Thermal Quantum Fields without Cut-offs in 1+1 Space-time Dimensions

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Abstract

We construct interacting quantum fields in 1+1 dimensional Minkowski space, representing neutral scalar bosons at positive temperature. Our work is based on prior work by Klein and Landau and Høegh-Krohn.

Contents

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1 Introduction

Constructive thermal field theory allows one to circumvent (at least in 1+1 space-time dimensions) the severe infrared problems (see e.g. [St]) of thermal perturbation theory. A class of models representing scalar neutral bosons with polynomial interactions was constructed by Høegh-Krohn [H-K] more than twenty years ago. Shortly afterwards, several related results on the construction of self-interacting thermal fields were announced by Fröhlich [Fr2].

Our first paper was devoted to the construction of neutral and charged thermal fields with *spatially cutoff* interactions in 1+1 space–time dimensions, using the notion of *stochastically positive* KMS systems due to Klein and Landau [KL1].

The construction of interacting thermal quantum fields without cutoffs presented here includes several of the original ideas of Høegh-Krohn [H-K], but instead of starting from the interacting system in a box we start from the *Araki-Woods representation* for the free thermal system in infinite volume. This 'algebraic' approach eliminates some cumbersome limiting procedures present in Høegh-Krohn's work due to the introduction of boxes. We provide complete proofs for a number of statements which where only touched upon in Høegh-Krohn's work. The list of 'new' contributions contains the Wick (re-)ordering with respect to different covariance functions, the existence of interacting sharp-time fields, the identification of local algebras, the existence and uniqueness of the solution of Høegh-Krohn's time dependent heat-equation, local normality of the interacting KMS state, uniqueness of the weak* accumulation point of the sequence of approximating KMS states, and a number of inequalities that enter into a rigorous construction at several points. Although some of our results were probably already known by the experts (most of our work is based on results by Glimm and Jaffe, Høegh-Krohn, Fröhlich, Klein and Landau, and Simon) more than twenty years ago, we feel that it is worth while to present the arguments in full detail.

We will provide a detailed description of the content of this paper in the next subsection. But before we do so, we give a rough outline of the main ideas.

Let $\mathfrak{h}$ and ϵ denote the one-particle Hilbert space and the one-particle energy for a single neutral scalar boson. On the Weyl algebra $\mathcal{W}(\mathfrak{h})$ we define a quasi-free (τ°, β)-KMS state ω°_β for the time evolution $\{\tau^\circ_t\}_{t\in\mathbb{R}}$ by

$$\omega^\circ_\beta\big(W(h)\big) := \mathrm{e}^{-\frac{1}{4}(h,(1+2\rho)h)}, \quad \tau^\circ_t\big(W(h)\big) = W(\mathrm{e}^{\mathrm{i}t\epsilon}h),\ h\in\mathfrak{h},\ t\in\mathbb{R},$$

where $\rho := (\mathrm{e}^{\beta\epsilon}-1)^{-1}$, $\beta > 0$.

A convenient realization of the GNS representation associated to the pair $\big(\mathcal{W}(\mathfrak{h}), \omega_\beta^\circ\big)$ is the *Araki-Woods representation* defined by:

$$\begin{aligned}
&\mathcal{H}_{AW} := \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}}),\\
&\Omega_{AW} := \Omega,\\
&\pi_{AW}(W(h)) = W_{AW}(h) := W_{\mathrm{F}}\big((1+\rho)^{\frac{1}{2}} h \oplus \overline{\rho}^{\frac{1}{2}}\overline{h}\big), \quad h \in \mathfrak{h}.
\end{aligned}$$

Here $\overline{\mathfrak{h}}$ is the conjugate Hilbert space to $\mathfrak{h}$, $W_F(.)$ denotes the Fock Weyl operator on $\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ and $\Omega \in \Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ is the Fock vacuum. The von Neumann algebra generated by $\{\pi_{AW}(W(h)) \mid h \in \mathfrak{h}\}$ is denoted by $\mathcal{R}_{AW}$. The *local von Neumann algebra* generated by $\{\pi_{AW}(W(h)) \mid h \in \mathfrak{h}_I\}$ is denoted by $\mathcal{R}_{AW}(I)$. Here $I \subset \mathbb{R}$ is an open and bounded interval and $\mathfrak{h}_I$ will be defined in (6.4).

Since ω_β° is τ°-invariant, there exists a standard implementation (see [DJP]) of the time evolution in the representation π_{AW}:

$$\mathrm{e}^{iL_{AW}t}\pi_{AW}(A)\Omega_{AW} := \pi_{AW}\big(\tau_t^\circ(A)\big)\Omega_{AW} \quad \text{and} \quad L_{AW}\Omega_{AW} = 0.$$

The generator L_{AW} of the free time evolution is called the (free) Liouvillean.

Euclidean techniques were used in our first paper to define the operator sum

$$H_l := L_{AW} + \int_{-l}^{l} :P(\phi(0,x)):_{C_0} \,\mathrm{d}x$$

and to show that H_l is essentially selfadjoint.

Using Trotter's product formula as in [GJ2], a finite propagation speed argument shows that

$$\tau_t^l(A) = \mathrm{e}^{iH_lt}A\mathrm{e}^{-iH_lt}$$

is independent of l for $t \in \mathbb{R}$ and $A \in \mathcal{R}_{AW}(I)$ fixed, if I is bounded and l is sufficiently large. Thus there exists a limiting dynamics τ such that

$$\lim_{l\to\infty}\big\|\tau_t^l(A) - \tau_t(A)\big\| = 0 \tag{1.1}$$

for all $A \in \mathcal{R}_{AW}(I)$, I bounded. This norm convergence extends to the *norm closure*

$$\mathcal{A} := \overline{\bigcup_{I\subset\mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)}$$

of the *local von Neumann algebras.* The C^*-algebra $\mathcal{A}$ is called the *algebra of local observables.*

It follows from general results of [KL1] that the vector $\Omega_l \in \mathcal{H}_{AW}$,

$$\Omega_l := \frac{\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}}{\|\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}\|}, \tag{1.2}$$

induces a (τ^l,β)-KMS state ω_l for the W^*-dynamical system $(\mathcal{A},\tau^l)$. Equation (1.2) should be compared with similar expressions which are well-known (see e.g. [BR, Theorem 5.4.4]) for bounded perturbations and which have recently been derived for a class of unbounded perturbations in [DJP, Theorem 5.6].

The existence of weak limit points (which are states) of the net $\{\omega_l\}_{l>0}$ is a consequence of the Banach-Alaoglu theorem (see [BR, Theorem 2.3.15]).

That fact that all limit states satisfy the *KMS condition* w.r.t. the pair $(\mathcal{A},\tau)$ follows from (1.1), which itself is a consequence of finite propagation speed.

Since $\mathcal{A}$ is the norm closure of the *weakly closed* local algebras, all limit points are *locally normal* KMS states w.r.t. the Araki-Woods representation [TW].

To prove that there is only one accumulation point is more delicate. Following Høegh-Krohn [H-K] we will use *Nelson symmetry* to relate the interacting vacuum theory on the circle to the interacting thermal model on the real line.

1.1 Content of this paper

In Section 2 we recall the notions of *stochastically positive KMS systems* and associated *generalized path spaces*, due to Klein and Landau [KL1]. The property corresponding to stochastic positivity in the 0-temperature case is called *Nelson-Symanzik positivity.*

In Subsection 2.1 we recall the characterization of the thermal equilibrium states of a dynamical system $(\mathcal{B}, \tau)$ by the *KMS condition* and the definition of *Euclidean Green's functions.* The notion of a *stochastically positive KMS systems* $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ rests on the introduction of a distinguished abelian sub-algebra $\mathcal{U}$ of the observable algebra $\mathcal{B}$. In our case this algebra will be the algebra generated by the time-zero fields.

In Subsection 2.2 we recall the notion of a *generalized path space* $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$. It consists of a probability space (Q, Σ, μ), a distinguished sub σ-algebra Σ_0, a one-parameter group $t \mapsto U(t)$ of automorphisms of $L^\infty(Q, \Sigma, \mu)$ such that $\Sigma = \bigvee_{t \in \mathbb{R}} U(t)\Sigma_0$ and a reflection R, acting as an automorphism on $L^\infty(Q, \Sigma, \mu)$ such that $R^2 = \mathbb{1}$, $RU(t) = U(-t)R$.

Klein and Landau (see [KL1]) have shown that for $\beta > 0$ there is a one to one correspondence between stochastically positive β-KMS systems and β-periodic OS-positive path spaces (for $\beta = \infty$ the object associated to an OS-positive path space is called a *positive semigroup structure*, see [K]). The role of OS-positivity is to ensure the positivity of the inner product in the Hilbert space $\mathcal{H}$ on which the real time quantum fields act.

The *reconstruction theorem* provides a concrete realization of the *GNS triple* $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated to the pair $(\mathcal{B}, \omega)$. The *Liouvillean* L implements the time evolution in the GNS representation π_ω.

In Subsection 2.3 we recall some results from [KL1] (with some improvements in [GeJ]) concerning perturbations of generalized path spaces obtained from Feynman-Kac-Nelson kernels. The main examples of FKN kernels are those obtained from a selfadjoint operator V on the physical Hilbert space $\mathcal{H}_\omega$, which is affiliated to $\mathcal{U} \cong L^\infty(K, \nu_\omega)$. If $\mathrm{e}^{-\beta V} \in L^1(K, \nu_\omega)$ and

$$V \in L^p(K, \nu_\omega), \quad \mathrm{e}^{-\frac{\beta}{2}V} \in L^q(K, \nu_\omega) \quad \text{for} \quad p^{-1} + q^{-1} = \frac{1}{2}, \quad 2 \le p, q \le \infty,$$

then the operator sum $L + V$ is essentially selfadjoint on $\mathcal{D}(L) \cap \mathcal{D}(V)$ and the perturbed time-evolution τ_V on $\mathcal{B}$ is given by

$$\tau_{V,t}(B) = \mathrm{e}^{\mathrm{i}t\overline{L+V}} B \mathrm{e}^{-\mathrm{i}t\overline{L+V}}.$$

The KMS state ω_V for the pair $(\mathcal{B}, \tau_V)$ is the vector state induced by

$$\Omega_V := \frac{\mathrm{e}^{-\frac{\beta}{2}\overline{L+V}}\Omega_\omega}{\|\mathrm{e}^{-\frac{\beta}{2}\overline{L+V}}\Omega_\omega\|}.$$

The *Liouvillean* L_V for the perturbed β-KMS system $(\mathcal{B}_V, \tau_V, \omega_V)$ equals $\overline{L + V - JVJ}$. (J denotes the modular conjugation associated to the pair $(\mathcal{B}, \Omega_\omega)$). It satisfies

$$\mathrm{e}^{itL_V} A\Omega_V = \tau_{V,t}(A)\Omega_V \text{ and } L_V\Omega_V = 0.$$

In Section 3 we recall some standard facts about Gaussian measures on distribution spaces and fix some notation. *Gaussian measures* are reviewed in Subsection 3.2. *Sharp-time free fields* are introduced in Subsection 3.3. If the space dimension d is 1, then it is possible to define similarly *sharp-space free fields*. This is done in Subsection 3.4.

In Section 4 we recall two well known path spaces supported by $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \mathrm{d}\phi_C)$, where S_β is the circle of length β. In Subsection 4.1 we identify the generalized path space on $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \mathrm{d}\phi_C)$ corresponding to the free massive scalar field on the circle S_β at temperature 0.

In Subsection 4.2 we identify the generalized path space on $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \mathrm{d}\phi_C)$ corresponding to the free massive scalar field on the real line $\mathbb{R}$ at temperature β^{-1}. The physical Hilbert space associated to this path space can be unitarily identified with the Fock space $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The KMS vector Ω_{AW} is identified with the Fock vacuum vector Ω in $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The dynamics τ° can be unitarily implemented in π_{AW}: The (free) Liouvillean L_{AW} is identified with $\mathrm{d}\Gamma(\epsilon \oplus -\overline{\epsilon})$.

In Section 5 we describe perturbations of the two path spaces defined in Subsects. 4.1 and 4.2. The perturbed path spaces are obtained from FKN kernels corresponding to $P(\phi)_2$ interactions.

In Subsection 5.1 we recall some well known facts concerning the Wick ordering of Gaussian random variables. In 1+1 space-time dimensions Wick ordering is sufficient to eliminate the UV divergences of polynomial interactions. As it turns out, the leading order in the UV divergences is independent of the temperature. Thus it is a matter of convenience whether one uses the thermal covariance function C_0 or the vacuum covariance function C_{vac} to define the Wick ordering.

In Subsection 5.2 the $P(\phi)_2$ model on the circle S_β at temperature 0 is discussed. It is specified by the formal interaction

$$V_{\mathsf{C}} = \int_{S_\beta} :P(\phi(t,0)):_{C_\beta} \,\mathrm{d}t.$$

Here $P(\lambda)$ is a real valued polynomial, which is bounded from below. The time-evolution $x \mapsto \mathrm{e}^{ixH_{\mathsf{C}}^{\mathrm{ren}}}$ is generated by $H_{\mathsf{C}}^{\mathrm{ren}} := H_{\mathsf{C}} - E_{\mathsf{C}}$, where $E_{\mathsf{C}} := \inf(\sigma(H_{\mathsf{C}}))$ and

$$H_{\mathsf{C}} = \overline{\mathrm{d}\Gamma\big((D_t^2 + m^2)^{\frac{1}{2}}\big) + V_{\mathsf{C}}}.$$

The operator H_{C} is bounded from below and has a unique vacuum state $\omega_{\mathsf{C}}(\,.\,) = (\Omega_{\mathsf{C}},\, .\, \Omega_{\mathsf{C}})$ such that $(\Omega_{\mathsf{C}}, \Omega) > 0$ and $H_{\mathsf{C}}^{\mathrm{ren}}\Omega_{\mathsf{C}} = 0$. The renormalized energy operator $H_{\mathsf{C}}^{\mathrm{ren}}$ is called the *$P(\phi)_2$ Hamiltonian on the circle S_β*.

Some bounds are provided in Proposition 5.4, which are used in the sequel to prove the existence of interacting sharp-time fields.

The *spatially cutoff $P(\phi)_2$ model on the real line $\mathbb{R}$ at temperature β^{-1}* is discussed in Subsection 5.3. It is specified by the formal interaction

$$V_l = \int_{-l}^{l} :P(\phi(0,x)):_{C_0} \,\mathrm{d}x.$$

Here $P(\lambda)$ is once again a real valued polynomial, which is bounded from below, and $l \in \mathbb{R}^+$ is a spatial cutoff parameter. The perturbed KMS state ω_l turns out to be normal w.r.t. the Araki-Woods representation π_{AW}. In fact, it is the vector state induced by

$$\Omega_l := \frac{\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}}{\|\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}\|},$$

where H_l is the selfadjoint operator $H_l := \overline{L_{AW} + V_l}$. The perturbed time-evolution on $\mathcal{B}$ is given by

$$\tau_t^l(B) := \mathrm{e}^{\mathrm{i}tH_l} B \mathrm{e}^{-\mathrm{i}tH_l}, \quad B \in \mathcal{B}.$$

The following consequence of Lemma 5.3 will be important in Section 7:

$$\mathrm{e}^{-\int_{-\beta/2}^{\beta/2} U(t) \int_{-l}^{l} :P(\phi(0,x)):_{C_0} \mathrm{d}x\mathrm{d}t} = \mathrm{e}^{-\int_{-l}^{l} U_{\mathrm{c}}(x) \int_{S_\beta} :P(\phi(t,0)):_{C_\beta} \mathrm{d}t\mathrm{d}x}. \tag{1.3}$$

The analog of (1.3) in the zero temperature case is called *Nelson symmetry* (see e.g. [Si]).

The thermodynamic limit is discussed in Section 6. We prove that the limits

$$\lim_{l\to+\infty} \tau_t^l(A) =: \tau_t(A) \quad \text{and} \quad \lim_{l\to+\infty} \omega_l(A) =: \omega_\beta(A)$$

exist for A in the C^*-algebra of local observables $\mathcal{A}$ and that $(\mathcal{A}, \tau, \omega_\beta)$ is a β-KMS system, describing the *translation invariant $P(\phi)_2$ model at temperature β^{-1}*.

In Subsection 6.1 we recall localization properties of the classical solutions of the Klein-Gordon equation.

In Subsection 6.2 we introduce the net of local algebras $I \to \mathcal{R}_{AW}(I)$ for the free thermal field: for a bounded open interval $I \subset \mathbb{R}$, the symbol $\mathcal{R}_{AW}(I)$ denotes the von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I\}$. By a result of Araki [Ar1], the local von Neumann algebras for the free thermal scalar field are regular from the inside and from the outside:

$$\bigcap_{J\supset\overline{I}} \mathcal{R}_{AW}(J) = \mathcal{R}_{AW}(I) = \bigvee_{\overline{J}\subset I} \mathcal{R}_{AW}(J).$$

Moreover, if I is bounded, then the local algebra $\mathcal{R}_{AW}(I)$ is $*$-isomorphic to the unique hyper-finite factor of type III_1.

In Subsection 6.3 the existence of the limiting dynamics is discussed. For $t \in \mathbb{R}$ fixed, the norm limit

$$\lim_{l\to\infty} \tau_t^l(B) =: \tau_t(B)$$

exists for all B in

$$\mathcal{A} := \overline{\bigcup_{I\subset\mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)},$$

where the I's are open and bounded. *Finite propagation speed* is used to show that $\tau_t^l(B)$, for $B \in \mathcal{R}_{AW}(I)$ and $|t| \le T$, is independent of l for $l > |I| + T$. The proof uses *Trotter's product formula*, which requires that $L_{AW} + V_l$ is essentially self-adjoint on $\mathcal{D}(L_{AW}) \cap \mathcal{D}(V_l)$.

In order to apply the results of Section 7 to the C^*-algebra $\mathcal{A}$, it is necessary to identify the local von Neumann algebra $\mathcal{R}_{AW}(I)$ with the von Neumann algebra obtained by applying the interacting dynamics τ to the local abelian algebra of time-zero fields. This is done in Subsection 6.4: for $I \subset \mathbb{R}$ a bounded open interval, we denote by $\mathcal{U}_{AW}(I)$ the abelian von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I,\ h \text{ real valued}\}$. We denote by $\mathcal{B}_\alpha(I)$ the von Neumann algebra generated by

$$\big\{\tau_t(A) \mid A \in \mathcal{U}_{AW}(I),\ |t| < \alpha\big\}.$$

We set $\mathcal{B}(I) := \bigcap_{\alpha>0} \mathcal{B}_\alpha(I)$ and show that $\mathcal{B}(I) = \mathcal{R}_{AW}(I)$.

Taking the existence of the interacting path space (which we will construct in Section 7) for granted, we show that the net $\{\omega_l\}_{l>0}$ has a unique accumulation point. This is done in Subsection 6.5, using the identification of algebras established in the previous subsection. Thus

$$\mathrm{w}-\lim_{l\to+\infty} \omega_l =: \omega_\beta \text{ exists on } \mathcal{A}.$$

The state ω_β is a (τ, β)-KMS state on $\mathcal{A}$. It follows from a result of Takesaki and Winnink [TW] that ω_β is *locally normal*, i.e., if I is an open and bounded interval, then $\omega_{\beta|\mathcal{R}_{AW}(I)}$ is normal w.r.t. the Araki-Woods representation; thus $\omega_{\beta|\mathcal{R}_{AW}(I)}$ is also normal with respect to the Fock representation. Moreover, ω_β is invariant under spatial translations and satisfies the *space-clustering property*:

$$\lim_{x\to\infty} \omega_\beta(A\alpha_x(B)) = \omega_\beta(A)\omega_\beta(B),\ A, B \in \mathcal{A}.$$

Finally, the main results of this paper, namely the explicit construction of the translation invariant $P(\phi)_2$ model at positive temperature is given in Section 7. Following ideas of Høegh-Krohn [H-K], *Nelson symmetry* is used to establish the existence of the model in the thermodynamic limit.

The first step is to construct the *interacting path space* supported by $\mathcal{S}'_{\mathrm{I\!R}}(S_\beta \times \mathrm{I\!R})$ describing the translation invariant $P(\phi)_2$ model at temperature β^{-1}.

Following Høegh-Krohn [H-K] we consider the operator $W_{[-\infty,\infty]}(f)$ solving the time-dependent heat equation

$$\frac{\mathrm{d}}{\mathrm{d}b} W_{[a,b]}(f) = W_{[a,b]}(f)\big(-H_{\mathrm{C}}^{\mathrm{ren}} + \mathrm{i}\phi(f_b)\big), \quad a \le b,$$

where $f_b(\cdot) := f(\cdot, b) \in \mathcal{S}_{\mathrm{I\!R}}(S_\beta)$ for $f \in \mathcal{S}_{\mathrm{I\!R}}(S_\beta \times \mathrm{I\!R})$. We show that for $f \in C^\infty_{0\,\mathrm{I\!R}}(S_\beta \times \mathrm{I\!R})$

$$\lim_{l\to+\infty} \int \mathrm{e}^{\mathrm{i}\phi(f)} \mathrm{d}\mu_l = (\Omega_{\mathrm{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathrm{C}})$$

exists and that the map

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{I\!R}}(S_\beta \times \mathrm{I\!R}) & \to & \mathrm{I\!R}^+ \\ f & \mapsto & \big(\Omega_{\mathrm{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathrm{C}}\big) \end{array}$$

is the generating functional of a *Borel probability measure* μ on (Q, Σ). The measure μ is invariant under space translations, time translations and time-reflection.

In Subsection 7.2 we prove the *existence of interacting sharp-time fields.* (Note that the necessary bounds (5.9) depend on the dimension of space-time.) This result allows us to equip the probability space (Q, Σ, μ) with an *OS-positive β-periodic path space structure*:

- $U(t)$ is the group of transformations generated by the time translations $\mathfrak{T}_s$ induced on Q by the map $(t, x) \mapsto (t + s, x)$;
- R is the transformation generated by the (euclidean) time reflection at $t = 0$;
- Σ_0 is the sub$-\sigma$-algebra of Σ generated by the functions $\{\phi(0, h) \mid h \in \mathcal{S}_{\mathrm{I\!R}}(\mathrm{I\!R})\}$.

In Subsection 7.3 some properties of the *associated interacting β-KMS system* $(\mathcal{B}, \mathcal{U}, \tilde{\tau}, \tilde{\omega})$ are discussed. We prove the *convergence of sharp-time Schwinger functions* and show that

$$\tilde{\omega}\big(\alpha_x(W_{AW}(h))\big) = \tilde{\omega}\big(W_{AW}(h)\big)$$

for all $x \in \mathrm{I\!R}$ and

$$\lim_{x\to\infty} \tilde{\omega}\big(W_{AW}(h)\alpha_x(W_{AW}(g))\big) = \tilde{\omega}\big(W_{AW}(h)\big)\tilde{\omega}\big(W_{AW}(g)\big)$$

for $h, g \in C^\infty_{0\,\mathrm{I\!R}}(\mathrm{I\!R})$.

In Appendix A we discuss the abstract time-dependent heat equation

$$\left\{ \begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t} U(t, s) = -\big(H + \mathrm{i}R(t)\big)U(t, s),\ s \le t, \\ U(s, s) = \mathbb{1}. \end{array} \right. \tag{1.4}$$

Here $H \ge 0$ is a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $R(t)$, $t \in \mathrm{I\!R}$, is a family of closed operators with $\mathcal{D}(H^\gamma) \subset \mathcal{D}(R(t))$ for some $0 \le \gamma < 1$. We show that there exists a unique solution $U(t, s)$ such that $U(s, s) = \mathbb{1}$ and

$$U(t, r)U(r, s) = U(t, s) \text{ for } s \le r \le t.$$

In Subsection A.2 we consider the dissipative case when $R(t)$ is selfadjoint for $t \in \mathbb{R}$. We establish an approximation of $U(t,s)$ by time-ordered products and prove some bounds on $U(t,s)$, which are used in the main text to show the existence of interacting sharp-time fields and the convergence of sharp-time Schwinger functions.

Finally we establish a lemma which is used in the main text to prove spatial clustering for the translation invariant $P(\phi)_2$ model at temperature β^{-1}.

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2 Stochastically positive KMS systems and generalized path spaces

In this section we briefly recall the notions of *stochastically positive KMS systems* and associated *generalized path spaces*, due to Klein and Landau [KL1]. We will also need the corresponding notions at 0-temperature, which can be found in [K].

