Non Linear Eigenvalue Problems

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Abstract

In this paper we consider generalized eigenvalue problems for a family of operators with a polynomial dependence on a complex parameter. This problem is equivalent to a genuine non self-adjoint operator. We discuss here existence of non trivial eigenstates for models coming from analytic theory of smoothness for P.D.E. We shall review some old results and present recent improvements on this subject.

1 Introduction

The problem considered in this paper has two very different origins. The first, from the historical point of view, concerns *Dissipative Problems in Mechanics.* Let us consider the second order differential equation

$$Au'' + Bu' + Cu = 0, \tag{1.1}$$

where the unkonwn function u is defined on $\mathbb{R}$ with values in some Hilbert space $\mathcal{H}$ and $u' = \frac{du}{dt}$. Equation 1.1 is a model in mechanics for small oscillations of a continuum system in the presence of an impedence force [19].
Now looking for stationary solutions of (1.1), that means $u(t) = u_0 \mathrm{e}^{\lambda t}$, we have the following equation

$$(\lambda^2 A + \lambda B + C)u_0 = 0 \tag{1.2}$$

So equation (1.2) is a non linear eigenvalue problem in the parameter $\lambda \in \mathbb{C}$. Existence of non null solutions for (1.2) is a non trivial problem. For $B \neq 0$ this problem is equivalent to a true non-selfajoint linear eigenvalue problem (see section II of this paper) and even the existence of one solution for one complex number may be a difficult problem. But by adding suitable conditions on A, B, C several authors, [19, 17, 9, 20] have proved the existence of a total set of generalized eigenfunctions for (1.2).

I start to work on this subject 25 years ago for a completely different raison. At that time B. Helffer asked Pham The Lai and me if the following non linear eigenvalue problem has at least a solution (λ, u) where $\lambda \in \mathbb{C}$ and u in the Schwartz space $\mathcal{S}(\mathbb{R})$, $u \neq 0$,

$$\left(D_x^2 + (x^2 - \lambda)^2\right) u = 0, \tag{1.3}$$

where $D_x = \dfrac{\partial}{i\partial x}$. Equation (1.3) is connected with analytic hypoellipticity for operators like sum of squares of analytic vector fields $X_1, \cdots, X_r$, defined in an open set of $\mathbb{R}^n$, satisfying the Hörmander's condition: there exists an integer N such that the iterated brackets of the fields X_j of length less than N span a vector space of dimension n at each point. This is a sufficient condition for C^∞-hypoellipticity for operator $A = \sum X_j^2$ [16], i.e if Au is C^∞ in the open set ω then u is also C^∞ in the open set ω. When the coefficients of the X_j are real-analytic in ω, satisfying Hörmander condition, and Au is real-analytic in ω, is it true that u is real-analytic in ω? This is the analytic-hypoellipticity problem.
The general answer is no. The first example was given by Baouendi-Goulaouic [2] with the following system in $\mathbb{R}^3$,

$$X_1 = \frac{\partial}{\partial x_1},\ X_2 = \frac{\partial}{\partial x_2},\ X_3 = x_1 \frac{\partial}{\partial x_3}. \tag{1.4}$$

For this example, Baouendi-Goulaouic have constructed a solution u, non analytic at 0, such that $Au = 0$, by using non trivial solutions of the equation $(D_{x_1}^2 + x_1^2 + \lambda^2)v = 0$ which exist for $\lambda = i\sqrt{2j+1}$, $j \in \mathbb{N}$, as it is well known for harmonic oscillators.
In 1978, B. Helffer has proposed another example of sum of squares of vector fields which are hypoelliptic but not analytic-hypoelliptic: $A = D_{x_1}^2 + \left(x_1^2 D_{x_2} - D_{x_3}\right)^2$. The Baouendi-Goulaouic construction of non analytic solutions at point $(0, 0, 0)$ for A is also possible if (1.3) has a non trivial solution (for a generalization of this method see [11]). But this problem is less obvious than for harmonic oscillators. In [25] we have given a positive answer to the question and we have proved furthermore that there exists a total set of generalized eigenfunctions. Our proof uses pseudodifferential technics (parametrices) and spectral analysis. Nowadays, two other proofs of this result are known.
M. Christ [5], using O.D.E techniques and Wronskian arguments, has extended our result to the equation

