various forms, has been used extensively in localization proofs for various types of one-dimensional random Schrödinger operators. It combines positivity of the Lyapunov exponent with polynomial boundedness of generalized eigenfunctions (Theorem 3.2) and spectral averaging. While implicit in the literature even earlier, this strategy was first explicitly spelled out in [SW].

Let  $\bar{\Omega} = \mathbb{T}^{\mathbb{Z} \setminus \{-1, 0\}}$ ,  $\bar{\mathbb{P}} = \otimes_{k \in \mathbb{Z} \setminus \{-1, 0\}} \nu$  and  $\bar{\omega} = (\dots, \bar{\omega}_{-3}, \bar{\omega}_{-2}, \bar{\omega}_{-1}, \bar{\omega}_1, \bar{\omega}_2, \dots)$ . We will write  $(\bar{\omega}, \theta_{-1}, \theta_0)$  for  $(\dots, \bar{\omega}_{-3}, \bar{\omega}_{-2}, \theta_{-1}, \theta_0, \bar{\omega}_1, \bar{\omega}_2, \dots)$ .

By construction  $\gamma_{(\bar{\omega}, \theta_{-1}, \theta_0)}(e^{i\alpha})$  is independent of  $(\theta_{-1}, \theta_0)$ . It follows from Proposition 3.1 that for any  $\alpha \in \mathbb{T}$ , there exists  $\bar{\Omega}(\alpha) \subset \bar{\Omega}$  with  $\bar{\mathbb{P}}(\bar{\Omega}(\alpha)) = 1$  such that

$$\gamma_{(\bar{\omega}, \theta_{-1}, \theta_0)}(e^{i\alpha}) = \gamma(e^{i\alpha}) > 0 \quad (4.1)$$

for all  $(\theta_{-1}, \theta_0)$  and all  $\bar{\omega} \in \bar{\Omega}(\alpha)$ . Hence, by Fubini applied to  $\bar{\mathbb{P}} \times |\cdot|$ , we get the existence of  $\bar{\Omega}_0 \in \bar{\Omega}$  with  $\bar{\mathbb{P}}(\bar{\Omega}_0) = 1$  such that for every  $\bar{\omega} \in \bar{\Omega}_0$  there is  $A_{\bar{\omega}} \in \mathbb{T}$  with  $|A_{\bar{\omega}}| = 0$  and

$$\gamma_{(\bar{\omega}, \theta_{-1}, \theta_0)}(e^{i\alpha}) > 0 \quad \text{for all } (\theta_{-1}, \theta_0) \text{ and all } \alpha \in A_{\bar{\omega}}^C. \quad (4.2)$$

Here  $|\cdot|$  denotes Lebesgue-measure on  $\mathbb{T}$ .

Let us show that  $A_{\bar{\omega}}^C$  is a support of the spectral resolution of  $U_{(\bar{\omega}, \theta_{-1}, \theta_0)}$  for Lebesgue almost every  $(\theta_{-1}, \theta_0)$ .

We introduce the spectral measures  $\mu_{\omega}^j$  associated with  $U_{\omega} = \int_{\mathbb{T}} e^{i\alpha} dE_{\omega}(\alpha)$  defined for all  $j \in \mathbb{Z}$  and all Borel sets  $\Delta \in \mathbb{T}$  by

$$\mu_{\omega}^j(\Delta) = \langle j | E_{\omega}(\Delta) | j \rangle. \quad (4.3)$$

By construction, the variation of a random phase at one site is described by a rank one perturbation. More precisely, dropping the subscript  $\omega$  temporarily, we define  $\hat{D}$  by taking  $\theta_0 = 0$  in the definition of  $D$ :

$$\hat{D} = e^{i\theta_0|0\rangle\langle 0|} D = D + |0\rangle\langle 0|(1 - e^{-i\theta_0}), \quad (4.4)$$

so that, with the obvious notations,

$$\hat{U} = \hat{D}S = e^{i\theta_0|0\rangle\langle 0|} U. \quad (4.5)$$

As for rank one perturbations of self-adjoint operators, a spectral averaging formula holds in the unitary case. In particular, see [C] and [B], for any  $f \in L^1(\mathbb{T})$ ,

$$\int_{\mathbb{T}} d\theta_0 \int_{\mathbb{T}} f(\alpha) d\mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^0(\alpha) = \int_{\mathbb{T}} f(\alpha) d\alpha. \quad (4.6)$$

By applying this to the characteristic function of  $A_{\bar{\omega}}$  we get

$$0 = |A_{\bar{\omega}}| = \int_{\mathbb{T}} \mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^0(A_{\bar{\omega}}) d\theta_0, \quad (4.7)$$

implying that  $\mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^0(A_{\bar{\omega}}) = 0$  for every  $\theta_{-1}$  and Lebesgue-a.e.  $\theta_0$ . Similarly, we get  $\mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^{-1}(A_{\bar{\omega}}) = 0$  for every  $\theta_0$  and Lebesgue-a.e.  $\theta_{-1}$ .