2.1 Stochastically positive KMS systems

Let $\mathcal{B}$ be a C^*-algebra and let $\{\tau_t\}_{t\in\mathbb{R}}$ be a one parameter group of $*$-automorphisms of $\mathcal{B}$. We recall that a state ω on $\mathcal{B}$ is a (τ,β)-*KMS state* or $(\mathcal{B},\tau,\omega)$ is a $\beta-$*KMS system*, if for each pair $A, B \in \mathcal{B}$ there exists a function $F_{A,B}(z)$ holomorphic in the strip $I_\beta^+ = \{z \in \mathbb{C} \mid 0 < \mathrm{Im} z < \beta\}$ and continuous on $\overline{I_\beta^+}$ such that

$$F_{A,B}(t) = \omega(A\tau_t(B)) \quad \text{and} \quad F_{A,B}(t+\mathrm{i}\beta) = \omega(\tau_t(B)A) \quad \forall t \in \mathbb{R}.$$

For $A_i \in \mathcal{B}$ and $t_i \in \mathbb{R}$, $1 \le i \le n$, the *Green's functions* are defined as follows:

$$G(t_1,\dots,t_n;A_1,\dots,A_n) := \omega\big(\prod_{i=1}^n \tau_{t_i}(A_i)\big).$$

It is well known (see [Ar2, Ar3]) that the Green's functions are holomorphic in

$$I_\beta^{n+} := \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid \mathrm{Im} z_i < \mathrm{Im} z_{i+1},\ \mathrm{Im} z_n - \mathrm{Im} z_1 < \beta\},$$

continuous on $\overline{I_\beta^{n+}}$ and bounded there by $\prod_1^n \|A_i\|$. Therefore one can define the *Euclidean Green's functions*:

$${}^E G(s_1,\dots,s_n;A_1,\dots,A_n) := G(\mathrm{i}s_1,\dots,\mathrm{i}s_n;A_1,\dots,A_n) \text{ for } s_1 \le \dots \le s_n,\ s_n - s_1 \le \beta.$$

The following class of β-KMS systems has been introduced by Klein and Landau [KL1].

Definition 2.1 *Let $(\mathcal{B},\tau,\omega)$ be a β-KMS system and let $\mathcal{U} \subset \mathcal{B}$ be an abelian *-sub-algebra. The KMS system $(\mathcal{B},\mathcal{U},\tau,\omega)$ is* stochastically positive *if*

- (i) *the C^*-algebra generated by $\bigcup_{t\in\mathbb{R}} \tau_t(\mathcal{U})$ is equal to $\mathcal{B}$;*
- (ii) *the Euclidean Green's functions ${}^E G(s_1,\dots,s_n;A_1,\cdots,A_n)$ are positive for all $A_1,\dots,A_n$ in $\mathcal{U}^+ = \{A \in \mathcal{U} \mid A \ge 0\}$.*

In applications it is more convenient to use a version of stochastic positivity, which is adapted to von Neumann algebras.

Definition 2.2 *Let $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ be a weakly closed abelian sub-algebra of $\mathcal{B}$. Assume that the dynamics $\tau: \mathcal{B} \to \mathcal{B}$ is given by*

$$\tau_t(B) := \mathrm{e}^{\mathrm{i}tL} B \mathrm{e}^{-\mathrm{i}tL}, \;\; B \in \mathcal{B},$$

where L is a selfadjoint operator on $\mathcal{H}$. Moreover, assume that ω is a β-KMS state for the W^*-dynamical system $(\mathcal{B}, \tau)$. Then the KMS system $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ is stochastically positive *if*

(i) *the von Neumann algebra generated by $\bigcup_{t\in \mathbb{R}} \tau_t(\mathcal{U})$ is equal to $\mathcal{B}$;*
(ii) *the Euclidean Green's functions ${}^E G(s_1, \ldots, s_n; A_1, \cdots, A_n)$ are positive for all $A_1, \ldots, A_n$ in $\mathcal{U}^+ = \{A \in \mathcal{U} \mid A \geq 0\}$.*

2.2 Generalized path spaces

Stochastically positive β-KMS systems can be associated to *generalized path spaces* (see [KL1], [K]). Let us first recall some terminology.

If Ξ_i, for i in an index set I, is a family of subsets of a set Q, then we denote by $\bigvee_{i\in I} \Xi_i$ the σ-algebra generated by the sets $\bigcup_{i\in J} \Xi_i$ where J runs over all countable subsets of I.

Definition 2.3 *A* generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ *consists of*

(i) *a probability space (Q, Σ, μ);*
(ii) *a distinguished sub σ-algebra $\Sigma_0 \subset \Sigma$;*
(iii) *a one-parameter group $\mathbb{R} \ni t \mapsto U(t)$ of measure preserving automorphisms of $L^\infty(Q, \Sigma, \mu)$, strongly continuous in measure, such that $\Sigma = \bigvee_{t\in\mathbb{R}} U(t)\Sigma_0$;*
(iv) *a measure preserving automorphism R of $L^\infty(Q, \Sigma, \mu)$ such that $R^2 = \mathbb{1}$, $RU(t) = U(-t)R$ and $RE_0 = E_0 R$, where E_0 is the conditional expectation with respect to Σ_0.*

A path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is said to be *supported* by the probability space (Q, Σ, μ).

It follows from (iii) and (iv) that $U(t)$ extends to a strongly continuous group of isometries of $L^p(Q, \Sigma, \mu)$ and R extends to an isometry of $L^p(Q, \Sigma, \mu)$ for $1 \leq p < \infty$.

We say that the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is *β-periodic* for $\beta > 0$ if $U(\beta) = \mathbb{1}$. On a β-periodic path space one can consider the one-parameter group $U(t)$ as being indexed by the circle $S_\beta = [-\beta/2, \beta/2]$.

For $I \subset \mathbb{R}$, we denote by E_I the conditional expectation with respect to the σ-algebra $\Sigma_I := \bigvee_{t\in I} \Sigma_t$.

Definition 2.4 (0-temperature case): *A generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is* OS-positive, *if $E_{[0,+\infty[} R E_{[0,+\infty[} \geq 0$ as an operator on $L^2(Q, \Sigma, \mu)$.*

(Positive temperature case): *A β-periodic path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is* OS-positive, *if $E_{[0,\beta/2]} R E_{[0,\beta/2]} \geq 0$ as an operator on $L^2(Q, \Sigma, \mu)$.*

For simplicity of notation we will consider β as a parameter in $]0, +\infty]$, the case $\beta = +\infty$ corresponding to the 0-temperature case.

It is shown in [KL1] that for $\beta > 0$ there is a one to one correspondence between stochastically positive β-KMS systems and β-periodic OS-positive path spaces. For $\beta = \infty$ the object associated to an OS-positive path space is called a *positive semigroup structure* (see [K]).

Let us describe in more details one part of this correspondence, which is an example of a *reconstruction theorem.* Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS-positive path space, β-periodic if $\beta < \infty$. We set

$$\mathcal{H}_{\mathrm{OS}} := L^2(Q, \Sigma_{[0,\beta/2]}, \mu).$$

Let $\mathcal{N} \subset \mathcal{H}_{\mathrm{OS}}$ be the kernel of the positive quadratic form

$$(\psi, \psi) := \int_Q \overline{\psi} R\psi \mathrm{d}\mu.$$

Then the *physical Hilbert space* is

$$\mathcal{H} := \text{ completion of } \mathcal{H}_{\mathrm{OS}}/\mathcal{N},$$

where the completion is done with respect to the positive definite scalar product $(.,.)$. Let us denote by $\mathcal{V}$ the canonical map $\mathcal{V}: \mathcal{H}_{\mathrm{OS}} \to \mathcal{H}_{\mathrm{OS}}/\mathcal{N}$. Then in $\mathcal{H}$ there is the *distinguished unit vector*

$$\Omega := \mathcal{V}1,$$

where $1 \in \mathcal{H}_{\mathrm{OS}}$ is the constant function equal to 1 on Q.

For $A \in L^\infty(Q, \Sigma_0, \mu)$ one defines $\tilde{A} \in \mathcal{B}(\mathcal{H})$ by

$$\tilde{A}\mathcal{V}\psi := \mathcal{V}A\psi. \tag{2.1}$$

(Note that multiplication by A preserves $\mathcal{N}$, since A is by assumption Σ_0 measurable). One denotes by $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ the abelian von Neumann algebra $\mathcal{U} := \{\tilde{A} \mid A \in L^\infty(Q, \Sigma_0, \mu)\}$. It is shown in [KL1, K] that the map $A \mapsto \tilde{A}$ is a weakly continuous *-isomorphism between $L^\infty(Q, \Sigma_0, \mu)$ and $\mathcal{U}$.

Finally, setting $\mathcal{M}_t = L^2(Q, \Sigma_{[0,\beta/2-t]}, \mu)$ for $0 \le t \le \beta/2$ and $\mathcal{D}_t = \mathcal{V}\mathcal{M}_t$, one can define $P(s): \mathcal{D}_t \to \mathcal{H}$ for $0 \le s \le t$ by

$$P(s)\mathcal{V}\psi := \mathcal{V}U(s)\psi,\ \psi \in \mathcal{M}_t.$$

The triple $(P(t), \mathcal{D}_t, \beta/2)$ forms a *local symmetric semigroup* (see [Fr1, KL3]), and there exists a unique selfadjoint operator L on $\mathcal{H}$ such that $P(s)u = \mathrm{e}^{-sL}u$ for $u \in \mathcal{D}_t$ and $0 \le s \le t$. The selfadjoint operator constructed in this way is said to be *associated* to the local symmetric semigroup $(P(t), \mathcal{D}_t, \beta/2)$.

Next one defines:

- $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ as the von Neumann algebra generated by $\{\mathrm{e}^{\mathrm{i}tL}A\mathrm{e}^{-\mathrm{i}tL} \mid t \in \mathbb{R}, A \in \mathcal{U}\}$;
- $\tau: t \mapsto \tau_t$ as the weakly continuous group of $*$-automorphisms of $\mathcal{B}$, which is given by

  $$\tau_t(B) = \mathrm{e}^{\mathrm{i}tL}B\mathrm{e}^{-\mathrm{i}tL}$$

  for $t \in \mathbb{R}$ and $B \in \mathcal{B}$;
- ω as the vector state on $\mathcal{B}$ given by $\omega(B) = (\Omega, B\Omega)$ for $B \in \mathcal{B}$.

It is shown in [KL1] that $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ is a stochastically positive β-KMS system. The relationship between the two objects is fixed by the following identity:

$${}^{E}G(s_1, \dots, s_n; \tilde{A}_1, \dots, \tilde{A}_n) = \int_Q (\prod_{i=1}^{n} U(s_i)A_i)\mathrm{d}\mu \tag{2.2}$$

for $A_i \in L^\infty(Q, \Sigma_0, \mu)$, $1 \le i \le n$, and $s_1 \le \cdots \le s_n$, $s_n - s_1 \le \beta$.

2.3 Perturbations of generalized path spaces

We now describe perturbations of generalized path spaces obtained from a Feynman-Kac-Nelson kernel. Unless stated otherwise, we will consider the case $\beta < \infty$.

Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS-positive β-periodic path space. Let V be a selfadjoint operator on $\mathcal{H}$, which is affiliated to $\mathcal{U}$. Using the isomorphism between $\mathcal{U}$ and $L^\infty(Q, \Sigma_0, \mu)$ we can view V as a real Σ_0-measurable function on Q, which we still denote by V.

Assume that $V \in L^1(Q, \Sigma_0, \mu)$ and $\exp(-\beta V) \in L^1(Q, \Sigma_0, \mu)$. Then (see [KL1] or [GeJ, Proposition 6.2]) the function $F := \exp\big(-\int_{-\beta/2}^{\beta/2} U(t)V\mathrm{d}t\big)$ belongs to $L^1(Q, \Sigma, \mu)$. One can hence define the perturbed measure $\mathrm{d}\mu_V := (\int_Q F\mathrm{d}\mu)^{-1}F\mathrm{d}\mu$. The perturbed path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_V)$ is OS-positive and β-periodic (see [KL1]). Hence we can associate to this perturbed path space a stochastically positive β-KMS system $(\mathcal{B}_V, \mathcal{U}_V, \tau_V, \omega_V)$.

The following concrete realization of the perturbed β-KMS system $(\mathcal{B}_V, \mathcal{U}_V, \tau_V, \omega_V)$ has been obtained in [KL1] (with some improvements in [GeJ]):

- the physical Hilbert space $\mathcal{H}_V$ obtained from the reconstruction theorem outlined in the previous subsection is equal to the physical Hilbert space $\mathcal{H}$ of the unperturbed β-KMS system $(\mathcal{B}, \mathcal{U}, \tau, \omega)$;
- the von Neumann algebra $\mathcal{B}_V$ and the abelian algebra $\mathcal{U}_V$ are equal to $\mathcal{B}$ and $\mathcal{U}$, respectively;
- the operator sum $L + V$ is essentially selfadjoint on $\mathcal{D}(L) \cap \mathcal{D}(V)$ and if $H_V := \overline{L + V}$, then the perturbed time-evolution τ_V on $\mathcal{B}$ is given by

$$\tau_{V,t}(B) = \mathrm{e}^{\mathrm{i}tH_V} B \mathrm{e}^{-\mathrm{i}tH_V}, \quad B \in \mathcal{B};$$

- the distinguished vector Ω of the unperturbed KMS system belongs to $\mathcal{D}\big(\mathrm{e}^{-\frac{\beta}{2}H_V}\big)$ and the perturbed KMS state ω_V is given by $\omega_V(B) = (\Omega_V, B\Omega_V)$, where

$$\Omega_V := \frac{\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega}{\|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|}.$$

The following result is shown in [GeJ, Theorem 6.12]: If $\mathrm{e}^{-\beta V} \in L^1(Q, \Sigma_0, \mu)$ and

$$V \in L^p(Q, \Sigma_0, \mu), \quad \mathrm{e}^{-\frac{\beta}{2}V} \in L^q(Q, \Sigma_0, \mu) \quad \text{for} \quad p^{-1} + q^{-1} = \frac{1}{2}, \quad 2 \le p, q \le \infty,$$

then the operator sum $H_V - JVJ$ is essentially selfadjoint and the *Liouvillean* L_V (for a general definition of Liouvilleans see, e.g., [DJP]) for the perturbed β-KMS system $(\mathcal{B}_V, \tau_V, \omega_V)$ is equal to $\overline{H_V - JVJ}$. Here J denotes the modular conjugation associated to the pair $(\mathcal{B}, \Omega)$.

2.4 Perturbed dynamics associated to FKN kernels

Let us describe in more details the construction of $H_V = \overline{L + V}$ given in [KL1] which is based on the Feynman-Kac-Nelson formula. Note that the results of this subsection are also valid in the 0-temperature case $\beta = +\infty$. Let V be a real Σ_0-measurable function such that $V \in L^1(Q, \Sigma_0, \mu)$ and $\mathrm{e}^{-TV} \in L^1(Q, \Sigma_0, \mu)$ for some $T > 0$ if $\beta = \infty$ and for $T = \beta$ if $\beta < \infty$. Set

$$F_{[0,s]} := \mathrm{e}^{-\int_0^s U(t)V\mathrm{d}t}, \quad 0 \le s \le \inf(T, \beta)/2,$$

which belongs to $L^2(Q, \Sigma_{[0,\inf(T,\beta)/2]}, \mu)$. The family $\{F_{[0,s]}\}_{0 \le s \le \inf(T,\beta)/2}$ is called a *Feynman-Kac-Nelson kernel.*

For $0 \le t \le \inf(T, \beta)/2$ we set

$$\mathcal{M}_t := \text{ linear span of } \bigcup_{0 \le s \le \inf(T,\beta)/2 - t} F_{[0,s]} L^\infty(Q, \Sigma_{[0,\inf(T,\beta)/2-t]}, \mu)$$

and

$$U_V(s)\colon \begin{array}{ccc} \mathcal{M}_t & \to & L^2(Q, \Sigma_+, \mu) \\ \psi & \mapsto & F_{[0,s]}U(s)\psi, \end{array} \quad 0 \le s \le t.$$

Setting finally

$$\mathcal{D}_t = \mathcal{V}(\mathcal{M}_t), \tag{2.3}$$

one can show that

$$\begin{array}{rccc} P_V(s)\colon & \mathcal{D}_t & \to & \mathcal{H} \\ & \mathcal{V}(\psi) & \mapsto & \mathcal{V}(F_{[0,s]}U(s)\psi) \end{array}$$

is a well defined linear operator, and that $\big(P_V(t), \mathcal{D}_t, \inf(T,\beta)/2\big)$ is a local symmetric semigroup on $\mathcal{H}$. Now let H_V be the unique selfadjoint operator associated to the local symmetric semigroup $(\mathcal{D}_t, P_V(t), \inf(T,\beta)/2)$. It follows (see [KL1]) that $H_V = \overline{L+V}$.

In the sequel we will need the following result.

Proposition 2.5 *Let $V \in L^2(Q,\Sigma_0,\mu)$ be a real function such that $\mathrm{e}^{-TV} \in L^1(Q,\Sigma_0,\mu)$ for some $T > 0$ and $V_n := V\mathbb{1}_{\{|V|\le n\}}$ for $n \in \mathbb{N}$. Denote by L the selfadjoint operator on $\mathcal{H}$ associated to the OS-positive path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$. Let H_n be the closure of $L+V-V_n$. Then*

$$\mathrm{e}^{-\mathrm{i}tL} = \text{s-}\lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}tH_n}, \ \ t \in \mathbb{R}.$$

Note that the selfadjoint operators H_n are associated to local symmetric semigroups $\big(P_n(t), \mathcal{D}_t^{(n)}, T/2\big)$ obtained from the FKN kernels

$$F^{(n)}_{[0,s]} := \mathrm{e}^{-\int_0^s U(t)(V-V_n)\mathrm{d}t},$$

and the operator L is associated to the local symmetric semigroup $\big(P_\infty(t), \mathcal{D}_t, T/2\big)$ obtained from the FKN kernels $F^{(\infty)}_{[0,s]} = 1$.

Proof. We first claim that

$$\sup_{0\le s\le T/2} \|F^{(n)}_{[0,s]} - 1\|_{L^1(Q,\Sigma,\mu)} \to 0 \text{ for } n \to \infty. \tag{2.4}$$

In order to prove (2.4), we recall the following bound from [KL4, Theorem 6.2 (i)]:

$$\|\mathrm{e}^{-\int_a^b U(t)V\mathrm{d}t}\|_{L^p(Q,\Sigma,\mu)} \le \|\mathrm{e}^{-(b-a)V}\|_{L^p(Q,\Sigma,\mu)}, \ \ 1 \le p < \infty. \tag{2.5}$$

Now let W be a real measurable function on Q. Using $1 - \mathrm{e}^{-a} = a\int_0^1 \mathrm{e}^{-\theta a}\mathrm{d}\theta$ we find

$$1 - \mathrm{e}^{-\int_0^s U(t)W\mathrm{d}t} = \int_0^s U(t)W\mathrm{d}t \int_0^1 \mathrm{e}^{-\theta\int_0^s U(t)W\mathrm{d}t}\mathrm{d}\theta.$$

This yields

$$\begin{array}{rcl} \|1 - \mathrm{e}^{-\int_0^s U(t)W\mathrm{d}t}\|_{L^1} & \le & |s|\,\|W\|_{L^2}\int_0^1 \|\mathrm{e}^{-\theta\int_0^s U(t)W\mathrm{d}t}\|_{L^2}\mathrm{d}\theta \\ & \le & |s|\,\|W\|_{L^2}\int_0^1 \|\mathrm{e}^{-\theta sW}\|_{L^2}\mathrm{d}\theta \\ & \le & |s|\,\|W\|_{L^2}\big(1 + \int_0^1 \|\mathrm{e}^{\theta sW_-}\|_{L^2}\mathrm{d}\theta\big) \\ & \le & \frac{T}{2}\|W\|_{L^2}\big(1 + \|\mathrm{e}^{TW_-}\|_{L^1}\big), \end{array}$$

where $W_- = \sup(0,-W)$ denotes the negative part of W. In the first line we have used the Cauchy-Schwarz inequality and the fact that $U(t)$ is unitary on $L^2(Q,\Sigma,\mu)$, in the second line the estimate (2.5).

By assumption $V \in L^2(Q,\Sigma,\mu)$ and $\mathrm{e}^{-TV} \in L^1(Q,\Sigma,\mu)$. Thus $V - V_n \to 0$ in $L^2(Q,\Sigma,\mu)$ and $\mathrm{e}^{T(V-V_n)_-} \to 0$ in $L^1(Q,\Sigma,\mu)$. Applying the above bound for $W = V - V_n$, we obtain (2.4).

Before we finish the proof, we extract a Lemma.

Lemma 2.6 *Let $\left(P_n(t), \mathcal{D}_t^{(n)}, T\right)$ for $n \in \mathbb{N} \cup \{\infty\}$ be a family of local symmetric semigroups on a Hilbert space $\mathcal{H}$. Let H_n, $n \in \mathbb{N} \cup \{\infty\}$, denote the associated selfadjoint operators.*

Assume that there exists a family $\{\mathcal{L}_t\}$ for $0 < t \leq T' \leq T$ of subspaces of $\mathcal{H}$ with

$$\mathcal{L}_t \subset \mathcal{D}_t^{(n)}, \quad \bigcup_{0<t\leq T'} \mathcal{L}_t \text{ dense in } \mathcal{H}. \tag{2.6}$$

Assume moreover that

$$\lim_{n\to\infty} (\Psi, P_n(s)\Psi) = (\Psi, P_\infty(s)\Psi), \ \Psi \in \mathcal{L}_t, \ 0 \leq s \leq t \leq T', \tag{2.7}$$

$$\sup_n \sup_{0\leq s\leq t} (\Psi, P_n(s)\Psi) < \infty, \ \Psi \in \mathcal{L}_t, \ 0 \leq t \leq T'. \tag{2.8}$$

Then $\text{s-}\lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}tH_n} = \mathrm{e}^{-\mathrm{i}tH_\infty}$ *for all $t \in \mathbb{R}$.*

Proof. Let us fix $0 < t \leq T'$ and $\Psi \in \mathcal{L}_t$. From [KL3, Lemma 1], we know that there exist positive measures $\{\nu_n\}$ on $\mathbb{R}$ such that

$$(\Psi, P_n(s)\Psi) = \left\| P_n\left(\frac{s}{2}\right)\Psi \right\|^2 = \int_{\mathbb{R}} \mathrm{e}^{-sa} \mathrm{d}\nu_n(a), \ \ 0 \leq s \leq t.$$

Moreover, one has (see [KL3, Lemma 1])

$$(\Psi, \mathrm{e}^{-\mathrm{i}yH_n}\Psi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ya} \mathrm{d}\nu_n(a).$$

Set

$$f_n(z) := \int_{\mathbb{R}} \mathrm{e}^{-za} \mathrm{d}\nu_n(a), \ \ z \in]0, t[+\mathrm{i}\mathbb{R}.$$

The family $\{f_n\}$ is uniformly bounded on $]0, t[+\mathrm{i}\mathbb{R}$ by (2.8) and converges pointwise to f_∞ on $]0, T[$ by (2.7). Applying Lemma B.3 we conclude that $f_n(z)$ converges to $f_\infty(z)$ for all $z \in \mathrm{i}\mathbb{R}$. This implies that on $\mathcal{L}_t$

$$\mathrm{w} - \lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}yH_n} = \mathrm{e}^{-\mathrm{i}yH_\infty} \ \ \forall y \in \mathbb{R}.$$

Since by hypothesis $\bigcup_{0<t\leq T} \mathcal{L}_t$ is dense in $\mathcal{H}$ and for unitary operators weak convergence implies strong convergence, this completes the proof of the lemma $\square$.

Proof of Proposition 2.5 (second part). Let us now fix a convenient family of subspaces $\mathcal{L}_t$. For $0 < t < T/4$ we set $\mathcal{L}_t = \mathcal{V}\mathcal{R}_t$, where $\mathcal{R}_t$ equals $L^\infty(Q, \Sigma_{[0,\beta/2-t]}, \mu)$ if $\beta < \infty$ and $\mathcal{R}_t$ equals $L^\infty(Q, \Sigma_{[0,+\infty[}, \mu)$ if $\beta = +\infty$. Clearly $\mathcal{L}_t$ is included in the spaces $\mathcal{D}_t^{(n)}$ defined in (2.3) (with V replaced by $V - V_n$). Moreover, $\bigcup_{0<t\leq T/4} \mathcal{L}_t$ is dense in $\mathcal{H}$, hence hypothesis (2.6) of Lemma 2.6 is satisfied. Let us now fix some $\Psi \in \mathcal{L}_t$, i.e., $\Psi = \mathcal{V}\psi$ for some $\psi \in \mathcal{R}_t$. Using (2.4) we obtain that

$$\lim_{n\to\infty} (\Psi, P_n(s)\Psi) = (\Psi, P_\infty(s)\Psi) \text{ for } 0 \leq s \leq t$$

and $\sup_n \sup_{0\leq s\leq t} (\Psi, P_n(s)\Psi) < \infty$. Hence hypotheses (2.7) and (2.8) of Lemma 2.6 are satisfied. Thus we can apply Lemma 2.6 and this completes the proof of Proposition 2.5 $\square$.

3 Gaussian measures

In this Section we recall some standard facts about Gaussian measures on distribution spaces.

3.1 Distribution spaces

Let $S_\beta = [-\beta/2, \beta/2]$ (with end points identified) be the circle of length $\beta > 0$. Points in $S_\beta \times \mathbb{R}^d$, $d \geq 1$, will be denoted by (t, x).

The Fréchet space of Schwartz functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}(\mathbb{R}^d)$. For coherence of notation, the Fréchet space $\mathcal{D}(S_\beta)$ of smooth periodic functions on S_β will also be denoted by $\mathcal{S}(S_\beta)$.

