Fractional Moment Estimates for Random Unitary Operators

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Abstract

We consider unitary analogs of $d-$dimensional Anderson models on $l^2(\mathbb{Z}^d)$ defined by the product $U_\omega = D_\omega S$ where S is a deterministic unitary and D_ω is a diagonal matrix of i.i.d. random phases. The operator S is an absolutely continuous band matrix which depends on parameters controlling the size of its off-diagonal elements. We adapt the method of Aizenman-Molchanov to get exponential estimates on fractional moments of the matrix elements of $U_\omega(U_\omega - z)^{-1}$, provided the distribution of phases is absolutely continuous and the parameters correspond to small off-diagonal elements of S. Such estimates imply almost sure localization for U_ω.

1 Introduction

Unitary operators displaying a band structure with respect to a distinguished basis appear in the description of the long time properties of certain quantum dynamical systems. For example, such operators on $l^2(\mathbb{N})$ are used to model the dynamics of an electron in a ring threaded by a time dependent magnetic flux. In some regime of the physical parameters, certain phases of the matrix elements can be considered as random variables. These models are useful for numerical investigations. See [BB], [BHJ] and references therein for details on the model and more on quantum dynamical systems.

Unitary operators with a similar band structure appear naturally in the study of orthogonal polynomials on the unit circle S^1 with respect to a measure $d\mu$ on the torus $\mathbb{T}$, see [S1]. Indeed, it is shown in [CMV] that multiplication by $e^{i\alpha} \in S^1$ on $L^2(\mathbb{T}, d\mu(\alpha))$ expressed in a certain basis of orthonormal polynomials is represented by such a band matrix in $l^2(\mathbb{N})$. This construction is simpler than the earlier Hessenberg form of the matrix representation of this unitary operator provided in [GT]. The spectral analysis of the unitary operator therefore yields informations on the polynomials. Considering some phases as random amounts to considering certain types of random polynomials.

The spectral analysis of a certain set of deterministic and random unitary operators with a band structure is undertaken in [BHJ] and [J]. This set contains the examples mentionned above as particular cases. In the random cases studied in these two papers, the operators considered consist in matrices on $l^2(\mathbb{Z})$ (which are unitarily equivalent to matrices) of the following form: $U_\omega = D_\omega S$ where S is a deterministic unitary and D_ω is a diagonal matrix of random phases, see [J]. The operator S is an absolutely continuous band matrix which depends on a parameter $t \in]0,1[$ which controls the size of its off-diagonal elements, see Section 2. When the phases are i.i.d random variables, typical results obtained for discrete one-dimensional random Schrödinger operators are shown in [BHJ] and [J] to hold in the unitary setting as well. For instance, the availability of a transfer matrix formalism to express generalized eigenvectors allows to introduce a Lyapunov exponent, to prove a unitary version of Ishii-Pastur Theorem, and get absence of absolutely continuous spectrum [BHJ]. A density of states can be introduced and a Thouless formula is proven in [J]. Related analyses in the framework of orthogonal polynomials on the unit circle are provided in [GT], [T], [S1].

In the present paper, we introduce a natural generalization of such unitary operators to higher dimensions, i.e. to $l^2(\mathbb{Z}^d)$, $d \geq 1$, in analogy with the self-adjoint Anderson model. The construction is motivated by the structure of U_ω given as a product of a diagonal random operator D_ω times a deterministic unitary S. This structure is a natural transposition to the unitary setting of that of the Anderson model consisting in the sum of a diagonal random potential and the deterministic discrete Laplacian. The extension is straightforward and consists in matrices U_ω of the form $D_\omega S$, acting on $l^2(\mathbb{Z}^d)$, where the infinite matrices D_ω and S have similar properties with respect to the canonical basis of $l^2(\mathbb{Z}^d)$, see Section 2. In particular, we assume the phases in the diagonal of D_ω are i.i.d. with an absolutely continuous distribution, and the operator S depends now on a set of d parameters $(t_1, t_2, \cdots, t_d)$ which control the size of its off-diagonal elements.