Therefore, for all  $\bar{\omega} \in \bar{\Omega}_0$ , there exists  $J_{\bar{\omega}} \subset \mathbb{T}^2$  such that  $|J_{\bar{\omega}}^C| = 0$  and

$$(\theta_{-1}, \theta_0) \in J_{\bar{\omega}} \Rightarrow \mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^j(A_{\bar{\omega}}) = 0, \quad j \in \{-1, 0\}. \quad (4.8)$$

Fix  $\bar{\omega} \in \bar{\Omega}_0$  and  $(\theta_{-1}, \theta_0) \in J_{\bar{\omega}}$  and consider  $\omega = (\bar{\omega}, \theta_{-1}, \theta_0)$ . Below we prove

**Lemma 4.1** *The subspace  $\text{Span}\{|-1\rangle, |0\rangle\}$  is cyclic for  $U_{\omega}$ .*

Therefore, we deduce from (4.8) that  $E_{\omega}(A_{\bar{\omega}}) = 0$ . If  $S_{\omega}$  is the set from Sh'nol's Theorem 3.2, we conclude that  $S_{\omega} \cap A_{\bar{\omega}}^C$  is a support for  $E_{\omega}(\cdot)$ .

Let  $\alpha \in S_{\omega} \cap A_{\bar{\omega}}^C$ . By Theorem 3.2,  $U_{\omega}\psi = e^{i\alpha}\psi$  has a non-trivial polynomially bounded solution  $\psi$ . On the other hand, by (4.2),  $\gamma_{\omega}(e^{i\alpha}) > 0$ . Thus, by Osceledec's Theorem, every solution which is polynomially bounded at  $+\infty$  necessarily has to decay exponentially, and the same holds at  $-\infty$ . Thus  $\psi$  decays exponentially at  $+\infty$  and  $-\infty$ , and therefore is in

$l^2(\mathbb{Z})$  and an eigenfunction of  $U_\omega$ . We have shown that every  $\alpha \in S_\omega \cap A_{\bar{\omega}}^C$  is an eigenvalue of  $U_\omega$ . As  $l^2(\mathbb{Z})$  is separable, it follows that  $S_\omega \cap A_{\bar{\omega}}^C$  is countable. Therefore  $E_\omega(\cdot)$  has countable support and thus  $U_\omega$  is pure point spectrum, in particular

$$\sigma_{sc}(U_\omega) = \emptyset \quad \text{for every } \omega \in \Omega_0 := \{(\bar{\omega}, \theta_{-1}, \theta_0) : \bar{\omega} \in \bar{\Omega}_0, (\theta_{-1}, \theta_0) \in J_{\bar{\omega}}\}. \quad (4.9)$$

From  $|J_{\bar{\omega}}^C| = 0$  and the non-triviality of the a.c. component of  $\nu$  we have

$$(\nu \times \nu)(J_{\bar{\omega}}) \geq (\nu_{ac} \times \nu_{ac})(J_{\bar{\omega}}) = (\nu_{ac} \times \nu_{ac})(\mathbb{T}^2) > 0. \quad (4.10)$$

As  $\bar{\mathbb{P}}(\bar{\Omega}_0) = 1$ , we conclude from (4.9) and (4.10) that

$$\mathbb{P}(\sigma_{sc}(U_\omega) = \emptyset) \geq \mathbb{P}(\Omega_0) = \int_{\bar{\Omega}_0} d\bar{\mathbb{P}}(\bar{\omega})(\nu \times \nu)(J_{\bar{\omega}}) > 0. \quad (4.11)$$

By the discussion in Section 2 we know that spectral types are almost surely deterministic, thus  $\Sigma_{sc} = \emptyset$ . We already know  $\Sigma_{ac} = \emptyset$  from (3.20). This proves that  $U_\omega$  has almost surely pure point spectrum.