In addition, we denote by $\mathcal{S}(S_\beta \times \mathbb{R}^d)$ the Fréchet space of Schwartz functions on $S_\beta \times \mathbb{R}$, i.e., the space of smooth functions on $S_\beta \times \mathbb{R}^d$, which are β-periodic in t and such that for all $p \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$

$$\left|(1+|x|)^{|\alpha|}\partial_t^p \partial_x^\alpha f(t,x)\right| \leq C_{p,\alpha}\,.$$

We will denote by $\mathcal{S}'(\mathbb{R}^d)$, $\mathcal{S}'(S_\beta)$ and $\mathcal{S}'(S_\beta \times \mathbb{R}^d)$ the duals of $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}(S_\beta)$ and $\mathcal{S}(S_\beta \times \mathbb{R}^d)$. The spaces of real elements in these spaces will be denoted by $\mathcal{S}'_{\mathbb{R}}(\mathbb{R}^d)$, $\mathcal{S}'_{\mathbb{R}}(S_\beta)$ and $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}^d)$.

We set $D_t = \mathrm{i}^{-1}\partial_t$ and $D_x = \mathrm{i}^{-1}\partial_x$, and we will denote by D_t^2 the selfadjoint operator on $L^2(S_\beta)$ defined by

$$D_t^2 := -\partial_t^2, \quad \mathcal{D}(D_t^2) := \left\{u \in L^2(S_\beta) \mid \partial_t^2 u \in L^2(S_\beta),\, u(0) = u(\beta)\right\}.$$

We denote by $D_t^2 + D_x^2$ the selfadjoint operator on $L^2(S_\beta \times \mathbb{R}^d)$ with domain

$$\mathcal{D}(D_t^2 + D_x^2) := \left\{u \in L^2(S_\beta \times \mathbb{R}^d) \mid (D_t^2 + D_x^2)u \in L^2(S_\beta \times \mathbb{R}^d),\, u \text{ is } \beta\text{-periodic in } t\right\}.$$

We denote by $\mathcal{S}(\mathbb{Z} \times \mathbb{R}^d)$ the Fréchet space of sequences $\{u_n\}_{n\in\mathbb{N}}$ with values in $\mathcal{S}(\mathbb{R}^d)$ such that

$$\sum_{n\in\mathbb{Z}} |n|^p \|(D_x^2 + x^2)^{p/2} u_n\|_{L^2(\mathbb{R}^d)} < \infty \quad \forall p \in \mathbb{N}.$$

We now fix the notation concerning partial Fourier transforms. We first define the (unitary) partial Fourier transform with respect to t:

$$\begin{array}{rccc} \mathcal{F}_t\colon & \mathcal{S}(S_\beta \times \mathbb{R}^d) & \to & \mathcal{S}(\mathbb{Z} \times \mathbb{R}^d) \\ & u & \mapsto & \{\hat{u}_n\} \end{array},$$

where $\hat{u}_n(x) = \beta^{-\frac{1}{2}} \int_{S_\beta} \mathrm{e}^{-\mathrm{i}\nu_n t} u(t,x)\mathrm{d}t$. (The coefficients $\nu_n = 2\pi n/\beta$, $n \in \mathbb{N}$, are called in physics *Matsubara frequencies*). Its inverse is

$$u(t,x) = \beta^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\nu_n t}\hat{u}_n(x).$$

The (unitary) partial Fourier transform with respect to x is

$$\begin{array}{rccc} \mathcal{F}_x\colon & \mathcal{S}(S_\beta \times \mathbb{R}^d) & \to & \mathcal{S}(S_\beta \times \mathbb{R}^d) \\ & u & \mapsto & \hat{u} \end{array},$$

where $\hat{u}(t,p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot p} u(t,x)\mathrm{d}x$. Its inverse is

$$u(t,x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot p}\hat{u}(t,p)\mathrm{d}p.$$

For later use we fix two approximations of the Dirac δ functions in t and x. We set, for $k \in \mathbb{N}$,

$$\delta_k(s) := \beta^{-1} \sum_{|n|\leq k} \mathrm{e}^{i\nu_n s} \quad \text{and} \quad \delta_k(x) := k\chi(kx),$$

where χ is a function in $C^\infty_{0\,\mathbb{R}}(\mathbb{R}^d)$ with $\int \chi(x)\mathrm{d}x = 1$.

3.2 Gaussian measures

We set

$$C(f,g) = \big(f,(D_t^2+D_x^2+m^2)^{-1}g\big),\ \ f,g\in\mathcal{S}(S_\beta\times\mathbb{R}^d), \tag{3.1}$$

where $(.,.)$ is the scalar product on $L^2(S_\beta\times\mathbb{R}^d)$.

Let $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta\times\mathbb{R}^d)$ and let Σ be the Borel σ-algebra on Q. If $f\in\mathcal{S}_{\mathbb{R}}(S_\beta\times\mathbb{R}^d)$, then $\phi(f)$ denotes the coordinate function

$$\begin{array}{rccc}\phi(f)\colon & Q & \to & \mathbb{C} \\ & q & \mapsto & \langle q,f\rangle\end{array}.$$

Let F be a Borel function on $\mathbb{R}$. Then $F(\phi(f))$ denotes the function

$$\begin{array}{rccc}F(\phi(f))\colon & Q & \to & \mathbb{C} \\ & q & \mapsto & F\big(\langle q,f\rangle\big)\end{array}.$$

We denote by $\mathrm{d}\phi_C$ the Gaussian measure on (Q,Σ) with covariance C defined by

$$\int_Q \mathrm{e}^{\mathrm{i}\phi(f)}\mathrm{d}\phi_C = \mathrm{e}^{-C(f,f)/2},\quad f\in\mathcal{S}_{\mathbb{R}}(S_\beta\times\mathbb{R}^d). \tag{3.2}$$

We have

$$\int_Q \phi(f)^p\mathrm{d}\phi_C = \left\{\begin{array}{l} 0,\ p\text{ odd}, \\ (p-1)!!C(f,f)^{p/2},\ p\text{ even},\end{array}\right. \tag{3.3}$$

where $n!! = n(n-2)(n-4)\cdots 1$. One easily deduces from (3.3) that $\mathrm{e}^{\phi(f)}\in L^1(Q,\Sigma,\mathrm{d}\phi_C)$ if $f\in\mathcal{S}_{\mathbb{R}}(S_\beta\times\mathbb{R}^d)$.

The cylindrical functions $F\big(\phi(f_1),\dots,\phi(f_n)\big)$, $f_i\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}\times S_\beta)$, F a Borel function on $\mathbb{R}^n$ and $n\in\mathbb{N}$, are dense in $L^p(Q,\Sigma,\mathrm{d}\phi_C)$ for $1\le p<\infty$.

3.3 Sharp-time fields

We now recall some standard results about the existence of *sharp-time fields.* We will make use of the following well known identity (see [KL2]):

$$\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\frac{\mathrm{e}^{i\nu_n t}}{\nu_n^2+\epsilon^2} = \frac{\mathrm{e}^{-|t|\epsilon}+\mathrm{e}^{-(\beta-|t|)\epsilon}}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})}\ \text{ for }\epsilon>0,\ \nu_n=\frac{2\pi n}{\beta},\ \ 0\le|t|\le\beta. \tag{3.4}$$

For $h_1,h_2\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$, $0\le t_1,t_2\le\beta$, and $k\in\mathbb{N}$

$$\begin{array}{rl} & C\big(\delta_k(.-t_1)\otimes h_1,\delta_k(.-t_2)\otimes h_2\big) \\ = & \beta^{-1}\sum_{|n|\le k}\mathrm{e}^{i\nu_n(t_1-t_2)}\big(\hat{h}_{1n},(\nu_n^2+D_x^2+m^2)^{-1}\hat{h}_{2n}\big)_{L^2(\mathbb{R}^d)}.\end{array}$$

Using (3.4) we see that

$$\lim_{k\to\infty}C\big(\delta_k(.-t_1)\otimes h_1,\delta_k(.-t_2)\otimes h_2\big) = \Big(h_1,\frac{\mathrm{e}^{-|t_2-t_1|\epsilon}+\mathrm{e}^{-(\beta-|t_2-t_1|)\epsilon}}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})}h_2\Big)_{L^2(\mathbb{R}^d)},$$

where $\epsilon := (D_x^2+m^2)^{\frac{1}{2}}$.

Using (3.3) this implies that, for $h\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$ and $t\in S_\beta$ fixed, the sequence of functions $\{\phi(\delta_k(.-t)\otimes h)\}_{k\in\mathbb{N}}$ is Cauchy in $\bigcap_{1\le p<\infty}L^p(Q,\Sigma,\mathrm{d}\phi_C)$.

We set

$$\phi(t,h) := \lim_{k\to\infty}\phi(\delta_k(.-t)\otimes h) \tag{3.5}$$

and

$$C_0(t_1,h_1,t_2,h_2) := \Big(h_1, \frac{\mathrm{e}^{-|t_2-t_1|\epsilon} + \mathrm{e}^{-(\beta-|t_2-t_1|)\epsilon}}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R}^d)}. \tag{3.6}$$

We note that $\phi(t,h)$ belongs to $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$. For later use we define the *temperature* β^{-1} *covariance on* $\mathbb{R}^d$:

$$C_0(h_1,h_2) := \Big(h_1, \frac{(1+\mathrm{e}^{-\beta\epsilon})}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R}^d)},\ \ h_1,h_2\in\mathcal{S}(\mathbb{R}^d). \tag{3.7}$$

3.4 Sharp-space fields

If $d=1$, then it is possible to define similarly *sharp-space fields*. We first recall another well-known identity, which is analogous to (3.4):

$$(2\pi)^{-1}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}px}}{p^2+b^2}\mathrm{d}p = \frac{\mathrm{e}^{-b|x|}}{2b} \text{ for } b>0,\ x\in\mathbb{R}. \tag{3.8}$$

For $g_1,g_2\in\mathcal{S}_{\mathbb{R}}(S_\beta)$ and $x_1,x_2\in\mathbb{R}$ one has

$$\begin{aligned} & C\big(g_1\otimes\delta_k(.-x_1), g_2\otimes\delta_k(.-x_2)\big) \\ = \ & \textstyle\int_{\mathbb{R}} \hat{\chi}^2(\frac{p}{k})\mathrm{e}^{\mathrm{i}p(x_1-x_2)}\big(g_1,(D_t^2+p^2+m^2)^{-1}g_2\big)_{L^2(S_\beta)}\mathrm{d}p. \end{aligned} \tag{3.9}$$

Using (3.8) and $\hat{\chi}(0)=(2\pi)^{-\frac{1}{2}}$ we find

$$\lim_{k\to\infty} C\big(g_1\otimes\delta_k(.-x_1), g_2\otimes\delta_k(.-x_2)\big) = \Big(g_1, \frac{\mathrm{e}^{-|x_1-x_2|b}}{2b} g_2\Big)_{L^2(S_\beta)}, \tag{3.10}$$

where $b := (D_t^2+m^2)^{\frac{1}{2}}$. Now we can use (3.3) again: for $g\in\mathcal{S}_{\mathbb{R}}(S_\beta)$ and $x\in\mathbb{R}$ fixed, the sequence of functions $\{\phi\big(g\otimes\delta_k(.-x)\big)\}_{k\in\mathbb{N}}$ is Cauchy in $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$.

We set

$$\phi(g,x) := \lim_{k\to\infty}\phi\big(g\otimes\delta_k(.-x)\big)$$

and

$$C_\beta(g_1,x_1,g_2,x_2) := \Big(g_1, \frac{\mathrm{e}^{-|x_1-x_2|b}}{2b} g_2\Big)_{L^2(S_\beta)}. \tag{3.11}$$

We note that $\phi(g,x)$ belongs to $\bigcap_{1\le k<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$. For later use we define the *0-temperature covariance on* S_β:

$$C_\beta(g_1,g_2) := \Big(g_1, \frac{1}{2b} g_2\Big)_{L^2(S_\beta)},\ g_1,g_2\in\mathcal{S}(S_\beta). \tag{3.12}$$

3.5 Some elementary properties

From (3.3), (3.6) and (3.11) we deduce that the maps

$$\begin{array}{ccc} H_{\mathbb{R}}^{-1}(S_\beta\times\mathbb{R}^d) & \to & \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C) \\ f & \mapsto & \phi(f) \end{array}, \tag{3.13}$$

$$\begin{array}{ccc} S_\beta\times H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{R}^d) & \to & \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C) \\ (t,h) & \mapsto & \phi(t,h) \end{array} \tag{3.14}$$

and

$$\begin{array}{ccc} H_{\mathbb{R}}^{-\frac{1}{2}}(S_\beta)\times\mathbb{R} & \to & \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C) \\ (g,x) & \mapsto & \phi(g,x) \end{array} \tag{3.15}$$

are continuous.

For $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$, $t \in S_\beta$ and $x \in \mathbb{R}$ we set

$$f_t\colon \begin{array}{ccc} \mathbb{R} & \to & \mathbb{C} \\ x & \mapsto & f(t,x) \end{array}, \qquad f_x\colon \begin{array}{ccc} S_\beta & \to & \mathbb{C} \\ t & \mapsto & f(t,x) \end{array}.$$

We note that $f_t \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $f_x \in \mathcal{S}_{\mathbb{R}}(S_\beta)$.

Lemma 3.1 *If $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$, then the following identity holds on $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\int_{\mathbb{R}} \phi(f_x, x)\mathrm{d}x = \int_{S_\beta} \phi(t, f_t)\mathrm{d}t = \phi(f).$$

Proof. Let $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and $k \in \mathbb{N}$. The map

$$\begin{array}{ccc} \mathbb{R} & \to & H^{-1}(S_\beta \times \mathbb{R}) \\ x & \mapsto & f_x \otimes \delta_k(. - x) \end{array}$$

is continuous. Since $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$, the bound $\|f_x \otimes \delta_k(. - x)\|_{H^{-1}(S_\beta \times \mathbb{R})} \in O\big(|x|^{-\infty}\big)$ holds true. Hence by (3.15) the map

$$\begin{array}{ccc} \mathbb{R} & \to & \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C) \\ x & \mapsto & \phi\big(f_x \otimes \delta_k(. - x)\big) \end{array}$$

is continuous and $\big\|\phi\big(f_x \otimes \delta_k(. - x)\big)\big\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)} \in O\big(|x|^{-\infty}\big)$. Therefore $\int_{\mathbb{R}} \phi\big(f_x \otimes \delta_k(. - x)\big)\mathrm{d}x$ is well defined as an element of $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. Moreover,

$$\int_{\mathbb{R}} \phi\big(f_x \otimes \delta_k(. - x)\big)\mathrm{d}x = \phi\Big(\int_{\mathbb{R}} f_x \otimes \delta_k(. - x)\mathrm{d}x\Big) = \phi(f * \delta_k),$$

where the convolution product $*$ acts only in the space variable x. Since $\lim_{k\to\infty} f * \delta_k = f$ holds in $H^{-1}(S_\beta \times \mathbb{R})$, we obtain from (3.13)

$$\lim_{k\to\infty} \int_{\mathbb{R}} \phi\big(f_x \otimes \delta_k(. - x)\big)\mathrm{d}x = \phi(f) \quad \text{in} \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C).$$

It follows from (3.9) and (3.10) that

$$\lim_{k\to\infty} \sup_{x \in \mathbb{R}} |x|^N \big\|\phi\big(f_x \otimes \delta_k(. - x)\big) - \phi(f_x, x)\big\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)} = 0$$

for $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and $N \in \mathbb{N}$. Hence

$$\lim_{k\to\infty} \int_{\mathbb{R}} \phi\big(f_x \otimes \delta_k(. - x)\big)\mathrm{d}x = \int_{\mathbb{R}} \phi(f_x, x)\mathrm{d}x.$$

This proves the first identity of the lemma. The second one can be shown by similar arguments $\Box$.

4 Path spaces supported by $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \Sigma, \mathrm{d}\phi_C)$

In this section we recall two well known path spaces supported by $(Q, \Sigma, \mathrm{d}\phi_C)$. The first is associated to the free neutral scalar field of mass m on S_β at temperature 0; the second is associated to the free neutral scalar field of mass m on $\mathbb{R}$ at temperature β^{-1}.

We recall that (t, x) denotes a point in $S_\beta \times \mathbb{R}$, and refer to t as the (euclidean) time and to x as space variable. The time translation induced on Q by the map $(t, x) \mapsto (t+s, x)$ will be denoted by $\mathfrak{t}_s\colon Q \to Q$ and the spatial translations induced on Q by the map $(t, x) \mapsto (t, x+y)$ will be denoted by $\mathfrak{a}_y\colon Q \to Q$.

4.1 The free massive euclidean field on the circle at 0-temperature

In this subsection we identify the generalized path space on $(Q, \Sigma, \mathrm{d}\phi_C)$ corresponding to the free massive scalar field on the circle S_β at temperature 0.

Let $\Sigma_0^{\mathtt{C}}$ be the sub σ-algebra of Σ generated by the functions $\{\phi(g,0) \mid g \in \mathcal{S}_{\mathbb{R}}(S_\beta)\}$.

We denote by $\{U_{\mathtt{C}}(x)\}_{x\in\mathbb{R}}$ the 1-parameter group generated by the spatial translations $\{\mathfrak{a}_x\}_{x\in\mathbb{R}}$. More precisely, if $F: Q \to \mathbb{C}$ is a function on Q, then $U_{\mathtt{C}}(x)F(q) := F\big(\mathfrak{a}_{-x}(q)\big)$ for $q \in Q$. Applying (3.2) we see that $x \mapsto U_{\mathtt{C}}(x)$ is a strongly continuous unitary group on $L^2(Q, \Sigma, \mathrm{d}\phi_C)$, and hence extends to a group of measure-preserving automorphisms of $L^\infty(Q, \Sigma, \mathrm{d}\phi_C)$ which is continuous in measure.

Let $r_{\mathtt{C}}: Q \to Q$ be the space reflection around $x = 0$. We denote by $R_{\mathtt{C}}$ the measure preserving transformation of $(Q, \Sigma, \mathrm{d}\phi_C)$ generated by $r_{\mathtt{C}}$.

For $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ we have

(4.1) $$U_{\mathtt{C}}(x)\phi(g,0) = \phi(g,x).$$

Using then Lemma 3.1, we see that $\Sigma = \bigvee_{x\in\mathbb{R}} U_{\mathtt{C}}(x)\Sigma_0^{\mathtt{C}}$.

Hence $(Q, \Sigma, \Sigma_0^{\mathtt{C}}, U_{\mathtt{C}}(x), R_{\mathtt{C}}, \mathrm{d}\phi_C)$ is a generalized path space. Moreover, it is OS-positive (see e.g. [KL2]).

It describes the *free neutral scalar euclidean field of mass m on the circle S_β at temperature* 0.

Let us now briefly describe a well-known concrete form of the physical objects associated to this path space by the reconstruction theorem. Let $H^{-\frac{1}{2}}(S_\beta)$ be the Sobolev space of order $-\frac{1}{2}$ equipped with its canonical complex structure i and scalar product $(h_1, (2b)^{-1}h_2)_{L^2(S_\beta)}$, where $b = (D_t^2 + m^2)^{\frac{1}{2}}$. Then the physical Hilbert space can be unitarily identified with the bosonic Fock space $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$ over $H^{-\frac{1}{2}}(S_\beta)$. The distinguished unit vector $\Omega_{\mathtt{C}}^\circ := \mathcal{V}1$ is identified with the Fock vacuum Ω in $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$. The (free) Hamiltonian is

$$H_{\mathtt{C}}^\circ = \mathrm{d}\Gamma(b).$$

The abelian von Neumann algebra $\mathcal{U}_{\mathtt{C}}$ obtained from the reconstruction theorem can be identified with the von Neumann algebra generated by $\{W_F(g) \mid g \in H_{\mathbb{R}}^{-\frac{1}{2}}(S_\beta)\}$. In fact, if $A = \mathrm{e}^{\mathrm{i}\phi(g,0)}$ for $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$, then the operator $\tilde{A}$ defined in (2.1) is identified with the Fock Weyl operator $W_F(g) = \mathrm{e}^{\mathrm{i}\phi_F(g)}$ on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$.

4.2 The free massive euclidean field on $\mathbb{R}$ at temperature β^{-1}

We now identify the generalized path space on $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \Sigma, \mathrm{d}\phi_C)$ corresponding to the free massive scalar euclidean field on $\mathbb{R}$ at temperature β^{-1}.

Let Σ_0 be the sub σ-algebra of Σ generated by the functions $\{\phi(0,h) \mid h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\}$. We denote by $\{U(t)\}_{t\in S_\beta}$ the one parameter group generated by $\{\mathfrak{r}_t\}_{t\in S_\beta}$. If $F: Q \to \mathbb{C}$ is a function on Q, then $U(t)F(q) := F\big(\mathfrak{r}_{-t}(q)\big)$ for $q \in Q$. Using (3.2) we see that $t \mapsto U(t)$ is a strongly continuous β-periodic unitary group on $L^2(Q, \Sigma, \mathrm{d}\phi_C)$. Hence it extends to a group of measure-preserving automorphisms of $L^\infty(Q, \Sigma, \mathrm{d}\phi_C)$ which is continuous in measure.

Let r be the (euclidean) time reflection around $t = 0$. We denote by R the measure preserving transformation of $(Q, \Sigma, \mathrm{d}\phi_C)$ generated by r.

For $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ we have

(4.2) $$U(t)\phi(0,h) = \phi(t,h).$$

Again by Lemma 3.1, we see that $\Sigma = \bigvee_{t\in S_\beta} U(t)\Sigma_0$. Hence $(Q, \Sigma, \Sigma_0, U(t), R, \mathrm{d}\phi_C)$ is a generalized path space. Moreover, it is β-periodic and OS-positive (see e.g. [KL2]). It describes the *free neutral scalar field of mass m on $\mathbb{R}$ at temperature β^{-1}*.

We now describe a well known concrete form of the β-KMS system associated to the generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mathrm{d}\phi_C)$. Let $\mathfrak{h} := H^{-\frac{1}{2}}(\mathbb{R})$ be the Sobolev space of order $-\frac{1}{2}$, equipped with its canonical complex structure i and scalar product $(h_1, h_2) = (h_1, (2\epsilon)^{-1}h_2)_{L^2(\mathbb{R})}$, where $\epsilon = (D_x^2 + m^2)^{\frac{1}{2}}$. On $\mathfrak{h}$ we consider the unitary dynamics $\mathrm{e}^{-\mathrm{i}t\epsilon}$.

On the Weyl algebra $\mathcal{W}(\mathfrak{h})$ we define a state ω_β° and a one-parameter group of automorphisms $\{\tau_t^\circ\}_{t\in\mathbb{R}}$ by

$$\omega_\beta^\circ\big(W(h)\big) := \mathrm{e}^{-\frac{1}{4}(h,(1+2\rho)h)}, \quad \tau_t^\circ\big(W(h)\big) := W(\mathrm{e}^{\mathrm{i}t\epsilon}h),\ h \in \mathfrak{h},\ t \in \mathbb{R}, \tag{4.3}$$

where $\rho := (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$, $\beta > 0$. It can be easily seen that ω_β° is a quasi-free (τ°, β)-KMS state on $\mathcal{W}(\mathfrak{h})$.

Let us now recall some terminology. If $\mathfrak{h}$ is a complex vector space, then the *conjugate vector space* $\overline{\mathfrak{h}}$ is the real vector space $\mathfrak{h}$ equipped with the complex structure $-\mathrm{i}$. We will denote by $\mathfrak{h} \ni h \mapsto \overline{h} \in \overline{\mathfrak{h}}$ the (anti-linear) identity operator. If $a \in \mathcal{L}(\mathfrak{h})$, then we denote by $\overline{a} \in \mathcal{L}(\overline{\mathfrak{h}})$ the operator $\overline{a}\overline{h} := \overline{ah}$. If $\mathfrak{h}$ is a Hilbert space, then $\overline{\mathfrak{h}}$ is equipped with the Hilbert space structure $(\overline{h_1}, \overline{h_2}) := (h_2, h_1)$.

We recall a convenient realization of the GNS representation associated to $\big(\mathcal{W}(\mathfrak{h}), \omega_\beta^\circ\big)$, which is called the *right Araki-Woods representation.* It is specified by setting

$$\begin{aligned}
&\mathcal{H}_{AW} := \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}}),\\
&\Omega_{AW} := \Omega,\\
&\pi_{AW}(W(h)) = W_{AW}(h) := W_{\mathrm{F}}\big((1+\rho)^{\frac{1}{2}}h \oplus \overline{\rho}^{\frac{1}{2}}\overline{h}\big), \quad h \in \mathfrak{h}.
\end{aligned}$$

Here $W_F(.)$ denotes the Fock Weyl operator on $\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ and $\Omega \in \Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ is the Fock vacuum.

The physical Hilbert space associated to the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mathrm{d}\phi_C)$ can be unitarily identified with $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The distinguished vector $\mathcal{V}1$ is identified with the Fock vacuum vector Ω in $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The Liouvillean L_{AW} satisfies

$$\mathrm{e}^{iL_{AW}t}\pi_{AW}(A)\Omega_{AW} = \pi_{AW}\big(\tau_t^\circ(A)\big)\Omega_{AW} \quad \text{and} \quad L_{AW}\Omega_{AW} = 0,$$

and can be identified with $\mathrm{d}\Gamma(\epsilon \oplus -\overline{\epsilon})$.

The abelian von Neumann algebra $\mathcal{U}_{AW}$ obtained by the reconstruction theorem can be identified with the abelian von Neumann algebra generated by $\{W_{AW}(h) \mid h \in H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{R})\}$. In fact, if $A = \mathrm{e}^{\mathrm{i}\phi(0,h)}$ for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, then the operator $\tilde{A}$ defined in (2.1) is identified with the Weyl operator $W_{AW}(h) = \mathrm{e}^{\mathrm{i}\phi_{AW}(h)}$ on $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$.