Once defined, these random operators call for an analysis of their spectral properties. In the self-adjoint case, the localization properties of the d-dimensional Anderson model can be conveniently proven for large disorder by means of the fractional moment method of Aizenman and Molchanov [AM] and the Simon-Wolff criterion [SW]. Our main result, Theorem 2.1 below, is an exponential estimate on the fractional moments of the matrix elements of $U_\omega(U_\omega - z)^{-1}$, uniform in z, obtained by an adaptation to the unitary setting of the Aizenman-Molchanov method. Our estimate holds for a range of parameters $(t_1, t_2, \dots, t_d)$ such that the off-diagonal elements of S are small enough. This last condition is the equivalent in our setting of the large disorder assumption made in the self-adjoint case. Then we apply the unitary version of the Simon-Wolff criterion proven by Combescure in [C] to derive localization for U_ω in Corollary 2.1, for the same range of parameters.

2 The Model and Main Result

We denote by $|k\rangle = |k_1, k_2, \cdots, k_d\rangle$ the unit vector at site $k \in \mathbb{Z}^d$, so that $\{|k\rangle\}_{k\in\mathbb{Z}^d}$ form an orthonormal basis of $L^2(\mathbb{Z}^d)$. We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is identified with $\{\mathbb{T}^{\mathbb{Z}^d}\}$, $\mathbb{T}$ being the torus, and $\mathbb{P} = \otimes_{k\in\mathbb{Z}^d}\mathbb{P}_k$, where $\mathbb{P}_k = \mathbb{P}_0$ for any $k \in \mathbb{Z}^d$ is a probability distributions on $\mathbb{T}$, and $\mathcal{F}$ the σ-algebra generated by the cylinders. We introduce a set of random vectors on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_k : \Omega \to \mathbb{T}, \quad \text{s.t.} \quad \theta_k(\omega) = \omega_k, \quad k \in \mathbb{Z}^d. \tag{2.1}$$

These random vectors $\{\theta_k\}_{k\in\mathbb{Z}^d}$ are thus i.i.d on $\mathbb{T}$.

In the one dimensional case, $d = 1$, we consider unitary operators of the form

$$U_\omega = D_\omega S_0, \quad \text{with } D_\omega = \text{ diag } \{e^{-i\theta_k(\omega)}\} \tag{2.2}$$

and

$$S_0 = \begin{pmatrix} \ddots & rt & -t^2 & & & \\ & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -tr & r^2 & -rt & \\ & & & rt & r^2 & \\ & & & -t^2 & -tr & \ddots \end{pmatrix}, \tag{2.3}$$

where the translation along the diagonal is fixed by $\langle \varphi_{2k-2} | S_0 \varphi_{2k} \rangle = -t^2$, $k \in \mathbb{Z}$. The parameters t and r are linked by $r^2 + t^2 = 1$ to ensure unitarity. We shall sometimes write $S_0(t)$ to emphasize this dependence. The spectrum of $S_0(t)$ is purely absolutely continuous and consists in the set

$$\sigma(S_0(t)) = \Sigma_0(t) = \{e^{\pm i \arccos(1 - t^2(1+\cos(y)))}, y \in \mathbb{T}\}. \tag{2.4}$$

For this and other properties of S_0, relations between U_ω with the physical model alluded to in Section 1 or links with orthogonal polynomials, see [J]. Note that the band structure (2.3) is the simplest one a unitary operator can take without being trivial from the point of view of its spectrum, [BHJ].

To deal with d-dimensional operators, we introduce the following natural generalization of (2.2) to $l^2(\mathbb{Z}^d)$. We consider the unitary