We still need to show that almost surely all eigenfunctions decay exponentially. To this end, note that we actually have shown above that the event ‘‘all eigenvectors of  $U_\omega$  decay at the rate of the Lyapunov exponent’’ has positive probability (as this is true for all  $\omega \in \Omega_0$ ). For the case of ergodic one-dimensional Schrödinger operators Kotani and Simon show in Theorem A.1 of [KS] that this event has probability 1 or 0. In fact, only measurability needs to be shown as the event is invariant under the ergodic transformation  $W$ . The proof of this fact provided in [KS] carries over to our model. Let us only note that, due to Lemma 4.1, we may use  $\rho_\omega = \mu_\omega^{-1} + \mu_\omega^0$  as spectral measures in their argument. This completes the proof of Theorem 2.1 up to the

**Proof of Lemma 4.1:** We drop the sub(super)scripts  $\omega$  in this proof. We have to show that any vector  $|k\rangle$ ,  $k \in \mathbb{Z}$  can be written as a linear combination of the vectors  $U^n|j\rangle$ ,  $n \in \mathbb{Z}$ ,  $j = -1$  and  $j = 0$ . We compute from (2.2) and its adjoint

$$U|-1\rangle = e^{-i\theta-2}rt|-2\rangle + e^{-i\theta-1}r^2|-1\rangle + e^{-i\theta_0}rt|0\rangle - e^{-i\theta_1}t^2|1\rangle \quad (4.12)$$

$$U|0\rangle = -e^{-i\theta-2}t^2|-2\rangle - e^{-i\theta-1}rt|-1\rangle + e^{-i\theta_0}r^2|0\rangle - e^{-i\theta_1}rt|1\rangle \quad (4.13)$$

$$U^{-1}|0\rangle = e^{i\theta_0}(rt|-1\rangle + r^2|0\rangle + rt|1\rangle - t^2|2\rangle) \quad (4.14)$$

$$U^{-1}|-1\rangle = e^{i\theta_1}(-t^2|-3\rangle - rt|-2\rangle + r^2|-1\rangle - rt|2\rangle). \quad (4.15)$$

Hence, using  $t^2 + r^2 = 1$ ,

$$|1\rangle = \frac{e^{i\theta_1}}{t} \left( e^{-i\theta_0}r|0\rangle - (tU|-1\rangle + rU|0\rangle) \right) \quad (4.16)$$

$$|-2\rangle = \frac{e^{i\theta-2}}{t} \left( rU|-1\rangle - tU|0\rangle - e^{-i\theta-1}r|-1\rangle \right). \quad (4.17)$$

Therefore, using (4.16) in (4.14), suitable linear combinations of  $|-1\rangle$ ,  $|0\rangle$ ,  $U|-1\rangle$ ,  $U|0\rangle$  and  $U^{-1}|0\rangle$  yield  $|2\rangle$ . Similarly,  $|-3\rangle$  can be obtained as a linear combination of  $|-1\rangle$ ,  $|0\rangle$ ,  $U|-1\rangle$ ,  $U|0\rangle$  and  $U^{-1}|-1\rangle$  using (4.17) in (4.15). These manipulations lead us from the indices  $(-1, 0)$  to the set  $(1, 2)$  in one direction and  $(-3, -2)$  in the other direction. Due to the shape of  $U$ , we can iterate the process to reach any vector. ■



## 5 Half-lattice operators

In this section we indicate how to adapt the results above to random unitary matrices similar to (2.2), but defined on  $l^2(\mathbb{N}_0)$ , in the same spirit as in [BHJ].

Let  $S^+$  be the unitary defined on  $l^2(\mathbb{N}_0)$ ,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ , in the canonical basis  $\{|j\rangle\}_{j \in \mathbb{N}_0}$  by the matrix

$$S^+ = \begin{pmatrix} -r & rt & -t^2 & & & \\ t & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -rt & r^2 & -rt & \\ & & & tr & r^2 & \\ & & & -t^2 & -rt & \ddots \end{pmatrix}, \quad (5.18)$$

where the dots mean repetition of the last  $4 \times 2$  block, as in (2.3). Then one considers

$$U_\omega^+ = D_\omega^+ S^+ \quad \text{with} \quad D_\omega^+ = \text{diag} \{e^{-i\theta_k^\omega}\}, \quad (5.19)$$

where the random phases  $\theta_k^\omega$  are given by (2.1), for  $k \in \mathbb{N}_0$ . The generalized eigenvectors  $\psi = \sum_{k \geq 0} c_k |k\rangle$  defined by