$$\left(D_x^2 + (x^m - \lambda)^2\right) u = 0 \tag{1.5}$$

for every $m \in \mathbb{N}$, $m \geq 2$.
Let us remark that if $m = 1$, for every $\lambda \in \mathbb{C}$, the equation (1.5) has only the zero solution because of translation invariance.
Recently Chanillo-Helffer-Laptev [4] have given a proof using a very different and elegant method involving trace inequalities and the Lidskii theorem concerning

the trace of operators. In the paper [4] the authors consider a more general problem, in several real variables ($x \in \mathbb{R}^n$). Let us introduce the following family of differential operators,

$$L_P(\lambda) = -\triangle + (P(x) - \lambda)^2 \tag{1.6}$$

where P is a polynomial of degree $m \geq 2$ such that the homogeneous part P_m of P satisfies $P_m(x) > 0$ for every $x \in \mathbb{R}^n \backslash \{0\}$ (in other word we say that P is a positive-elliptic polynomial).
In [4] the authors prove existence of non trivial solutions for $1 \leq n \leq 3$ assuming that m is large enough.
In January 2003 Bernard Helffer gave in Nantes a lecture concerning the work [4]. After that, Bernard, Xue Ping (Wang) and me, have improved in [15] the results of [4] by making a semi-classical analysis of the traces identities coming from the Lidskii theorem. The main result proved in [15] is the following

Theorem 1.1 *Assume that n is even and that P is a positive-elliptic polynomial of degree $m \geq 2$.*
Then there exists $\lambda \in \mathbb{C}$ and $u \in \mathcal{S}(\mathbb{R}^n)$, $u \neq 0$, such that $L_P(\lambda)u = 0$.

Remark 1.2 *Our proof gives an infinite number of solutions but it is not known, if the solutions of (1.5) span the whole Hilbert space $L^2(\mathbb{R}^n)$.*
On the other side for n odd, $n \geq 3$, the problem of existence of non zero solutions is still open. It seems reasonable to conjecture that such solutions always exist. It is true for $n = 1$ and for some cases if $n = 3$.
Another difficult but interesting problem would be to localize in the complex plane these possible eigenvalues λ. We shall give a very partial result at the end of this paper.

In this paper we want to explain in more details some results concerning these non linear eigenvalue problems and to give the main steps of their proofs. We also explain an approach to prove the above conjecture in odd dimension, $n \geq 3$ (see also [24]).

2 Functional Analysis approach of the problem

We start here with a more general problem. Let be $k \in \mathbb{N}$, $n \geq 1$, and a pencil $L(\lambda)$ of operators in the Hilbert space $\mathcal{H}$ defined by

$$L(\lambda) = H_0 + \lambda H_1 + \cdots + \lambda^{k-1} H_{k-1} + \lambda^k \mathbf{1} \tag{2.7}$$

Let us assume the following properties:
(P-1) H_0 is a self-adjoint, positive operator, with domain $D(H_0)$ in $\mathcal{H}$.
(P-2) For every $0 \leq j \leq k-1$, $H_j H_0^{(j-k/)k}$ and $H_0^{(j-k)/k} H_j$ are bounded operators in $\mathcal{H}$.
(P-3) $H_0^{-1/k}$ is in some Schatten class $\mathcal{C}^p$ for $p > 0$ (for the definition of Schatten classes see [10]).

Then $L(\lambda)$, $\lambda \in \mathbb{C}$, is a family of closed operators on the domain $D(H_0)$. Moreover the index of $L(\lambda)$ is 0 and $\lambda \mapsto L(\lambda)^{-1}$ is a meromorphic mapping from $\mathbb{C}$ into the Banach space $\mathcal{C}(\mathcal{H})$ of compact operators in $\mathcal{H}$. So λ is a pôle for L if and only if $L(\lambda)$ is not injective in $D(H_0)$. But according to a well known trick the poles of L can be identified with the eigenvalues of a non-self-adjoint operator.
Let us define the $k \times k$ matrix of operators

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -H_0 & -H_1 & -H_2 & \dots & -H_{k-1} \end{pmatrix}. \tag{2.8}$$

