SMILANSKY'S MODEL OF IRREVERSIBLE QUANTUM GRAPHS, I: THE ABSOLUTELY CONTINUOUS SPECTRUM

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Abstract. In the model suggested by Smilansky [7] one studies an operator describing the interaction between a quantum graph and a system of K one-dimensional oscillators attached at several different points in the graph. The present paper is the first one in which the case $K > 1$ is investigated. For the sake of simplicity we consider $K = 2$, but our argument is of a general character. In this first of two papers on the problem, we describe the absolutely continuous spectrum. Our approach is based upon scattering theory.

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1. Introduction

In the paper [7] U. Smilansky suggested a mathematical model to which he gave the name "Irreversible quantum graph". In this model one studies the interaction between a quantum graph and a finite system of one-dimensional oscillators attached at several different points in the graph. Recall that the term "quantum graph" usually stands for a metric graph Γ equipped with a self-adjoint differential operator acting on $L^2(\Gamma)$; see the survey paper [4] and references therein. In our case this operator will be the Laplacian $-\Delta$.

In Smilansky's model one initially deals with two independent dynamical systems. One of the systems acts in $L^2(\Gamma)$ and its Hamiltonian is the Laplacian. Another system acts in the space $L^2(\mathbb{R}^K)$, $K \geq 1$ and is generated by the Hamiltonian $H_{osc} = \sum_{k=1}^K h_k$ where

$$h_k = \frac{\nu_k^2}{2}\left(-\frac{\partial^2}{\partial {q_k}^2} + q_k^2\right), \qquad k = 1, \dots, K;$$

in [7] the oscillators are written in a slightly different form; one form reduces to another by scaling. In what follows the points in Γ are denoted by x and the points in $\mathbb{R}^K$ by $\mathbf{q} = (q_1, \dots, q_K)$.

Consider now the operator

1.1 (1.1) $$\mathbf{A}_0 = -\Delta \otimes I + I \otimes H_{osc}$$

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in the space $L^2(\Gamma \times \mathbb{R}^K)$. It is defined by the differential expression

1.2
$$\mathcal{A}U = -\Delta_x U + \frac{1}{2}\sum_{k=1}^{K} \nu_k^2\big(-\frac{\partial^2 U}{\partial {q_k}^2} + q_k^2 U\big) \tag{1.2}$$

and is self-adjoint on the natural domain. The terms in (1.1) do not interact with each other.

Interaction is introduced with the help of a system of "matching conditions" on the derivative U'_x at some points $o_1, \dots, o_K \in \Gamma$. One says that the k-th oscillator is attached to the graph at the point o_k. The condition at the point o_k is

1.4
$$[U'_x](o_k, \mathbf{q}) = \alpha_k q_k U(o_k, \mathbf{q}), \qquad k = 1, \dots, K, \tag{1.3}$$

where $[f'_x](.)$ stands for the expression appearing in the Kirchhoff condition, well known in the theory of electric networks. When $\Gamma = \mathbb{R}$ (which is the only case we deal with in the main body of the paper), $[f'_x](.)$ is the jump of the derivative,

1.5
$$[f'_x](o) = f'_x(o+) - f'_x(o-). \tag{1.4}$$

The real parameter α_k in (1.3) expresses the strength of interaction between the quantum graph and the oscillator h_k. The case $\alpha_1 = \ldots = \alpha_K = 0$ corresponds to the operator $\mathbf{A}_0$ as in (1.1).

Sometimes we shall denote by α, ν the multi-dimensional parameters $\alpha = \{\alpha_1, \dots, \alpha_K\}$, $\nu = \{\nu_1, \dots, \nu_K\}$. Let $\mathbf{A}_{\alpha;\nu} = \mathbf{A}_{\alpha_1,\dots,\alpha_K;\nu_1,\dots,\nu_K}$ stand for the operator defined by the differential expression (1.2) and the conditions (1.3). Usually, the values of ν_k are fixed and we exclude them from the notation. On the other hand, we use the notation $\mathbf{A}_{\Gamma;\alpha;\nu}$ for this operator when it is necessary to reflect its dependence on the graph.

The problem to be considered is the description of the spectrum of the dynamical system generated by the Hamiltonian $\mathbf{A}_{\alpha;\nu}$. More specifically, it is to construct the self-adjoint realization of $\mathbf{A}_{\alpha;\nu}$ as an operator in the Hilbert space $L^2(\Gamma \times \mathbb{R}^K)$ and to describe its spectrum.

Up until now, the problem has only been investigated for the simplest case $K = 1$. The first results were obtained in the paper [7] by Smilansky. Then a detailed study of the problem was carried out in the papers [8], [9] and [5]. In [10], along with some new results, a detailed survey of the current state of the problem is given.

On first sight, the problem might seem amenable to the perturbation theory of quadratic forms. Indeed, the spectrum $\sigma(\mathbf{A}_0)$ can be easily described by separation of variables and the perturbation in the quadratic form, which appears when passing from $\mathbf{A}_0$ to $\mathbf{A}_\alpha$ with $\alpha \neq 0$, seems not to be too strong. However, this is not so: this perturbation turns out to be only form-bounded but not form-compact, which makes it impossible to apply the standard techniques. So, the problem requires certain specific tools which were developed in [8] – [10] and [5]. The most important of these tools is the systematic use of Jacobi matrices.

It was found in the above mentioned papers on the one-oscillator problem that the character of the spectrum strongly depends on the size of α: there exists some $\alpha^* > 0$ such that the absolutely continuous spectrum $\sigma_{a.c.}(\mathbf{A}_\alpha)$ coincides with $\sigma_{a.c.}(\mathbf{A}_0)$ if $|\alpha| < \alpha^*$ (in particular, it is absent if the graph is compact) and fills the whole of $\mathbb{R}$ if $|\alpha| > \alpha^*$. The dependence of the structure of the point spectrum $\sigma_p(\mathbf{A}_\alpha)$ on α is also well understood.

This is the first of two papers on the problem for $K > 1$ and in it we study the absolutely continuous spectrum; in our other paper [1] the point spectrum is investigated. This division is natural, since the technical tools used in each part are different. We address the simplest situation, when $\Gamma = \mathbb{R}$ and $K = 2$, but our argument is of a rather general character and we firmly believe that it applies to a wide class of graphs and to any K. However, in the general case, the calculations become more complicated and this obscures the main features of the argument.

We first describe informally the main ideas lying behind our approach.

The effect of adding one more oscillator to a system with K oscillators is twofold. Firstly, the total dimension of the set $\Gamma \times \mathbb{R}^K$ increases by one which certainly affects the spectrum. Secondly, there is some effect coming from the additional matching condition (1.3) at the point o_{K+1}. This second effect disappears if we take $\alpha_{K+1} = 0$. Indeed, then the variable q_{K+1} can be separated and the operator decomposes into the orthogonal sum of simpler operators. More exactly, denote by $\widetilde{\mathbf{A}}$ the operator which corresponds to the configuration with the $(K+1)$-th oscillator removed,

$$\widetilde{\mathbf{A}} = \mathbf{A}_{\alpha_1,\dots,\alpha_K;\nu_1,\dots,\nu_K}.$$

Then it is easy to see that

1.6
$$\mathbf{A}_{\alpha_1,\dots,\alpha_K,0;\nu_1,\dots,\nu_K,\nu_{K+1}} = \sum_{n\in\mathbb{N}_0}^{\oplus} (\widetilde{\mathbf{A}} + \nu_{K+1}^2(n+1/2)). \tag{1.5}$$

This orthogonal decomposition yields the complete description of the spectrum of the operator on the left-hand side, provided that the spectrum of $\widetilde{\mathbf{A}}$ is known.

The key observation which allows one to solve the general problem is that the interaction between the oscillators attached at different points is weak. For $K = 2$ this observation leads to the conclusion that the study of $\sigma(\mathbf{A}_{\alpha_1,\alpha_2;\nu_1,\nu_2})$ can be reduced to the same problem for the operators $\mathbf{A}_{\alpha_1,0;\nu_1,\nu_2}$ and $\mathbf{A}_{0,\alpha_2;\nu_1,\nu_2}$. Due to the equality (1.5) this reduces the problem to the study of the spectra of two operators, $\mathbf{A}_{\alpha_1;\nu_1}$ and $\mathbf{A}_{\alpha_2;\nu_2}$, each corresponding to the case of only one oscillator. Since the latter case is already well understood, we obtain the desired results for our more complicated case.