$$U_\omega^+ \psi = e^{i\alpha} \psi \quad (5.20)$$

give rise to the same dynamical system (3.14) on the coefficients  $c_k$ . Starting from  $(c_1, c_2)^T$ , we have

$$\begin{pmatrix} c_{2(k+1)-1} \\ c_{2(k+1)} \end{pmatrix} = T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)) \begin{pmatrix} c_{2k-1} \\ c_{2k} \end{pmatrix}, \quad k = 1, 2, \dots \quad (5.21)$$

where the transfer matrix  $T(\theta, \eta)$  is given by (3.11) and  $\theta_k^\omega(\alpha) = \theta_k^\omega + \alpha$  as before. This relation must be supplemented by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_0 \begin{pmatrix} (e^{-i(\theta_1+\alpha)} + r e^{-i(\theta_1-\theta_0)})/t \\ (e^{-i(\theta_1+\alpha)} + r e^{-i(\theta_1-\theta_0)})r/t^2 - (r + e^{i(\alpha+\theta_0)})/t^2 \end{pmatrix}, \quad (5.22)$$

where  $c_0$  is free, because of the boundary condition at component 0. The (forward) Lyapunov exponent  $\gamma(e^{i\alpha})$  corresponding to (5.21) defined by (3.16) exists almost surely and the conclusions of Theorems 3.2 and 3.3 remain true for  $U_\omega^+$ .

The first difference/simplification with respect to the operator  $U_\omega$  defined on the whole lattice is that  $U_\omega^+$  admits a cyclic vector

**Lemma 5.1** *The vector  $|0\rangle$  is cyclic for  $U_\omega^+$ .*

**Proof:** One first checks that  $|1\rangle = \frac{e^{i\theta_1}}{t} (U_\omega^+ |0\rangle + r e^{-i\theta_0} |0\rangle)$ , and then one concludes as in the proof of Lemma 4.1. ■

As a consequence, we get

**Theorem 5.1** *Let  $U_\omega^+$  be defined by (5.18), (5.19) and (2.1). If the distribution  $d\nu$  of the i.i.d. phases possesses a non-trivial absolutely continuous component and its support has non-empty interior, then  $U_\omega^+$  is pure point almost surely, with exponentially decaying eigenfunctions.*

**Proof:** The proof is virtually the same as that of Theorem 2.1. Due to cyclicity of  $|0\rangle$  it suffices to average over the single phase  $\theta_0$ , which leads to some simplifications. ■

## 6 Application to OPUC

The previous extension of our result to  $l^2(\mathbb{N}_0)$  was aimed to pave way for the applications of our localization results to orthogonal polynomials on the unit circle (OPUC) with respect to an infinitely supported probability measure  $d\mu$ . Such polynomials  $\Phi_k$  are determined via the Szego recursion  $\Phi_{k+1}(z) = z\Phi_k(z) - \bar{\alpha}_k\Phi_k^*(z)$ ,  $\Phi_0 = 1$ , by a sequence of complex valued coefficients  $\{\alpha_k\}_{k \in \mathbb{N}_0}$ , such that  $|\alpha_k| < 1$ , called Verblunski coefficients, which also characterize the measure  $d\mu$ , see [S1]. This latter relation is encoded in a five diagonal unitary matrix  $C$  on  $l^2(\mathbb{N}_0)$  representing multiplication by  $z \in S^1$ : The measure  $d\mu$  arises as the spectral measure  $\mu(\Delta) = \langle 0|E(\Delta)|0\rangle$  of the cyclic vector  $|0\rangle$  of  $C$ . This matrix is the equivalent of the Jacobi matrix in the case of orthogonal polynomials with respect to a measure on the real axis, and it is called the CMV matrix, after [CMV].

In case the Verblunski coefficients all have the same modulus, i.e.

$$\alpha_k = re^{i\eta_k}, \quad k = 0, 1, \dots \quad (6.23)$$

the corresponding CMV matrix reads

$$C = \begin{pmatrix} re^{-i\eta_0} & rte^{-i\eta_1} & t^2 & & & & & & \\ & t & -r^2e^{i(\eta_0-\eta_1)} & -rte^{i\eta_0} & & & & & \\ & & rte^{-i\eta_2} & -r^2e^{i(\eta_1-\eta_2)} & rte^{-i\eta_3} & t^2 & & & \\ & & t^2 & -rte^{i\eta_1} & -r^2e^{i(\eta_2-\eta_3)} & -rte^{i\eta_2} & & & \\ & & & & rte^{-i\eta_4} & -r^2e^{i(\eta_3-\eta_4)} & & & \\ & & & & t^2 & -rte^{i\eta_3} & & & \\ & & & & & & & & \ddots \end{pmatrix}. \quad (6.24)$$