A_L is a closed operator in the Hilbert space

$$\mathcal{K} = \prod_{1 \le j \le k} D\left(H_0^{(k-j)/k}\right), \tag{2.9}$$

with domain

$$D(A_L) = \prod_{0 \le j \le k-1} D\left(H_0^{(k-j)/k}\right). \tag{2.10}$$

A_L is invertible and A_L^{-1} is in the Schatten class $\mathcal{C}^p$. For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have

$$\{L(\lambda) \text{ is invertible}\} \Longleftrightarrow \{A_L - \lambda \mathbf{1} \text{ is invertible}\} \tag{2.11}$$

Moreover, if we write down the resolvent of A_L as a matrix operator,

$$(A_L - \lambda \mathbf{1})^{-1} = \{r_{j,\ell}(\lambda)\}_{0 \le j, \le \ell \le k-1} \tag{2.12}$$

then we have $L(\lambda)^{-1} = -r_{0,k-1}(\lambda)$.
Let us denote by sp$[L]$ the set of eigenvalues of L and if $\lambda_0 \in$ sp$[L]$, $\mathcal{E}_{\lambda_0}[L]$ denotes the generalized eigenspace for the eigenvalue λ_0, defined by Keldysh [17], as the linear space span by the solutions $u_0, u_1, \cdots, u_\ell$ of the following system of equations ($\ell \in \mathbb{N}$),

$$L(\lambda_0)u_0 = 0 \tag{2.13}$$

$$L(\lambda_0)u_\ell + \frac{dL(\lambda_0)}{d\lambda}u_{\ell-1} + \cdots + \frac{d^\ell L(\lambda_0)}{d\lambda^\ell}u_0 = 0. \tag{2.14}$$

The following result is proved in [17, 25]

Lemma 2.1 *If the linear space* $\bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}_\lambda[A_L]$ *is dense in* $\mathcal{H}$ *then* $\bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}_\lambda[L]$ *is dense in* $\mathcal{K}$.

This lemma is useful because we can apply known results for non self-adjoint operator to our non linear eigenvalues problem.

Theorem 2.2 (Dunford-Schwartz, [8]) *Assume that there exist rays Ξ_j, $1 \leq j \leq J$ in the complex plane $\mathbb{C}$, starting from the origin, such that the angle between two consecutive rays is strictly smaller than π/p and there exist positive real numbers ρ, R, such that*

$$\|L(\lambda)\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(|\lambda|^\rho), \text{ for } |\lambda| \geq R, \ \lambda \in \cup_{1\leq j\leq J}\Xi_j. \tag{2.15}$$

Then $\bigoplus_{\lambda\in\mathbb{C}} \mathcal{E}_\lambda[L]$ *is dense in* $\mathcal{H}$.

Following [25], we can apply the above functional analysis result to quadratic pencils $L_m(\lambda) = D_x^2 + (x^m - \lambda)^2$, for $m \in \mathbb{N}$, m even, $m \geq 2$.

Proposition 2.3 $L_m(\lambda)$ *has a complete system of generalized eigenfunctions in* $L^2(\mathbb{R})$. *These functions are in the Schwartz space* $\mathcal{S}(\mathbb{R})$.

Sketch of Proof:
We have here $L(\lambda) = D_x^2 + x^{2m} - 2\lambda x^m + \lambda^2$ and $H_0^{-1/2}$ is in $\mathcal{C}^p$ for every $p > \dfrac{m+1}{m}$. So the angle condition of the Dunford-Schwartz Theorem is here $\theta < \dfrac{m\pi}{m+1}$. Let us denote $\Xi_\alpha = \{re^{i\alpha}, r \geq 0\}$. Then proposition 2.3 is a consequence of the following result:

Lemma 2.4 $L(\lambda)^{-1}$ *exists on* Ξ_0 *and on* Ξ_α *for every* $\alpha \in]\pi/2, \pi] \cup [-\pi, -\pi/2[$ *and satisfies the estimates*

$$\|L(\lambda)^{-1}\| = \mathcal{O}\left(|\lambda|^{1/2m}\right) \text{ for } \lambda \in \Xi_0, \tag{2.16}$$

$$\|L(\lambda)^{-1}\| = \mathcal{O}\left(|\lambda|^{-2}\right) \text{ for } \lambda \in \Xi_\alpha, \ \alpha \in]\pi/2, \pi] \cup [-\pi, -\pi/2[. \tag{2.17}$$

Sketch of Proof of the Lemma 2.4
The estimate on Ξ_0 comes from a direct computation. Estimate on Ξ_α can be proved by pseudodifferential technics (cf [25]) or also by direct estimates. ∎