An accurate realization of this idea is different for the point spectrum and for the absolutely continuous spectrum. In the present paper we concentrate on the absolutely continuous spectrum. Here an important correction to the above scheme is necessary: the study of $\sigma_{a.c.}(\mathbf{A}_{\alpha_1,\alpha_2;\nu_1,\nu_2})$ does not reduce to the study of $\sigma_{a.c.}(\mathbf{A}_{\alpha_1;\nu_1})$ and $\sigma_{a.c.}(\mathbf{A}_{\alpha_1;\nu_1})$ for the same graph Γ. Rather, we have to divide Γ into two parts, $\Gamma = \Gamma_1 \cup \Gamma_2$ in such a way that $o_j \in \Gamma_j$ and $o_j \notin \Gamma_{3-j}$. Then $\sigma_{a.c.}(\mathbf{A}_{\Gamma;\alpha_1,\alpha_2;\nu_1,\nu_2})$ can be expressed in terms of $\sigma_{a.c.}(\mathbf{A}_{\Gamma_j;\alpha_j;\nu_j})$, $j = 1, 2$. The paper [1] is devoted to the study of the point spectrum. There such a partition of Γ is unnecessary.

We use the following notation. We write $\mathbb{N}_0$ for the set $\{0, 1, \dots\}$. The diagonal operator in an appropriate ℓ^2-space, with the diagonal elements $a_0, a_1, \dots$, is denoted by $\operatorname{diag}\{a_n\}$. We apply similar notation for the block-diagonal operators. The notation $\mathfrak{J}(\{a_n\}, \{b_n\})$ stands for the Jacobi matrix whose non-zero entries are $j_{n,n} = a_n$ and $j_{n,n+1} = j_{n+1,n} = b_n$. If $\mathbf{A}$ is a self-adjoint operator in a Hilbert space, then $\sigma(\mathbf{A}), \sigma_{a.c}(\mathbf{A}), \sigma_p(\mathbf{A})$ stand for its spectrum, absolutely continuous (a.c.) spectrum and point spectrum respectively. We use the symbol $\mathfrak{m}_{a.c}(\lambda; \mathbf{A})$ for the multiplicity function of the a.c. spectrum. The symbol $\mathfrak{S}_1$ stands for the trace class of compact operators.

Other necessary notations are introduced in the course of the presentation.

2. Statement of the problem. Results

st1

2.1. **The operator $\mathbf{A}_\alpha$.** As was mentioned in the introduction, we present our argument for the graph $\Gamma = \mathbb{R}$ and $K = 2$. We choose the points $o_1 = 1, o_2 = -1$ and denote the coordinates in $\mathbb{R}^2$ by q_+, q_- and the parameters by $\alpha = \{\alpha_+, \alpha_-\}$, $\nu = \{\nu_+, \nu_-\}$. The Laplacian on Γ is just the operator $-d^2/dx^2$ with the Sobolev space $H^2(\mathbb{R})$ as the operator domain. The operator $\mathbf{A}_{\alpha,\nu}$ acts in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3)$ and is defined by the differential expression

st.1
$$\mathcal{A}U = \mathcal{A}_\nu U = -U''_{x^2} + \frac{\nu_+^2}{2}(-U''_{q_+^2} + q_+^2 U) + \frac{\nu_-^2}{2}(-U''_{q_-^2} + q_-^2 U) \tag{2.1}$$

and the matching conditions (cf. (1.4))

st2
$$[U'_x](\pm 1, q_+, q_-) = \alpha_\pm q_\pm U(\pm 1, q_+, q_-). \tag{2.2}$$

So, in the notation of the introduction, we are dealing with the operator

$$\mathbf{A}_{\mathbb{R};\alpha_+,\alpha_-;\nu_+,\nu_-}.$$

However, as a rule we use the shortened notation $\mathbf{A}_\alpha$. Note that the replacement $\alpha_\pm \mapsto -\alpha_\pm$ corresponds to the change of variables $q_\pm \mapsto -q_\pm$ which does not affect the spectrum. For this reason, we discuss only $\alpha_\pm \geq 0$.

The structure of the differential expression $\mathcal{A}$ makes it natural to decompose the function U in a double series in terms of the normalized Hermite functions χ_n, namely

sa.0
$$U(x, q_+, q_-) = \sum_{m,n\in\mathbb{N}_0} u_{m,n}(x)\chi_m(q_+)\chi_n(q_-), \tag{2.3}$$

which is hereafter represented by $U \sim \{u_{m,n}\}$. The mapping $U \mapsto \{u_{m,n}\}$ is an isometry of the space $L^2(\mathbb{R}^3)$ onto the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{N}_0^2; L^2(\mathbb{R}))$. We evidently have $\mathcal{A}U \sim \{L_{m,n}u_{m,n}\}$ where

sa.1
$$(L_{m,n}u)(x) = -u''(x) + r_{m,n}u(x), \qquad x \neq \pm 1; \tag{2.4}$$

sa.2
$$r_{m,n} = \nu_+^2(m + 1/2) + \nu_-^2(n + 1/2), \qquad m, n \in \mathbb{N}_0. \tag{2.5}$$

The conditions at $x = \pm 1$ reduce to

sa.3
$$\begin{aligned} [u'_{m,n}](1) &= \frac{\alpha_+}{\sqrt{2}}\left(\sqrt{m+1}\,u_{m+1,n}(1) + \sqrt{m}\,u_{m-1,n}(1)\right); \\ [u'_{m,n}](-1) &= \frac{\alpha_-}{\sqrt{2}}\left(\sqrt{n+1}\,u_{m,n+1}(-1) + \sqrt{n}\,u_{m,n-1}(-1)\right). \end{aligned} \tag{2.6}$$

To derive the conditions (2.6) from (2.2), one uses the recurrency equation for the functions χ_n,

$$\sqrt{n+1}\,\chi_{n+1}(q) - \sqrt{2}q\chi_n(q) + \sqrt{n}\,\chi_{n-1}(q) = 0.$$

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2.2. **Operator $\mathbf{A}_0$.** The operator $\mathbf{A}_0 := \mathbf{A}_{0,0;\nu_+,\nu_-}$ admits separation of variables and we get

add1
$$\mathbf{A}_0 = \sum_{m,n}^{\oplus}(\mathbf{H}_0 + r_{m,n}), \tag{2.7}$$

where $\mathbf{H}_0$ is the self-adjoint operator $-d^2/dx^2$ in $L^2(\mathbb{R})$. This leads to the complete description of the spectrum $\sigma(\mathbf{A}_0)$, namely, that it is purely a.c. and fills the half-line $[r_{0,0}, \infty) = [(\nu_+^2 + \nu_-^2)/2, \infty)$. The expression for the multiplicity function $\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_0)$ immediately follows from (2.7), but is omitted.

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2.3. **Domain of $\mathbf{A}_\alpha$.** It is convenient to describe the domain of the self-adjoint realization of the operator $\mathbf{A}_\alpha$ in terms of the decomposition (2.3). Define the set $\mathcal{D}_\alpha$ as follows.

d **Definition 2.1.** An element $U \sim \{u_{m,n}\}$ lies in $\mathcal{D}_\alpha$ if and only if
1. $u_{m,n} \in H^1(\mathbb{R})$ for all m, n.
2. For all m, n the restriction of $u_{m,n}$ to each interval $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ lies in H^2 and moreover,

$$\sum_{m,n} \int_{\mathbb{R}} |L_{m,n} u_{m,n}|^2 dx < \infty.$$

3. The conditions (2.6) are satisfied.

Along with the set $\mathcal{D}_\alpha$, define its subset

$$\mathcal{D}_\alpha^\bullet = \{U \in \mathcal{D}_\alpha : U \sim \{u_{m,n}\} \text{ finite}\}$$

where by *finite* we mean that the sequence has only a finite number of non-zero components. Denote by $\mathbf{A}_\alpha^\bullet$ the operator in $\mathfrak{H} = L^2(\mathbb{R}^3)$, defined by the system (2.4) on the domain $\mathcal{D}_\alpha^\bullet$,

l1 **Lemma 2.2.** *The operator $\mathbf{A}_\alpha^\bullet$ is symmetric in $\mathfrak{H}$. Its adjoint coincides with the operator $\mathbf{A}_\alpha$ considered on the domain $\mathcal{D}_\alpha$:*

$$(\mathbf{A}_\alpha^\bullet)^* = \mathbf{A}_\alpha.$$

The proof is a straightforward modification of that for (5.2) in [5].

sa.thm **Theorem 2.3.** *For any $\alpha_+, \alpha_- \geq 0$ the operator $\mathbf{A}_\alpha$ is self-adjoint.*

The proof is given in section 4. Theorem 2.3 and Lemma 2.2 show that $\mathbf{A}_\alpha$ is the unique natural self-adjoint realization of the operator, defined by the differential expression (2.1) and the matching conditions (2.2).

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2.4. **Absolutely continuous spectrum of the operator $\mathbf{A}_\alpha$.** Below we construct an operator $\mathbf{A}_\alpha^\circ$ whose a.c. spectrum admits a complete description. Then we show that the a.c. spectra of both operators $\mathbf{A}_\alpha$ and $\mathbf{A}_\alpha^\circ$ coincide, including the multiplicities.