$$U_\omega = D_\omega S \quad \text{on} \quad l^2(\mathbb{Z}^d), \tag{2.5}$$

where D_ω is diagonal again

$$D_\omega |k\rangle = e^{-i\theta_k(\omega)} |k\rangle \tag{2.6}$$

whereas the deterministic part is defined by

$$S = S_1 \otimes \cdots \otimes S_d. \tag{2.7}$$

That is, we view $l^2(\mathbb{Z}^d)$ as $\otimes_{j=1}^d l^2(\mathbb{Z})$ so that $|k\rangle \simeq |k_1\rangle \otimes \cdots \otimes |k_d\rangle$ and S_j acts on $|k_j\rangle$ as S_0 in (2.3). We shall identify S_j with $\mathbb{I} \otimes \ldots \mathbb{I} \otimes S_j \otimes \mathbb{I} \cdots \otimes \mathbb{I}$. A natural symmetric choice consists in taking the same parameter t for each unitary $S_j(t)$ appearing in the definition of $S = S(t)$. But we can naturally consider non-symmetric cases characterized by a set of parameters $t = (t_1, t_2, \cdots, t_d) \in]0,1[^d$ to construct the unitary operator $S(t) = S_1(t_1) S_2(t_2) \cdots S_d(t_d)$. Note that one gets rightaway that $S(t)$ is purely absolutely continuous and

$$\sigma(S(t)) = \Sigma_0(t_1) \times \Sigma_0(t_2) \times \cdots \times \Sigma_0(t_d). \tag{2.8}$$

Moreover, with the norm $|x| = \max_{j=1,\cdots,d} |x_j|$, $x \in \mathbb{R}^d$, we have the band structure

$$\langle k | S j \rangle = 0 \quad \text{if} \quad |j - k| > 2. \tag{2.9}$$

Remarks:
i) In this definition, S plays the role of the free d-dimensional Laplacian in the self-adjoint case. Therefore, in the same way the Laplacian can be written as a sum of commuting one-dimensional Laplacians, S is defined as a product of commuting unitary operators.
ii) Our construction of S gives a band structure to U_ω. However, our results do not require such a structure, see below.
iii) Note that as $|t| \to 0$, $S(t)$ tends to the identity operator.

Our main result is an estimate on the fractional moments of, essentially, the matrix elements of the resolvent of U_ω.

Theorem 2.1 *Let U_ω be defined by (2.5, 2.6, 2.7). Assume that $\{\theta_k(\omega)\}_{k\in\mathbb{Z}^d}$ are i.i.d. and distributed according to the probability measure $d\nu(\theta)=\tau(\theta)d\theta$, where $\tau\in L^\infty(\mathbb{T})$. Let $s\in]0,1[$. There exists $t_0(s)>0$ small enough and $0<K(s)<\infty$ such that if $|t|<t_0(s)$, there exists $\gamma(s,t)>0$ so that for any $j,k\in\mathbb{Z}^d$ and for any $z\in\mathbb{C}$,*

$$\mathbb{E}(|\langle j|U_\omega(U_\omega-z)^{-1}k\rangle|^s)\leq K(s)e^{-\gamma(s,t)|j-k|}. \tag{2.10}$$

Remarks:
i) The Theorem is true for more general deterministic unitary operators S than (2.7). The only requirement is that for some $\gamma(s)>0$,

$$\sup_k\sum_{j\neq k}|\langle Sk|j\rangle|^s e^{\gamma|k-j|}<C_\nu^{(1)}(s)\inf_k|\langle Sk|k\rangle|^s, \tag{2.11}$$

where $C_\nu^{(1)}(s)$ is defined in (4.9) and depends on s and on ν only. This condition corresponds to the large disorder assumption in the self-adjoint case.
ii) The random variables $\theta_k(\omega)$ need not be independent, and their distribution can be more general, see [AM]. However, we stick to the present hypotheses for simplicity.

Corollary 2.1 *Consider $U_\omega=D_\omega S(t)$ under the hypotheses of Theorem 2.1. Then, if $|t|<t_0(s)$,*

$$\sigma(U_\omega)\ \ \textit{is pure point almost surely.}$$

Note:
As this paper was being completed, the preprint [S2] appeared. It announces that estimates of the type (2.10) are proven by Stoiciu in the realm of orthogonal polynomials on the unit circle and proves that dynamical localization is a consequence of these estimates in this set up.

The rest of the paper is organized as follows. The next Section describes the effect of changing a phase at one site in terms of rank one perturbations in order to derive formulas for later use. Then we prove Theorem 2.1 along the lines of [AM], [AG] in Section 4. The Corollary on localization is proven in Section 5. An Appendix containing some technical material closes the paper.

3 Rank One Perturbations

By construction, the variation of a random phase at one site is described by a rank one perturbation. As randomness plays no particular role here, we drop the ω's in the notation.