The angle condition is more difficult to check in higher dimension n. The reason is the following. For $L_P(\lambda) = -\triangle + (P(x) - \lambda)^2$ we have $H_0 = -\triangle + P^2(x)$, where $P(x)$ is like $|x|^m$. Because of eigenvalue asymptotics (see for exemple [26]) we have $H_0^{-1/2}$ is in $\mathcal{C}^p$ for every $p > \dfrac{n(m+1)}{m}$, this is optimal, so the angle condition to apply the Dunford-Schwartz theorem is $\theta < \dfrac{m\pi}{n(m+1)}$.
Then we need resolvent estimates on closer and closer rays when n increases.
It is the reason why the approach proposed by Chanillo-Helffer-Laptev is very useful. The basic idea is the following. Let us recall the Lidskii Theorem [10]. Let $\mathcal{H}$ be an Hilbert space and T an operator in $\mathcal{H}$ in the Schatten class $\mathcal{C}^1$. Then we have

$$\text{Tr }(T) = \sum_{\lambda\in\text{sp}[T]} \lambda \tag{2.18}$$

where sp$[T]$ is the set of eigenvalues of T, with their multiplicities.
So to prove that $L(\lambda)$ has a non empty spectrum it is sufficient to prove that

$$\text{Tr } (A_L)^{-\ell} \neq 0 \text{ for some } \ell \geq p \tag{2.19}$$

The property (2.19) is the core of the method of [4]. For simplicity let us explain this method in more details in the one dimension case, for L_m $(m \geq 2)$. For the computations it is more convenient to conjugate A_L by a unitary operator such that we get an operator $\tilde{A}_L$ acting in the Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. For $L = L_m$ we denote $A_m = \tilde{A}_{L_m}$ and we easily have

$$A_m = \begin{pmatrix} 0 & H_0^{1/2} \\ -H_0^{1/2} & -H_1 \end{pmatrix}, \tag{2.20}$$

where $H_0 = D_x^2 + x^{2m}$ and $H_1 = -2x^m$.
So we have

$$A_m^{-2} = \begin{pmatrix} B^2 - C^2 & -BC \\ CB & -C^2 \end{pmatrix} \tag{2.21}$$

where $C = H_0^{-1/2}$, $B = -H_0^{-1/2} H_1 H_0^{-1/2}$. Hence we have

$$\text{Tr } \left(A_m^{-2}\right) = \text{ Tr } \left(B^2 - 2C^2\right). \tag{2.22}$$

By scaling we have, for every $\gamma > 0$,

$$\text{Tr } \left(D_x^2 + \gamma x^{2m}\right)^{-1} = \gamma^{1/(m+1)} \text{ Tr } \left(D_x^2 + \gamma x^{2m}\right)^{-1}. \tag{2.23}$$

Then, computing the derivative at $\gamma = 1$ on each side, using the Cauchy-Schwarz inequality, $|\text{ Tr } (M^2)| \leq \text{ Tr } (MM^*)$ for $M = H_0^{-1} H_1$, we get

$$\text{Tr } (B^2) \leq \frac{4}{m+1} \text{ Tr } (C^2). \tag{2.24}$$

So the conclusion follows, for $m \geq 2$, with

$$\text{Tr } \left(A_m^{-2}\right) \leq \left(\frac{4}{m+1} - 2\right) \text{ Tr } \left(H_0^{-1}\right) < 0. \tag{2.25}$$

In [4] the authors have used the same method for $n = 2, 3$, by computing $\text{Tr } \left(A_L^{-4}\right)$. The proof is much more tricky and give the expected conclusion for $m \geq 6$. We shall see in the next section that by adding some semiclassical ingredients in the Chanillo-Helffer-Laptev approach as we did in [15], it is possible to improve their result in the even dimension case.