Now, changing from the canonical basis  $\{|j\rangle\}_{j \in \mathbb{N}_0}$  to  $\{e^{i\beta_j}|j\rangle\}_{j \in \mathbb{N}_0}$  by means of the unitary  $B$  defined by  $B|j\rangle = e^{i\beta_j}|j\rangle$ ,  $j = 0, 1, \dots$ , we get

$$B^{-1}CB = \begin{pmatrix} re^{-i\eta_0} & rte^{-i\eta_1}e^{i(\beta_1-\beta_0)} & t^2e^{i(\beta_2-\beta_0)} & & \\ te^{i(\beta_0-\beta_1)} & -r^2e^{i(\eta_0-\eta_1)} & -rte^{i\eta_0}e^{i(\beta_2-\beta_1)} & & \\ & rte^{-i\eta_2}e^{i(\beta_1-\beta_2)} & -r^2e^{i(\eta_1-\eta_2)} & & \\ & t^2e^{i(\beta_1-\beta_3)} & -rte^{i\eta_1}e^{i(\beta_2-\beta_3)} & \ddots & \end{pmatrix}. \quad (6.25)$$

Then, by choosing the  $\beta_j$ 's suitably, the matrix (6.25) becomes the negative of a matrix of the form (5.19):

$$-U^+ = \begin{pmatrix} re^{-i\theta_0} & -re^{-i\theta_0}t & t^2e^{-i\theta_0} & & & & & & \\ -te^{-i\theta_1} & -r^2e^{-i\theta_1} & rte^{-i\theta_1} & & & & & & \\ & -rte^{-i\theta_2} & -r^2e^{-i\theta_2} & -rte^{-i\theta_2} & t^2e^{-i\theta_2} & & & & \\ & t^2e^{-i\theta_3} & rte^{-i\theta_3} & -r^2e^{-i\theta_3} & rte^{-i\theta_3} & & & & \\ & & & -tre^{-i\theta_4} & -r^2e^{-i\theta_4} & & & & \\ & & & t^2e^{-i\theta_5} & rte^{-i\theta_5} & \ddots & & & \end{pmatrix}. \quad (6.26)$$

Here, as seen from the diagonal elements, the phases  $\theta_k$  are given in terms of the phases of the Verblunski coefficients by (set  $\eta_{-1} = 0$ )

$$\theta_k = \eta_k - \eta_{k-1}, \quad k = 0, 1, 2, \dots, \quad (6.27)$$

or, equivalently

$$\eta_k = \theta_k + \theta_{k-1} + \dots + \theta_0, \quad k = 0, 1, 2, \dots \quad (6.28)$$

The terms in  $t^2$  require

$$\begin{aligned}\beta_1 - \beta_0 &= \theta_1 + \pi \\ \beta_{2k+1} - \beta_{2k-1} &= \theta_{2k+1}, \quad k = 1, 2, \dots, \\ \beta_{2k+2} - \beta_{2k} &= -\theta_{2k}, \quad k = 0, 1, \dots,\end{aligned}\tag{6.29}$$

where  $\beta_0$  is free. Explicitly, for  $k \geq 0$ ,

$$\begin{aligned}\beta_{2k+1} &= \theta_{2k+1} + \theta_{2k-1} + \dots + \theta_1 + \beta_0 + \pi \\ \beta_{2k+2} &= -(\theta_{2k} + \theta_{2k-2} + \dots + \theta_0).\end{aligned}\tag{6.30}$$

It is straightforward to check that (6.27) and (6.29) form a consistent choice in the sense that all terms in  $rt$  in (6.25) and (6.26) agree. Assuming the  $\theta_k$ 's are i.i.d. random variables, Theorem 5.1 applies to this case and yields

**Proposition 6.1** *Let  $\alpha_k(\omega)_{k \in \mathbb{N}_0}$  be random Verblunski coefficients of the form*

$$\alpha_k(\omega) = r e^{i\eta_k(\omega)}, \quad 0 < r < 1, \quad k = 0, 1, 2, \dots\tag{6.31}$$

whose phases are distributed on  $\mathbb{T}$  according to

$$\eta_k(\omega) \sim d\nu * d\nu * \dots * d\nu, \quad (k+1 \text{ convolutions})\tag{6.32}$$

and  $d\nu$  is a probability measure with non-trivial a.c. component and such that its support has non-empty interior. Then, the random measure  $d\mu_\omega$  on  $S^1$  with respect to which the corresponding random polynomials  $\Phi_{k,\omega}$  are orthogonal is almost surely pure point.