Let $j\in\mathbb{Z}^d$ be fixed. We define $\hat{D}$ by taking $\theta_j=0$ in the definition of D:

$$\hat{D}=e^{i\theta_j|j\rangle\langle j|}D=D+|j\rangle\langle j|(1-e^{-i\theta_j})\equiv D+|j\rangle\langle j|\eta_j,\quad\text{with}\quad\eta_j=1-e^{-i\theta_j}, \tag{3.1}$$

so that, with the obvious notations,

$$\hat{U}=\hat{D}S=e^{i\theta_j|j\rangle\langle j|}U=U+|j\rangle\langle j|S\eta_j. \tag{3.2}$$

Let $z\notin S^1$. By the first resolvent identity, we have

$$\begin{aligned}(\hat{U}-z)^{-1}-(U-z)^{-1} &= -(\hat{U}-z)^{-1}|j\rangle\langle j|\eta_j S(U-z)^{-1}\\ &= -(U-z)^{-1}|j\rangle\langle j|\eta_j S(\hat{U}-z)^{-1}.\end{aligned} \tag{3.3}$$

Therefore, $F(z) = S(U-z)^{-1}$ and $\hat{F}(z) = S(\hat{U}-z)^{-1}$ satisfy

$$\hat{F}(z) - F(z) = -\eta_j \hat{F}(z)|j\rangle\langle j|F(z) = -\eta_j F(z)|j\rangle\langle j|\hat{F}(z). \tag{3.4}$$

It is readily checked that this implies

$$F(z) = \hat{F}(z) + \frac{\eta_j}{1-\eta_j \langle j|\hat{F}(z)j\rangle}\hat{F}(z)|j\rangle\langle j|\hat{F}(z). \tag{3.5}$$

Hence, with the notation $F(j,k;z) = \langle j|F(z)k\rangle$ and similarly with $\hat{F}(z)$, for any $j \in \mathbb{Z}^d$,

$$F(j,k;z) = \frac{\hat{F}(j,k;z)}{1-\eta_j \hat{F}(j,j;z)}. \tag{3.6}$$

We emphasize that in the relation above, the operator $\hat{F}(z)$ depends on j fixed. Note also that $F(j,k;z) = e^{i\theta_j}\langle j|U(U-z)^{-1}k\rangle$, so that it is equivalent to deal with $F(z)$ or $U(U-z)^{-1}$ as far as the modulus of matrix elements is concerned. We choose to deal with $F(z)$ because of the simple relation (3.4).

4 Estimates on Fractional Moments

The Aizenman-Molchanov approach of localization for self-adjoint operators consists in deriving exponential estimates on the expectation of fractional powers of the matrix elements of the resolvent that are uniform in the spectral parameter [AM]. We conduct a similar analysis on the matrix elements of $F(z)$ to prove Theorem 2.1, following the original strategy and [AG].

We restore the dependence in the disorder ω in the notation at this point and we derive the equation satisfied by the matrix elements $F_\omega(k,j;z)$, $z \notin S^1$. We have

$$\mathbb{I} = (U_\omega - z)(U_\omega - z)^{-1} = (U_\omega - z)S^* F_\omega(z) = (D_\omega - zS^*)F_\omega(z). \tag{4.1}$$

Taking matrix elements, this yields

$$\begin{aligned} \delta_{jk} &= \langle k|(D_\omega - zS^*)F_\omega(z)j\rangle = e^{-i\theta_k(\omega)}\langle k|F_\omega(z)j\rangle - z\langle Sk|F_\omega(z)j\rangle \\ &= e^{-i\theta_k(\omega)}F_\omega(k,j;z) - z\sum_{l\in\mathbb{Z}^d}\langle Sk|l\rangle F_\omega(l,j;z). \end{aligned} \tag{4.2}$$

The diagonal elements of $S = S(t)$ are constant and given by

$$\langle Sk|k\rangle = (1-t_1^2)(1-t_2^2)\cdots(1-t_d^2) = r_1^2 r_2^2 \cdots r_d^2 \equiv \rho_d(t). \tag{4.3}$$