As a first step, let us consider two operators, $\mathbf{A}_{\alpha_+}^+$ and $\mathbf{A}_{\alpha_-}^-$. The operator $\mathbf{A}_{\alpha_+}^+$, say, acts in the space $L^2(\mathbb{R}_+ \times \mathbb{R}^2)$ and is defined by the differential expression (2.1), the matching condition (2.2) at the point $o_1 = 1$ and the Dirichlet condition $U(0, q_+, q_-) = 0$. The definition of $\mathbf{A}_{\alpha_-}^-$ is similar, with $\mathbb{R}_+$ replaced by $\mathbb{R}_-$ and the point $o_1 = 1$ by $o_2 = -1$. By separation of variables, the operators $\mathbf{A}_{\alpha_\pm}^\pm$ can be identified with the orthogonal sum of simpler operators:

one.2 (2.8)
$$\begin{aligned} \mathbf{A}_{\alpha_+}^+ &= \sum_{n \in \mathbb{N}_0}^{\oplus} \big(\mathbf{A}_{\mathbb{R}_+;\alpha_+;\nu_+} + \nu_-^2(n + 1/2)\big), \\ \mathbf{A}_{\alpha_-}^- &= \sum_{m \in \mathbb{N}_0}^{\oplus} \big(\mathbf{A}_{\mathbb{R}_-;\alpha_-;\nu_-} + \nu_+^2(m + 1/2)\big). \end{aligned}$$

Hence, both operators are self-adjoint. The direct sum

one.3 (2.9) $$\mathbf{A}^\circ_\alpha = \mathbf{A}^\circ_{\alpha_+,\alpha_-;\nu_+,\nu_-} := \mathbf{A}^+_{\alpha_+} \oplus \mathbf{A}^-_{\alpha_-}$$

is a self-adjoint operator in the original Hilbert space $\mathfrak{H}$.

The following theorem is the main result of the paper. Its formulation involves the notion of wave operator, which is one of the basic notions in mathematical scattering theory; see e.g. [3], [6] or [11].

one.t1 **Theorem 2.4.** *For each of the pairs* $(\mathbf{A}_\alpha,\ \mathbf{A}^\circ_\alpha), (\mathbf{A}^\circ_\alpha,\ \mathbf{A}_\alpha)$, *there exist complete isometric wave operators. In particular, the absolutely continuous parts of* $\mathbf{A}_\alpha$ *and* $\mathbf{A}^\circ_\alpha$ *are unitarily equivalent.*

Theorem 2.4 and the formulae (2.8), (2.9) reduce the study of $\sigma_{a.c.}(\mathbf{A}_\alpha)$ to the similar problem for the case of only one oscillator. The latter problem was solved in [5] and [10]. The next statement collects, for the particular case we need, the results of section 3 in [10]; see also Theorem 5.1 and remarks in section 9 of [5]. In both papers it was assumed that $\nu = 1$, and we arrive at the formulation below via scaling. By default, we take $\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}) = 0$ if $\lambda \notin \sigma_{a.c.}(\mathbf{A})$.

prop **Proposition 2.5.** (The case of one oscillator.) *Let* $\Gamma = \mathbb{R}_+$ *and* $o = 1$, *or* $\Gamma = \mathbb{R}_-$ *and* $o = -1$. *Then*

1) $$\sigma_{a.c.}(\mathbf{A}_{0;\nu}) = [\nu^2/2, \infty);$$
$$\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_{0;\nu}) = n \text{ for } -\nu^2/2 \le \lambda - \nu^2 n < \nu^2/2,\ n \in \mathbb{N};$$

2) *if* $0 < \alpha < \nu\sqrt{2}$, *then*
$$\sigma_{a.c.}(\mathbf{A}_{\alpha;\nu}) = \sigma_{a.c.}(\mathbf{A}_{0;\nu}) = [\nu^2/2, \infty);$$
$$\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_{\alpha;\nu}) = \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_{0;\nu});$$

3) *if* $\alpha = \nu\sqrt{2}$, *then*
$$\sigma_{a.c.}(\mathbf{A}_{\alpha;\nu}) = [0, \infty); \qquad \mathfrak{m}_{\mathrm{a.c.}}(\lambda; \mathbf{A}_{\alpha;\nu}) = \mathfrak{m}_{\mathrm{a.c.}}(\lambda; \mathbf{A}_{0;\nu}) + 1,\ \forall \lambda \ge 0;$$

4) *if* $\nu\sqrt{2} < \alpha < \infty$, *then*
$$\sigma_{a.c.}(\mathbf{A}_{\alpha;\nu}) = \mathbb{R}; \qquad \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_{\alpha;\nu}) = \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_{0;\nu}) + 1,\ \forall \lambda \in \mathbb{R}.$$

Now we are in a position to present the final formula for the function $\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}^\circ_\alpha)$, and thus for our original operator $\mathbf{A}_\alpha$.

add3 (2.10) $$\begin{aligned}\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_\alpha) = &\sum_{n \in \mathbb{N}_0} \mathfrak{m}_{a.c.}(\lambda - \nu_-^2(n + 1/2); \mathbf{A}_{\mathbb{R}_+;\alpha_+;\nu_+}) \\ &+ \sum_{m \in \mathbb{N}_0} \mathfrak{m}_{a.c.}(\lambda - \nu_+^2(m + 1/2); \mathbf{A}_{\mathbb{R}_-;\alpha_-;\nu_-}).\end{aligned}$$

Combining the equality (2.10) with Proposition 2.5, we obtain the following description of the a.c. spectrum of the operator $\mathbf{A}_\alpha$ for any $\alpha_+, \alpha_- \ge 0$.

one.t2a **Theorem 2.6.** *Let* $\mathbf{A}_\alpha = \mathbf{A}_{\alpha;\nu}$ *be the self-adjoint operator defined by the differential expression* (2.1) *on the operator domain* $\mathcal{D}_\alpha$.
1) *If* $\alpha_\pm/\nu\pm < \sqrt{2}$, *then*

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = [r_{0,0}, \infty) = [(\nu_+^2 + \nu_-^2)/2, \infty).$$

2) *Let* $\alpha_+/\nu_+ = \sqrt{2}$ *and* $\alpha_-/\nu_- < \sqrt{2}$, *or* $\alpha_-/\nu_- = \sqrt{2}$ *and* $\alpha_+/\nu_+ < \sqrt{2}$. *Then*

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = [\nu_-^2/2, \infty) \qquad \text{or} \qquad \sigma_{a.c.}(\mathbf{A}_\alpha) = [\nu_+^2/2, \infty)$$

respectively.
3) *Let* $\alpha_+/\nu_+ = \alpha_-/\nu_- = \sqrt{2}$, *then*

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = [0, \infty).$$

In all the cases 1 – 3 *the multiplicity function* $\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_\alpha)$, *given by the equality* (2.10), *is finite for all* $\lambda \in \sigma_{a.c.}(\mathbf{A}_\alpha)$.

4) *Let* $max(\alpha_+/\nu_+, \alpha_-/\nu_-) > \sqrt{2}$. *Then*

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = \mathbb{R}, \qquad \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_\alpha) \equiv \infty.$$

In connection with this theorem, we would like to emphasize that the existence of the wave operators established in Theorem 2.4 gives much more information about the operator $\mathbf{A}_\alpha$ than just the description of its a.c. spectrum.

For the proof of Theorem 2.4 we use the following classical result due to Kato, see Theorem 6.5.1 and Remark 6.5.2 in [11].

kato **Proposition 2.7.** *Let* $\mathbf{A}, \mathbf{A}^\circ$ *be self-adjoint operators in a Hilbert space. Suppose that for some natural number* p *the inclusion*

$$(\mathbf{A}^\circ - \Lambda)^{-p} - (\mathbf{A}^\circ - \Lambda)^{-p} \in \mathfrak{S}_1$$

is satisfied for all non-real $\Lambda \in \mathbb{C}$. *Then the complete isometric wave operators exist for both pairs* $\mathbf{A}, \mathbf{A}^\circ$ *and* $\mathbf{A}^\circ, \mathbf{A}$.

In our case the conditions of Proposition 2.7 turn out to be fulfilled with $p = 3$. This is the result of the following statement whose proof is our main technical goal in this paper.

one.t2 **Theorem 2.8.** *For any non-real* $\Lambda \in \mathbb{C}$ *one has*

goal (2.11)
$$(\mathbf{A}_\alpha^\circ - \Lambda)^{-3} - (\mathbf{A}_\alpha - \Lambda)^{-3} \in \mathfrak{S}_1.$$

Theorem 2.4 is a direct consequence of Theorem 2.8.

The proof of Theorem 2.8 is rather long and requires some preparatory work.

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3. Auxiliary material

In this section we present some elementary technical material concerning the equations

res.1g (3.1) $$-u'' + \zeta^2 u = f,$$

res.1go (3.2) $$-v'' + \zeta^2 v = 0.$$

where $\zeta = \gamma + i\delta$ is a complex parameter. We need this material for the proofs of both our main technical results, Theorem 2.3 and Theorem 2.8.

We assume that $\gamma > 0$, and are mainly interested in estimates which are uniform with respect to ζ.