**Remark:** Other localization results for random polynomials on the unit circle, [S2], [T], [Su] are proven for independent Verblunski coefficients. Moreover, the results of [Su] and [S2] require rotational invariance of the distribution of the Verblunski coefficients in the unit disk. By contrast, the proposition above holds for strongly correlated random Verblunski coefficients.

## 7 Proof of Proposition 3.1

We want to apply Fürstenberg's Theorem. Since the latter is stated for real valued matrices, we proceed as in [BHJ] and introduce the mapping  $\tau : M_2(\mathbb{C}) \rightarrow M_4(\mathbb{R})$  defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} \Re(a)I + \Im(a)J & \Re(b)I + \Im(b)J \\ \Re(c)I + \Im(c)J & \Re(d)I + \Im(d)J \end{pmatrix},\tag{7.33}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\tag{7.34}$$

This mapping is a homeomorphism from  $M_2(\mathbb{C})$  to  $\tau(M_2(\mathbb{C}))$  and, in particular, a group homeomorphism from the set of matrices in  $M_2(\mathbb{C})$  with determinant of modulus one to the set of matrices in  $M_4(\mathbb{R})$  with determinant of modulus one.

Let  $G_{\alpha,\nu}$  be the closed group generated by the matrices  $T(\theta, \eta)$  from (3.11), where  $\theta$  and  $\eta$  vary in  $\alpha + \text{supp } \nu$ . As discussed in [BHJ], Fürstenberg's Theorem implies that  $\gamma(e^{i\alpha}) > 0$  if it can be shown that  $G_{\alpha,\nu}$  is strongly-irreducible and non-compact. As  $\text{supp } \nu$  has non-empty interior and thus  $G_{\alpha,\nu}$  has a non-trivial connected component it suffices to show

**Lemma 7.1** *The group  $\tau(G_{\alpha,\nu})$  is irreducible and non-compact.*

To show the first statement, one uses arguments of the same type as those developed in [BHJ]. The set  $\alpha + \text{supp } \nu$  contains a non-empty open interval  $\mathcal{I}$  and it will suffice to only work with  $\theta, \eta \in \mathcal{I}$ .

Let us assume there exists a strict subspace  $V$  of  $\mathbb{R}^4$  which is invariant under  $\tau(T(\theta, \eta))$ , for all  $(\theta, \eta) \in \mathcal{I} \times \mathcal{I}$ . The mapping  $\tau(T)$  is smooth in  $\mathcal{I} \times \mathcal{I}$  and it follows by approximating derivatives with finite differences that  $V$  is also left invariant by  $\partial_\theta^r \partial_\eta^s \tau(T(\theta, \eta))$  for all  $r, s \in \mathbb{N}_0$  and for all  $(\theta, \eta) \in \mathcal{I} \times \mathcal{I}$ .

In particular, we compute for  $\eta = \theta \in \mathcal{I}$ ,

$$\tau(T(\theta, \theta)) = A_0 + A_1 \sin(\theta) + A_2 \cos(\theta), \quad (7.35)$$

$$\partial_\theta \tau(T(\theta, \eta))|_{\eta=\theta} = B_0 + B_1 \sin(\theta) + B_2 \cos(\theta), \quad (7.36)$$

with

$$A_0 = \begin{pmatrix} 0 & 0 & r/t & 0 \\ 0 & 0 & 0 & r/t \\ r/t & 0 & 2r^2/t^2 & 0 \\ 0 & r/t & 0 & 2r^2/t^2 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & 0 & r/t \\ -1 & 0 & -r/t & 0 \\ 0 & r/t & 0 & -1 \\ -r/t & 0 & 1 & 0 \end{pmatrix}, \quad (7.37)$$

$A_2 = -(A_0 + \mathbb{I})$ , and

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & r/t \\ 0 & 0 & -r/t & 0 \\ 0 & 0 & 0 & r^2/t^2 \\ 0 & 0 & -r^2/t^2 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/t^2 & 0 \\ 0 & 0 & 0 & 1/t^2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/t^2 \\ 0 & 0 & 1/t^2 & 0 \end{pmatrix}. \quad (7.38)$$

It follows that  $V$  is invariant under  $A_0, A_1, A_2, B_0, B_1$  and  $B_2$  (for example by differentiating the right hand sides of (7.35) and (7.36) two more times). Since these matrices are real (anti) self-adjoint, they leave  $V^\perp$  invariant as well. Hence,  $V$  and  $V^\perp$  are generated by real eigenvectors of these matrices, if they are diagonalizable over  $\mathbb{R}$ . Note that  $B_1$  is diagonal in the canonical basis denoted by  $\{e_j\}_{j \in \{1,2,3,4\}}$ .