Separating the index $l = k$ from the other l's we get for all $j \neq k$ and $0 \neq z \notin S^1$

$$F_\omega(k,j;z)\left(e^{-i\theta_k(\omega)}z^{-1} - \rho_d(t)\right) = \sum_{l\neq k}\langle Sk|l\rangle F_\omega(l,j;z). \tag{4.4}$$

Note that the off-diagonal elements $k \neq l$ satisfy

$$\langle Sk|l\rangle = \Pi_{j=1}^d \langle S_j(t_j)k_j|l_j\rangle = O(|t|), \quad \text{with} \quad |t| = \max(t_1,\cdots,t_d), \tag{4.5}$$

since for one j at least, $k_j \neq l_j$, so that there is at least a factor t_j in the product, whereas

$$\langle Sk|k\rangle = 1 + O(|t|^2) < 1. \tag{4.6}$$

At this point, we mimick [AM] and [AG]. We take $s \in]0,1[$ and try to get estimates on the expectation of $|F_\omega(k,j;z)|^s$. Using $|\sum_j a_j|^s \leq \sum_j |a_j|^s$, we infer from (4.4)

$$|F_\omega(k,j;z)|^s \left| e^{-i\theta_k(\omega)} z^{-1} - \rho_d(t) \right|^s \leq \sum_{l \neq k} |\langle Sk|l\rangle|^s |F_\omega(l,j;z)|^s, \quad j \neq k. \tag{4.7}$$

Taking expectation and making use of the identity (3.6) (with k in place of j), this yield

$$\mathbb{E}\left(\sum_{l \neq k} |\langle Sk|l\rangle|^s |F_\omega(l,j;z)|^s \right) \geq \mathbb{E}\left(\frac{|\hat{F}_\omega(k,j;z)|^s \left| e^{-i\theta_k(\omega)} z^{-1} - \rho_d(t) \right|^s}{|1 - \eta_k \hat{F}_\omega(k,k;z)|^s} \right). \tag{4.8}$$

In order to get estimates uniform in z, we need to get rid of the factor $\left| e^{-i\theta_k(\omega)} z^{-1} - \rho_d(t) \right|^s$. This is done by means of a decoupling lemma similar to the one proven in [AM] for the self-adjoint setting. Recall that $d\nu(\theta)$ defined on $\mathbb{T}$ is the common distribution of the i.i.d. phases $\{\theta_k(\omega)\}_{k \in \mathbb{Z}^d}$. As $\hat{F}_\omega$ is independent of $\theta_k(\omega)$, we shall first average over $\theta_k(\omega)$ and make use of a unitary version of the decoupling Lemma.

Lemma 4.1 (Decoupling Lemma) *Assume $d\nu(\theta) = \tau(\theta)d\theta$, where $\tau \in L^\infty(\mathbb{T})$ is such that $\int_{\mathbb{T}} d\nu(\theta) = 1$. Then, for any $0 < s < 1$, there exists a constant $0 < C_\nu^{(1)}(s) < \infty$ such that for all $\alpha, \beta \in \mathbb{C}$*

$$\int_{\mathbb{T}} d\nu(\theta) \left| \frac{e^{\pm i\theta} - \alpha}{e^{\pm i\theta} - \beta} \right|^s \geq C_\nu^{(1)}(s) \int_{\mathbb{T}} d\nu(\theta) \frac{1}{|e^{\pm i\theta} - \beta|^s}. \tag{4.9}$$

Moreover, there exists $0 < C_\nu^{(2)}(s) < \infty$ such that for all $\beta \in \mathbb{C}$

$$\int_{\mathbb{T}} d\nu(\theta) \frac{1}{|e^{\pm i\theta} - \beta|^s} \leq C_\nu^{(2)}(s). \tag{4.10}$$