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3.1. **Homogeneous equation.** Let $\mathfrak{F}_\zeta$ be the two-dimensional space of functions on $\mathbb{R}$ which are continuous, vanish as $|x| \to \infty$, and for $x \neq \pm 1$ satisfy the equation (3.2). We choose the following basis $\varphi_\zeta^+, \varphi_\zeta^-$ in $\mathfrak{F}_\zeta$:

$$\varphi_\zeta^+(x) = \begin{cases} 0, & x < -1, \\ \frac{\sinh \zeta(x+1)}{\sinh 2\zeta}, & -1 \le x \le 1; \\ e^{-\zeta(x-1)}, & x > 1; \end{cases} \qquad \varphi_\zeta^-(x) = \varphi_\zeta^+(-x).$$

Then

1 (3.3) $$\varphi_\zeta^+(1) = \varphi_\zeta^-(-1) = 1, \qquad \varphi_\zeta^+(-1) = \varphi_\zeta^-(1) = 0.$$

Just for this reason this basis is more convenient than the "natural" basis consisting of the functions $e^{-\zeta|x\pm1|}$. Note also that

$$\left[(\varphi_\zeta^\pm)'\right](\pm 1) = -\frac{2\zeta}{1-e^{-4\zeta}}, \qquad \left[(\varphi_\zeta^\pm)'\right](\mp 1) = \frac{2\zeta e^{-2\zeta}}{1-e^{-4\zeta}}$$

and hence, for all $v \in \mathfrak{F}_\zeta$,

1c (3.4) $$\begin{aligned} [v'](1) &= -\frac{2\zeta}{1-e^{-4\zeta}} \left(v(1) - e^{-2\zeta} v(-1)\right), \\ [v'](-1) &= -\frac{2\zeta}{1-e^{-4\zeta}} \left(v(-1) - e^{-2\zeta} v(1)\right). \end{aligned}$$

A standard calculation shows that for the norm and scalar product in $L^2(\mathbb{R})$,

1f (3.5) $$\begin{gathered} \|\varphi_\zeta^+\|^2 = \|\varphi_\zeta^-\|^2 = \frac{1}{2\gamma} + \frac{\gamma^{-1}\sinh 4\gamma - \delta^{-1} \sin 4\delta}{4(\sinh^2 2\gamma + \sin^2 2\delta)} = \gamma^{-1} + o(e^{-4\gamma}), \\ (\varphi_\zeta^+, \varphi_\zeta^-) = O(e^{-2\gamma}), \qquad \gamma \to \infty. \end{gathered}$$

This shows that for γ large the chosen basis is "almost orthogonal". It follows that the two-sided estimate

1x (3.6) $$c_0^{-1}\gamma\|v\|^2 \le |C_+|^2 + |C_-|^2 \le c_0\gamma\|v\|^2, \qquad v = C_+\varphi_\zeta^+ + C_-\varphi_\zeta^- \in \mathfrak{F}_\zeta$$

with some $c_0 > 1$ is satisfied uniformly in any half-plane $\gamma = \operatorname{Re}\zeta \ge \gamma_0 > 0$.

Now we turn to the subspace $\mathfrak{F}^\circ_\zeta$ formed by the functions $v \in \mathfrak{F}_\zeta$, satisfying an additional condition $v(0) = 0$. The functions

$$\varphi^{\circ,+}_\zeta(x) = \begin{cases} 0, & x < 0, \\ \frac{\sinh \zeta x}{\sinh \zeta}, & 0 \le x \le 1, \\ e^{-\zeta(x-1)}, & x > 1; \end{cases} \qquad \varphi^{\circ,-}_\zeta(x) = \varphi^{\circ,+}_\zeta(-x)$$

form a natural basis in $\mathfrak{F}^\circ_\zeta$. For the functions $\varphi^{\circ,\pm}_\zeta$ the equalities (3.3) are satisfied, and instead of (3.4) we have

1ac (3.7)
$$[v'](\pm 1) = -\frac{2\zeta}{1 - e^{-2\zeta}} v(\pm 1), \qquad \forall v \in \mathfrak{F}^\circ_\zeta.$$

Similarly to (3.5), we find that

$$\|\varphi^{\circ,\pm}_\zeta\|^2 = \gamma^{-1} + O(e^{-2\gamma}), \qquad \left(\varphi^{\circ,+}_\zeta, \varphi^{\circ,-}_\zeta\right) = O(e^{-\gamma}), \qquad \gamma \to \infty.$$

As a consequence, we conclude that an analogue of (3.6), with the functions $\varphi^\pm_\zeta$ replaced by $\varphi^{\circ,\pm}_\zeta$, is valid for $v \in \mathfrak{F}^\circ_\zeta$.

A straightforward calculation shows also that

2c (3.8)
$$\|\varphi^{\circ,\pm}_\zeta - \varphi^\pm_\zeta\| = O(e^{-\gamma}), \qquad \gamma \to \infty.$$

nhom

3.2. **Non-homogeneous equation.** Here we discuss the equation (3.1) without the matching conditions at $x = \pm 1$ or, equivalently, under the conditions of the type (2.6) with $\alpha = 0$. Then the solution is given by

res.1 (3.9)
$$u_\zeta(x) = (2\zeta)^{-1} \int_{\mathbb{R}} e^{-\zeta|x-t|} f(t)dt.$$

The solution of the same equation (3.1) subject to the condition $u(0) = 0$ is

res.1b (3.10)
$$u^\circ_\zeta(x) = \begin{cases} (2\zeta)^{-1} \int_{\mathbb{R}_+} \left(e^{-\zeta|x-t|} - e^{-\zeta(x+t)}\right) f(t)dt, & x > 0; \\ (2\zeta)^{-1} \int_{\mathbb{R}_-} \left(e^{-\zeta|x-t|} - e^{\zeta(x+t)}\right) f(t)dt, & x < 0. \end{cases}$$

The difference $u^\circ_\zeta - u_\zeta$ is given by a rank one operator,

res.1f (3.11)
$$u^\circ_\zeta(x) - u_\zeta(x) = -(2\zeta)^{-1} g_\zeta(x) \int_{\mathbb{R}} f(t) g_\zeta(t)dt, \qquad g_\zeta(x) = e^{-\zeta|x|}.$$

Note that $\|g_\zeta\|^2 = \gamma^{-1}$.

add

3.3. **Dependence on the additional parameters.** We are particularly interested in the case when ζ depends on two parameters $r \in \mathbb{R}$ and $\Lambda \in \mathbb{C}$, where $r \ge r_0 > 0$, $\Lambda \notin \mathbb{R}_+$:

2 (3.12)
$$\zeta = \zeta_r(\Lambda) := \gamma_r(\Lambda) + i\delta_r(\Lambda) = \sqrt{r - \Lambda}.$$

We select the branch of the square root in (3.12) to have

$$\operatorname{Re} \zeta_r(\Lambda) > 0, \qquad \operatorname{Im} \Lambda \cdot \operatorname{Im} \zeta_r(\Lambda) \le 0.$$

For Λ fixed all the points $\zeta_r(\Lambda)$ lie in some half-plane $\operatorname{Re}\zeta_r(\Lambda) \geq \gamma_0(\Lambda) > 0$, hence (3.6) is satisfied. It is clear that $\gamma_r(\Lambda) \sim r^{1/2}$ as $r \to \infty$. Therefore, for any $\Lambda \notin \mathbb{R}$ there exists a constant $c_1 = c_1(\Lambda) > 1$ such that

2r (3.13)
$$c_1^{-1} r^{1/2} \|v\|^2 \leq |C_+|^2 + |C_-|^2 \leq c_1 r^{1/2} \|v\|^2, \\ \forall v = C_+\varphi_\zeta^+ + C_-\varphi_\zeta^- \in \mathfrak{F}_\zeta, \qquad \zeta = \zeta_r(\Lambda).$$

pr

4. Self-adjointness: proof of Theorem 2.3

According to the general theory of self-adjoint operators, we must show that the equation $\mathbf{A}_\alpha V = \Lambda V$ has only the trivial solution for some (and then all) $\Lambda \in \mathbb{C}_\pm$. To simplify our notation, we shall denote

sa.4 (4.1)
$$\zeta_{m,n}(\Lambda) = \zeta_{r_{m,n}}(\Lambda) = (r_{m,n} - \Lambda)^{1/2}, \qquad \varphi^\pm_{m,n}(x;\Lambda) = \varphi^\pm_{\zeta_{m,n}(\Lambda)}(x).$$