The von Neumann algebra $\mathcal{B}_{AW}$ generated by $\bigcup_{t\in\mathbb{R}} \tau_t^\circ(\mathcal{U}_{AW})$ can be identified with the von Neumann algebra $\mathcal{R}_{AW}$ generated by $\{W_{AW}(h) \mid h \in H^{-\frac{1}{2}}(\mathbb{R})\}$.

5 Perturbations of path spaces

In this section we describe perturbations of the two path spaces defined in Subsects. 4.1 and 4.2, obtained from FKN kernels corresponding to $P(\phi)_2$ interactions.

5.1 Interaction terms

We recall some well known facts concerning the Wick ordering of Gaussian random variables. Let (K, ν) be a probability space and X a real vector space equipped with a positive quadratic form $f \mapsto c(f, f)$ called a *covariance.* Let $f \mapsto \phi(f)$ be a $\mathbb{R}$-linear map from X into the space of real measurable functions on K.

The *Wick ordering* $:\phi(f)^n:_c$ with respect to the covariance c is defined by the following generating series:

$$:\mathrm{e}^{\alpha\phi(f)}:_c := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :\phi(f)^n:_c = \mathrm{e}^{\alpha\phi(f)}\mathrm{e}^{-\frac{\alpha^2}{2}c(f,f)}. \tag{5.1}$$

Thus

$$:\phi(f)^n:_c = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m!)}\phi(f)^{n-2m}\Big(-\frac{1}{2}c(f,f)\Big)^m, \tag{5.2}$$

where [.] denotes the integer part.

Lemma 5.1

(i) *For $f \in L^1(S_\beta \times \mathbb{R}) \cap L^2(S_\beta \times \mathbb{R})$ the following limit exists in $\bigcap_{1\le p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\lim_{(k,k')\to\infty} \int_{S_\beta \times \mathbb{R}} f(t,x) :\phi\big(\delta_k(.-t)\otimes\delta_{k'}(.-x)\big)^n:_C \mathrm{d}t\mathrm{d}x.$$

It will be denoted by $\int_{S_\beta\times\mathbb{R}} f(t,x) :\phi(t,x)^n:_C \mathrm{d}t\mathrm{d}x$.

(ii) *For $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the following limit exists in $\bigcap_{1\le p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\lim_{k\to\infty} \int_{\mathbb{R}} h(x) :\phi(0, \delta_k(.-x))^n:_{C_0} \mathrm{d}x.$$

It will be denoted by $\int_{\mathbb{R}} h(x) :\phi(0,x)^n:_{C_0} \mathrm{d}x$.

(iii) *For $g \in L^1(S_\beta) \cap L^2(S_\beta)$ the following limit exists in $\bigcap_{1\le p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\lim_{k'\to\infty} \int_{S_\beta} g(t) :\phi(\delta_k(.-t), 0)^n:_{C_\beta} \mathrm{d}t.$$

It will be denoted by $\int_{S_\beta} g(t) :\phi(t,0)^n:_{C_\beta} \mathrm{d}t$.

We recall that the covariances C, C_0 and C_β have been defined in (3.1), (3.7) and (3.12), respectively. In Lemma 5.1 the probability space is $(Q, \Sigma, \mathrm{d}\phi_C)$ and the real vector spaces are equal to $\mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$, $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $\mathcal{S}_{\mathbb{R}}(S_\beta)$, respectively.

Proof. The proof is straightforward, adapting standard arguments (see e.g. [Si], [GeJ, Section 9]) used for the spatially cutoff $P(\phi)_2$ model at 0-temperature $\Box$.

Remark 5.2 *If $P = P(\lambda)$ is a polynomial, then the functions*

$$\int_{S_\beta\times\mathbb{R}} f(t,x) :P(\phi(t,x)):_C \mathrm{d}t\mathrm{d}x, \quad \int_{\mathbb{R}} h(x) :P(\phi(0,x)):_{C_0} \mathrm{d}x \quad \textit{and} \quad \int_{S_\beta} g(t) :P(\phi(t,0)):_{C_\beta} \mathrm{d}t$$

are well defined, by linearity. It can be easily shown (see [GeJ, Proposition 8.4]) using the so-called Wick reordering identities *that there exists a linear invertible map between polynomials*

$$P \mapsto \tilde{P}$$

with $\mathrm{deg}P = \mathrm{deg}\tilde{P}$, $\mathrm{deg}(P-\tilde{P}) \le \mathrm{deg}(P) - 1$ *such that*

$$\int_{\mathbb{R}} h(x) :P(\phi(0,x)):_{C_0} \mathrm{d}x = \int_{\mathbb{R}} h(x) :\tilde{P}(\phi(0,x)):_{\mathrm{vac}} \mathrm{d}x.$$

Here $: :_{\mathrm{vac}}$ denotes Wick ordering with respect to the 0-temperature covariance $(h, \frac{1}{2\epsilon}h)_{L^2(\mathbb{R})}$.

Lemma 5.3 *Let P be a polynomial, $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g \in L^1(S_\beta) \cap L^2(S_\beta)$. Set*

$$\begin{aligned} V_0(h) &:= \textstyle\int_{\mathbb{R}} h(x) :P(\phi(0,x)):_{C_0} \mathrm{d}x, \\ V_\beta(g) &:= \textstyle\int_{S_\beta} g(t) :P(\phi(t,0)):_{C_\beta} \mathrm{d}t, \end{aligned} \tag{5.3}$$

as functions on Q.

Then

$$(5.4)\quad \int_{S_\beta} g(t)U(t)V_0(h)\mathrm{d}t = \int_{S_\beta\times\mathrm{I\!R}} \big(g(t)\otimes h(x)\big) :P(\phi(t,x)):_C \,\mathrm{d}t\mathrm{d}x = \int_{\mathrm{I\!R}} h(x)U_{\mathsf{C}}(x)V_\beta(g)\mathrm{d}x$$

as functions on Q.

Proof. Let W be a function in $L^p(Q,\Sigma,\mathrm{d}\phi_C)$ for some $1\le p<\infty$. The one parameter groups $\{U(t)\}_{t\in S_\beta}$ and $\{U_{\mathsf{C}}(x)\}_{x\in\mathrm{I\!R}}$ are strongly continuous groups of isometries of $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$. Therefore the functions $\int_{\mathrm{I\!R}} h(x)U_{\mathsf{C}}(x)W\mathrm{d}x$ and $\int_{S_\beta} g(t)U(t)W\mathrm{d}t$ belong to $L^p(Q,\Sigma,\mathrm{d}\phi_C)$.

Together with Lemma 5.1 this implies that all three functions given in (5.4) belong to $L^p(Q,\Sigma,\mathrm{d}\phi_C)$. Let us now prove that they are identical. By linearity, we may assume that $P(\lambda)=\lambda^n$. Using Lemma 5.1 and the Wick identity (5.2), it follows that

$$\int_{S_\beta\times\mathrm{I\!R}} \big(g(t)\otimes h(x)\big) :P(\phi(t,x)):_C \,\mathrm{d}t\mathrm{d}x = \lim_{(k,k')\to\infty} F(k,k') \text{ in } L^p(Q,\Sigma,\mathrm{d}\phi_C),$$

where

$$F(k,k') = \sum_{m=0}^{[n/2]} \frac{n!\big(-\frac{1}{2}C(\delta_{k,k'},\delta_{k,k'})\big)^m}{m!(n-2m)!} \int_{S_\beta\times\mathrm{I\!R}} \big(g(t)\otimes h(x)\big)\phi\big(\delta_k(.-t)\otimes\delta_{k'}(.-x)\big)^m \mathrm{d}t\mathrm{d}x$$

and $\delta_{k,k'}(t,x):=\delta_k(t)\otimes\delta_{k'}(x)$. Since

$$\lim_{k\to\infty} C(\delta_{k,k'},\delta_{k,k'}) = C_0(\delta_{k'},\delta_{k'}),$$

the definition given in (3.5) of sharp-time fields implies that

$$\lim_{k\to\infty} F(k,k') = \int_{S_\beta} g(t)V_{k'}(t,h)\mathrm{d}t \ \text{ in } \ L^p(Q,\Sigma,\mathrm{d}\phi_C),$$

where

$$V_{k'}(t,h) = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m)!}\Big(-\frac{1}{2}C_0(\delta_{k'},\delta_{k'})\Big)^m \int_{\mathrm{I\!R}} h(x)\phi\big(t,\delta_{k'}(.-x)\big)^m\mathrm{d}x.$$

Note that (4.2) implies $V_{k'}(t,h)=U(t)V_{k'}(0,h)$. By Lemma 5.1 (ii) we know that

$$\lim_{k'\to\infty} V_{k'}(0,h) = \int_{\mathrm{I\!R}} h(x) :P(\phi(0,x)):_{C_0} \mathrm{d}x \ \text{ in } \ L^p(Q,\Sigma,\mathrm{d}\phi_C)$$

and hence

$$\lim_{k\to\infty}\int_{S_\beta} g(t)V_{k'}(t,h)\mathrm{d}t = \int_{S_\beta} g(t)U(t)V_0(h)\mathrm{d}t \ \text{ in } \ L^p(Q,\Sigma,\mathrm{d}\phi_C).$$

Applying Lemma B.1 with $E=L^p(Q,\Sigma,\mathrm{d}\phi_C)$ we obtain the first identity in (5.4). The second identity follows by the same argument, taking first the limit $k'\to\infty$ and using then that

$$\lim_{k'\to\infty} C(\delta_{k,k'},\delta_{k,k'}) = C_\beta(\delta_k,\delta_k)\,\square.$$

5.2 The $P(\phi)_2$ model on the circle S_β at temperature 0

Let $P(\lambda)$ be a real valued polynomial, which is bounded from below. The $P(\phi)_2$ *model on the circle* S_β is specified by the formal interaction term

$$V_{\mathsf{C}} := V_\beta\big(1_{[-\beta/2,\beta/2]}\big) = \int_{S_\beta} :P(\phi(t,0)):_{C_\beta} \, \mathrm{d}t.$$

This expression can be given two equivalent meanings: first of all, as recalled in Lemma 5.1, it can be viewed as a Σ_0^{C} measurable function $V_{\mathsf{C}} \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma_0^{\mathsf{C}},\mathrm{d}\phi_C)$. Secondly, V_{C} can be considered as a selfadjoint operator on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$ affiliated to the abelian algebra $\mathcal{U}_{\mathsf{C}}$. More precisely, for $t \in S_\beta$ and $\Lambda \gg 1$ an UV cutoff parameter, we define an approximation $h_{\Lambda,t} \in H^{-\frac{1}{2}}(S_\beta)$ of the Dirac delta-function $\delta(.-t) \in H^{-\frac{1}{2}}(S_\beta)$ by

$$h_{\Lambda,t} := 1\!\!1_{[0,\Lambda]}(b)\delta(.-t) \in H^{-\frac{1}{2}}(S_\beta),$$

where $b = (D_t^2+m^2)^{\frac{1}{2}}$. Setting $\phi_\Lambda(t,0) := \phi_F(h_{\Lambda,t})$ one obtains by well-known arguments that

$$V_{\mathsf{C}} = \lim_{\Lambda\to\infty} \int_{S_\beta} :P(\phi_\Lambda(t,0)):_{C_\beta} \, \mathrm{d}t$$

on a dense set of vectors in $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$. Since $h_{\Lambda,t} \in H_{\mathbb{R}}^{-\frac{1}{2}}(S_\beta)$ is a real valued function, it is easy to see that V_{C} is a selfadjoint operator affiliated to $\mathcal{U}_{\mathsf{C}}$.

It is then easy to verify, by adapting well-known results for the spatially cutoff $P(\phi)_2$ model on the real line $\mathbb{R}$ at 0-temperature (see [S-H.K]) that $V_{\mathsf{C}} \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma_0^{\mathsf{C}},\mathrm{d}\phi_C)$ and $\mathrm{e}^{-TV_{\mathsf{C}}} \in L^1(Q,\Sigma_0^{\mathsf{C}},\mathrm{d}\phi_C)$ for all $T>0$. Now consider, for $0 \le b-a < \infty$,

$$G_{[a,b]} := \mathrm{e}^{-\int_a^b U_{\mathsf{C}}(x)V_{\mathsf{C}}\,\mathrm{d}x} \tag{5.5}$$

as a function on Q. It follows from Jensen's inequality (see [KL4, Theorem 6.2]) that

$$\|G_{[a,b]}\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)} \le \|\mathrm{e}^{-(b-a)V_{\mathsf{C}}}\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)} \tag{5.6}$$

and hence $G_{[a,b]} \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$. From the results recalled in Subsection 2.3, we obtain a selfadjoint operator

$$H_{\mathsf{C}} = \overline{\mathrm{d}\Gamma(b)+V_{\mathsf{C}}}$$

on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$ associated to the FKN kernel $\{G_{[0,s]}\}$. The Hamiltonian H_{C} is called the $P(\phi)_2$ *Hamiltonian on the circle* S_β.

Proposition 5.4 *The Hamiltonian H_{C} is bounded from below and has a unique normalized ground state such that $(\Omega_{\mathsf{C}},\Omega)>0$. We set*

$$\omega_{\mathsf{C}}(\,.\,) = (\Omega_{\mathsf{C}},\,.\,\Omega_{\mathsf{C}}).$$

Moreover, for $c \gg 1$,

$$\|\phi_F(g)(H_{\mathsf{C}}+c)^{-\frac{1}{2}}\| \le C\|g\|_{H^{-\frac{1}{2}}(S_\beta)}, \tag{5.7}$$

$$\pm\phi_F(g) \le C\|g\|_{H^{-\frac{1}{2}}(S_\beta)}(H_{\mathsf{C}}+c)^{\frac{1}{2}} \tag{5.8}$$

and

$$\pm\phi_F(g) \le C\|g\|_{H^{-1}(S_\beta)}(H_{\mathsf{C}}+c) \tag{5.9}$$

for all $g \in H^{-\frac{1}{2}}(S_\beta)$. As before, $W_F(g) = \mathrm{e}^{\mathrm{i}\phi_F(g)}$ is the Fock Weyl operator on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$.

Proof. The existence and uniqueness of the vacuum state can be shown by following the proofs of the corresponding results for spatially cutoff $P(\phi)_2$ models. For example, one easily obtains (see e.g. [Si, Theorem V.20] or [DG, Theorem 6.4 (ii)]) that

$$(\mathrm{d}\Gamma(b)+1) \leq C(H_{\mathrm{C}}+c) \text{ for } c \gg 1. \tag{5.10}$$

Since $\mathrm{d}\Gamma(b)$ has compact resolvent on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$, it follows that H_{C} is bounded from below with a compact resolvent and hence has a ground state. The uniqueness of the vacuum (i.e., the ground state of H_{C}) follows from a Perron-Frobenius argument (see e.g. [Si, Theorem V.17]). Since $b \geq m > 0$, we see that it suffices to check (5.7) and (5.8), with H_{C} replaced by the number operator N, which is immediate. To prove (5.9) we use (5.10) and the well known bound (see e.g. [Ge, Appendix])

$$\pm\phi_F(g) \leq \|b^{-\frac{1}{2}}g\|_{H^{-\frac{1}{2}}(S_\beta)}(\mathrm{d}\Gamma(b)+1)\,\square.$$

Without proof we quote the following result (see [HO]).

Theorem 5.5 *Let $H_{\mathrm{C}}^{\mathrm{ren}} := H_{\mathrm{C}} - E_{\mathrm{C}}$, where $E_{\mathrm{C}} := \inf(\sigma(H_{\mathrm{C}}))$ and let P_{C} denote the generator of the translations along the circle S_β. The joint spectrum of $H_{\mathrm{C}}^{\mathrm{ren}}$ and P_{C} is purely discrete and is contained in the forward light cone.*

Consequently the correlation function

$$(t,x) \mapsto \big(\Omega_{\mathrm{C}}, A\mathrm{e}^{ixH_{\mathrm{C}}^{\mathrm{ren}}+itP_{\mathrm{C}}}B\Omega_{\mathrm{C}}\big), \quad A, B \in \mathcal{B}\big(\Gamma(H^{-\frac{1}{2}}(S_\beta))\big),$$

allows an analytic continuation to the tube $\mathbb{R}^2 + iV_+$, where $V_+ := \{(t,x) \mid |t| < x;\ x > 0\}$ denotes the forward light cone (with t and x reversed, due to our conventions).

5.3 The spatially cutoff $P(\phi)_2$ model on $\mathbb{R}$ at temperature β^{-1}

Let $P(\lambda)$ be a real valued polynomial, which is bounded from below (as in Subsection 5.2), and let $l \in \mathbb{R}^+$ be a spatial cutoff parameter. The *spatially cutoff $P(\phi)_2$ model on $\mathbb{R}$* is specified by the formal interaction term (see (5.3))

$$V_l := V_0\big(\mathbb{1}_{[-l,l]}\big) = \int_{-l}^{l} :P(\phi(0,x)):_{C_0}\, \mathrm{d}x.$$

Again this formal expression can be given two equivalent meanings: first of all, as recalled in Lemma 5.1, it can be viewed as a Σ_0-measurable function $V_l \in \bigcap_{1\leq p<\infty} L^p(Q,\Sigma_0,\mathrm{d}\phi_C)$. Secondly, V_l can be considered as a selfadjoint operator on $\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ affiliated to the abelian von Neumann algebra $\mathcal{U}_{AW}$. As in Subsection 5.2 we define an approximation $h_{\Lambda,x} \in H^{-\frac{1}{2}}(\mathbb{R})$ of the Dirac delta-function $\delta(.-x) \in H^{-\frac{1}{2}}(\mathbb{R})$. For $x \in \mathbb{R}$ and $\Lambda \gg 1$ we set

$$h_{\Lambda,x} := \mathbb{1}_{[0,\Lambda]}(\epsilon)\delta(.-x) \in H^{-\frac{1}{2}}(\mathbb{R})$$

and introduce cutoff fields $\phi_\Lambda(0,x) := \phi_{AW}(h_{\Lambda,x})$, where $\phi_{AW}(h)$ is the selfadjoint field operator associated to $W_{AW}(h)$, $h \in \mathfrak{h}$.

As before, the limit

$$V_l = \lim_{\Lambda\to\infty}\int_{-l}^{l} :P(\phi_\Lambda(0,x)):_{C_0}\,\mathrm{d}x \tag{5.11}$$

exists on a dense set of vectors in $\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$. Since $h_{\Lambda,x} \in H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{R})$, one obtains that V_l is a selfadjoint operator affiliated to $\mathcal{U}_{AW}$.

Adapting well known arguments (see [GeJ, Section 8.2]) it can be shown that $\mathrm{e}^{-TV_l} \in L^1(Q, \Sigma_0, \mu)$ for all $T > 0$. Consequently, we can associate to V_l the FKN kernel

$$F^l_{[a,b]} := \mathrm{e}^{-\int_a^b U(t)V_l \mathrm{d}t}, \quad 0 \leq b - a \leq \beta,$$

and the measure

$$\mathrm{d}\mu_l := \frac{F^l_{[-\beta/2,\beta/2]}\mathrm{d}\phi_C}{\int_Q F^l_{[-\beta/2,\beta/2]}\mathrm{d}\phi_C}.$$

The generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_l)$ is β-periodic and OS-positive. The associated β-KMS system is called the *spatially cutoff $P(\phi)_2$ model on $\mathbb{R}$ at temperature β^{-1}*. Applying the abstract results recalled in Subsection 2.3, we obtain the following facts:

- the physical Hilbert space $\mathcal{H}_{V_l}$ is equal to $\mathcal{H}_{AW} = \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$;
- the W^*-algebra $\mathcal{B}_{V_l}$ and the abelian algebra $\mathcal{U}_{V_l}$ are equal to $\mathcal{R}_{AW}$ and $\mathcal{U}_{AW}$, respectively;
- the operator sum $L_{AW} + V_l$ is essentially selfadjoint on $\mathcal{D}(L_{AW}) \cap \mathcal{D}(V_l)$ and if $H_l := \overline{L_{AW} + V_l}$, then the perturbed time-evolution on $\mathcal{B}$ is given by $\tau^l_t(B) := \mathrm{e}^{\mathrm{i}tH_l} B \mathrm{e}^{-\mathrm{i}tH_l}$, $B \in \mathcal{B}$;
- the GNS vector $\Omega_{AW} \in \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ belongs to $\mathcal{D}\big(\mathrm{e}^{-\frac{\beta}{2}H_l}\big)$ and the perturbed KMS state ω_l is given by $\omega_l(B) = (\Omega_l, B\Omega_l)$, where $\Omega_l := \|\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}\|^{-1}\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}$.

The following consequence of Lemma 5.3 will be important in Section 7:

$$F^l_{[-\beta/2,\beta/2]} = G_{[-l,l]}, \tag{5.12}$$

where $G_{[a,b]}$ was defined in (5.5). The analog identity in the temperature zero case is called *Nelson symmetry.*

6 The thermodynamic limit

In this section we prove that the limits

$$\lim_{l \to +\infty} \tau^l_t(A) =: \tau_t(A) \quad \text{and} \quad \lim_{l \to +\infty} \omega_l(A) =: \omega_\beta(A)$$

exist for A in the C^*-algebra of local observables $\mathcal{A}$ and that $(\mathcal{A}, \tau, \omega_\beta)$ is a β-KMS system, describing the *translation invariant $P(\phi)_2$ model at temperature β^{-1}*.

6.1 Preparations

We first recall a well known relationship between $\mathrm{e}^{-\mathrm{i}t\epsilon}$ and the Klein-Gordon equation: let

$$\begin{array}{rccl} U\colon & H^{-\frac{1}{2}}(\mathbb{R}) & \to & H^{-\frac{1}{2}}_{\mathbb{R}}(\mathbb{R}) \oplus H^{\frac{1}{2}}_{\mathbb{R}}(\mathbb{R}). \\ & h & \mapsto & (\mathrm{Re}h, \epsilon^{-1}\mathrm{Im}h) = (\varphi, \pi). \end{array} \tag{6.1}$$

(Note that U is $\mathbb{R}$-linear but not $\mathbb{C}$-linear). Then

$$U\mathrm{e}^{-\mathrm{i}t\epsilon} = T(t)U, \text{ where } T(t)(\varphi, \pi) = (\varphi_t, \partial_t\varphi_t), \tag{6.2}$$

and φ_t is the solution of the Klein-Gordon equation

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 + m^2\right)\varphi_t = 0, \\ \varphi_{t=0} = \varphi, \quad \left(\partial_t\varphi\right)_{t=0} = -\epsilon^2\pi. \end{cases}$$

Moreover if $h_i \in H^{-\frac{1}{2}}(\mathbb{R})$ and $Uh_i = (\varphi_i, \pi_i)$ for $i = 1, 2$, then

$$\sigma(h_1, h_2) := \mathrm{Im}(h_1, h_2)_{H^{-\frac{1}{2}}(\mathbb{R})} = \int_{\mathbb{R}} \varphi_1(x)\pi_2(x) - \pi_1(x)\varphi_2(x)\mathrm{d}x. \tag{6.3}$$

For $I \subset \mathbb{R}$ a bounded open interval we define the real vector subspace $\mathfrak{h}_I$ of $\mathfrak{h}$

$$\mathfrak{h}_I := \{h \in \mathfrak{h} \mid \operatorname{supp} Uh \subset I \times I\}. \tag{6.4}$$

It follows from (6.2) that $\mathrm{i}\epsilon: \mathcal{D}(\epsilon) \cap \mathfrak{h}_I \to \mathfrak{h}_I$, and hence $(1 + \alpha\epsilon^2)^{-1}: \mathfrak{h}_I \to \mathfrak{h}_I$ for $\alpha > 0$. In particular $\mathcal{D}(\epsilon) \cap \mathfrak{h}_I$ is dense in $\mathfrak{h}_I$. Moreover (6.3) shows that $\mathfrak{h}_I$ and $\mathfrak{h}_J$ are orthogonal for the symplectic form σ if $I \cap J = \emptyset$.

6.2 The net of local algebras

We start by recalling a result of Araki [Ar1, Thm. 1] which will be useful later on. Let us recall a standard notation: If $\mathcal{H}_1$, $\mathcal{H}_2$ are two vector subspaces of a Hilbert space $\mathcal{H}$, then $\mathcal{H}_1 \vee \mathcal{H}_2$ denotes $\overline{\mathcal{H}_1 + \mathcal{H}_2}$. If $\mathcal{R}_1$, $\mathcal{R}_2$ are two *-sub-algebras of $\mathcal{B}(\mathcal{H})$, then $\mathcal{R}_1 \vee \mathcal{R}_2$ denotes the von Neumann algebra generated by $\mathcal{R}_1 \cup \mathcal{R}_2$.