3 A Semiclassical Analysis of the problem

Let us consider first the quadratic pencil L_P where P is a positive-elliptic polynomial of degree $m \geq 2$ in $\mathbb{R}^n$, $n \geq 2$. For simplicity we assume that P is homogeneous. By the scaling transformation $x = \tau y$ with $\hbar = \tau^{1-m}$ and $\mu = \dfrac{\lambda}{\tau^m} + 1$

we can see that $L_P(\lambda)$ is unitary equivalent to the semiclassical Hamiltonian $\hat{H}(\mu)$ where

$$\hat{H}(\mu) = -\hbar^2 \triangle_y + (P(y) + 1 - \mu)^2 \tag{3.26}$$

So $\hat{H}(\mu)$ is the $\hbar$-Weyl operator with the symbol $H(\mu, y, \eta) = \eta^2 + (P(y) + 1 - \mu)^2$. For semiclassical analysis tools and $\hbar$- Weyl quantization we refer to [27] for scalar symbols and to [18] for matrix symbols. Here we use the notation $\hat{H}$ for the $\hbar$-Weyl quantization of the symbol H.
As above, to the semiclassical quadratic pencil $\hat{H}(\mu)$ is associated a non self-adjoint matricial operator $\hat{A}_P$ in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,

$$\hat{A}_P = \begin{pmatrix} 0 & \hat{H}_0^{1/2} \\ -\hat{H}_0^{1/2} & -\hat{H}_1 \end{pmatrix} \tag{3.27}$$

where $\hat{H}_0 = -\hbar^2 \triangle_y + (P(y) + 1)^2$. The $\hbar$-symbol $A_P(y, \eta)$ of $\hat{A}_P$ has two eigenvalues

$$\mu_\pm(y, \eta) = P(y) + 1 \pm i|\eta| \tag{3.28}$$

So using standard methods in semiclassical and spectral analysis adapted from R. Seeley [29] and [26], the authors get the following result

Theorem 3.1 *For every real number* $s < -\dfrac{n(m+1)}{m}$*, in the semiclassical regime* $\hbar \searrow 0$*, we have,*

$$\mathrm{Tr}\left(\hat{A}_P^s\right) \asymp \sum_{j \geq 0} c_{j,s} \hbar^{j-n} \tag{3.29}$$

with

$$\begin{aligned} c_{0,s} &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \Re\left(\mu_+(x, \xi)\right)^s dx d\xi, && (3.30) \\ c_{1,s} &= 0, && (3.31) \\ c_{2,s} &= << \text{something computable} >> && (3.32) \end{aligned}$$

So using Lidskii Theorem to prove that $L_P(\lambda)$ has a non empty spectrum, it is enough to prove that there exist $s < -\dfrac{n(m+1)}{m}$ and $j \in \mathbb{N}$ such that $c_{j,s} \neq 0$. The main result in [15] is that this can be checked if n is even.

Lemma 3.2 *If* $m \geq 2$ *and* n *is even then for every* $s < -\dfrac{n(m+1)}{m}$*,* $c_{0,s} \neq 0$*.*

Proof
We have

$$\int_{\mathbb{R}^{2n}} \mu_+(x, \xi)^s dx d\xi = \int_{\mathbb{R}^n} (P(x) + 1)^{s+n} \int_{\mathbb{R}^n} (1 + i|\eta|)^s d\eta \tag{3.33}$$

So we have to compute $f_s(i)$ where

$$f_s(\alpha) = \int_{\mathbb{R}^n} (1+\alpha|\eta|)^s d\eta. \tag{3.34}$$

By scaling and analytic extension we easily get $f_s(\alpha) = \alpha^{-n} f(1)$ for $\alpha \in \mathbb{C}$, $\alpha \notin]-\infty, 0]$. Then $\Re f_s(i) = \cos\left(\frac{n\pi}{2}\right) f_s(1)$ and $f_s(1) > 0$. So the conclusion follows. ∎

Remark 3.3 *We conjecture that for n odd, $n \geq 3$, there always exists $j \geq 1$ and $s < -\frac{n(m+1)}{m}$ such that $c_{j,s} \neq 0$. To check this it is necessary to perform algebraic computations which are under investigation in [1]. A similar method was used in [24] to prove existence of resonances for matrix Schrödinger operators.*