If $V \sim \{v_{m,n}\}$, then each function $v_{m,n}$ can be written as

sa.4y (4.2)
$$v_{m,n}(x) = r_{m,n}^{1/4}\left(C^+_{m,n}\varphi^+_{m,n}(x;\Lambda) + C^-_{m,n}\varphi^-_{m,n}(x;\Lambda)\right),$$

with some coefficients $C^\pm_{m,n}$. We have inserted the factor $r_{m,n}^{1/4}$ in order that (cf. (3.13))

sa.4a (4.3)
$$\{C^+_{m,n}, C^-_{m,n}\} \in \ell^2 \iff \{V \in \mathfrak{H}\}.$$

The matching conditions (2.6) at $x = \pm 1$ yield an infinite system of homogeneous linear equations for the unknown coefficients $C^\pm_{m,n}$. Below we set $\mu_\pm = \sqrt{2}/\alpha_\pm$. Taking (3.4) into account, we get from the condition at $x = 1$:

sa.4b (4.4)
$$r_{m+1,n}^{1/4}(m+1)^{1/2}C^+_{m+1,n} + \frac{2\mu_+\zeta_{m,n}(\Lambda)r_{m,n}^{1/4}}{1 - e^{-4\zeta_{m,n}(\Lambda)}}\left(C^+_{m,n} - C^-_{m,n}e^{-2\zeta_{m,n}(\Lambda)}\right) \\ + r_{m-1,n}^{1/4} m^{1/2} C^+_{m-1,n} = 0.$$

It is convenient to multiply each equation by the factor $r_{m,n}^{1/4}$. Let us also denote

sa.5 (4.5)
$$q^+_{m,n} = m^{1/2} r_{m,n}^{1/4} r_{m-1,n}^{1/4}, \qquad q^-_{m,n} = n^{1/2} r_{m,n}^{1/4} r_{m,n-1}^{1/4}; \\ p_{m,n}(\Lambda) = \zeta_{m,n}(\Lambda) r_{m,n}^{1/2}.$$

The equation (4.4) and the similar equation coming from the condition (2.6) at $x = -1$ yield

sa.6 (4.6)
$$q^+_{m+1,n}C^+_{m+1,n} + \frac{2\mu_+ p_{m,n}(\Lambda)}{1 - e^{-4\zeta_{m,n}(\Lambda)}}\left(C^+_{m,n} - C^-_{m,n}e^{-2\zeta_{m,n}(\Lambda)}\right) + q^+_{m,n}C^+_{m-1,n} = 0,$$
$$q^-_{m,n+1}C^-_{m,n+1} + \frac{2\mu_- p_{m,n}(\Lambda)}{1 - e^{-4\zeta_{m,n}(\Lambda)}}\left(C^-_{m,n} - C^+_{m,n}e^{-2\zeta_{m,n}(\Lambda)}\right) + q^-_{m,n}C^-_{m,n-1} = 0.$$

Denote by $\mathcal{R} = \mathcal{R}(\Lambda)$ the infinite matrix which corresponds to this system. In view of (4.3), we consider $\mathcal{R}$ as an operator in the space

$$\mathfrak{G} = \ell^2(\mathbb{N}_0^2; \mathbb{C}^2).$$

Removing in (4.6) the exponentially small terms, we come to a simpler system

sa.7
$$q^+_{m+1,n} C^+_{m+1,n} + 2\mu_+ p_{m,n}(\Lambda) C^+_{m,n} + q^+_{m,n} C^+_{m-1,n} = 0; \tag{4.7}$$

sa.8
$$q^-_{m,n+1} C^-_{m+1,n} + 2\mu_- p_{m,n}(\Lambda) C^-_{m,n} + q^-_{m,n} C^-_{m,n-1} = 0. \tag{4.8}$$

Let $\mathcal{R}' = \mathcal{R}'(\Lambda)$ stand for the matrix which corresponds to the system (4.7) – (4.8), and also for the operator in $\mathfrak{G}$ generated by this matrix. The operator $\mathcal{R}'$ decomposes into an infinite family of simpler operators. First of all, the equations (4.7) (for $C^+_{m,n}$) and (4.8) (for $C^-_{m,n}$) are mutually independent. Further, fix any $n \in \mathbb{N}_0$. The equations in (4.7) which correspond to the chosen value of n form a linear system in $\ell^2(\mathbb{N}_0)$ with the Jacobi matrix

$$\mathcal{J}^+_n(\Lambda) = \mathcal{J}\left(\{2\mu_+ p_{m,n}(\Lambda)\}, \{q^+_{m,n}\}\right).$$

In the same way, the equations in (4.8), which correspond to the chosen value of m, form a linear system in $\ell^2(\mathbb{N}_0)$ with the Jacobi matrix

$$\mathcal{J}^-_m(\Lambda) = \mathcal{J}\left(\{2\mu_- p_{m,n}(\Lambda)\}, \{q^-_{m,n}\}\right).$$

The above reasoning shows that

sa.11
$$\mathcal{R}'(\Lambda) = \sum_n{}^{\oplus} \mathcal{J}^+_n(\Lambda) \oplus \sum_m{}^{\oplus} \mathcal{J}^-_m(\Lambda). \tag{4.9}$$

The original operator $\mathcal{R}$ can be written as

sa.11p
$$\mathcal{R}(\Lambda) = \mathcal{R}'(\Lambda) + \mathcal{N}(\Lambda) \tag{4.10}$$

where $\mathcal{N} = \mathcal{N}(\Lambda)$ is a block-diagonal matrix with 2×2-blocks:

sa.11n
$$N_{m,n}(\Lambda) = \frac{2p_{m,n}(\Lambda) e^{-2\zeta_{m,n}(\Lambda)}}{1 - e^{-4\zeta_{m,n}(\Lambda)}} \begin{pmatrix} \mu_+ e^{-2\zeta_{m,n}(\Lambda)} & -\mu_+ \\ -\mu_- & e^{-2\zeta_{m,n}(\Lambda)} \end{pmatrix}. \tag{4.11}$$

The last two equations elucidate the structure of the matrix $\mathcal{R}(\Lambda)$.

In the rest of the section we take $\Lambda = i\tau \in i\mathbb{R}$. We will show that each term on the right-hand side of (4.9) is an invertible operator in ℓ^2 and that the norms of $\|\mathcal{J}^\pm_k(i\tau)^{-1}\|$ are uniformly bounded. For this purpose, we note that

$$(2\mu_+)^{-1} \operatorname{Im} \mathcal{J}^+_n(i\tau) = (2\mu_-)^{-1} \operatorname{Im} \mathcal{J}^-_m(i\tau) = \operatorname{diag}\{p_{m,n}(i\tau)\}.$$

We have $p_{m,n}(i\tau) = \sqrt{r^2_{m,n} - i r_{m,n}\tau} = X + iY$ where

$$2Y^2 = (r^4_{m,n} + r^2_{m,n}\tau^2)^{1/2} - r^2_{m,n} = \frac{r_{m,n}\tau^2}{(r^2_{m,n} + \tau^2)^{1/2} + r_{m,n}} \geq 2c^2|\tau|.$$

The last inequality, with some constant $c > 0$, is valid for $|\tau| \geq \tau_0$ and for any $m, n \geq 0$; we have taken into account that $r_{m,n} \geq r_{0,0} = (\nu_+^2 + \nu_-^2)/2$.

By the well known estimate for the operators with sign-defined imaginary part, see e.g. Theorem IV.4.1 in [2], this implies that

$$\|\mathcal{J}_k^\pm(i\tau)^{-1}\| \leq \left(c\sqrt{|\tau|}\right)^{-1}, \qquad \forall k \in \mathbb{N}_0,$$

and therefore

sa.13 (4.12) $$\|\mathcal{R}'(\tau)^{-1}\| = \sup \|\mathcal{J}_k^\pm(\tau)^{-1}\| \leq \left(c\sqrt{|\tau|}\right)^{-1}. \tag{4.12}$$

The norms of the blocks $N_{m,n}(i\tau)$ in (4.11) are controlled by $|p_{m,n}(i\tau)|e^{-2\gamma_{m,n}(i\tau)}$ and hence, are bounded uniformly in $m, n \in \mathbb{N}_0$. Therefore,

$$\|\mathcal{R}(\pm i\tau) - \mathcal{R}'(\pm i\tau)\| \leq C = C(\tau_0).$$

Choosing $|\tau|$ large enough, we conclude from (4.12) that

$$\|\mathcal{R}(\pm i\tau) - \mathcal{R}'(\pm i\tau)\| < \|\mathcal{R}'(\pm i\tau)^{-1}\|^{-1}.$$

It follows that the operator $\mathcal{R}(\pm i\tau)$ has bounded inverse, and, in particular, the system (4.6) has only the trivial solution in $\mathfrak{G}$.

The proof of Theorem 2.3 is complete.

5. Representation of the resolvent $(\mathbf{A}_\alpha - \Lambda)^{-1}$

res

5.1. **Resolvent.** In order to prove Theorem 2.8, we need a convenient representation for both resolvents involved in (2.11). Here we do this for the operator $\mathbf{A}_\alpha$. We derive an analogue of the formula (6.6) (the *basic formula*) in [5] or, equivalently, (6.4) in [10]. However, there is an important difference between the techniques we employ here and those in [5] and [10]. The main goal in both cited papers was the direct study of the a.c. spectrum of the operator corresponding to the case of one oscillator. To achieve this objective, the behaviour of the resolvent as the spectral parameter approaches the real axis was carefully studied. What we do here is to apply scattering theory, and use the already known results of [5] and [10]. This makes our analysis much easier. We are able to work with the resolvents for a fixed value of the parameter Λ. We exclude Λ from the notation, unless to do so would be confusing.