Remarks:
i) A proof is provided in Appendix. We only note here that once the estimates hold for $e^{i\theta}$ in the integrand, they hold for $e^{-i\theta}$ by conjugation.
ii) A variant of the above result holds for more general distributions $d\nu(\theta)$ of phases, in the spirit of [AM], and [AG].
iii) As a first application we get the uniform bound

$$\begin{aligned} \mathbb{E}(|F_\omega(k,k;z)|^s) &= \mathbb{E}\left(\left| \frac{\hat{F}_\omega(k,k;z)}{1 - \hat{F}_\omega(k,k;z) + e^{-i\theta_k(\omega)} \hat{F}_\omega(k,k;z)} \right|^s \right) \qquad (4.11) \\ &= \mathbb{E}\left(\frac{1}{\left| (1 - \hat{F}_\omega(k,k;z)) \hat{F}_\omega(k,k;z)^{-1} + e^{-i\theta_k(\omega)} \right|^s} \right) \leq C_\nu^{(2)}(s). \end{aligned}$$

We apply now the decoupling Lemma to the RHS of (4.8) as follows. We can write

$$\frac{|\hat{F}_\omega(k,j;z)|^s \left| e^{-i\theta_k(\omega)} z^{-1} - \rho_d(t) \right|^s}{|1 - \hat{F}_\omega(k,k;z) + e^{-i\theta_k(\omega)} \hat{F}_\omega(k,k;z)|^s} = \tag{4.12}$$
$$\frac{\rho_d^s(t) |\hat{F}_\omega(k,j;z)|^s \left| \rho_d(t)^{-1} z^{-1} - e^{i\theta_k(\omega)} \right|^s}{|1 - \hat{F}_\omega(k,k;z)|^s \left| e^{i\theta_k(\omega)} + \hat{F}_\omega(k,k;z)(1 - \hat{F}_\omega(k,k;z))^{-1} \right|^s}.$$

Therefore, the average over $\theta_k(\omega)$ of the above yields the bound

$$\int_{\mathbb{T}} d\nu(\theta) \frac{|\hat{F}_\omega(k,j;z)|^s \left|e^{-i\theta_k(\omega)}z^{-1} - \rho_d(t)\right|^s}{|1 - \hat{F}_\omega(k,k;z) + e^{-i\theta_k(\omega)}\hat{F}_\omega(k,k;z)|^s} \geq \tag{4.13}$$
$$C_\nu^{(1)}(s)\rho_d^s(t) \int_{\mathbb{T}} d\nu(\theta) \frac{|\hat{F}_\omega(k,j;z)|^s}{|1 - \hat{F}_\omega(k,k;z) + e^{-i\theta_k(\omega)}\hat{F}_\omega(k,k;z)|^s},$$

where the last integrand coincides with $|F(k,j;z)|^s$. Therefore, inserting this in (4.8), we finally get for $j \neq k$ and any $s \in]0,1[$,

$$\sum_{l \neq k} |\langle Sk|l\rangle|^s \mathbb{E}(|F_\omega(l,j;z)|^s) \geq C_\nu^{(1)}(s)\rho_d^s(t)\mathbb{E}(|F_\omega(k,j;z)|^s). \tag{4.14}$$

This last formula is the key to the desired bound, due to the following Lemma, see [AM],[AG]. The proof of [AG] is repeated in Appendix, for completeness.

Lemma 4.2 *Let $f \in l^\infty(\mathbb{Z}^d)$ be non-negative and $\sigma : l^\infty(\mathbb{Z}^d) \to l^\infty(\mathbb{Z}^d)$ be a linear operator with kernel $\sigma(k,l) \geq 0$ such that $\sigma(k,k) = 0$ and*

$$\sup_k \sum_{l \neq k} \sigma(k,l) = N < \infty. \tag{4.15}$$

Fix a $j \in \mathbb{Z}^d$ and assume there exists some finite $C > 0$ such that f satisfies for any $k \neq j$

$$(\sigma f)(k) = \sum_{l \neq k} \sigma(k,l) f(l) \geq C f(k). \tag{4.16}$$

Then, if $N < C$, and if there exists $\gamma > 0$ such that

$$\sup_k \sum_{l \neq k} \sigma(k,l) e^{\gamma|k-l|} < C, \tag{4.17}$$

we have for any k,

$$f(k) \leq f(j) e^{-\gamma|j-k|}. \tag{4.18}$$

This proposition applies to $f(k) = \mathbb{E}(|F_\omega(k,j;z)|^s)$ and $\sigma(j,k) = |\langle Sj|k\rangle|^s$ with the constants