The above lemma gives much more than existence of at least one eigenstate. We shall see that there exists an infinite number of eigenstates and give an estimate of their density. It is known that existence of resonances can be proved as a consequence of a trace formula [31, 24]. The same method can be applied here, in an easier way, to estimate the number of eigenvalues. Let us write the generalized eingenvalues $\{\lambda_j\}_{j\geq 1}$ of L_P by increasing order of their modulus, repeated according their mutiplicities. Let us introduce the counting function

$$N_L(r) = \#\{j \geq 1,\ |\lambda_j| \leq r\}. \tag{3.35}$$

Proposition 3.4 *Under the assumption of Theorem 3.1 and Lemma 3.2, there exists a constant $C > 0$ such that for every $r \geq 1$ we have*

$$\frac{r^{n(m+1)/m}}{C} \leq N_L(r) \leq C r^{n(m+1)/m} \tag{3.36}$$

Proof

Let us denote $\theta = \frac{n(m+1)}{m}$ and fix an integer $k > \theta$. By a change of parameter, it results from Theorem 3.1 that we have

$$\sum_{j\geq 1} (t+\lambda_j)^{-k} = c_0 t^{\theta-k} + \mathcal{O}\left(t^{\theta-k-(m+1)/m}\right) \tag{3.37}$$

where $c_0 \neq 0$.
To find an upper bound we apply the Weyl-Ky Fan inequality [10]

$$\sum_{j\geq 1} (t+|\lambda_j|)^{-k} \leq \sum_{j\geq 1} (t+s_j)^{-k} \tag{3.38}$$

where $\{s_j\}_{j\geq 1}$ is the set of eigenvalues of $(A_L^* A_L)^{1/2}$. But we also have a trace formula for $(A_L^* A_L)^{1/2}$, so we have

$$\sum_{j\geq 1} (t+|\lambda_j|)^{-k} = \mathcal{O}(t^{\theta-k}) \tag{3.39}$$

which gives easily (taking $t = r$) the upper bound:

$$N_L(r) \leq C r^{n(m+1)/m} \tag{3.40}$$

For the lower bound, we first remark that for $t \geq t_0$, t_0 large enough, we have $|t + \lambda_j| \geq \dfrac{|\lambda_j| + t}{8}$. This is true because, for every $\varepsilon > 0$, we have $\arg(\lambda_j) \in [-\pi/2 - \varepsilon, \pi/2 + \varepsilon]$ for j large enough [25]. Then there exists $c_1 > 0$ such that

$$\sum_{j \geq 1} (t + |\lambda_j|)^{-k} \geq c_1 t^{\theta - k} \tag{3.41}$$

It is convenient to write the above inequality with Stieljès integral

$$\int_0^\infty (t + r)^{-k} dN_L(r) \geq c_1 t^{\theta - k} \tag{3.42}$$

Let $\gamma > 1$ be a large constant to be chosen later. We have, using the upper bound and an integration by part,

$$\int_{\gamma t}^\infty (t + r)^{-k} dN_L(r) \leq kC \left(\int_\gamma^\infty (1 + u)^{-k-1} u^\theta du \right) t^{\theta - k} \tag{3.43}$$

So we can choose γ large enough such that

$$\int_0^{\gamma t} (t + r)^{-k} dN_L(r) \geq \frac{c_1}{2} t^{\theta - k} \tag{3.44}$$

which easily gives the lower bound:

$$N_L(\gamma t) \geq \frac{c_1}{2} t^\theta. \tag{3.45}$$

∎

In the paper [15] we also consider the following quadratic pencils

$$L_{P,Q}(\lambda) = -\triangle + (P(x) - \lambda)^2 + Q(x)^2 \tag{3.46}$$

where we assume that P, Q are homogeneous polynomials of degree $m \geq 2$, $P \geq 0$, $P^2 + Q^2$ is elliptic and Q **is not identically 0** if n is odd. Then we can extend Proposition 3.4 to the corresponding counting function $N_{L_{P,Q}}(r)$.

Proposition 3.5 *The following estimates are satisfied*

$$\frac{1}{C} r^{n(m+1)/m} \leq N_{L_{P,Q}}(r) \leq C r^{n(m+1)/m}, \tag{3.47}$$

if one of the following condition is satisfied
(i) $m \geq 2$, $n = 1, 3$,
(ii) $n = 2$, $m \geq 3$,
(iii) $n = 2$, $m = 3$ *and the following technical condition*

$$\frac{(P+1)^2}{(P+1)^2 + Q^2} - \frac{3 - \sqrt{2}}{4} \tag{3.48}$$

is everywhere non negative, or everywhere non positive, on $\mathbb{R}^n$.

Sketch of Proof (see [15] for details).
We prove that the leading coefficient $c_{0,s}$ in the trace formula is not 0.