Let a function $F \in \mathfrak{H}$ have the decomposition $F \sim \{f_{m,n}\}$. For any $\alpha \geq 0$ let us denote

au0 (5.1) $$U_\alpha = \left(\mathbf{A}_\alpha - \Lambda\right)^{-1} F \sim \{u_{\alpha;m,n}\}, \qquad V \sim \{v_{m,n}\} = \{u_{\alpha;m,n} - u_{0;m,n}\}. \tag{5.1}$$

In the notation for v we do not reflect dependence on the parameter α.

The operator $(\mathbf{A}_0 - \Lambda)^{-1}$ can be written in an explicit form. The functions $u_{0;m,n}$ are given by the formula (3.9), with $\zeta = \zeta_{m,n}(\Lambda)$, see (4.1). It follows that

res.1amn
$$u_{0;m,n}(\pm 1) = (2\zeta_{m,n})^{-1} \int_{\mathbb{R}} e^{-\zeta_{m,n}|t\mp 1|} f_{m,n}(t)dt; \qquad [u'_{0;m,n}](\pm 1) = 0. \tag{5.2}$$

Each function $v_{m,n}(x)$ belongs to the space $\mathcal{F}_{\zeta_{m,n}}$. Hence, the equalities (3.4) are satisfied for it, again with $\zeta = \zeta_{m,n}(\Lambda)$. Using these equalities and taking into account that $[u'_{\alpha;m,n}](\pm 1) = [v'_{m,n}](\pm 1)$, we find from the matching condition in (2.6) at the point $x = 1$ that

res.3
$$\begin{aligned} &-\frac{2\zeta_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(v_{m,n}(1) - e^{-2\zeta_{m,n}} v_{m,n}(-1) \right) \\ &\qquad = \frac{\alpha_+}{\sqrt{2}} \left(\sqrt{m+1}\, u_{\alpha;m+1,n}(1) + \sqrt{m}\, u_{\alpha;m-1,n}(1) \right). \end{aligned} \tag{5.3}$$

As in section 4, we let $\mu_\pm = \sqrt{2}/\alpha_\pm$. Since $v_{m,n} = u_{\alpha;m,n} - u_{0;m,n}$, the equation (5.3) yields

res.3q
$$\begin{aligned} &\sqrt{m+1}\, u_{\alpha;m+1,n}(1) + \frac{2\mu_+ \zeta_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(u_{\alpha;m,n}(1) - e^{-2\zeta_{m,n}} u_{\alpha;m,n}(-1) \right) \\ &+ \sqrt{m}\, u_{\alpha;m-1,n}(1) = \frac{2\mu_+ \zeta_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(u_{0;m,n}(1) - e^{-2\zeta_{m,n}} u_{0;m,n}(-1) \right). \end{aligned} \tag{5.4}$$

The next step is the same normalization as in section 4. Denote

xz
$$X^\pm_{m,n} = r_{m,n}^{-1/4} u_{0;m,n}(\pm 1), \qquad Z^\pm_{m,n} = r_{m,n}^{-1/4} u_{\alpha;m,n}(\pm 1). \tag{5.5}$$

Each function $v_{m,n}$ can be represented as in (4.2), with $C^\pm_{m,n} = Z^\pm_{m,n} - X^\pm_{m,n}$. We use a shortened notation for the corresponding elements in $\mathfrak{G}$:

$$X = \{X^+_{m,n}, X^-_{m,n}\},\ Z = \{Z^+_{m,n}, Z^-_{m,n}\},\ C = \{C^+_{m,n}, C^-_{m,n}\}, \qquad m, n \in \mathbb{N}_0.$$

Multiplying each equation in (5.4) by $r_{m,n}^{1/4}$ and writing out the similar equations coming from the matching conditions at $x = -1$, we reduce the system to the form

res.3r
$$\begin{aligned} &q^+_{m+1,n} Z^+_{m+1,n} + \frac{2\mu_+ p_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(Z^+_{m,n} - e^{-2\zeta_{m,n}} Z^-_{m,n} \right) + q^+_{m,n} Z^+_{m-1,n} \\ &\qquad = \frac{2\mu_+ p_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(X^+_{m,n} - e^{-2\zeta_{m,n}} X^-_{m,n} \right); \\ &q^-_{m,n+1} Z^-_{m,n+1} + \frac{2\mu_- p_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(Z^-_{m,n} - e^{-2\zeta_{m,n}} Z^+_{m,n} \right) + q^-_{m,n} Z^-_{m-1,n} \\ &\qquad = \frac{2\mu_- p_{m,n}}{1 - e^{-4\zeta_{m,n}}} \left(X^-_{m,n} - e^{-2\zeta_{m,n}} X^+_{m,n} \right). \end{aligned} \tag{5.6}$$

This is the non-homogeneous counterpart of the system (4.6). In order to write it more conveniently, we need more notation. All the operators introduced below depend on Λ and we always assume that $\Lambda \notin \mathbb{R}$.

Define the operator $\mathcal{S} = \mathcal{S}(\Lambda) : \mathfrak{H} \to \mathfrak{G}$ by

$$\mathcal{S} : F \mapsto \left\{ \frac{r_{m,n}^{1/4}}{2} \int_{\mathbb{R}} e^{-\zeta_{m,n}|t-1|} f_{m,n}(t)dt,\ \frac{r_{m,n}^{1/4}}{2} \int_{\mathbb{R}} e^{-\zeta_{m,n}|t+1|} f_{m,n}(t)dt \right\}.$$

According to (5.2) and (5.5), this can be written as

cs

$$\mathcal{S} : F \mapsto \{p_{m,n}X^+_{m,n}, p_{m,n}X^-_{m,n}\}. \tag{5.7}$$

It follows from the Cauchy-Schwartz inequality and (3.13) that the operator $\mathcal{S}$ is bounded.

The diagonal operator

$$\mathcal{P} = \mathcal{P}(\Lambda) : \{X^+_{m,n}, X^-_{m,n}\} \mapsto \{p_{m,n}X^+_{m,n}, p_{m,n}X^-_{m,n}\}$$

acts in $\mathfrak{G}$ and is unbounded. Its inverse $\mathcal{P}^{-1}$ is a bounded operator.

Further, let $\mathcal{M} = \mathcal{M}(\Lambda)$ be the operator generated by the block-diagonal matrix, $\mathcal{M} = \operatorname{diag}\{M_{m,n}\}$, where

3e

$$M_{m,n} = (1 - e^{-4\zeta_{m,n}})^{-1} \begin{pmatrix} \mu_+ & -\mu_+ e^{-2\zeta_{m,n}} \\ -\mu_- e^{-2\zeta_{m,n}} & \mu_- \end{pmatrix}. \tag{5.8}$$

Evidently, $\mathcal{M}$ is bounded in $\mathfrak{G}$.

Finally, we let

t

$$\mathcal{T} = \mathcal{T}(\Lambda) : \{C^+_{m,n}, C^-_{m,n}\} \mapsto \{r_{m,n}^{1/4}(C^+_{m,n}\varphi^+_{m,n} + C^-_{m,n}\varphi^-_{m,n})\}. \tag{5.9}$$

This is a bounded operator acting from $\mathfrak{G}$ into $\mathfrak{H}$.

The system (5.6) can be written in the operator form

$$\mathcal{R}Z = 2\mathcal{M}\mathcal{S}F,$$

whence $Z = 2\mathcal{R}^{-1}\mathcal{M}\mathcal{S}F$. Here $\mathcal{R} = \mathcal{R}(\Lambda)$ is the operator in $\mathfrak{G}$ which corresponds to the left-hand side of the system (5.6), or, equivalently, of the homogeneous system (4.6). We also have $X = \mathcal{P}^{-1}\mathcal{S}F$, so that

$$C = Z - X = \left(2\mathcal{R}^{-1}\mathcal{M} - \mathcal{P}^{-1}\right)\mathcal{S}F.$$

If C is found from this equation, then evidently $U_\alpha - U_0 = \mathcal{T}C$.