Proposition 6.1 *Let X be a Hilbert space and let Z be a real vector subspace of X. Let $\mathcal{W}(Z) \subset \mathcal{W}(X)$ denote the C^*-algebra generated by $\{W(x) \mid x \in Z\}$ and let $\pi_F : \mathcal{W}(X) \to \mathcal{B}(\Gamma(X))$ be the Fock representation. Then*

$$\bigcap_\alpha \pi_F\big(\mathcal{W}(Z_\alpha)\big)'' = \pi_F\big(\mathcal{W}\left(\cap_\alpha Z_\alpha\right)\big)'', \quad \bigvee_\alpha \pi_F\big(\mathcal{W}(Z_\alpha)\big)'' = \pi_F\big(\mathcal{W}\left(\vee_\alpha Z_\alpha\right)\big)'' \tag{6.5}$$

and

$$\pi_F\big(\mathcal{W}(Z)\big)' = \pi_F\big(\mathcal{W}(Z^\perp)\big)'', \tag{6.6}$$

where Z_α is a family of real vector subspaces of X and $Z^\perp$ is the vector space orthogonal to Z for the symplectic form $\sigma(x_1, x_2) = \mathrm{Im}(x_1, x_2)$.

We now define the *net of local von Neumann algebras $I \to \mathcal{R}_{AW}(I)$* describing free thermal scalar bosons. Let $I \subset \mathbb{R}$ be a bounded open interval. We denote by $\mathcal{R}_{AW}(I)$ the von Neumann algebra generated by

$$\{W_{AW}(h) \mid h \in \mathfrak{h}_I\}.$$

Lemma 6.2

(i) *The local von Neumann algebras for the free thermal field are regular from the inside and regular from the outside:*

$$\bigcap_{J \supset \overline{I}} \mathcal{R}_{AW}(J) = \mathcal{R}_{AW}(I) = \bigvee_{\overline{J} \subset I} \mathcal{R}_{AW}(J);$$

(ii) *The net of local von Neumann algebras for the free thermal field is* additive*:*

$$\mathcal{R}_{AW}(I) = \bigvee_{J_i} \mathcal{R}_{AW}(J_i) \text{ if } I = \cup_i J_i;$$

(iii) *For each open and bounded interval I, the local observable algebra $\mathcal{R}_{AW}(I)$ is $*$-isomorphic to the unique hyper-finite factor of type III_1.*

Proof. Recalling the definition of the Araki-Woods representation we see that, with the notation introduced above,

$$\mathcal{R}_{AW}(I) = \pi_F\big(\mathcal{W}(Z_I)\big)'',$$

where $Z_I \subset \mathfrak{h} \oplus \overline{\mathfrak{h}}$ is the vector subspace

$$Z_I = \{(1+\rho)^{\frac{1}{2}} h \oplus \overline{\rho}^{\frac{1}{2}} \overline{h} \mid h \in \mathfrak{h}_I\}.$$

Clearly $\bigcap_{J \supset \overline{I}} Z_J = \bigvee_{\overline{J} \subset I} Z_J = Z_I$, which using (6.5) implies (i). Part (ii) is a direct consequence of (6.5). To prove (iii) we use (6.6) and (6.5) which implies that

$$\mathcal{R}_{AW}(I) \cap \mathcal{R}_{AW}(I)' = \pi_F\big(\mathcal{W}(Z_I \cap Z_I^{\perp})\big)'',$$

where $Z^{\perp}$ is the orthogonal space to Z in $\mathfrak{h} \oplus \overline{\mathfrak{h}}$ for the symplectic form $\sigma(f,g) = \mathrm{Im}(f,g)$ on $\mathfrak{h} \oplus \overline{\mathfrak{h}}$. We claim that

$$Z_I \cap Z_I^{\perp} = \{0\}, \tag{6.7}$$

which will imply that $\mathcal{R}_{AW}(I)$ is a factor. To prove our claim we pick $h \in \mathfrak{h}_I$ such that $(1+\rho)^{\frac{1}{2}} h \oplus \overline{\rho}^{\frac{1}{2}} \overline{h} \in Z_I^{\perp}$. This implies that $\mathrm{Im}(h,g) = 0$ for all $g \in \mathfrak{h}_I$. Hence to prove (6.7) it suffices to check that

$$\mathfrak{h}_I \cap \mathfrak{h}_I^{\perp} = \{0\}. \tag{6.8}$$

But if $h \in \mathfrak{h}_I \cap \mathfrak{h}_I^{\perp}$, we have $\mathrm{Im}(h, \mathrm{i}\epsilon(1+\alpha\epsilon^2)^{-1}h) = 0$ for $\alpha > 0$, since $\mathrm{i}\epsilon(1+\alpha\epsilon^2)^{-1}h \in \mathfrak{h}_I$ for $h \in \mathfrak{h}_I$. Letting $\alpha \to 0$ this yields $\mathrm{Re}(h, \epsilon h) = (h, \epsilon h) = 0$, since ϵ is selfadjoint. Using that $\epsilon \geq m > 0$ this implies that $h = 0$, which proves (6.8) and hence (6.7). Thus $\mathcal{R}_{AW}(I)$ is a factor, if I is bounded. Note that (6.8) shows that $\pi_F(\mathcal{W}(\mathfrak{h}_I))''$ is a factor, and it is well known (see e.g. [BD'AF][L] and lit. cit.) that $\pi_F(\mathcal{W}(\mathfrak{h}_I))''$ is $*$-isomorphic to the unique hyper-finite factor of type III_1. Thus Lemma 6.3 below completes the proof of the lemma $\Box$.

We now recall an easy fact about the restriction of the free KMS state ω_β° to the local algebras $\mathcal{W}(\mathfrak{h}_I)$.

Lemma 6.3 *Let $I \subset \mathbb{R}$ be a bounded open interval. Then the representations π_{AW} and π_F of $\mathcal{W}(\mathfrak{h}_I)$ are quasi-equivalent.*

Proof. Let $\mathfrak{h}$ be a Hilbert space and let $\epsilon \geq m > 0$ be a positive selfadjoint operator on $\mathfrak{h}$. Let ω_β° be the quasi free state on $\mathcal{W}(\mathfrak{h})$ defined by $\omega_\beta^\circ\big(W(h)\big) = \mathrm{e}^{-\frac{1}{4}(h,(1+2\rho)h)}$, where $\rho = (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$. Then it is well known that ω_β° is normal with respect to the Fock representation π_F of $\mathcal{W}(X)$ iff $\mathrm{Tr}\, \mathrm{e}^{-\beta\epsilon} < \infty$ (see e.g. [BR, Prop. 5.2.27]).

This fact implies that if $\mathfrak{h}_1 \subset \mathfrak{h}$ is a complex vector subspace, then the restriction of ω_β° to $\mathcal{W}(\mathfrak{h}_1)$ is π_F-normal iff $\mathrm{Tr}(E\mathrm{e}^{-\beta\epsilon}E) < \infty$, where E is the orthogonal projection onto $\mathfrak{h}_1$.

We will apply this remark to $\mathfrak{h} = H^{-\frac{1}{2}}(\mathbb{R})$, $\rho = (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$ and $\mathfrak{h}_1 = \mathbb{C}\mathfrak{h}_I$. Let E_I denote the orthogonal projection on $\mathbb{C}\mathfrak{h}_I$. Let $\chi \in C_{0\,\mathbb{R}}^{\infty}$ such that $\chi \equiv 1$ near I and $x = \mathrm{i}\partial_k$. If $h \in \mathfrak{h}_I$, then $\mathrm{Re}h = \chi(x)\mathrm{Re}h$ and $\mathrm{Im}h = \epsilon\chi(x)\epsilon^{-1}\mathrm{Im}h$. Using pseudodifferential calculus, we see that the operators $(1+|x|)^N\chi(x)$ and $(1+|x|)^N\epsilon\chi(x)\epsilon^{-1}$ are bounded on $H^{-\frac{1}{2}}(\mathbb{R})$ for all $N \in \mathbb{N}$. This implies that

$$\|(1+|x|)^N h\|_{H^{-\frac{1}{2}}(\mathbb{R})} \leq C\|h\|_{H^{-\frac{1}{2}}(\mathbb{R})}, \ h \in \mathfrak{h}_I. \tag{6.9}$$

Clearly (6.9) extends to $\mathbb{C}\mathfrak{h}_I$, which implies that $(1+|x|)^N E_I$ is bounded for all $N \in \mathbb{N}$. Since $\mathrm{e}^{-\beta\epsilon}(1+|x|)^{-N}$ is trace class for N large enough we see that $E_I\mathrm{e}^{-\beta\epsilon}E_I$ is trace class. Using the arguments given above we obtain that ω_β° restricted to $\mathcal{W}(\mathbb{C}\mathfrak{h}_I)$ (and hence also to $\mathcal{W}(\mathfrak{h}_I)$) is π_F-normal.

Finally we have seen in the proof of Lemma 6.2 that $\pi_F(\mathcal{W}(\mathfrak{h}_I))''$ is a factor, hence π_F is a factor representation of $\mathcal{W}(\mathfrak{h}_I)$. It is shown in [KR, Prop. 10.3.14] that if $\mathcal{R}$ is a C^*-algebra and π is a factor representation of $\mathcal{R}$, then π is quasi-equivalent to the GNS representation of any π-normal state ω. Since the restriction of π_{AW} to $\mathcal{W}(\mathfrak{h}_I)$ is the GNS representation for the quasi-free state ω_β°, this completes the proof of the lemma $\Box$.

6.3 Existence of the limiting dynamics

The C^*-algebra of local observables $\mathcal{A}$ is defined as follows:

$$\mathcal{A} := \overline{\bigcup_{I\subset\mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)},$$

where the union is over all open bounded intervals $I \subset \mathbb{R}$ and the symbol $\overline{\bigcup_{I\subset\mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)}$ denotes the C^*-inductive limit (see e.g. [KR, Proposition 11.4.1.]).

We denote by $\{\alpha_x\}_{x\in\mathbb{R}}$ the group of space translations on $\mathcal{A}$, defined by

$$\alpha_x\big(W_{AW}(h)\big) := W_{AW}(\mathrm{e}^{\mathrm{i}x.k}h),\ \ x \in \mathbb{R},$$

where k is the momentum operator acting on $\mathfrak{h} = H^{-\frac{1}{2}}(\mathbb{R})$.

Theorem 6.4 (Existence of limiting dynamics). *Let $I \subset \mathbb{R}$ be a bounded open interval. For $t \in \mathbb{R}$ fixed, the norm limit*

$$\lim_{l\to\infty} \tau_t^l(B) =: \tau_t(B)$$

exists for all $B \in \mathcal{R}_{AW}(I)$. The map $\tau: t \mapsto \tau_t$ defines a group of $*$-automorphisms of $\mathcal{A}$ such that $\tau_t \circ \alpha_x = \alpha_x \circ \tau_t$ for all $t, x \in \mathbb{R}$. Moreover,

$$\tau_t: \mathcal{R}_{AW}(I) \to \mathcal{R}_{AW}\big(I+]-t,t[\big). \tag{6.10}$$

Proof. The proof follows the well-known proof in the 0-temperature case, which is based on finite propagation speed (see [GJ2, Theorem 4.1.2]). To prove the existence of the limit and the group property, it suffices to show that $\tau_t^l(B)$, for $B \in \mathcal{R}_{AW}(I)$ and $|t| \leq T$, is independent of l for $l > |I| + T$.

It follows from (6.2) and Huygens principle that

$$\tau_t^\circ: \mathcal{R}_{AW}(I) \to \mathcal{R}_{AW}(I+]-t,t[). \tag{6.11}$$

Moreover (6.3) implies that $\mathcal{R}_{AW}(I_1) \subset \mathcal{R}_{AW}(I_2)'$, if $I_1 \cap I_2 = \emptyset$.

The dynamics τ_t^l is unitarily implemented by $\mathrm{e}^{\mathrm{i}tH_l}$, where $H_l = \overline{L_{AW} + V_l}$ for

$$V_l = \int_{]-l,l[} :P(\phi(0,x)):_{C_0} \mathrm{d}x.$$

Trotter's formula yields $\mathrm{e}^{\mathrm{i}tH_l} = \text{s-}\lim_{n\to\infty}(\mathrm{e}^{\mathrm{i}tL_{AW}/n}\mathrm{e}^{\mathrm{i}tV_l/n})^n$ and hence

$$\tau_t^l(A) = \text{s-}\lim_{n\to\infty} \big(\tau_{t/n}^\circ \circ \gamma_{t/n}^l\big)^n(A),\ \ A \in \mathcal{B}(\mathcal{H}_{AW}), \tag{6.12}$$

where $\gamma_t^l(A) := \mathrm{e}^{\mathrm{i}tV_l}A\mathrm{e}^{-\mathrm{i}tV_l}$. Note that for $l' > l$

$$V_{l'} = V_l + \int_{]-l',l'[\,\setminus\,]-l,l[} :P(\phi(0,x)):_{C_0} \mathrm{d}x.$$

Since $V_{l'} - V_l$ is affiliated to $\mathcal{R}_{AW}\big(]-l',l'[\setminus[-l,l]\big)$, we see that $\gamma_t^l = \gamma_t^{l'}$ on $\mathcal{R}_{AW}(I)$ for $l, l' > |I|$. Using (6.11) and (6.12), this implies that $\tau_t^l = \tau_t^{l'}$ on $\mathcal{R}_{AW}(I)$ for $|t| \leq T$ and $l, l' > |I| + T$. This proves our claim. The same argument using again (6.11) proves (6.10).

It remains to check that τ and α commute. Let $T > 0$ and I a bounded interval. For $|t| \leq T$ the time evolution is locally (i.e., applied to elements in $\mathcal{R}_{AW}(I)$) generated by H_l if $l > |I| + t$. Now α_x is implemented by $\mathrm{e}^{\mathrm{i}xP}$ with $P = \mathrm{d}\Gamma(k \oplus \overline{k})$. It follows that $\alpha_x \circ \tau_t \circ \alpha_x^{-1}$ is implemented by $\mathrm{e}^{\mathrm{i}tH_{l,x}}$ with $H_{l,x} = \mathrm{e}^{\mathrm{i}xP}H_l\mathrm{e}^{-\mathrm{i}xP}$. It is easy to see that

$$H_{l,x} = L_{AW} + \int_{]-l+x,l+x[} :P(\phi(0,x)):_{C_0} \mathrm{d}x.$$

By the same argument as above, τ_t is implemented by $\mathrm{e}^{\mathrm{i}tH_{l,x}}$ for $|t| \leq T$ if $l > |I| + |T| + |x|$, which implies that $\alpha_x \circ \tau_t \circ \alpha_x^{-1} = \tau_t$ $\square$.

6.4 An identification of local algebras

In order to apply the results of Section 7 to the algebra of local observables $\mathcal{A}$, it is necessary to identify the local Weyl algebra $\mathcal{R}_{AW}(I)$ with the von Neumann algebra $\mathcal{B}(I)$ obtained by applying the interacting dynamics τ to the local abelian algebra of time-zero fields $\mathcal{U}_{AW}(I)$. This is done in Proposition 6.5 below. Note that by similar arguments the corresponding result holds also in the 0-temperature case.

For $I \subset \mathbb{R}$ a bounded open interval, we denote by $\mathcal{U}_{AW}(I)$ the von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I,\ h \text{ real valued}\}$. Note that $\mathcal{U}_{AW}(I) \subset \mathcal{R}_{AW}(I)$ is abelian. We denote by $\mathcal{B}_\alpha(I)$ the von Neumann algebra generated by

$$\{\tau_t(A) \mid A \in \mathcal{U}_{AW}(I),\ |t| < \alpha\}. \tag{6.13}$$

Proposition 6.5 *Set* $\mathcal{B}(I) := \bigcap_{\alpha>0} \mathcal{B}_\alpha(I)$. *Then*

$$\mathcal{B}(I) = \mathcal{R}_{AW}(I).$$

Proof. Let us first prove that $\mathcal{B}(I) \subset \mathcal{R}_{AW}(I)$. Using (6.10) and $\mathcal{U}_{AW}(I) \subset \mathcal{R}_{AW}(I)$, we see that $\mathcal{B}_\alpha(I) \subset \mathcal{R}_{AW}\big(I+]-\alpha,\alpha[\big)$ for all $\alpha > 0$. According to Lemma 6.2 (i) this implies $\mathcal{B}(I) \subset \mathcal{R}_{AW}(I)$.

Let us now prove that $\mathcal{R}_{AW}(I) \subset \mathcal{B}(I)$. Using Lemma 6.2 (i) it suffices to show that for all $\overline{J} \subset I$ and $\alpha \ll 1$ one has

$$\mathcal{R}_{AW}(J) \subset \mathcal{B}_\alpha(I). \tag{6.14}$$

To this end we fix I and J with $\overline{J} \subset I$ and set $\delta = \frac{1}{2}\mathrm{dist}(J, I^c)$. We will first prove that

$$\mathrm{e}^{\mathrm{i}tL_{AW}} A \mathrm{e}^{-\mathrm{i}tL_{AW}} \in \mathcal{B}_\alpha(I),\ A \in \mathcal{U}_{AW}(J),\ |t| < \alpha, \tag{6.15}$$

if $\alpha < \delta$. The proof of Theorem 6.4 shows that for $|t| \le \delta$ the unitary group $\mathrm{e}^{\mathrm{i}tH_I}$, with $H_I := \overline{L_{AW} + V_I}$ and

$$V_I := \int_I :P(\phi(0,x)):\,\mathrm{d}x,$$

induces the correct dynamics τ on $\mathcal{R}_{AW}(J)$. Applying then Proposition 2.5, we obtain

$$\mathrm{e}^{\mathrm{i}tL_{AW}} = \text{s-}\lim_{n\to\infty} \mathrm{e}^{\mathrm{i}tH_I^{(n)}},\ t \in \mathbb{R},$$

for $H_I^{(n)} = \overline{L_{AW} + V_I - V_I^{(n)}}$, where $V_I^{(n)} = V_I 1\!\!1_{\{|V_I| \le n\}}$. Since $V_I^{(n)}$ is bounded,

$$H_I^{(n)} = \overline{L_{AW} + V_I} - V_I^{(n)} = H_I - V_I^{(n)},$$

and hence by Trotter's formula

$$\mathrm{e}^{\mathrm{i}tH_I^{(n)}} = \text{s-}\lim_{p\to\infty} \big(\mathrm{e}^{\mathrm{i}tH_I/p}\mathrm{e}^{-\mathrm{i}tV_I^{(n)}/p}\big)^p.$$

This yields, for $A \in \mathcal{R}_{AW}(J)$,

$$\mathrm{e}^{\mathrm{i}tL_{AW}} A \mathrm{e}^{-\mathrm{i}tL_{AW}} = \text{s-}\lim_{n\to\infty} \text{s-}\lim_{p\to\infty} \big(\mathrm{e}^{\mathrm{i}tH_I/p}\mathrm{e}^{-\mathrm{i}tV_I^{(n)}/p}\big)^p A \big(\mathrm{e}^{\mathrm{i}tV_I^{(n)}/p}\mathrm{e}^{-\mathrm{i}tH_I/p}\big)^p.$$

Using again Theorem 6.4 we obtain, for $|t| < \alpha$,

$$\big(\mathrm{e}^{\mathrm{i}tH_I/p}\mathrm{e}^{-\mathrm{i}tV_I^{(n)}/p}\big)^p A \big(\mathrm{e}^{\mathrm{i}tV_I^{(n)}/p}\mathrm{e}^{-\mathrm{i}tH_I/p}\big)^p = \big(\tau_{t/p} \circ \gamma_{t/p}^{(n)}\big)^p(A),$$

where $\gamma^{(n)}$ is the dynamics implemented by the unitary group $t \mapsto \mathrm{e}^{-\mathrm{i}tV_I^{(n)}}$. Since V_I is affiliated to $\mathcal{U}_{AW}(I)$, $\mathrm{e}^{-\mathrm{i}tV_I^{(n)}} \in \mathcal{U}_{AW}(I)$ and hence $\left(\tau_{t/p} \circ \gamma_{t/p}^{(n)}\right)^p(A) \in \mathcal{B}_\alpha(I)$ for $|t| < \alpha$. Since $\mathcal{B}_\alpha(I)$ is weakly closed, we obtain (6.15).

Let us now prove (6.14). Clearly the operators $W_{AW}(h)$ for $h \in \mathfrak{h}_J$ and h real valued belong to $\mathcal{U}_{AW}(J)$ and hence to $\mathcal{B}_\alpha(I)$. Let us now pick $h \in \mathfrak{h}_J \cap \mathcal{D}(\epsilon)$ and h real valued. (This is possible; see the discussion presented at the end of Subsection 6.1). Applying (6.15) to $A = W_{AW}(h)$, we obtain that $W_{AW}(\mathrm{e}^{\mathrm{i}t\epsilon}h) \in \mathcal{B}_\alpha(I)$ for $|t| < \alpha$. Hence $W_{AW}\left(t^{-1}(\mathrm{e}^{\mathrm{i}t\epsilon}h - h)\right) \in \mathcal{B}_\alpha(I)$ for $|t| < \alpha$. Letting $t \to 0$ and using the fact that the map $\mathfrak{h} \ni h \mapsto W_{AW}(h)$ is continuous for the strong operator topology, we obtain that $W_{AW}(\mathrm{i}\epsilon h) \in \mathcal{B}_\alpha(I)$. But any vector $h \in \mathfrak{h}_J$ can be approximated in norm by vectors of the form $h_1 + \mathrm{i}\epsilon h_2$, with $h_i \in \mathfrak{h}_J$ real and $h_2 \in \mathcal{D}(\epsilon)$. This implies that for all $h \in \mathfrak{h}_J$ the operators $W_{AW}(h)$ belong to $\mathcal{B}_\alpha(I)$ and hence $\mathcal{R}_{AW}(J) \subset \mathcal{B}_\alpha(I)$. This completes the proof of the proposition $\Box$.

6.5 Existence of the limiting state

Theorem 6.6 (Existence of limiting state). *Let $\{\omega_l\}_{l>0}$ be the family of (τ^l, β)-KMS states for the spatially cutoff $P(\phi)_2$ models constructed in Subsection 5.3.*

Then

$$\mathrm{w} - \lim_{l \to +\infty} \omega_l =: \omega_\beta \text{ exists on } \mathcal{A}.$$

The state ω_β on $\mathcal{A}$ has the following properties:

- (i) *ω_β is a (τ, β)-KMS state on $\mathcal{A}$;*
- (ii) *ω_β is* locally normal, *i.e., if I is an open and bounded interval, then $\omega_{\beta|\mathcal{R}_{AW}(I)}$ is normal w.r.t. the Araki-Woods representation;*
- (iii) *ω_β is invariant under spatial translations, i.e.,*

$$\omega_\beta(\alpha_x(A)) = \omega_\beta(A), \quad x \in \mathbb{R},\ A \in \mathcal{A};$$

- (iv) *ω_β has the* spatial clustering property, *i.e.,*

$$\lim_{x \to \infty} \omega_\beta(A\alpha_x(B)) = \omega_\beta(A)\omega_\beta(B) \quad \forall A, B \in \mathcal{A}.$$

Remark 6.7 *Let $\mathcal{R}$ be a C^*-algebra, $\pi_i : \mathcal{R} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, two quasi-equivalent representations of $\mathcal{R}$. Then there exists a $*$-isomorphism τ between $\pi_1(\mathcal{R})''$ and $\pi_2(\mathcal{R})''$ intertwining the two representations. This isomorphism is automatically weakly continuous. Therefore the representation π_2 extends uniquely from $\mathcal{R}$ to $\pi_1(\mathcal{R})''$ and is quasi-equivalent to the concrete representation of $\pi_1(\mathcal{R})''$ in $\mathcal{B}(\mathcal{H}_1)$.*

Applying this easy observation to the representations π_{AW} and π_F of $\mathcal{W}(\mathfrak{h}_I)$, which are quasi-equivalent by Lemma 6.3, we see that the Fock representation π_F extends by weak continuity from $\pi_{AW}(\mathcal{W}(\mathfrak{h}_I))$ to $\mathcal{R}_{AW}(I)$ and is quasi-equivalent to the Araki-Woods representation. Since two quasi-equivalent representations have the same set of normal states, we obtain that $\omega_{\beta|\mathcal{R}_{AW}(I)}$ is also normal with respect to the Fock representation.

Proof. The family $\{\omega_l\}_{l>0}$ of states on $\mathcal{A}$ is weak* compact by the Banach-Alaoglu theorem. Let ω_1 be one of the limit points of $\{\omega_l\}_{l>0}$. Then we can find a subnet[1] $\{\omega^r\}_{r \in R}$ such that $\omega_1 = \mathrm{w} - \lim_{r \in R} \omega^r$.