Remark 3.6 *The same results holds if P, Q are polynomials such that the assumptions are satisfied for their homogeneous part of degree m. Furthermore we can replace the homogeneity condition by a quasi-homogeneity condition like in the example $P(x_1, x_2) = x_1^2 + x_2^4$.*

4 Localization of some eigenvalues

We revisit here an example coming from a question that G. Métivier asked me twenty years ago. Let us consider the following quadratic pencil depending on a large parameter $\eta > 0$.

$$L_\eta(\lambda) = -\triangle_x + (P(x) - \lambda)^2 + \eta^2. \tag{4.49}$$

Assuming as before that P is an elliptic-positive homogeneous polynomial of degree $m \geq 2$. $L_\eta(\lambda)$ is conjugate to a semiclassical operator where $\hbar = \eta^{-(m+1)/m}$ and $\mu = \frac{\lambda}{\eta}$. More precisely, we have

$$L_\eta(\lambda) = \eta^2 \left(-\hbar^2 \triangle_y + (P(y) - \mu)^2 + 1\right). \tag{4.50}$$

The analogue of the trace Theorem 3.1 gives here for the leading coefficient

$$c_{0,s} = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} 2\Re \left(P(x) + i\sqrt{1+\xi^2}\right)^s \tag{4.51}$$

A direct computation, as we did in the proof of Lemma 3.2, gives

$$c_{s,o} = \gamma_s \cos\left(\frac{(n+sm)\pi}{2m}\right), \tag{4.52}$$

where $\gamma_s \neq 0$. So, for every $n \geq 1$, there exists $s < -\frac{n(m+1)}{m}$, such that $c_{0,s} \neq 0$. So we get

Proposition 4.1 *There exists $\eta_0 > 0$, large enough, such that, for every $\eta \geq \eta_0$, $L_\eta(\lambda)$ has a non empty spectrum.*

Let $\{\lambda_j(\eta)\}_{j\geq 1}$ be the sequence of eigenvalues of L_η, ordered by increasing modulus. So $\lambda_1(\eta)$ is a generalized eigenvalue of minimal modulus. The question of G. Métivier was about the behaviour of $\lambda_1(\eta)$ as $\eta \to +\infty$. The answer is

Proposition 4.2 *[28]*

$$\lim_{\eta \nearrow +\infty} \frac{\lambda_1(\eta)}{\eta} = \pm i \tag{4.53}$$

(We have $\pm i$ because λ is eigenvalue if and only if $\bar{\lambda}$ is eigenvalue).

Sketch of the Proof
On the semiclassical side we have to prove

$$\lim_{\hbar\searrow 0}\mu_0(\hbar)=\pm i \tag{4.54}$$

where $\mu_0(\hbar)$ is an eigenvalue of minimal modulus of $\hat{L}(\mu)=-\hbar^2\triangle_y+(P(y)-\mu)^2+1$. If $\hat{A}$ is the non self-adjoint matricial operator associated with $\hat{L}(\mu)$, let us introduce the family of complex variable functions

$$F_\hbar(z)=(2\pi\hbar)^n\ \mathrm{Tr}\left(\hat{A}-z\right)^{-N} \tag{4.55}$$

where N is chosen large enough. Let us denote $\Omega_r=\{z\in\mathbb{C},\ \ |z|<r\}$. The $\hbar$ principal symbol of $\hat{A}$ has the eigenvalues $\mu_\pm(x,\xi)=P(x)\pm i\sqrt{1+\xi^2}$. So, by standard parametrix construction, for every $\alpha>0$, there exits $\varepsilon>0$ such that for $\hbar<\varepsilon$, $F_\hbar(z)$ is holomorphic in the set $B_\alpha=\{z\in\mathbb{C},\ \ \Re z<1-\alpha\}\cup\{z\in\mathbb{C},\ \ |\Im z|<1-\alpha\}$. In particular $F_\hbar(z)$ is holomorphic in $\Omega_r:=\{z\in\mathbb{C},|z|z<r\}$ for $r<1$, and we have

$$\lim_{\hbar\searrow 0}F_\hbar(z)=F_{c\ell}(z) \tag{4.56}$$

where

$$F_{c\ell}(z)=\int_{\mathbb{R}^{2n}}[(\mu_+(x,\xi)-z)^{-N}+(\mu_-(x,\xi)-z)^{-N}]dxd\xi \tag{4.57}$$