Now it follows from the construction that

3f

$$(\mathbf{A}_\alpha - \Lambda)^{-1} - (\mathbf{A}_0 - \Lambda)^{-1} = \mathcal{T}\left(2\mathcal{R}^{-1}\mathcal{M} - \mathcal{P}^{-1}\right)\mathcal{S}. \tag{5.10}$$

This is the desired representation of the resolvent of the operator $\mathbf{A}_\alpha$.

matr

5.2. **On the matrix** $\mathcal{R}(\Lambda)$**.** It was shown in section 4 that for τ large enough the matrix $\mathcal{R}(i\tau)$ has a bounded inverse. This allowed us to conclude that the operator $\mathbf{A}_\alpha - \Lambda$ has a bounded inverse for all $\Lambda \notin \mathbb{R}$, and hence $\ker \mathcal{R}(\Lambda) = \{0\}$ for all such Λ. So, the operator $\mathcal{R}(\Lambda)^{-1}$ is well-defined. However, this does not imply automatically that this operator is bounded in $\mathfrak{G}$. We now show that this property is a direct consequence of the representation (5.10). Indeed, (5.10) implies that

$$2\mathcal{R}^{-1}\mathcal{M} - \mathcal{P}^{-1} = (\mathcal{T}^*\mathcal{T})^{-1}\mathcal{T}^*\left((\mathbf{A}_\alpha - \Lambda)^{-1} - (\mathbf{A}_0 - \Lambda)^{-1}\right)\mathcal{S}^*(\mathcal{S}\mathcal{S}^*)^{-1}.$$

It is easy to show that for $\Lambda \notin \mathbb{R}_+$ the operators $\mathcal{M}$ and $\mathcal{T}^*\mathcal{T}$ and $\mathcal{S}\mathcal{S}^*$ (acting in $\mathfrak{G}$) have bounded inverses. This yields the desired result.

6. Representation of the resolvent $(\mathbf{A}^\circ_\alpha - \Lambda)^{-1}$

reso

Our aim here is to derive an analogue of the representation (5.10) for the operator $\mathbf{A}^\circ_\alpha$. One possible way to proceed is to use the decompositions (2.8), (2.9). However, we prefer another way, one which parallels our argument in section 5. The calculations are easier for $\mathbf{A}^\circ_\alpha$ than for $\mathbf{A}_\alpha$.

For the objects related to the operator $\mathbf{A}^\circ_\alpha$ we use the notation

au00 (6.1) $$U^\circ_\alpha = \left(\mathbf{A}^\circ_\alpha - \Lambda\right)^{-1} F \sim \{u^\circ_{\alpha;m,n}\}, \qquad V^\circ \sim \{v^\circ_{m,n}\} = \{u^\circ_{\alpha;m,n} - u^\circ_{0;m,n}\}.$$

An analogue of (5.2) is given by

u1a (6.2) $$u^\circ_{0;m,n}(\pm 1) = (2\zeta_{m,n})^{-1}\int_{\mathbb{R}_\pm}\left(e^{-\zeta_{m,n}|t\mp 1|} - e^{-\zeta_{m,n}(1\pm t)}\right) f_{m,n}(t)dt.$$

Next, we derive an analogue of (5.3). Taking (3.7) and $[{u^\circ_{0;m,n}}'](\pm 1) = 0$ into account, we get from the matching condition at $x = 1$:

$$-\frac{2\zeta_{m,n}}{1 - e^{-2\zeta_{m,n}}} v^\circ_{m,n}(1) = \frac{\alpha_+}{\sqrt{2}}\left(\sqrt{m+1}\,u^\circ_{\alpha;m+1,n}(1) + \sqrt{m}\,u^\circ_{\alpha;m-1,n}(1)\right).$$

Since $v^\circ_{m,n} = u^\circ_{\alpha;m,n} - u^\circ_{0;m,n}$, we find, taking, as before, $\mu_\pm = \sqrt{2}/\alpha_\pm$:

3d (6.3) $$\sqrt{m+1}\,u^\circ_{\alpha;m+1,n}(1) + \frac{2\mu_+\zeta_{m,n}}{1 - e^{-2\zeta_{m,n}}} u^\circ_{\alpha;m,n}(1) + \sqrt{m}\,u^\circ_{\alpha;m-1,n}(1) = \frac{2\mu_+\zeta_{m,n}}{1 - e^{-2\zeta_{m,n}}} u^\circ_{0;m,n}(1).$$

This is much simpler than the system (5.4), which, of course, merely reflects the special structure of the operator $\mathbf{A}^\circ_\alpha$ as given by (2.9).

The normalization, as in section 5, reduces (6.3) and the similar equations for $x = -1$ to the form

3m (6.4)
$$\begin{aligned} q^+_{m+1,n} Z^{\circ,+}_{m+1,n} + \frac{2\mu_+ p_{m,n}}{1-e^{-2\zeta_{m,n}}} Z^{\circ,+}_{m,n} + q^+_{m-1,n} Z^{\circ,+}_{m-1,n} &= \frac{2\mu_+ p_{m,n}}{1-e^{-2\zeta_{m,n}}} X^{\circ,+}_{m,n}, \\ q^-_{m,n+1} Z^{\circ,-}_{m,n+1} + \frac{2\mu_- p_{m,n}}{1-e^{-2\zeta_{m,n}}} Z^{\circ,-}_{m,n} + q^-_{m,n+1} Z^{\circ,-}_{m,n-1} &= \frac{2\mu_- p_{m,n}}{1-e^{-2\zeta_{m,n}}} X^{\circ,-}_{m,n}. \end{aligned}$$

Here
$$X^{\circ,\pm}_{m,n} = r^{-1/4}_{m,n} u^\circ_{0;m,n}(\pm 1), \qquad Z^{\circ,\pm}_{m,n} = r^{-1/4}_{m,n} u^\circ_{\alpha;m,n}(\pm 1).$$
The coefficients $q^\pm_{m,n}$, $p_{m,n}$ are the same as in (5.6), being defined in (4.5). By (6.2), we have
$$\begin{aligned} 2p_{m,n} X^{\circ,+}_{m,n} &= \int_{\mathbb{R}_+} \left(e^{-\zeta_{m,n}|t-1|} - e^{-\zeta_{m,n}(t+1)}\right) f_{m,n}(t)dt, \\ 2p_{m,n} X^{\circ,-}_{m,n} &= \int_{\mathbb{R}_-} \left(e^{-\zeta_{m,n}|t+1|} - e^{\zeta_{m,n}(t-1)}\right) f_{m,n}(t)dt. \end{aligned}$$

Now we define analogues of the operators involved in the equality (5.6). First of all, $\mathcal{R}^\circ = \mathcal{R}^\circ(\Lambda)$ is the operator in $\mathfrak{G}$, defined by the infinite matrix which corresponds to the left-hand side of (6.4). The operator $\mathcal{R}^\circ$ can be written in the form similar to (4.10):

ro (6.5)
$$\mathcal{R}^\circ(\Lambda) = \mathcal{R}'(\Lambda) + \mathcal{N}^\circ(\Lambda)$$

where

ro0 (6.6)
$$\mathcal{N}^\circ(\Lambda) = \operatorname{diag}\{N^\circ_{m,n}\}, \qquad N^\circ_{m,n} = 2\frac{p_{m,n} e^{-2\zeta_{m,n}}}{1-e^{-2\zeta_{m,n}}} \begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix}.$$

The self-adjointness of the operator $\mathbf{A}^\circ_\alpha$ in $\mathfrak{H}$ implies that the operator $\mathcal{R}^\circ(\Lambda)$ is invertible for any $\Lambda \notin \mathbb{R}$.

The operator $\mathcal{S}^\circ = \mathcal{S}^\circ(\Lambda) : \mathfrak{H} \to \mathfrak{G}$ is a bounded operator defined by
$$\mathcal{S}^\circ : F \sim \{f_{m,n}\} \mapsto \{p_{m,n} X^{\circ,+}_{m,n}, p_{m,n} X^{\circ,-}_{m,n}\};$$
cf. (5.7).

The operator $\mathcal{M}^\circ = \mathcal{M}^\circ(\Lambda)$ is the bounded operator on $\mathfrak{G}$ of block-diagonal form

3ec (6.7)
$$\mathcal{M}^\circ = \operatorname{diag}\{M_{m,n}\}, \qquad M_{m,n} = (1-e^{-2\zeta_{m,n}})^{-1} \begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix}.$$

Finally, let
$$\mathcal{T}^\circ = \mathcal{T}^\circ(\Lambda) : \{C^+_{m,n}, C^-_{m,n}\} \mapsto \left\{r^{1/4}_{m,n}(C^+_{m,n}\varphi^{\circ,+}_{m,n} + C^+_{m,n}\varphi^{\circ,-}_{m,n})\right\}$$
This is a bounded operator acting from $\mathfrak{G}$ into $\mathfrak{H}$.

As in section 5, we can re-write the system (6.4) as

3fc (6.8)
$$(\mathbf{A}^\circ_\alpha - \Lambda)^{-1} - (\mathbf{A}^\circ_0 - \Lambda)^{-1} = \mathcal{T}^\circ\left(2(\mathcal{R}^\circ)^{-1}\mathcal{M}^\circ - \mathcal{P}^{-1}\right)\mathcal{S}^\circ.$$

Note that for any $\Lambda \neq \overline{\Lambda}$ the operator $\mathcal{R}^\circ(\Lambda)^{-1}$ is bounded. The proof is the same as for the operator $\mathcal{R}^{-1}(\Lambda)$, see section 5.2.