We claim that ω_1 is a (τ, β)-KMS state. Let $A, B \in \mathcal{A}$. Writing

$$\omega_1(A\tau_t(B)) - \omega^r(A\tau_t^{l_r}(B)) = (\omega_1 - \omega^r)(A\tau_t(B)) + \omega^r\left(A\tau_t(B) - A\tau_t^{l_r}(B)\right)$$

[1] A net $\{y_\beta\}_{\beta \in B}$ is a subnet of a net $\{x_\alpha\}_{\alpha \in A}$ if there exists a map $B \ni \beta \mapsto \alpha(\beta) \in A$ such that: i.) $y_\beta = x_{\alpha(\beta)}$ for all $\beta \in B$; ii.) for all $\alpha_0 \in A$ there exists some β_0 such that $\alpha(\beta) \geq \alpha_0$ whenever $\beta \geq \beta_0$.

and using that $\lim_{l\to\infty} \|\tau^l(A) - \tau_t(A)\| = 0$ for $A \in \mathcal{A}$ and $t \in \mathbb{R}$ fixed, we find

$$\omega_1(A\tau_t(B)) = \lim_{r\in R} \omega^r\big(A\tau_t^{l_r}(B)\big), \ \ t \in \mathbb{R}. \tag{6.16}$$

The same argument shows

$$\omega_1(\tau_t(B)A) = \lim_{r\in R} \omega^r\big(\tau_t^{l_r}(B)A\big), \ \ t \in \mathbb{R}. \tag{6.17}$$

Since the ω^r's are (τ^{l_r}, β)-KMS states there exist functions $F^r(z)$, which are holomorphic in $I_\beta^+ = \{0 < \mathrm{Im} z < \beta\}$ and continuous in $\overline{I_\beta^+}$, such that $F^r(t) = \omega^r\big(A\tau_t^{l_r}B\big)$ and $F^r(t + \mathrm{i}\beta) = \omega^r\big(\tau_t^{l_r}(B)A\big)$. Moreover, one has $\sup_{z\in I_\beta} |F^r(z)| \le \|A\|\|B\|$. Applying Vitali's theorem and possibly extracting a subnet, we know that $\lim_{r\to\infty} F^r(z) = F(z)$ exists and is holomorphic and bounded in I_β^+. By Lemma B.3, we obtain that F is continuous on $\overline{I_\beta^+}$ and

$$F(t) = \lim_{r\to\infty} F^r(t), \quad F(t + \mathrm{i}\beta) = \lim_{r\to\infty} F^r(t + \mathrm{i}\beta).$$

Using (6.16) and (6.17) this implies that ω_1 is a (τ, β)-KMS state.

We now apply a result of Takesaki and Winnink [TW]: clearly $I \to \{\mathcal{R}_{AW}(I)\}$ is a net of von Neumann algebras (see [TW, Section 2]). The algebras $\mathcal{R}_{AW}(I)$ are σ-finite, since the Hilbert space $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ on which they act is separable. Moreover, as factors on a separable infinite dimensional Hilbert space they are properly infinite. Applying [TW, Theorem 1], we obtain that the KMS state ω_1 is *normal* on $\mathcal{R}_{AW}(I)$.

Let us now show that all limit states are identical. Let us denote by $\mathcal{U}_0(I)$ the abelian C^*-algebra generated by

$$\big\{F(\phi_{AW}(h)) \mid h \in C_{0\,\mathbb{R}}^\infty(I),\ F \in C_0^\infty(\mathbb{R})\big\}$$

and by $\mathcal{T}_\alpha(I)$ the $*$-algebra generated by $\big\{\tau_t(A) \mid A \in \mathcal{U}_0(I),\ |t| < \alpha\big\}$.

From Theorem 6.4 and Proposition 7.6 we deduce that

$$\lim_{l\to\infty} \omega_l\Big(\prod_1^n \tau_{t_i}(A_i)\Big) = \tilde\omega\Big(\prod_1^n \tilde\tau_{t_i}(A_i)\Big) = \omega_1\Big(\prod_1^n \tau_{t_i}(A_i)\Big),\ \ A_i \in \mathcal{U}_0(I),\ t_i \in \mathbb{R},$$

where $\tilde\omega$ and $\tilde\tau$ are defined in Subsection 7.2. Therefore all weak accumulation points of $\{\omega_l\}_{l>0}$ coincide on the algebras $\mathcal{T}_\alpha(I) \subset \mathcal{R}_{AW}\big(I+]-\alpha,\alpha[\big)$. We note that $\mathcal{T}_\alpha(I)$ is weakly dense in the von Neumann algebra $\mathcal{B}_\alpha(I)$ defined in (6.13). Moreover, we have seen that all limit states are normal on the local algebras $\mathcal{R}_{AW}(I)$, I open and bounded. Therefore they coincide on the von Neumann algebras $\mathcal{B}_\alpha(I)$, and hence by Proposition 6.5 on $\mathcal{R}_{AW}(I)$. Consequently, they also coincide on the norm closure $\mathcal{A}$. Thus the weak* compact family $\{\omega_l\}_{l>0}$ has a unique accumulation point, which implies that

$$\omega_\beta := \mathrm{w} - \lim_{l\to\infty} \omega_l \text{ exists on } \mathcal{A}.$$

We have already seen that ω_β is a locally normal (τ, β)-KMS state on $\mathcal{A}$, which completes the proof of (i) and (ii). Property (iii) follows from the invariance of the state $\tilde\omega$ under space translations shown in Lemma 7.7 and the same density argument as above. It remains to prove (iv). Let $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$ denote the GNS objects associated to $(\mathcal{A}, \omega_\beta)$. The group $\{\alpha_x\}_{x\in\mathbb{R}}$ is implemented in $\mathcal{H}_\beta$ by a strongly continuous group of unitary operators $\{\mathrm{e}^{\mathrm{i}xP_\beta}\}_{x\in\mathbb{R}}$ with $P_\beta\Omega_\beta = 0$. Lemma 7.7 (ii) implies that, for $A, B \in \mathcal{T}_\alpha(I)$,

$$\lim_{x\to\infty} \big(\pi_\beta(A)\Omega_\beta, \mathrm{e}^{\mathrm{i}xP_\beta}\pi_\beta(B)\Omega_\beta\big) = \big(\pi_\beta(A)\Omega_\beta, \Omega_\beta)(\Omega_\beta, \pi_\beta(B)\Omega_\beta\big). \tag{6.18}$$

Since $\mathcal{R}_{AW}(I)$ is a factor, the representation π_β provides a weakly continuous *-isomorphism between $\mathcal{R}_{AW}(I)$ and $\mathcal{R}_\beta(I) = \pi_\beta(\mathcal{R}_{AW}(I)) = \pi_\beta(\mathcal{R}_{AW}(I))''$. Hence, by the same weak density argument as above, (6.18) extends to all $A, B \in \mathcal{R}_{AW}(I)$. Thus the space clustering property holds on $\mathcal{R}_{AW}(I)$ for all I, I open and bounded, and extends to $\mathcal{A}$ by norm density.

7 Construction of the interacting path space

In this section we construct the interacting path space supported by $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ describing the translation invariant $P(\phi)_2$ model at temperature β^{-1} and study some of its properties.

7.1 Construction of the interacting measure

Let $H_{\mathsf{C}}^{\rm ren} = H_{\mathsf{C}}^{\rm ren} - E_{\mathsf{C}}$ be the renormalized $P(\phi)_2$ Hamiltonian on S_β defined in Subsection 5.2. Let $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$. For $x \in \mathbb{R}$ the function f_x defined in Subsection 3.5 belongs to $\mathcal{S}_{\mathbb{R}}(S_\beta)$. We will apply the results of Appendix A to the selfadjoint operator $H = H_{\mathsf{C}}^{\rm ren}$, $R(x) = \phi_F(f_x)$ (replacing the variable t in Appendix A by the variable x).

It follows from the bound (5.7) in Proposition 5.4 and the fact that the map

$$\begin{array}{rcl} \mathbb{R} & \to & \mathcal{B}\big(\Gamma(H^{-\frac{1}{2}}(S_\beta))\big) \\ x & \mapsto & \phi(f_x)(H_{\mathsf{C}}^{\rm ren}+1)^{-\frac{1}{2}} \end{array}$$

is infinitely differentiable that the hypothesis (A.3) in Subsection A.1 is satisfied. Similarly, using the bound (5.8) and the fact that the map $x \mapsto \|f_x\|_{H^{-\frac{1}{2}}(S_\beta)}$ is in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we see that hypotheses (A.7) and (A.8) in Subsection A.2 is satisfied. Therefore we can apply all the abstract results from Subsections A.1 and A.2. In particular there exists a solution $U(b,a)$ of the time-dependent heat equation:

$$\frac{\mathrm{d}}{\mathrm{d}b}U(b,a) = (-H_{\mathsf{C}}^{\rm ren} + \mathrm{i}\phi_F(f_b))U(b,a),\ U(a,a) = \mathbb{1}.$$

We will set for $-\infty \le a \le b \le +\infty$:

$$W_{[a,b]}(f) := U(b,a)^*.$$

Proposition 7.1 *Let* $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ *and assume that* $\operatorname{supp} f \subset S_\beta \times [-a,a]$. *Then*

$$\int_Q \mathrm{e}^{\mathrm{i}\phi(f)} G_{[-l,l]} \mathrm{d}\phi_C = \mathrm{e}^{-2lE_{\mathsf{C}}}\big(\mathrm{e}^{-(l-a)H_{\mathsf{C}}^{\rm ren}}\Omega_{\mathsf{C}}^\circ, W_{[-a,a]}(f)\mathrm{e}^{-(l-a)H_{\mathsf{C}}^{\rm ren}}\Omega_{\mathsf{C}}^\circ\big),$$

where $\Omega_{\mathsf{C}}^\circ$ *is the free vacuum on* $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$.

Proof. Let us first introduce a notation which we will use throughout the proof. If A is a Σ-measurable function on Q, the image of A under $U_{\mathsf{C}}(x)$ for $x \in \mathbb{R}$ will be denoted by $U_{\mathsf{C}}(x)(A)$. On the other hand, the expression $U_{\mathsf{C}}(x)A$ will denote the operator product of the operator $U_{\mathsf{C}}(x)$ and the operator of multiplication by A, acting on $L^2(Q,\Sigma,\mathrm{d}\phi_C)$.

Using Lemma 3.1 we find

$$\mathrm{e}^{\mathrm{i}\phi(f)} = \mathrm{e}^{\mathrm{i}\int_{-a}^a \phi(f_x,x)\mathrm{d}x} = \mathrm{e}^{\mathrm{i}\int_{-a}^a U_{\mathsf{C}}(x)(\phi(f_x,0))\mathrm{d}x}.$$

We will approximate the above integral using Riemann sums. Let $n,p \in \mathbb{N}$ and $0 \le j \le 2np$. We set $x_j = -a + j\frac{a}{np}$ and $z_j = -a + [j/p]\frac{a}{n}$, where $[.]$ denotes the integer part.

It follows from (3.15) that the map $x \mapsto \phi(f_x,x) \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_c)$ is continuous. Therefore

$$\int_{-a}^a U_{\mathsf{C}}(x)\big(\phi(f_x,0)\big)\mathrm{d}x = \lim_{n,p\to\infty}\sum_{j=0}^{2np-1}(x_{j+1}-x_j)U_{\mathsf{C}}(x_j)\big(\phi(f_{z_j},0)\big)$$

in $\bigcap_{1\le p\le\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$ and hence

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\int_{-a}^a U_{\mathsf{C}}(x)(\phi(f_x,0))\mathrm{d}x} &= \lim_{n,p\to\infty}\prod_{j=0}^{2np-1}\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)U_{\mathsf{C}}(x_j)(\phi(f_{z_j},0))} \\ &= \lim_{n,p\to\infty}\prod_{j=0}^{2np-1}U_{\mathsf{C}}(x_j)\big(\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}\big) \end{aligned} \tag{7.1}$$

in $\bigcap_{1\leq p\leq\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C)$, where in the last line we use the fact that $U_{\mathsf{C}}(x_j)$ is an automorphism of $L^\infty(Q,\Sigma,\mathrm{d}\phi_C)$. Since $G_{[a,b]}$ is a FKN kernel,

$$\begin{aligned} G_{[-l,l]} &= G_{[-l,-a]}\Big[\prod_{j=0}^{2np-1} G_{[x_j,x_{j+1}]}\Big]G_{[a,l]} \\ &= G_{[-l,-a]}\Big[\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(G_{[0,x_{j+1}-x_j]}\big)\Big]G_{[a,l]}. \end{aligned}$$

Therefore

$$\begin{aligned} & G_{[-l,l]}\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}\big) \\ = \quad & G_{[-l,-a]}\Big[\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}G_{[0,x_{j+1}-x_j]}\big)\Big]G_{[a,l]}. \end{aligned}$$

Next let A_j, $0\leq j<2np-1$, be the multiplication operators by Σ-measurable functions. Using the identity $U_{\mathsf{C}}(x_j)(A_j)=U_{\mathsf{C}}(x_j)A_jU_{\mathsf{C}}(-x_j)$ and the fact that $U_{\mathsf{C}}(x)$ is an automorphism of $L^\infty(Q,\Sigma,\mathrm{d}\phi_C)$, we obtain as an operator identity on $L^2(Q,\Sigma,\mathrm{d}\phi_C)$:

$$\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)(A_j) = U_{\mathsf{C}}(x_0)\prod_{j=0}^{2np-1} A_jU_{\mathsf{C}}(x_{j+1}-x_j)U_{\mathsf{C}}(-x_{2np}).$$

In the above identity the product on the l.h.s. is the operator of multiplication by the product of the functions $U_{\mathsf{C}}(x_j)(A_j)$ and the product on the r.h.s. is an operator product. Using that $x_0=-x_{2np}=-a$ and that $U_{\mathsf{C}}(-a)^*=U_{\mathsf{C}}(a)$ we get

$$\begin{aligned} & \int_Q G_{[-l,l]}\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}\big)\mathrm{d}\phi_C \\ = \quad & \int_Q G_{[-l,-a]}U_{\mathsf{C}}(-a)\Big[\prod_{j=0}^{2np-1} \mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}U_{\mathsf{C}}(x_{j+1}-x_j)G_{[0,x_{j+1}-x_j]}\Big]U_{\mathsf{C}}(-a)G_{[a,l]}\mathrm{d}\phi_C \\ = \quad & \int_Q G_{[-l+a,0]}\Big[\prod_{j=0}^{2np-1} \mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}U_V(x_{j+1}-x_j)\Big]G_{[0,l-a]}\mathrm{d}\phi_C \end{aligned}$$

for $U_V(s)=G_{[0,s]}U_{\mathsf{C}}(s)$.

Let us now set for $0\leq k\leq 2n$, $y_k=-a+k\frac{a}{n}$. We note that $z_j=y_{[j/p]}$ and that $(x_{j+1}-x_j)=\frac{a}{np}=(y_{k+1}-y_k)/p$. We obtain that

$$\begin{aligned} & \int_Q G_{[-l,l]}\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}\big)\mathrm{d}\phi_C \\ = \quad & \int_Q G_{[-l+a,0]}\prod_{k=0}^{2n-1}\big(\mathrm{e}^{\mathrm{i}(y_{k+1}-y_k)\phi(f_{y_k},0)/p}U_V\big(\tfrac{y_{k+1}-y_k}{p}\big)\big)^pG_{[0,l-a]}\mathrm{d}\phi_C \\ = \quad & \int_Q R_{\mathsf{C}}(G_{[0,l-a]})\prod_{k=0}^{2n-1}\big(\mathrm{e}^{\mathrm{i}(y_{k+1}-y_k)\phi(f_{y_k},0)/p}U_V\big(\tfrac{y_{k+1}-y_k}{p}\big)\big)^pG_{[0,l-a]}\mathrm{d}\phi_C. \end{aligned}$$

Taking into account the construction of H_{C} recalled in Subsection 5.2 we find

$$\begin{aligned} & \int_Q G_{[-l,l]}\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(\mathrm{e}^{\mathrm{i}(x_{j+1}-x_j)\phi(f_{z_j},0)}\big)\mathrm{d}\phi_C \\ = \quad & \Big(\mathrm{e}^{-(l-a)H_{\mathsf{C}}}\Omega_{\mathsf{C}}^\circ,\prod_{k=0}^{2n-1}\big(\mathrm{e}^{\mathrm{i}(y_{k+1}-y_k)\phi(f_{y_k})/p}\mathrm{e}^{-(y_{k+1}-y_k)H_{\mathsf{C}}/p}\big)^p\mathrm{e}^{-(l-a)H_{\mathsf{C}}}\Omega_{\mathsf{C}}^\circ\Big) \\ = \quad & \mathrm{e}^{-2lE_{\mathsf{C}}}\Big(\mathrm{e}^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^\circ,\prod_{k=0}^{2n-1}\big(\mathrm{e}^{\mathrm{i}(y_{k+1}-y_k)\phi(f_{y_k})/p}\mathrm{e}^{-(y_{k+1}-y_k)H_{\mathsf{C}}^{\mathrm{ren}}/p}\big)^p\mathrm{e}^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^\circ\Big). \end{aligned}$$

Letting now n and p tend to ∞ and using Proposition A.5, we obtain the proposition $\Box$.

Theorem 7.2

(i) *Let* $f \in C^\infty_{0\,\mathbb{R}}(S_\beta \times \mathbb{R})$. *Then*

$$\lim_{l\to+\infty} \int \mathrm{e}^{\mathrm{i}\phi(f)}\mathrm{d}\mu_l = (\Omega_{\mathrm{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathrm{C}}),$$

where Ω_{C} *is the unique vacuum state of* H_{C}.

(ii) *The map*

$$\mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}) \ni f \mapsto (\Omega_{\mathrm{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathrm{C}})$$

is the generating functional of a Borel probability measure μ *on* (Q,Σ).

(iii) *The measure* μ *is invariant under space translations* $\{\mathfrak{a}_x\}_{x\in\mathbb{R}}$, *time translations* $\{\mathfrak{T}_t\}_{t\in S_\beta}$ *and the time reflection* r.

(iv) *The functions* $\phi(f)$ *belong to* $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mu)$ *for* $f \in \mathcal{S}_{\mathbb{R}}(S_\beta\times\mathbb{R})$. *Moreover,*

$$\int_Q \phi(f)^n \mathrm{d}\mu = n! \int_{-\infty<x_1\le\cdots\le x_n<\infty} (\Omega_{\mathrm{C}}, [\prod_1^{n-1} \phi(f_{x_k})\mathrm{e}^{-(x_{k+1}-x_k)H_{\mathrm{C}}^{\mathrm{ren}}}]\phi(f_{x_n})\Omega_{\mathrm{C}})\mathrm{d}x_1\ldots\mathrm{d}x_n.$$

(v) *Let* $f_i \in C^\infty_{0\,\mathbb{R}}(S_\beta\times\mathbb{R})$ *for* $1\le i\le n$. *Then*

$$\lim_{l\to+\infty}\int_Q \prod_{i=1}^n \phi(f_i)\mathrm{d}\mu_l = \int_Q \prod_{i=1}^n \phi(f_i)\mathrm{d}\mu.$$

Proof. Note first that applying Proposition 7.1 for $f=0$, we obtain $W_{[-a,a]}(0) = \mathrm{e}^{-2aH_{\mathrm{C}}^{\mathrm{ren}}}$:

$$\int_Q G_{[-l,l]}\mathrm{d}\phi_C = \mathrm{e}^{-2lE_{\mathrm{C}}}\big(\mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ, \mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ\big)..$$

Let $f\in C^\infty_{0\,\mathbb{R}}(S_\beta\times\mathbb{R})$ with $\operatorname{supp} f\subset S_\beta\times[-a,a]$ for some $a\in\mathbb{R}$. Using Proposition 7.1 we find

$$\int \mathrm{e}^{\mathrm{i}\phi(f)}\mathrm{d}\mu_l = \frac{\big(\mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ, W_{[-a,a]}(f)\mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ\big)}{(\mathrm{e}^{-lH_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ, \mathrm{e}^{-lH_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ)}.$$

Now $\lim_{l\to+\infty}\mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ = (\Omega_{\mathrm{C}},\Omega_{\mathrm{C}}^\circ)\Omega_{\mathrm{C}}$, where Ω_{C} is the eigenvector for the simple eigenvalue $\{0\}$ of $H_{\mathrm{C}}^{\mathrm{ren}}$. Thus

$$\lim_{l\to+\infty}\int \mathrm{e}^{\mathrm{i}\phi(f)}\mathrm{d}\mu_l = (\Omega_{\mathrm{C}}, W_{[-a,a]}(f)\Omega_{\mathrm{C}}),$$

Because $\operatorname{supp} f\subset S_\beta\times[-a,a]$, we see that $\big(\Omega_{\mathrm{C}}, W_{[s,t]}(f)\Omega_{\mathrm{C}}\big)$ is constant for $s\le -a$, $t\ge a$, which proves (i).

To prove (ii) we apply Minlos theorem (see e.g. [GV]). As a limit of functionals of Borel probability measures on (Q,Σ) the functional $f\mapsto \big(\Omega_{\mathrm{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathrm{C}}\big)$ is of positive type. It remains to show that the map

$$\begin{array}{ccc} \mathcal{S}(S_\beta\times\mathbb{R}) & \to & \mathbb{C} \\ f & \mapsto & (\Omega_{\mathrm{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathrm{C}}) \end{array}$$

is continuous. Using the bound (5.8) we obtain

$$\pm\big(\phi_F(f_{2,x}) - \phi_F(f_{1,x})\big) \leq Cr(x)(H_{\mathrm{C}}^{\mathrm{ren}} + 1)^{\frac{1}{2}} \text{ for } f_1,\ f_2 \in \mathcal{S}(S_\beta \times \mathbb{R}),$$

where $C > 0$ is some constant and

$$r(x) := \|(f_{2,x} - f_{1,x})\|_{H^{-\frac{1}{2}}(S_\beta)}.$$

Clearly $\|r\|_{L^2(\mathbb{R})} \leq C\|f_1 - f_2\|_p$, where $\|.\|_p$ is a Schwartz semi-norm on $\mathcal{S}(S_\beta \times \mathbb{R})$. Applying Lemma A.7 for $\delta = \frac{1}{2}$, we obtain

$$\big\|W_{[-\infty,+\infty]}(f_2) - W_{[-\infty,+\infty]}(f_1)\big\| \leq C\|f_2 - f_1\|_p,$$

which proves the desired continuity result.

Let us now verify (iii). The measure μ is invariant under time translations and time reflection as the weak limit of the time translation and time reflection invariant measures μ_l. The fact that μ is invariant under space translations follows directly from (i) and Remark A.9.

To prove (iv) we apply Lemma B.2, using the estimates in Proposition A.6 (ii). We obtain that $\phi(f) \in \bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mu)$. The formula in (iv) follows from Proposition A.6 (iii).

It remains to prove (v). Let $f \in C_{0\,\mathbb{R}}^\infty(S_\beta \times \mathbb{R})$ with $\operatorname{supp} f \subset S_\beta \times [-a, a]$. We consider the family of functions

$$u_l(\lambda) = \int \mathrm{e}^{\mathrm{i}\lambda\phi(f)}\mathrm{d}\mu_l \text{ for } \lambda \in \mathbb{C}.$$

Since $\mathrm{e}^{\phi(f)} \in \bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$ and $F^l_{[-\beta/2,\beta/2]} \in L^{1+\epsilon}(Q, \Sigma, \mathrm{d}\phi_C)$, the functions $u_l(\lambda)$ are entire and

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}u_l(0) = \mathrm{i}^n \int \phi(f)^n \mathrm{d}\mu_l. \tag{7.2}$$

Using Proposition 7.1 and $\lambda\phi(f) = \phi(\lambda f)$ for $\lambda \in \mathbb{R}$ we find

$$u_l(\lambda) = \frac{\big(W_{[-a,a]}(\lambda f)\mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ, \mathrm{e}^{-(l-a)H_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ\big)}{\|\mathrm{e}^{-lH_{\mathrm{C}}^{\mathrm{ren}}}\Omega_{\mathrm{C}}^\circ\|} \quad \text{for } \lambda \in \mathbb{R}.$$

The r.h.s. is an entire function by Lemma A.3. Therefore this identity extends to $\lambda \in \mathbb{C}$. Applying (5.8) and Proposition A.6 (i) with $\delta = 1/2$ we obtain

$$|u_l(\lambda)| \leq \mathrm{e}^{C|\mathrm{Im}\lambda|^2},\ l \in \mathbb{R}^+,\ \lambda \in \mathbb{C}. \tag{7.3}$$

We have seen above that

$$\lim_{l\to\infty} u_l(\lambda) = \int_Q \mathrm{e}^{\mathrm{i}\lambda\phi(f)}\mathrm{d}\mu \text{ for } \lambda \in \mathbb{R}.$$

By Vitali's theorem we obtain

$$\lim_{l\to\infty} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}u_l(0) = \mathrm{i}^n \int_Q \phi(f)^n \mathrm{d}\mu.$$

Using (7.2) and multi-linearity, this proves (v) □.

7.2 Existence and properties of sharp-time fields

Proposition 7.3 *Let $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $t \in S_\beta$. Then the sequence $\phi\big(\delta_k(.-t)\otimes h\big)$ is Cauchy in $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mu)$ and hence*

$$\phi(t,h) := \lim_{k\to\infty} \phi\big(\delta_k(.-t)\otimes h\big) \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mu).$$

Moreover, the map

$$\begin{array}{ccc} S_\beta & \to & \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mu) \\ t & \mapsto & \phi(t,h) \end{array}$$

is continuous for each $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$.