Now we shall complete the proof of the proposition by contradiction. Assume that there exists a sequence $\hbar_j$, $\lim_{j\to+\infty}\hbar_j=0$, such that $\hat{A}$ has no eigenvalues in a neighborhood of $\pm i$. It follows that $F_j(z):=F_{\hbar_j}(z)$, is holomorphic in a disc Ω_{r_1} for some $r_1>1$. Using Weyl-Ky Fan inequalities [10], we can see that F_j is a uniformly bounded sequence of holomorphic functions in Ω_{r_1}. Using Montel's Theorem, by taking a subsequence, we can assume that $\lim_{j\to+\infty}F_j=F_\infty$ exists and is holomorphic in Ω_{r_1}. Then $F_{c\ell}$ has an holomorphic extension in Ω_{r_1}. Hence we get a contradiction by computing

$$\lim_{s<1,s\to 1}|F_{c\ell}(is)|=+\infty \tag{4.58}$$

∎

We can apply the above result to improve a little bit a result of [15]

Corollary 4.3 *Let us assume that $P(x')$ and $Q(x'')$ are elliptic polynomials in $\mathbb{R}^{n'}$ of degree m', respectively in $\mathbb{R}^{n''}$ of degree m''. Then for every n', n'', $m'\geq 2$, the quadratic pencil $L(\lambda)=-\triangle_{x',x''}+Q^2(x'')+(P(x')-\lambda)^2$ has an infinite number of eigenvalues*

Proof
Let us remark that the self-adjoint operator $K:=-\triangle_{x''}+Q^2(x'')$ has a basis, $\{\varphi_j\}$, of eigenfunctions in $L^2(\mathbb{R}^{n'})$, with eigenvalues η_j such that $\lim_{j\to+\infty}\eta_j=+\infty$. So the corollary is a consequence of Proposition 4.2 ∎

We can get also estimates on the number of eigenvalues for the pencil $L_\eta(\lambda)$, in every dimension, for η large enough. Let us introduce $N_\eta(R) = \#\{j \geq 1,\ |\lambda_j(\eta)| \leq R\}$.

Proposition 4.4 *There exist $C > 1$, $R_0 > 0$, $\eta_0 > 0$ such that, for $R \geq R_0$, $\eta \geq \eta_0$ we have*

$$\frac{(\eta R)^{n(m+1)/m}}{C} \leq N_\eta(R) \leq C(\eta R)^{n(m+1)/m}. \tag{4.59}$$

Sketch of Proof
We follow the same method as for proving Proposition 3.4. For convenience, we work in the semiclassical side. Le us denote $\tilde{N}_\hbar(R) = \#\{j \geq 1,\ |\mu_j(\hbar)| \leq R\}$.
We prove first the upper bound. If $s_j(\hbar)$ denotes the eigenvalues of $(\hat{A}^*\hat{A})^{1/2}$, spectral and semiclassical analysis [27, 18] gives, that for some constants $K > 0$, $\varepsilon_0 > 0$, we have

$$\sum_{j\geq 1}(s_j(\hbar) + t)^{-k} \leq K\hbar^{-n}t^{\theta-k} \tag{4.60}$$

for $\hbar \leq \varepsilon_0$, $t \geq 1$. Then using Weyl-Ky Fan's inequality, we get as in Proposition3.4,

$$\tilde{N}_\hbar(R) \leq K\hbar^{-n}R^\theta. \tag{4.61}$$

For the lower bound, we first remark that from the trace formula (Theorem 3.1) we get for some $c_0 > 0$,

$$\sum_{j\geq 1}(|\mu_j(\hbar)| + t)^{-k} \geq c_0\hbar^{-n}t^{\theta-k} \tag{4.62}$$

So, for $\hbar$ small enough, we have

$$\int_{1/2}^{+\infty}(r + t)^{-k}d\tilde{N}_\hbar(r) \geq c_0\hbar^{-n}t^{\theta-k}. \tag{4.63}$$

Using the upper bound, we can choose $\gamma > 0$ large enough, such that, for $R \geq R_0$, we have

$$\int_{\gamma R}^{+\infty} r^{-k}d\tilde{N}_\hbar(r) \leq \frac{c_0}{2}\hbar^{-n}R^{\theta-k}, \tag{4.64}$$

which gives easily the lower bound

$$\tilde{N}_\hbar(\gamma R) \geq 2^{-k-1}c_0\hbar^{-n}R^\theta. \tag{4.65}$$

∎

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