If  $V$  is one-dimensional, it is generated by one vector which is either in the subspace  $\langle e_1, e_2 \rangle$  or in the  $\langle e_3, e_4 \rangle$ . As neither of these subspaces is invariant under  $A_1$ , this is impossible. The same is true for  $V^\perp$ , so that  $V$  must be two-dimensional.  $V$  cannot coincide with any of the previously considered subspaces, by the same argument, so the only possibility left is

$$V = \langle w_1, w_2 \rangle, \quad \text{where } w_1 = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix} \quad \text{and } \alpha, \beta, \gamma, \delta \in \mathbb{R}. \quad (7.39)$$

But, to have  $A_1 w_1 \in V$ , there must exist  $a, b \in \mathbb{R}$  such that  $A_1 w_1 = a w_1 + b w_2$ , i.e.

$$\begin{pmatrix} \beta \\ -\alpha \\ \beta r/t \\ -\alpha r/t \end{pmatrix} = \begin{pmatrix} a\alpha \\ a\beta \\ b\gamma \\ b\delta \end{pmatrix}. \quad (7.40)$$

The first two components imply  $(1 + a^2)\beta = 0$ , thus  $\beta = 0$  and  $\alpha = 0$ , which is absurd. Hence  $\tau(G_{\alpha,\nu})$  is irreducible.

Let us now turn to the second statement of Lemma 7.1. The non-compactness of  $\tau(G_{\alpha,\nu})$  and  $G_{\alpha,\nu}$  being equivalent, we can choose to work on  $G_{\alpha,\nu}$ . We will actually show a much stronger statement. For this, pick any fixed  $\theta$  and  $\eta$  on the torus with  $\theta \neq \eta$ . Consider the subgroup  $G(\theta, \eta)$  of  $GL(2, \mathbb{C})$  generated by  $T(\theta, \theta), T(\eta, \eta), T(\theta, \eta)$  and  $T(\eta, \theta)$ , where the latter are defined through (3.11). As  $\alpha + \text{supp } \nu$  contains at least two points, non-compactness of  $G_{\alpha,\nu}$  will follow from non-compactness of  $G(\theta, \eta)$ .

With the abbreviations  $x := e^{-i\theta}$  and  $z := e^{-i\eta}$  the first of them takes the form

$$T(\theta, \eta) = \begin{pmatrix} -z & \frac{r}{t}(\bar{x}z - z) \\ \frac{r}{t}(1 - z) & \frac{r^2}{t^2}(\bar{x}z + 1 - z) - \frac{1}{t^2}\bar{x} \end{pmatrix},$$

with  $\det T(\theta, \eta) = \bar{x}z$ , and similarly for the other three generating matrices.

Calculations (some lengthy) show that

$$C := T(\theta, \theta)T(\theta, \eta)^{-1} = \begin{pmatrix} x\bar{z} & 0 \\ \frac{r}{t}(x\bar{z} - 1) & 1 \end{pmatrix} \in G(\theta, \eta),$$

$$E := T(\eta, \theta)^{-1}T(\theta, \theta) = \begin{pmatrix} 1 & \frac{r}{t}(1 - \bar{x}z) \\ 0 & \bar{x}z \end{pmatrix} \in G(\theta, \eta),$$

$$L := CE = \begin{pmatrix} x\bar{z} & \frac{r}{t}(x\bar{z} - 1) \\ \frac{r}{t}(x\bar{z} - 1) & \bar{x}z - \frac{r^2}{t^2}|x\bar{z} - 1|^2 \end{pmatrix} \in G(\theta, \eta),$$

$$J := EC = \begin{pmatrix} x\bar{z} - \frac{r^2}{t^2}|\bar{x}z - 1|^2 & \frac{r}{t}(1 - \bar{x}z) \\ \frac{r}{t}(1 - \bar{x}z) & \bar{x}z \end{pmatrix} \in G(\theta, \eta).$$

Note that  $\det L = \det J = 1$  and that  $J^{-1} = L^*$ . Thus we get the self-adjoint element  $K := J^{-1}L$  of  $G(\theta, \eta)$ . In fact,  $K$  is positive definite and  $\det K = 1$ .