Proof. For $p \ge 1$ we have

$$\begin{array}{cl} & \int_Q \big(\phi(\delta_k(.-t)\otimes h) - \phi(\delta_{k'}(.-t)\otimes h)\big)^{2p} \mathrm{d}\mu \\ = & (-\mathrm{i})^{2p} \frac{\mathrm{d}^{2p}}{\mathrm{d}\lambda^{2p}} \Big(\Omega_{\mathsf{C}}, W_{[-\infty,+\infty]}\big(\lambda(\delta_k(.-t)\otimes h - \delta_{k'}(.-t)\otimes h)\big)\Omega_{\mathsf{C}}\Big)_{|\lambda=0}. \end{array}$$

If $f = \delta_k(.-t)\otimes h$, then for $x \in \mathbb{R}$ the function $f_x \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ is equal to $\delta_k(.-t)h(x)$. It follows then from the estimate (5.9) in Proposition 5.4 that

$$\pm\big(\phi_F(\delta_k(.-t)h(x)) - \phi_F(\delta_{k'}(.-t)h(x))\big) \le c\,\|\delta_k(.-t) - \delta_{k'}(.-t)\|_{H^{-1}(S_\beta)}\,|h(x)|\,\big(H_{\mathsf{C}}^{\mathrm{ren}}+1\big).$$

Applying now Lemma A.8 we obtain that

$$\begin{array}{cl} & \big\| \frac{\mathrm{d}^{2p}}{\mathrm{d}\lambda^{2p}} W_{[-\infty,+\infty]}\big(\lambda(\delta_k(.-t)\otimes h - \delta_{k'}(.-t)\otimes h)\big)\big\| \\ \le & c_p \|\delta_k(.-t) - \delta_{k'}(.-t)\|^{2p}_{H^{-1}(S_\beta)}\, \|h\|_\infty^{2p}\, \mathrm{e}^{\|h\|_1\|h\|_\infty^{-1}}. \end{array} \tag{7.4}$$

Since $\delta_k(.-t)$ converges to $\delta(.-t)$ in $H^{-1}(S_\beta)$, we see that $\phi\big(\delta_k(.-t)\otimes h\big)$ is Cauchy in $L^{2p}(Q,\Sigma,\mu)$. A similar argument shows that $t \mapsto \phi(t,h) \in L^{2p}(Q,\Sigma,\mu)$ is continuous, using the fact that $t \mapsto \delta(.-t) \in H^{-1}(S_\beta)$ is continuous □.

Using the existence of sharp-time fields, we can equip the probability space (Q,Σ,μ) with an OS-positive β-periodic path space structure: We recall that $U(t)$ is the group of transformations generated by the (euclidean) time translations $\mathfrak{r}_t$ and R is the transformation generated by time reflection, and Σ_0 is the sub$-\sigma$-algebra of Σ generated by the functions $\{\phi(0,h) \mid h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\}$.

Theorem 7.4 *$(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is an OS-positive β-periodic generalized path space.*

Proof. Since the measure μ is invariant under time translations and time reflection, we see that $U(t)$ and R are measure preserving automorphisms of $L^\infty(Q,\Sigma,\mu)$. Proposition 7.3 implies that the map $S_\beta \ni t \mapsto \mathrm{e}^{\mathrm{i}\phi(t,h)} \in L^2(Q,\Sigma,\mu)$ is continuous. Hence $U(t)$ is a strongly continuous group on $L^2(Q,\Sigma,\mu)$. This implies that $U(t)$ is strongly continuous in measure on $L^\infty(Q,\Sigma,\mu)$. Clearly it is β-periodic.

The generalized path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is OS-positive, since μ is the weak limit of the measures μ_l, which are associated to OS-positive path spaces. Finally we have already seen that $\Sigma = \bigvee_{t\in S_\beta}\Sigma_t$. This completes the proof of the theorem □.

By the reconstruction theorem, we obtain a stochastically positive β-KMS system

$$(\tilde{\mathcal{B}}, \tilde{\mathcal{U}}, \tilde{\tau}, \tilde{\omega})$$

which describes the *translation invariant $P(\phi)_2$ model at temperature β^{-1}.*

7.3 Properties of the interacting β-KMS system

We first prove the convergence of sharp-time Schwinger functions.

Proposition 7.5 *Let $h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$ and $t_i \in S_\beta$ for $1 \le i \le n$. Then*

$$\lim_{l\to\infty}\int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)}\mathrm{d}\mu_l = \int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)}\mathrm{d}\mu.$$

Proof. Let $a>0$ such that $\operatorname{supp} h_j \subset [-a,a]$. By Proposition 7.3, we know that

$$\phi(t_j,h_j) = \lim_{k\to\infty}\phi\big(\delta_k(.-t_j)\otimes h_j\big) \text{ in } L^1(Q,\Sigma,\mu).$$

After extracting a subsequence, this implies that

$$\phi(t_j,h_j) = \lim_{k\to\infty}\phi\big(\delta_k(.-t_j)\otimes h_j\big) \text{ pointwise } \mu \text{ a.e. on } Q,$$

and hence

$$\begin{aligned}\int_Q \textstyle\prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)}\mathrm{d}\mu = {} & \lim_{k\to\infty}\int_Q \textstyle\prod_1^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-t_j)\otimes h_j)}\mathrm{d}\mu \\ = {} & \lim_{k\to\infty}\big(\Omega_{\mathbb{C}}, W_{[-a,a]}\big(\textstyle\sum_1^n \delta_k(.-t_j)\otimes h_j\big)\Omega_{\mathbb{C}}\big),\end{aligned} \tag{7.5}$$

by Theorem 7.2 (i). Note that for all $l>0$

$$\phi(t_j,h_j) = \lim_{k\to\infty}\phi\big(\delta_k(.-t_j)\otimes h_j\big) \text{ in } L^1(Q,\Sigma,\mu_l),$$

because this convergence holds in $L^2(Q,\Sigma,\mathrm{d}\phi_C)$ and

$$\mathrm{d}\mu_l := \frac{G_{[-l,l]}\mathrm{d}\phi_C}{\int_Q G_{[-l,l]}\mathrm{d}\phi_C},$$

where $G_{[-l,l]} \in L^2(Q,\Sigma,\mathrm{d}\phi_C)$ as a consequence of (5.6). By the same arguments as above, we obtain

$$\int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)}\mathrm{d}\mu_l = \lim_{k\to\infty}\frac{\big(\mathrm{e}^{-(l-a)H_{\mathrm{c}}^{\mathrm{ren}}}\Omega_{\mathbb{C}}^\circ, W_{[-a,a]}\big(\sum_1^n\delta_k(.-t_j)\otimes h_j\big)\mathrm{e}^{-(l-a)H_{\mathrm{c}}^{\mathrm{ren}}}\Omega_{\mathbb{C}}^\circ\big)}{\|\mathrm{e}^{-lH_{\mathrm{c}}^{\mathrm{ren}}}\Omega_{\mathbb{C}}^\circ\|^2}.$$

Let us denote by $F(k,l)$ the quantity on the r.h.s.. Applying (7.4) we obtain

$$\lim_{k\to\infty}F(k,l) = \int_Q\prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)}\mathrm{d}\mu_l \text{ uniformly w.r.t. } l.$$

As we have seen, Theorem 7.2 (i) implies

$$\lim_{l\to\infty}F(k,l) = \int_Q\prod_1^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-t_j)\otimes h_j)}\mathrm{d}\mu.$$

Applying now Lemma B.1 (ii) and using (7.5) we obtain the proposition $\square$.

Let us denote by $\mathcal{U}_0 \subset \mathcal{B}\big(\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})\big)$ the C^*-algebra generated by

$$\big\{F\big(\phi_{AW}(h_1),\dots,\phi_{AW}(h_n)\big) \mid h_i \in C^\infty_{0\,\mathbb{R}}(\mathbb{R}),\ F\in C_0^\infty(\mathbb{R}^n),\ n\in\mathbb{N}\big\}.$$

The isomorphism between $L^\infty(Q,\Sigma_0,\mathrm{d}\phi_C)$ and $\mathcal{U}_{AW}$, which we recalled in Subsection 2.3, maps the operator $F\big(\phi_{AW}(h_1),\dots,\phi_{AW}(h_n)\big)$ onto the function $F\big(\phi(0,h_1),\dots,\phi(0,h_n)\big)$. This function is Σ_0-measurable. We will still denote by A the image of such a function A in the abelian algebra $\tilde{\mathcal{U}}$ provided by the reconstruction theorem for the translation invariant $P(\phi)_2$ model.

Proposition 7.6 *Let $A_i \in \mathcal{U}_0$ and $t_i \in \mathbb{R}$, $1 \le i \le n$. Then*

$$\lim_{l \to +\infty} \omega_l\Big(\prod_1^n \tau^l_{t_i}(A_i)\Big) = \tilde{\omega}\Big(\prod_1^n \tilde{\tau}_{t_i}(A_i)\Big).$$

Proof. Let us fix $A_i \in \mathcal{U}_0$ and set

$$G^l(t_1, \dots, t_n) := \omega_l\Big(\prod_1^n \tau^l_{t_i}(A_i)\Big) \quad \text{and} \quad G(t_1, \dots, t_n) = \tilde{\omega}\Big(\prod_1^n \tilde{\tau}_{t_i}(A_i)\Big).$$

Due to the KMS condition, the functions G^l and G are holomorphic in

$$I_\beta^{n+} = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \mathrm{Im} z_i < \mathrm{Im} z_{i+1},\ \mathrm{Im} z_n - \mathrm{Im} z_1 < \beta\big\},$$

continuous on $\overline{I_\beta^{n+}}$ and bounded by $\prod_1^n \|A_i\|$.

We first claim that

$$\lim_{l \to \infty} G^l(\mathrm{i}s_1, \dots, \mathrm{i}s_n) = G(\mathrm{i}s_1, \dots, \mathrm{i}s_n) \text{ for } s_1 \le \dots \le s_n,\ s_n - s_1 \le \beta. \tag{7.6}$$

Using Proposition 7.5 and the identity (2.2) we see that (7.6) holds for $A_j = \mathrm{e}^{\mathrm{i}\phi_{AW}(h_j)}$, $h_j \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$. Using functional calculus we can extend (7.6) to arbitrary $A_j \in \mathcal{U}_0$.

Let us now consider, for $s_2 \le \dots \le s_n$ and $s_n - s_2 \le \beta$, the functions

$$u_l(z) := G^l(z, \mathrm{i}s_2, \dots, \mathrm{i}s_n),$$

which are holomorphic in $\{0 < \mathrm{Im} z < s_2\}$ and continuous on $\{0 \le \mathrm{Im} z \le s_2\}$. Since the family $\{u_l\}$ is uniformly bounded, we can apply Lemma B.3. It follows that

$$\lim_{l \to +\infty} G^l(t_1, \mathrm{i}s_2, \dots, \mathrm{i}s_n) = G(t_1, \mathrm{i}s_2, \dots, \mathrm{i}s_n)$$

for $s_2 \le \dots \le s_n$, $s_n - s_2 \le \beta$ and $t_1 \in \mathbb{R}$. Iterating this argument, we obtain

$$\lim_{l \to +\infty} G^l(t_1, \dots, t_n) = G(t_1, \dots, t_n).$$

This completes the proof of the proposition $\Box$.

Let us denote by $\{\alpha_x\}_{x \in \mathbb{R}}$ the group of space translations on $\mathcal{U}_0$ defined by $\alpha_x\big(W_{AW}(h)\big) = W_{AW}\big(h(\,.\, - x)\big)$ for $h \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$.

Lemma 7.7 *Let $A_j \in \mathcal{U}_0$ and $t_j \in \mathbb{R}$, $1 \le j \le n$. Set $A = \prod_{j=1}^k \tau_{t_j}(A_j)$ and $B = \prod_{j=k+1}^n \tau_{t_j}(A_j)$. It follows that*

(i) $\tilde{\omega}(\alpha_x(A)) = \tilde{\omega}(A)$ *for all* $x \in \mathbb{R}$*;*
(ii) $\lim_{x \to \infty} \tilde{\omega}(A \alpha_x(B)) = \tilde{\omega}(A)\tilde{\omega}(B)$.

Proof. Property (i) follows directly from the invariance of the measure μ under the space translations $\{\alpha_x\}_{x \in \mathbb{R}}$ shown in Theorem 7.2 (iii). It remains to prove (ii). We set

$$\begin{aligned} G_x(t_1, \dots, t_n) &:= \tilde{\omega}\big(\textstyle\prod_{j=1}^l \tau_{t_j}(A_j) \prod_{j=l+1}^n \alpha_x \circ \tau_{t_j}(A_j)\big), \\ G_\infty(t_1, \dots, t_n) &:= \tilde{\omega}\big(\textstyle\prod_{j=1}^l \tau_{t_j}(A_j)\big) \cdot \tilde{\omega}\big(\textstyle\prod_{j=l+1}^n \tau_{t_j}(A_j)\big). \end{aligned}$$

Due to the KMS condition, the functions G_x and G_∞ are holomorphic in I_β^{n+} and bounded by $\prod_{j=1}^n \|A_j\|$. We claim that, for $s_1 \le \dots \le s_n$ and $s_n - s_1 \le \beta$,

$$\lim_{x \to \infty} G_x(\mathrm{i}s_1, \dots, \mathrm{i}s_n) = G_\infty(\mathrm{i}s_1, \dots, \mathrm{i}s_n). \tag{7.7}$$

Let us prove (7.7). Let us first assume that $A_j = \mathrm{e}^{\mathrm{i}\phi(0,h_j)}$ for $h_j \in C_{0\,\mathbb{R}}^{\infty}(\mathbb{R})$. Then

$$\begin{aligned} G_x(\mathrm{i}s_1,\dots,\mathrm{i}s_n) &= \int_Q \prod_{j=1}^{l} \mathrm{e}^{\mathrm{i}\phi(\delta(.-s_j)\otimes h_j)} \prod_{j=l+1}^{n} \mathrm{e}^{\mathrm{i}\phi(\delta(.-s_j)\otimes h_j(\,.\,-x))}\mathrm{d}\mu, \\ G_\infty(\mathrm{i}s_1,\dots,\mathrm{i}s_n) &= \int_Q \prod_{j=1}^{l} \mathrm{e}^{\mathrm{i}\phi(\delta(.-s_j)\otimes h_j)}\mathrm{d}\mu \times \int_Q \prod_{j=l+1}^{n} \mathrm{e}^{\mathrm{i}\phi(\delta(.-s_j)\otimes h_j)}\mathrm{d}\mu. \end{aligned}$$

By Proposition 7.3 we have

$$\begin{aligned} G_x(\mathrm{i}s_1,\dots,\mathrm{i}s_n) &= \lim_{k\to+\infty} \int_Q \prod_{j=1}^{l} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j)} \prod_{j=l+1}^{n} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j(\,.\,-x))}\mathrm{d}\mu, \\ G_\infty(\mathrm{i}s_1,\dots,\mathrm{i}s_n) &= \lim_{k\to\infty} \int_Q \prod_{j=1}^{l} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j)}\mathrm{d}\mu \times \int_Q \prod_{j=l+1}^{n} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j)}\mathrm{d}\mu. \end{aligned}$$

From Theorem 7.2 we get

$$\begin{aligned} &\int_Q \prod_{j=1}^{l} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j)} \prod_{j=l+1}^{n} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j(\,.\,-x))}\mathrm{d}\mu \\ &= \big(\Omega_{\mathsf{C}}, W_{[-\infty,+\infty]}(R_{1,k} + \mathfrak{a}_x(R_{2,k}))\Omega_{\mathsf{C}}\big) \end{aligned}$$

and

$$\begin{aligned} &\int_Q \prod_{j=1}^{l} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j)}\mathrm{d}\mu \times \int_Q \prod_{j=l+1}^{n} \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-s_j)\otimes h_j)}\mathrm{d}\mu \\ &= \big(\Omega_{\mathsf{C}}, W_{[-\infty,+\infty]}(R_{1,k})\Omega_{\mathsf{C}}\big)\cdot\big(\Omega_{\mathsf{C}}, W_{[-\infty,+\infty]}(R_{2,k})\Omega_{\mathsf{C}}\big), \end{aligned}$$

where

$$R_{1,k} = \phi\big(\sum_{j=1}^{l} \delta_k(.-s_j)\otimes h_j\big) \text{ and } R_{2,k} = \phi\big(\sum_{j=l+1}^{n} \delta_k(.-s_j)\otimes h_j\big).$$

As before (see Section 4.1), the group of spatial translations induced on Q by the map $(t,y)\mapsto(t,y+x)$ has been denoted by $\{\mathfrak{a}_x\}_{x\in\mathbb{R}}$. Applying Lemma A.10 we find

$$\Big|\big(\Omega_{\mathsf{C}}\,,\,W\big(R_{1,k} + \mathfrak{a}_x(R_{2,k})\big)\Omega_{\mathsf{C}}\big) - \big(\Omega_{\mathsf{C}}\,,\,W(R_{1,k})\Omega_{\mathsf{C}}\big)\big(\Omega_{\mathsf{C}}, W(R_{2,k})\Omega_{\mathsf{C}}\big)\Big| \le \mathrm{e}^{-(|x|-C)a},$$

where $a>0$ is the spectral gap of $H_{\mathsf{C}}^{\mathrm{ren}}$ and $W(.) := W_{[-\infty,+\infty]}(.)$. Letting $k\to\infty$ and using Proposition 7.3 we obtain

$$\big|G_x(\mathrm{i}s_1,\dots,\mathrm{i}s_n) - G_\infty(\mathrm{i}s_1,\dots,\mathrm{i}s_n)\big| \le \mathrm{e}^{-(|x|-C)a}.$$

Using functional calculus, we conclude that (7.7) holds for all $A_j \in \mathcal{U}_0$. To complete the proof of the lemma, we can now argue as in the proof of Proposition 7.6, using Lemma B.3 $\square$.

A A time-dependent heat equation

Let $H \ge 0$ be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and let $R(t)$, $t\in\mathbb{R}$, be a family of closed operators with $\mathcal{D}(H^\gamma)\subset\mathcal{D}(R(t))$ for some $0\le\gamma<1$. We consider the following time-dependent heat equation:

$$\text{(A.1)} \qquad \left\{\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = -\big(H + \mathrm{i}\lambda R(t)\big)U(t,s),\ s\le t, \\ U(s,s) = \mathbb{1}. \end{array}\right.$$

This equation is (formally) equivalent to the following integral equation:

$$\text{(A.2)} \qquad U(t,s) = \mathrm{e}^{-(t-s)H} - \mathrm{i}\lambda\int_s^t \mathrm{e}^{-(t-\tau)H}R(\tau)U(\tau,s)\mathrm{d}\tau.$$

In the main text we only use the results of this section in the *dissipative* case, i.e., when $R(t)$ is selfadjoint for all $t\in\mathbb{R}$. However, part of the results are valid and will be proved in the general case.

The solution of (A.1) will be denoted by $U(t,s)$ or $U_\lambda(t,s)$. If we want to display its dependence on the family $R(t)$, then the solution of (A.1) will be denoted by $U(t,s;R)$.

A.1 Existence of solutions

We assume that the maps

$$\text{(A.3)} \qquad \begin{array}{ccc} \mathbb{R} & \to & \mathcal{B}(\mathcal{H}) \\ t & \mapsto & R(t)(H+1)^{-\gamma} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{R} & \to & \mathcal{B}(\mathcal{H}) \\ t & \mapsto & R^*(t)(H+1)^{-\gamma} \end{array}$$

are Hölder continuous of some order $\epsilon > 0$.

In the sequel we will use the following result.

Lemma A.1 *Assume (A.3). Then*

$$\text{(A.4)} \qquad \|\mathrm{e}^{-(t-\tau)H}R(\tau)\| \le c_\gamma \|R(\tau)^*(H+1)^{-\gamma}\| \left(|t-\tau|^{-\gamma}+1\right) \quad \forall \tau \le t.$$

Proof. We have $\|\mathrm{e}^{-(t-\tau)H}R(\tau)\| \le \|R(\tau)^*(H+1)^{-\gamma}\|\|(H+1)^\gamma \mathrm{e}^{-(t-\tau)H}\|$. This proves the lemma, using

$$\text{(A.5)} \qquad |(\lambda+1)^\gamma \mathrm{e}^{-s\lambda}| \le c_\gamma(s^{-\gamma}+1) \quad \text{for} \quad s,\ \lambda \ge 0\, \square\,.$$

The following result is shown in [H, Theorem 7.1.3].

Proposition A.2 *There exists a unique solution $U(t,s)$ of (A.1) such that*

- (i) $U(s,s) = \mathbb{1}$ *and* $U(t,r)U(r,s) = U(t,s)$ *for* $s \le r \le t$;
- (ii) $t \mapsto U(t,s) \in \mathcal{B}(\mathcal{H})$ *is strongly continuous in* $[s,+\infty[$ *and strongly differentiable in* $]s,+\infty[$.

Lemma A.3 *The map $\lambda \mapsto U_\lambda(.,s)\Psi \in C\big([s,+\infty[,\mathcal{H}\big)$ is entire analytic for each $\Psi \in \mathcal{H}$.*

Proof. Let $\Psi \in \mathcal{H}$. For $s \le T < \infty$, set $V(t)\Psi = \mathrm{e}^{-(t-s)H}\Psi$ and

$$\|U(.,s)\Psi\| := \sup_{t\in[s,T]} \|U(t,s)\Psi\|_{\mathcal{H}}.$$

If we define a map

$$K\colon \begin{array}{ccc} C([s,T],\mathcal{H}) & \to & C([s,T],\mathcal{H}) \\ W(.) & \mapsto & -\mathrm{i}\int_s^{(.)} \mathrm{e}^{-(.-\tau)H}R(\tau)W(\tau)\mathrm{d}\tau, \end{array}$$

then the integral equation (A.2) can be rewritten as $(1-K)(U(\,.\,,s)\Psi) = V(\,.\,)\Psi$. Now (A.4) implies

$$\|KU(t,s)\Psi\| \le \|U(.,s)\Psi\| \sup_{\tau\in[s,T]} \|R^*(\tau)(H+1)^{-\gamma}\| \int_s^t \left(|t-\tau|^{-\gamma}+1\right)\mathrm{d}\tau,$$

and hence

$$\|KU(\,.\,,s)\Psi\| \le c \sup_{\tau\in[s,T]} \|R^*(\tau)(H+1)^{-\gamma}\|\, |T-s|^{1-\gamma}\|U(\,.\,,s)\Psi\|,$$

which shows that $K \in \mathcal{B}\big(C([s,T],\mathcal{H})\big)$. Then $U_\lambda(.,s)\Psi$ solves $(1-\lambda K)\big(U_\lambda(.,s)\Psi\big) = V(.)\Psi$, which implies that $\lambda \mapsto U_\lambda(\,.\,,s)\Psi$ is entire analytic $\square$.

A.2 The dissipative case

We now consider the dissipative case when $R(t)$ is selfadjoint for $t \in \mathbb{R}$. We first prove a result about approximation by time-ordered products. We will make use of an extension of Gronwall's inequality to integral equations, shown in [H, Lemma 7.1.1].

Lemma A.4 *Let $b \geq 0$ and $\gamma > 0$. Let $a(t)$ and $u(t)$ be non negative locally integrable functions on $s \leq t \leq T < \infty$ such that*

$$u(t) \leq a(t) + b \int_s^t (t-\tau)^{-\gamma} u(\tau) \mathrm{d}\tau \text{ for } s \leq t \leq T.$$

Then

$$u(t) \leq a(t) + c b^{(1-\gamma)^{-1}} \int_s^t E(t-\tau) a(\tau) \mathrm{d}\tau \text{ for } s \leq t \leq T,$$

where $|E(r)| \leq c_T(|r|^{-\gamma})$ on $[0, T-s]$.

Proposition A.5 *Assume that $R(t)$ is selfadjoint and that (A.3) holds.*
Then for $s \leq t$ there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} p_n = +\infty$ such that

$$U(t,s) = \text{s-}\lim_{n\to\infty} \prod_{n-1}^{0} \left(\mathrm{e}^{-(t_{j+1}-t_j)H/p_n} \mathrm{e}^{-\mathrm{i}(t_{j+1}-t_j)R(t_j)/p_n} \right)^{p_n},$$

where $t_j = s + \frac{(t-s)j}{n}$ for $0 \leq j \leq n$.

Proof. Let R_i, $i = 1, 2$, be two families of closed operators satisfying (A.3) and let $U^{(i)}(t,s)$ be the associated propagators. Then

$$\begin{aligned} U^{(1)}(t,s) - U^{(2)}(t,s) = & -\mathrm{i} \textstyle\int_s^t \mathrm{e}^{-(t-\tau)H} R_1(\tau) \big(U^{(1)}(\tau,s) - U^{(2)}(\tau,s) \big) \mathrm{d}\tau \\ & -\mathrm{i} \textstyle\int_s^t \mathrm{e}^{-(t-\tau)H} \big(R_1(\tau) - R_2(\tau) \big) U^{(2)}(\tau,s) \mathrm{d}\tau. \end{aligned}$$

This implies, using Lemma A.1, that

$$\begin{aligned} & \| U^{(1)}(t,s) - U^{(2)}(t,s) \| \\ \leq \; & c_\gamma \textstyle\int_s^t (|t-\tau|^{-\gamma} + 1) \, \|(H+1)^{-\gamma} R_1(\tau)\| \, \|U^{(1)}(\tau,s) - U^{(2)}(\tau,s)\| \mathrm{d}\tau \\ & + c_\gamma \textstyle\int_s^t (|t-\tau|^{-\gamma} + 1) \, \big\| (H+1)^{-\gamma} \big(R_1(\tau) - R_2(\tau) \big) \big\| \, \|U^{(2)}(\tau,s)\| \mathrm{d}\tau. \end{aligned}$$

We now apply Gronwall's inequality, as given in Lemma A.4, with

$$\begin{aligned} & b = \sup_{s\leq t\leq T} \|(H+1)^{-\gamma} R_1(t)\|, \\ & a(t) \equiv a = c_T \sup_{s\leq t\leq T} \big\| (H+1)^{-\gamma} \big(R_1(t) - R_2(t) \big) \big\| \times \sup_{s\leq t\leq T} \|U^{(2)}(t,s)\|. \end{aligned}$$

We obtain

$$\begin{aligned} & \sup_{s\leq t\leq T} \|U^{(1)}(t,s) - U^{(2)}(t,s)\| \\ \leq \; & c_T \sup_{s\leq t\leq T} \|(H+1)^{-\gamma} R_1(t)\|^{(1-\gamma)^{-1}} \\ & \times \sup_{s\leq t\leq T} \big\| (H+1)^{-\gamma} \big(R_1(t) - R_2(t) \big) \big\| \times \sup_{s\leq t\leq T} \|U^{(2)}(t,s)\|. \end{aligned} \tag{A.6}$$

Let us now prove the proposition. For $s < t$ fixed, $n \in \mathbb{N}$, we set

$$t_j := s + \frac{(t-s)j}{n}, \;\; 0 \leq j \leq n, \quad R_n(\tau) = \sum_{n=0}^{n-1} \mathbb{1}_{[t_j, t_{j+1}[}(\tau) R(t_j).$$

Note that $H + \mathrm{i}R(t_j)$ with domain $\mathcal{D}(H)$ is the generator of a C_0-semigroup of contractions, since it is closed and maximal accretive, using (A.3).