7. Proof of Theorem 2.8.

`pr0.0`

7.1. **The case $\alpha = 0$.** Here we show that

`goal0`

$$(\mathbf{A}_0^\circ - \Lambda)^{-3} - (\mathbf{A}_0 - \Lambda)^{-3} \in \mathfrak{S}_1. \tag{7.1}$$

Recall that in the notation of (5.1) and (6.1)

$$(\mathbf{A}_0 - \Lambda)^{-1}F = \{u_{0,m,n}\}, \qquad (\mathbf{A}_0^\circ - \Lambda)^{-1}F = \{u_{0,m,n}^\circ\}$$

where the functions $u_{0,m,n}, u_{0,m,n}^\circ$ are given by the equations (3.9) and (3.10) respectively, with $\zeta = \zeta_{m,n}$. So, both operators are diagonal. Denote by $\Phi_{m,n}$, $\Phi_{m,n}^\circ$ their components, and let

`pr0.qq`

$$\begin{aligned} \mathbf{Q} = \operatorname{diag}\{Q_{m,n}\} =& (\mathbf{A}_0^\circ - \Lambda)^{-1} - (\mathbf{A}_0 - \Lambda)^{-1}, \\ Q_{m,n} =& \Phi_{m,n}^\circ - \Phi_{m,n}. \end{aligned} \tag{7.2}$$

According to (3.11), each $Q_{m,n}$ is a rank one operator:

`p2`

$$Q_{m,n} : f_{m,n}(x) \mapsto -(2\zeta_{m,n})^{-1} g_{m,n}(x) \int_{\mathbb{R}} g_{m,n}(t) f_{m,n}(t)dt \tag{7.3}$$

where $g_{m,n}(x) = e^{-\zeta_{m,n}|x|}$. It follows from (7.3) that

$$\|Q_{m,n}\| = (2|\zeta_{m,n}| \operatorname{Re} \zeta_{m,n})^{-1} \le C r_{m,n}^{-1}, \qquad C = C(\Lambda).$$

The norms of $\Phi_{m,n}$ and $\Phi_{m,n}^\circ$ can be easily estimated (actually, $\|\Phi_{m,n}\|$ can be calculated explicitly, since this is a convolution operator). By the "Schur test", the norm $\|\mathbf{K}\|$ of an integral operator in L^2 with the kernel $K(x,t)$ can be estimated as

$$\|\mathbf{K}\|^2 \le \sup_t \int |K(x,t)|dx \sup_x \int |K(x,t)|dt.$$

Applying this to the operators (3.9) and (3.10), we find that

$$\|\Phi_{m,n}\|, \ \|\Phi_{m,n}^\circ\| \le C r_{m,n}^{-1}.$$

Furthermore, the components of the operator $(\mathbf{A}_0^\circ - \Lambda)^{-3} - (\mathbf{A}_0 - \Lambda)^{-3}$ are

$$\Phi_{m,n}^2 Q_{m,n} + \Phi_{m,n} Q_{m,n} \Phi_{m,n}^\circ + Q_{m,n} (\Phi_{m,n}^\circ)^2.$$

The norm of this operator does not exceed $3C^3 r_{m,n}^{-3}$ and, since its rank is no greater than 3, its trace class norm does not exceed $9C^3 r_{m,n}^{-3}$. By (2.5), these numbers form a convergent double series, and hence, (7.1) is established.

Note that the exponent 3 in (7.1) can not be replaced by 2.

e

7.2. **Difference between the right-hand sides in (5.10), (6.8).** To shorten our notation, let us denote

end.0 (7.4) $$\mathbf{H}=\mathcal{T}\left(2\mathcal{R}^{-1}\mathcal{M}-\mathcal{P}^{-1}\right)\mathcal{S},\qquad \mathbf{H}^{\circ}=\mathcal{T}^{\circ}\left((2\mathcal{R}^{\circ})^{-1}\mathcal{M}^{\circ}-\mathcal{P}^{-1}\right)\mathcal{S}^{\circ}.$$

Here we show that

end.1 (7.5) $$\Psi:=\mathbf{H}^{\circ}-\mathbf{H}=\in\mathfrak{S}_1.$$

Since all the inverse operators appearing in (7.4) are bounded, we only need to check that

$$\mathcal{T}^{\circ}-\mathcal{T},\ \mathcal{S}^{\circ}-\mathcal{S},\ \mathcal{M}^{\circ}-\mathcal{M},\ \mathcal{R}^{\circ}-\mathcal{R}\in\mathfrak{S}_1.$$

Here each operator has a block-diagonal structure, with (2×2)-blocks, and it is sufficient to estimate the operator norm of each block and to verify that the corresponding series converge.

For the operators $\mathcal{T}^{\circ}-\mathcal{T}$ and $\mathcal{S}^{\circ}-\mathcal{S}$ the result immediately follows from the definitions of the operators involved and the estimate (3.8). For the operator $\mathcal{M}^{\circ}-\mathcal{M}$ the result is evident from the comparison of (5.8) and (6.7). Finally, for $\mathcal{R}^{\circ}-\mathcal{R}$ the result is implied by (4.10) and (6.5), if we take into account evident estimates of the norms of blocks $N_{m,n}$ in (4.11) and $N^{\circ}_{m,n}$ in (6.6).

end

7.3. **End of the proof.** Unfortunately, the desired inclusion (2.11) is not implied by (7.1) and (7.5) automatically and we need an extra argument in order to finalize the proof.

Let us denote $\mathbf{G}=(\mathbf{A}_0-\Lambda)^{-1}$. Using also the notation $\mathbf{Q}$, Ψ as in (7.2), (7.5), we can re-write the equalities (5.10) and (6.8) as

$$(\mathbf{A}_\alpha-\Lambda)^{-1}=\mathbf{G}+\mathbf{H},\qquad (\mathbf{A}^{\circ}_\alpha-\Lambda)^{-1}=\mathbf{G}+\mathbf{H}+\mathbf{Q}+\Psi.$$

We already know that $\Psi\in\mathfrak{S}_1$ and $(\mathbf{G}+\mathbf{Q})^3-\mathbf{G}^3\in\mathfrak{S}_1$. Therefore, the following equality is satisfied modulo a trace class correction:

$$\begin{aligned}(\mathbf{A}^{\circ}_\alpha-\Lambda)^{-1}-(\mathbf{A}_\alpha-\Lambda)^{-1}&=(\mathbf{G}+\mathbf{Q}+\mathbf{H}+\Psi)^3-(\mathbf{G}+\mathbf{H})^3\\ &=(\mathbf{G}+\mathbf{Q}+\mathbf{H})^3-(\mathbf{G}+\mathbf{H})^3\ (\mod\mathfrak{S}_1)\\ &=\left((\mathbf{G}+\mathbf{Q})^2-\mathbf{G}^2\right)\mathbf{H}+(\mathbf{G}+\mathbf{Q})\mathbf{H}(\mathbf{G}+\mathbf{Q})-\mathbf{GHG}\\ &+\mathbf{H}\left((\mathbf{G}+\mathbf{Q})^2-\mathbf{G}^2\right)+\mathbf{QH}^2+\mathbf{HQH}+\mathbf{H}^2\mathbf{Q}\ (\mod\mathfrak{S}_1).\end{aligned}$$

Removing the parentheses, we come to the sum where each term involves one of the products $\mathbf{QH}$, $\mathbf{HQ}$, $\mathbf{QGH}$, $\mathbf{HGQ}$. Taking into account the structure of the operator $\mathbf{H}$, we see that it is sufficient for us to prove that the operators

fin (7.6) $$\mathbf{Q}\mathcal{T},\ \mathcal{S}\mathbf{Q},\ \mathbf{Q}(\mathbf{A}_0-\Lambda)^{-1}\mathcal{T},\ \mathcal{S}(\mathbf{A}_0-\Lambda)^{-1}\mathbf{Q}$$

are trace class. All these operators have block-diagonal form, with the blocks given by explicit formulae implied by the corresponding definitions. For instance, according to (5.9) the operator $\mathbf{Q}\mathfrak{T}$ transforms the number sequence $\{C^+_{m,n}, C^-_{m,n}\}$ into the sequence of functions $\{w_{m,n}\}$ where

$$w_{m,n}(x) = -\frac{r_{m,n}^{1/4}}{2\zeta_{m,n}} g_{m,n}(x) \int_{\mathbb{R}} \left(C^+_{m,n}\varphi^+_{m,n}(t) +, C^-_{m,n}\varphi^-_{m,n}(t) \right) g_{m,n}(t) dt.$$

An elementary calculation shows that the integral here is of order $O(e^{-\gamma_{m,n}})$. This happens because the function $g_{m,n}(t)$, see (7.3), is concentrated around the point $t = 0$, while $\varphi^{\pm}_{m,n}(t)$ is concentrated around $t = \pm 1$, and all the three functions decay exponentially when t moves away from the corresponding center. Clearly, this estimate implies that $\mathbf{Q}\mathfrak{T} \in \mathfrak{S}_1$. The proofs for the other operators in (7.6) are similar.

The proof of is complete.

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