More calculation shows that

$$\begin{aligned} \text{tr } K &= 1 + \frac{2r^2}{t^2}|x\bar{z} - 1|^2 + \left| x\bar{z} - \frac{r^2}{t^2}|x\bar{z} - 1|^2 \right|^2 \\ &= 2 + \frac{r^2}{t^4}|x\bar{z} - 1|^4. \end{aligned}$$

As  $\theta \neq \eta$  and therefore  $x\bar{z} \neq 1$  we conclude that  $\text{tr } K > 2$ . Positivity of  $K$  implies that it has an eigenvalue strictly bigger than 1. Thus, containing all powers of  $K$ , the group  $G(\theta, \eta)$  is non-compact. ■

## References

- [AM] Aizenman, M., Molchanov, S.: Localization at large disorder and at extreme energies: an elementary derivation, *Commun. Math. Phys.* **157**, 245-278, (1993).
- [B] Bourget, O.: Singular continuous Floquet operator for periodic Quantum systems, *J. Math. Anal. Appl.* **301**, 65-83, (2005).

- [BB] Blatter, G., Browne, D.: Zener tunneling and localization in small conducting rings, *Phys. Rev. B* **37**, 3856, (1988).
- [BHJ] Bourget, O., Howland, J.S., Joye, A.: Spectral Analysis of Unitary Band Matrices, *Commun. Math. Phys.* **234**, 191-227 (2003). .
- [C] Combesure, M.: Spectral Properties of a Periodically Kicked Quantum Hamiltonian, *J. Stat. Phys.* **59**, 679-690, (1990).
- [CFKS] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: *Schrödinger Operators*, Springer Verlag, 1987.
- [CKM] Carmona, R., Klein A., Martinelli, F.: Anderson localization for Bernoulli and other singular potentials, *Commun. Math. Phys.* **108**, 41-66, (1987).
- [CL] Carmona, R., Lacroix, J.: *Spectral theory of random Schrodinger Operators*, Birkhäuser, 1990.
- [CMV] Cantero, M.J., Moral, L., Velázquez, L.: Five-Diagonal Matrices and Zeros of Orthogonal Polynomials on the Unit Circle, *Linear Algebra Appl.* **362**, 29-56, (2003).
- [DSS] Damanik, D., Sims, R., Stolz, G.: Localization of one-dimensional, continuum, Bernoulli-Anderson models, *Duke Math. J.* **114**, 59-100, (2002).
- [GT] Geronimo, J.S., Teplyaev, A.: A Difference Equation Arising from the Trigonometric Moment Problem Having Random Reflection Coefficients-An Operator Theoretic Approach, *J. Func. Anal.* **123**, 12-45, (1994).
- [HS] Hamza, E., Stolz, G.: in preparation
- [J1] Joye, A.: Density of States and Thouless Formula for Random Unitary Band Matrices, *Ann. Henri Poincaré* **5**, 347-379, (2004).
- [J2] Joye, A.: Fractional Moment Estimates for Random Unitary Band Matrices, *Lett. Math. Phys.*, to appear.
- [KS] Kotani, S., Simon, B.: Localization in general one-dimensional random systems, *Commun. Math. Phys.* **112**, 103-119, (1987).
- [S1] Simon, B.: Orthogonal Polynomials on the Unit Circle, Vol. 1 and 2, *AMS Colloquium Series, American Mathematical Society, Providence, RI*, to appear.
- [S2] Simon, B.: Aizenman's Theorem for Orthogonal Polynomials on the Unit Circle, preprint, mp-arc 04-386.
- [S3] Simon, B.: OPUC on one foot, preprint, mp-arc 05-78.
- [SW] Simon, B., Wolff, T.: Singular Continuous Spectrum under Rank One Perturbations and Localization for Random Hamiltonians, *Commun. Pure Appl. Math.* **39**, 75-90, (1986).
- [Su] Stoiciu, M.: The statistical distribution of the zeros of random paraorthonormal polynomials on the unit circle, preprint, mp-arc 04-405.
- [T] Teplyaev, A. V.: the Pure Point Spectrum of Random Polynomials orthogonal on the Circle, *Soviet. Math. Dokl.* **44**, 407-411, (1992).