If $U^{(n)}(t,s)$ is the solution of (A.1) for the piecewise constant family of operators $\{R_n(t)\}$, then one can easily verify that:

$$U^{(n)}(t,s) = \prod_{n-1}^{0} \mathrm{e}^{-(t_{j+1}-t_j)(H+\mathrm{i}R_n(t_j))}.$$

Since $\mathbb{R} \ni t \mapsto (H+1)^{-\gamma}R(t) \in \mathcal{B}(\mathcal{H})$ is continuous, we conclude that

$$\lim_{n\to\infty} \sup_{s\le t\le T} \left\| (H+1)^{-\gamma}\big(R(t) - R_n(t)\big) \right\| = 0.$$

Using (A.6) we get

$$\lim_{n\to\infty} \sup_{s\le t\le T} \|U(t,s) - U^{(n)}(t,s)\| = 0.$$

Applying next [Ch] we obtain

$$\mathrm{e}^{-(t_{j+1}-t_j)(H+\mathrm{i}R(t_j))} = \text{s-}\lim_{p\to\infty} \left(\mathrm{e}^{-(t_{j+1}-t_j)H/p} \mathrm{e}^{-\mathrm{i}(t_{j+1}-t_j)R(t_j)/p} \right)^p.$$

Using the fact that $\mathrm{e}^{-\tau(H+\mathrm{i}R(t_j))}$, $\mathrm{e}^{-\tau H}$ and $\mathrm{e}^{-\mathrm{i}\tau R(t_j)}$ are all contractions, we conclude that there exists a sequence $p_n \to \infty$ such that

$$U(t,s) = \text{s-}\lim_{n\to\infty} \prod_{n-1}^{0} \left(\mathrm{e}^{-(t_{j+1}-t_j)H/p_n} \mathrm{e}^{-\mathrm{i}(t_{j+1}-t_j)R(t_j)/p_n} \right)^{p_n}.$$

This completes the proof of the proposition $\square$.

Proposition A.6 *Assume that $R(t)$ is selfadjoint and satisfies (A.3). Assume moreover that the function*

(A.7) $$t \mapsto \|(H+1)^{-\gamma}R(t)\| \text{ is in } L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

and

(A.8) $$\pm R(t) \le r(t)(H+1)^\delta, \quad 0 \le \delta < 1,$$

for some $r \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the limit $U_\lambda(+\infty,-\infty) := \mathrm{w}-\lim_{(t,s)\to(+\infty,-\infty)} U_\lambda(t,s)$ exists for all $\lambda \in \mathbb{C}$.

(i) *the function $\mathbb{C} \ni \lambda \mapsto U_\lambda(+\infty,-\infty)$ is entire and satisfies*

$$\|U_\lambda(+\infty,-\infty)\| \le \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}}} \quad \forall \lambda \in \mathbb{C};$$

(ii) *the derivatives w.r.t. λ are uniformly bounded:*

$$\sup_{\lambda\in\mathbb{R}} |\partial_\lambda^n U_\lambda(+\infty,-\infty)| < \infty \quad \forall n \in \mathbb{N};$$

(iii) *for $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ the derivatives at $\lambda = 0$ are given by the following formula:*

$$\begin{aligned} &\tfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U_\lambda(+\infty,-\infty)_{|\lambda=0} \\ = \;& n!(-\mathrm{i})^n \int_{-\infty<t_1\le\cdots\le t_n<\infty} 1\!\!1_{\{0\}}(H) \left(\textstyle\prod_n^2 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H} \right) R(t_1) 1\!\!1_{\{0\}}(H) \mathrm{d}t_1 \ldots \mathrm{d}t_n. \end{aligned}$$

Proof. Let $\Psi \in \mathcal{H}$. Using Proposition A.2 and the fact that $R(t)$ is selfadjoint we obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\|U_\lambda(t,s)\Psi\|^2 &= -2\mathrm{Re}\big(U_\lambda(t,s)\Psi,(H+\mathrm{i}\lambda R(t))U_\lambda(t,s)\Psi\big)\\ &= -2\big(U_\lambda(t,s)\Psi,(H-\mathrm{Im}\lambda R(t))U_\lambda(t,s)\Psi\big).\end{aligned}$$

Now

$$H-\mathrm{Im}\lambda R(t) \geq H - |\mathrm{Im}\lambda|\, r(t)(H+1)^\delta \geq c\big(|\mathrm{Im}\lambda|\, r(t)\big)^{(1-\delta)^{-1}},$$

since $\inf_{s\geq 0}\ s-t(s+1)^\delta = -c\,t^{(1-\delta)^{-1}}$ for $t\geq 0$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U_\lambda(t,s)\Psi\|^2 \leq c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}} r(t)^{(1-\delta)^{-1}}\|U_\lambda(t,s)\Psi\|^2,$$

and hence

$$\|U_\lambda(t,s)\Psi\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}}\int_s^t r(\tau)^{(1-\delta)^{-1}}\mathrm{d}\tau}\|\Psi\|.$$

Since $r\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ we have $r^{(1-\delta)^{-1}}\in L^1(\mathbb{R})$, which yields

$$\sup_{s\leq t}\|U_\lambda(t,s)\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}}}\quad \forall\lambda\in\mathbb{C}. \tag{A.9}$$

Let us now prove that

$$\mathrm{w}-\lim_{(t,s)\to(+\infty,-\infty)} U_\lambda(t,s) \text{ exists for all } \lambda\in\mathbb{R}. \tag{A.10}$$

For $\Psi\in\mathcal{H}$, $\Phi\in\mathcal{D}(H)$ and $0\leq\gamma<1$ we find

$$\big(\Phi,(U_\lambda(t,s)-\mathrm{e}^{-(t-s)H})\Psi\big) = -\mathrm{i}\lambda\int_s^t\Big(\mathrm{e}^{-(t-\tau)H}(H+1)^\gamma\Phi,(H+1)^{-\gamma}R(\tau)U(\tau,s)\Psi\Big)\mathrm{d}\tau.$$

Using dominated convergence and hypothesis (A.7) we obtain the existence of

$$\lim_{(t,s)\to(+\infty,-\infty)}\big(\Phi,U_\lambda(t,s)\Psi\big) \text{ for } \Psi\in\mathcal{H} \text{ and } \Phi\in\mathcal{D}(H^\gamma).$$

Applying a density argument and the uniform bound (A.9) this proves (A.10).

Now $\{\lambda\mapsto U_\lambda(t,s)\mid s\leq t\}$ is a locally uniformly bounded family of entire functions. Applying Lemma B.3 and (A.10) we obtain that

$$U_\lambda(+\infty,-\infty) = \mathrm{w}-\lim_{(t,s)\to(+\infty,-\infty)} U_\lambda(t,s)$$

exists for all $\lambda\in\mathbb{C}$. Moreover, the map $\lambda\mapsto U_\lambda(+\infty,-\infty)$ is entire and

$$\|U_\lambda(+\infty,-\infty)\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}}}\quad\forall\lambda\in\mathbb{C}.$$

If $f(z)$ is a bounded holomorphic function in a strip $\{|\mathrm{Im} z|<a\}$, then it follows easily from Cauchy's formula that $\sup_{x\in\mathbb{R}}|\partial_x^n f(x)|<\infty$ for all $n\in\mathbb{N}$. This completes the proof of (ii).

Let us now prove (iii). Set, as in Subsection A.1,

$$\begin{aligned} K\colon\ C([s,T],\mathcal{H}) &\to C([s,T],\mathcal{H})\\ W(.) &\mapsto -\mathrm{i}\int_s^{(.)}\mathrm{e}^{-(.-\tau)H}R(\tau)W(\tau)\mathrm{d}\tau,\end{aligned}$$

and $V(t)\Psi = \mathrm{e}^{-(t-s)H}\Psi$. From the integral equation $(\mathbb{1}-\lambda K)U_\lambda(.,s)\Psi = V(.)\Psi$ we deduce that

$$\begin{aligned}\tfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n}U_\lambda(t,t_0)_{|\lambda=0}\Psi &= n!K^nV(t)\Psi\\ &= n!(-\mathrm{i})^n\int_{t_0\leq t_1\leq\cdots\leq t_n\leq t}\mathrm{e}^{-(t-t_n)H}\Big[\textstyle\prod_n^1 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H}\Big]\Psi\mathrm{d}t_1\ldots\mathrm{d}t_n.\end{aligned}$$

The function $\mathbb{C} \ni \lambda \mapsto U_\lambda(t,s)\Psi$ is entire and uniformly bounded in $\{|\mathrm{Im}\lambda| \leq a\}$ for $-\infty < s \leq t < +\infty$. Therefore

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U_\lambda(+\infty,-\infty)\Psi = \lim_{(t,s)\to(+\infty,-\infty)} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U_\lambda(t,s)\Psi.$$

Setting $t_{n+1} = t$ we find

$$\left\| \mathrm{e}^{-(t-t_n)H} \prod_n^1 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H} \right\| \leq c \prod_n^1 \left(|t_{k+1}-t_k|^{-\gamma}+1\right) \|(H+1)^{-\gamma} R(t_k)\|.$$

From Lebesgue dominated convergence we deduce that

$$\lim_{s\to-\infty} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U_\lambda(t,s)_{|\lambda=0} = n!(-\mathrm{i})^n \int\limits_{-\infty<t_1\leq\cdots\leq t_n\leq t} \left[\prod_n^1 \mathrm{e}^{-(t_{k+1}-t_k)H} R(t_k)\right] 1\!\!1_{\{0\}}(H)\mathrm{d}t_1\ldots\mathrm{d}t_n.$$

A similar argument yields

$$\begin{aligned} & \lim_{t\to+\infty} \lim_{s\to-\infty} \tfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U_\lambda(t,s)_{|\lambda=0} \\ = \; & n!(-\mathrm{i})^n \int\limits_{-\infty<t_1\leq\cdots\leq t_n\leq+\infty} 1\!\!1_{\{0\}}(H) \left[\textstyle\prod_n^2 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H}\right] R(t_1) 1\!\!1_{\{0\}}(H)\mathrm{d}t_1\ldots\mathrm{d}t_n. \end{aligned}$$

Applying Lemma B.1 we obtain (iii) $\Box$.

We will use the following lemma to show that the limiting functional obtained in Theorem 7.2 defines a Borel measure on $\mathcal{S}'(S_\beta \times \mathbb{R})$.

Lemma A.7 *Let $R_i(t)$, $i=1,2$, be selfadjoint families satisfying (A.3) and (A.7). Assume that*

$$\pm\left(R_1(t) - R_2(t)\right) \leq r(t)(H+1)^\delta,\ 0 \leq \delta < 1,$$

for $r \in L^{(1-\delta)^{-1}}(\mathbb{R})$. Then

$$\|U(+\infty,-\infty;R_2) - U(+\infty,-\infty;R_1)\| \leq c\|r\|_{(1-\delta)^{-1}}.$$

Proof. Let us denote by $Z_\lambda(t,s)$ the operator $U(t,s;R_1+\lambda(R_2-R_1))$. By the same arguments as used in the proof of Proposition A.6, we see that $\lambda \mapsto Z_\lambda(t,s)$ is an entire analytic function, which satisfies the bound

$$\|Z_\lambda(t,s)\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^\gamma \|r\|_\gamma^\gamma} \quad \text{for } \lambda \in \mathbb{C} \text{ and } \gamma = (1-\delta)^{-1}.$$

As in Proposition A.6, the limit of $Z_\lambda(t,s)$ when $(t,s) \to (+\infty,-\infty)$ exists for $\lambda \in \mathbb{R}$ fixed. Applying again Vitali's theorem, we obtain the existence of $Z_\lambda(+\infty,-\infty)$ for all $\lambda \in \mathbb{C}$, and the bound

$$\|Z_\lambda(+\infty,-\infty)\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^\gamma \|r\|_\gamma^\gamma} \quad \forall \lambda \in \mathbb{C}.$$

Applying Cauchy's formula on the circle of radius R centered around $\lambda \in \mathbb{R}$ yields

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda} Z_\lambda(+\infty,-\infty)\right| \leq R^{-1}\mathrm{e}^{cR^\gamma \|r\|_\gamma^\gamma}.$$

Optimizing this bound w.r.t. R we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda} Z_\lambda(+\infty,-\infty)\right| \leq c\|r\|_\gamma.$$

Integrating in λ from 0 to 1 we obtain the lemma $\Box$.

A.3 Some additional results

We now prove some bounds on $U(t,s)$, which we use in the main text to show the existence of sharp-time fields and the convergence of sharp-time Schwinger functions.

Lemma A.8 *Let $R_i(t)$, $i = 1, 2$, be two families of selfadjoint operators satisfying (A.3) and (A.7). Assume that*

(A.11) $$\pm\big(R_2(t) - R_1(t)\big) \le r(t)(H+1) \text{ for } r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}).$$

Set $Z_\lambda(t,s) := U\big(t,s; R_1 + \lambda(R_2 - R_1)\big)$ for $-\infty \le s \le t \le +\infty$. Then

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} Z_\lambda(t,s) \right\| \le n!\, \|r\|_\infty^n\, \mathrm{e}^{\|r\|_1 \|r\|_\infty^{-1}}.$$

Proof. Since $R_i(t)$ satisfy (A.3) and (A.7), the function $\lambda \mapsto Z_\lambda(t,s)$ is entire. We still denote by $Z_\lambda(t,s)$ its extension to $\lambda \in \mathbb{C}$. As in the proof of Proposition A.6, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z_\lambda(t,s)\Psi\|^2 = -2\Big(Z_\lambda(t,s)\Psi, \big(H - \mathrm{Im}\lambda(R_2(t) - R_1(t))\big) Z_\lambda(t,s)\Psi\Big).$$

Now

$$H - \mathrm{Im}\lambda\big(R_2(t) - R_1(t)\big) \ge H - r(t)\,|\mathrm{Im}\lambda|\,(H+1) \ge -r(t)\,|\mathrm{Im}\lambda|$$

for $|\mathrm{Im}\lambda| \le \|r\|_\infty^{-1}$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z_\lambda(t,s)\Psi\|^2 \le 2\,|\mathrm{Im}\lambda|\, r(t)\|\Psi\|^2 \text{ for } |\mathrm{Im}\lambda| \le \|r\|_\infty^{-1}.$$

Hence

$$\|Z_\lambda(t,s)\| \le \mathrm{e}^{|\mathrm{Im}\lambda|\|r\|_1} \text{ for } |\mathrm{Im}\lambda| \le \|r\|_\infty^{-1}.$$

We apply Cauchy's formula on a circle of radius $\|r\|_\infty^{-1}$ and obtain

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} Z_\lambda(t,s) \right\| \le n!\, \|r\|_\infty^n\, \mathrm{e}^{\|r\|_1 \|r\|_\infty^{-1}} \text{ for } \lambda \in \mathbb{R}.$$

This completes the proof of the lemma □.

Finally we prove a lemma which is used in the main text to prove spatial clustering.

Remark A.9 *Let $t_0 \in \mathbb{R}$ and define the time-translated family by $\xi_{t_0}\big(R(t)\big) := R(t - t_0)$. Then clearly*

$$U\big(t,s; \xi_{t_0}(R)\big) = U(t - t_0, s - t_0; R) \text{ for } -\infty < s \le t < +\infty.$$

Letting $(s,t) \to (-\infty, +\infty)$ we obtain that $U\big(+\infty, -\infty; \xi_{t_0}(R)\big) = U(+\infty, -\infty; R)$.

Lemma A.10 *Assume that 0 is a simple eigenvalue of H and that H has a spectral gap, i.e.,*

$$]0, a] \cap \sigma(H) = \emptyset \text{ for some } a > 0.$$

Let $\{R_1(t)\}$, $\{R_2(t)\}$ be two selfadjoint families of operators satisfying (A.3) and (A.7) with $R_i(t) \equiv 0$ for $|t| \ge T$. If Ω is a normalized ground state of H, then

$$\big|\big(\Omega, U^\infty(R_1 + \xi_t(R_2))\Omega\big) - \big(\Omega, U^\infty(R_1)\Omega\big)\big(\Omega, U^\infty(R_2)\Omega\big)\big| \le \mathrm{e}^{-(|t| - 2T)a}$$

for $|t| > 2T$, where $U^\infty(R) := U(+\infty, -\infty; R)$.

Proof. It suffices to consider the case $t > 0$. Using the group property and considering the supports of $R_i(.)$, we find

$$\begin{aligned} U\big(t, s; R_1 + \xi_{t_0}(R_2)\big) &= U\big(t, t_0 - T; \xi_{t_0}(R_2)\big)\mathrm{e}^{-(t_0-2T)H}U(T, s; R_1) \\ &= U(t - t_0, -T; R_2)\mathrm{e}^{-(t_0-2T)H}U(T, s; R_1) \end{aligned} \tag{A.12}$$

for $s \leq -T$, $t_0 > 2T$ and $t > t_0 + T$. Since H has a spectral gap of length a,

$$\big\|\mathrm{e}^{-(t_0-2T)H} - |\Omega\rangle\langle\Omega|\,\big\| \leq \mathrm{e}^{-(t_0-2T)a}. \tag{A.13}$$

Moreover, since $H\Omega = 0$ and $\operatorname{supp} R_i(.) \subset [-T, T]$,

$$\begin{aligned} \big(\Omega, U(t - t_0, -T; R_2)\Omega\big) &= \big(\Omega, U(t - t_0, s; R_2)\Omega\big), \\ \big(\Omega, U(T, s; R_2)\Omega\big) &= \big(\Omega, U(t, s; R_2)\Omega\big). \end{aligned} \tag{A.14}$$

Combining (A.12), (A.13), (A.14) and letting $(t, s) \to (+\infty, -\infty)$ we obtain the lemma □.

B Miscellaneous results

Lemma B.1 *Let $F\colon \mathbb{R}^2 \to E$ be a map with value in a metric space E.*

(i) *Assume that*

$$\lim_{k,k'\to\infty} F(k, k') = F_\infty \text{ exists,}$$
$$\lim_{k'\to\infty} F(k, k') = G(k) \text{ exists } \forall k \in \mathbb{N},$$
$$\lim_{k\to\infty} G(k) = G_\infty \text{ exists.}$$

Then $F_\infty = G_\infty$.

(ii) *Assume that*

$$\lim_{k'\to\infty} F(k, k') = G(k) \text{ exists,}$$
$$\lim_{k\to\infty} F(k, k') = F(k') \text{ exists and the convergence is uniform w.r.t. } k',$$
$$\lim_{k\to\infty} G(k) = G_\infty \text{ exists.}$$

Then $\lim_{k'\to\infty} F(k') = G_\infty$.

The proof is easy and left to the reader.

Lemma B.2 *Let (Q, Σ, μ) be a probability space. Let f be a real measurable function on Q and set*

$$C(t) := \int_Q \mathrm{e}^{\mathrm{i}tf}\mathrm{d}\mu.$$

Then $f \in \bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mu)$ if and only if $\sup_{t\in\mathbb{R}} |\partial_t^n C(t)| < \infty$ for all $n \in \mathbb{N}$. If this is the case, then

$$\partial_t^n C(t) = \mathrm{i}^n \int_Q f^n \mathrm{e}^{\mathrm{i}tf}\mathrm{d}\mu.$$

Proof. The $\Rightarrow$ part and the formula for $\partial_t^n C(t)$ is obvious by differentiating under the integral sign. It remains to prove the $\Leftarrow$ part. Let $\chi(\tau) = \mathrm{e}^{-\tau^2/2}$ and let $p \geq 1$. By monotone convergence it suffices to prove that

$$\sup_{n\in\mathbb{N}} \int_Q f^{2p}\chi\left(\frac{f}{n}\right)\mathrm{d}\mu < \infty$$

in order to show that $f \in L^{2p}(Q, \Sigma, \mu)$. We have

$$\tau^{2p}\chi\left(\frac{\tau}{n}\right) = \frac{n^{2p+1}}{2\pi}\int \mathrm{e}^{\mathrm{i}t\tau}\left(\partial_t^{2p}\hat{\chi}(nt)\right)\mathrm{d}t.$$

Hence

$$\begin{aligned}\int_Q f^{2p}\chi\left(\tfrac{f}{n}\right)\mathrm{d}\mu &= \tfrac{n^{2p+1}}{2\pi}\int_Q\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tf}\left(\partial_t^{2p}\hat{\chi}(nt)\right)\mathrm{d}t\mathrm{d}\mu\\ &= \tfrac{n^{2p+1}}{2\pi}\int_{\mathbb{R}} C(t)\left(\partial_t^{2p}\hat{\chi}(nt)\right)\mathrm{d}t\\ &= \tfrac{(-1)^{2p}}{2\pi}n\int_{\mathbb{R}}\left(\partial_t^{2p}C(t)\right)\hat{\chi}(nt)\mathrm{d}t,\end{aligned}$$

using Fubini's theorem and integrating by parts $2p$ times. Since $\hat{\chi} \in L^1(\mathbb{R})$ and $\partial_t^{2p}C$ is uniformly bounded, we obtain that $\sup_{n\in\mathbb{N}}\int_Q f^{2p}\chi(n^{-1}f)\mathrm{d}\mu < \infty$, which completes the proof of the lemma $\Box$.

Lemma B.3 *Let I be a directed set and $\{u_\alpha\}_{\alpha\in I}$ a net of functions which are holomorphic in an open set $\Omega \subset \mathbb{C}$.*

(i) *Assume that the family $\{u_\alpha\}$ is locally uniformly bounded in Ω and that there exists a set $\Gamma \subset \Omega$ having an accumulation point in Ω such that*

$$\lim_{\alpha\in I} u_\alpha(z) \text{ exists for } z \in \Gamma.$$

Then $\lim_{\alpha\in I} u_\alpha = u$ exists in the compact-open topology on Ω and u is a holomorphic function in Ω.

(ii) *Assume moreover that Ω is bounded with a smooth boundary and that*

$$\sup_{\alpha\in I}\sup_{z\in\Omega}|u_\alpha(z)| < \infty.$$

Then u is continuous on $\overline{\Omega}$ and $\lim_{\alpha\in I}\sup_{z\in\partial\Omega}|u_\alpha(z) - u(z)| = 0$.

Proof. Let us first prove (i). By Vitali's theorem the family $\{u_\alpha\}$ is compact for the compact-open topology. Let $\{u_\beta\}_{\beta\in J}$ be a subnet converging to a continuous function u. Assume that the net $\{u_\alpha\}_{\alpha\in I}$ does not converge to u. Then there exists a bounded open set $\Omega_1 \subset \Omega$ and a subnet $\{u_\gamma\}_{\gamma\in J_1}$ such that $\sup_{z\in\Omega_1}|u_\gamma(z) - u(z)| \geq \epsilon_0 > 0$ for $\gamma \in J_1$. Applying again Vitali's theorem to the net $\{u_\gamma\}_{\gamma\in J_1}$, we obtain another subnet $\{u_\delta\}_{\delta\in J_2}$ such that $\lim_{\delta\in J_2} u_\delta = v$, with $v \neq u$. But u and v are holomorphic in Ω, as limits of holomorphic functions for the compact-open topology and coincide on Γ by hypothesis. Since Γ has an accumulation point in Ω, we have $u = v$ which gives a contradiction.

Let us now prove (ii). Assume the contrary and let $\{u_\beta\}_{\beta\in J}$ be a subnet such that

$$\inf_{\beta\in J}\sup_{z\in\partial\Omega}|u_\beta(z) - u(z)| \geq \epsilon > 0.$$

Since $\Delta u = 0$ in Ω, we see that u belongs to the Sobolev space $H^2(\Omega)$. Using that $\Delta u_\beta = 0$ in Ω and the fact that the family $\{u_\beta\}_{\beta\in J}$ is uniformly bounded in Ω, we obtain similarly that $\{u_\beta\}_{\beta\in J}$ is a bounded family in $H^2(\Omega)$. Hence (i) implies $\lim_{\beta\in J} u_\beta = u$ in $\mathcal{D}'(\Omega)$. Finally we note that the injection $H^2(\Omega) \to H^{3/2}(\Omega)$ is compact. Extracting again a subnet, we obtain $\lim_{\gamma\in J_1} u_\gamma = u$ in $H^{3/2}(\Omega)$. Together with the trace theorem this implies that $\lim_{\gamma\in J_1} u_\gamma = u$ in $H^1(\partial\Omega)$ and hence in $C(\partial\Omega)$. This gives a contradiction $\Box$.

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