$$C = C_\nu^{(1)}(s)\rho_d^s(t), \quad \text{and} \quad N = \sup_k \sum_{j \neq k} |\langle Sk|j\rangle|^s, \tag{4.19}$$

for small enough values of $|t|$. Indeed, for $z \notin S^1$, we have the *a priori* bound

$$|F_\omega(k,j;z)| = |\langle j|S(U_\omega - z)^{-1}k\rangle| \leq 1/\operatorname{dist}(z, S^1), \tag{4.20}$$

showing that f is in l^∞. Moreover, if $|t|$ is small enough, we get from (4.3) and (4.5) that

$$N = \sup_k \sum_{j \neq k} |\langle Sk|j\rangle|^s = O(|t|^s) < C_\nu^{(1)}(s)\rho_d^s(t) = C_\nu^{(1)}(s) + O(|t|^2) = C. \tag{4.21}$$

Finally, as the sum defining N in (4.19) carries over a finite number of indices only, for such values of $|t|$, there exists a $\gamma = \gamma(s,t) > 0$ so that (4.17) holds true. With the uniform bound on $\mathbb{E}(|F_\omega(j,j;z)|^s)$ derived in (4.11), and by the fact that D_ω is diagonal, this ends the proof of Theorem 2.1. ■

5 Localization

We spell out here a spectral consequence of the estimates derived in Theorem 2.1 by proving Corollary 2.1. We do this by applying the unitary version of the Simon-Wolff criterion [SW] for localization presented by Combescure in [C], see also [T].

We need some preliminary estimates. Let us introduce for $z \notin S^1$,

$$H_\omega(z) = U_\omega(U_\omega - z)^{-1}. \tag{5.1}$$

We choose $j = 0$ in the definition (3.2). By the Spectral Theorem ,

$$H_\omega(z) = \int_{\mathbb{T}} \frac{dE_\omega(\alpha)e^{i\alpha}}{e^{i\alpha} - z}, \tag{5.2}$$

where $E_\omega(\alpha)$ is the spectral family associated with U_ω. Therefore, the spectral measure associated with $|0\rangle$

$$d\mu_\omega(\alpha) = d\langle 0|E_\omega(\alpha)0\rangle = d\|E_\omega(\alpha)0\rangle\|^2 \tag{5.3}$$

is such that

$$\langle 0|H_\omega(z)0\rangle = \int_{\mathbb{T}} \frac{d\mu_\omega(\alpha)e^{i\alpha}}{e^{i\alpha} - z}. \tag{5.4}$$

Thus, for $z < 1$,

$$\|H_\omega(z)0\rangle\|^2 = \langle 0|H_\omega^*(\bar{z})H_\omega(z)0\rangle = \int_{\mathbb{T}} \frac{d\mu_\omega(\alpha)}{|e^{i\alpha} - z|^2}. \tag{5.5}$$

Introducing the Poisson integral of a measure $d\mu$

$$P[d\mu](z) = \int_{\mathbb{T}} \frac{d\mu(\alpha)(1 - |z|^2)}{|e^{i\alpha} - z|^2} \geq 0, \quad |z| < 1, \tag{5.6}$$

the identity above for $z = re^{i\theta}, r < 1$ can be cast under the form

$$\begin{aligned} \|H_\omega(re^{i\theta})0\rangle\|^2 &= P[d\mu_\omega](re^{i\theta}) + \int_{\mathbb{T}} \frac{d\mu_\omega(\alpha)r^2}{1 + r^2 - 2r\cos(\alpha - \theta)} && (5.7) \\ &\equiv P[d\mu_\omega](re^{i\theta}) + B_\omega(r, \theta). && (5.8) \end{aligned}$$

We know that the following limit exists and is finite for a.e. $\theta \in \mathbb{T}$ with respect to $d\theta/2\pi$

$$\lim_{r \to 1^-} P[d\mu_\omega](re^{i\theta}) = \frac{d\mu_\omega(\theta)}{d\theta}. \tag{5.9}$$

Since

$$r \mapsto \frac{r^2}{1 + r^2 - 2r\cos(\alpha - \theta)} \quad \text{is positive, monotone increasing,} \tag{5.10}$$

then

$$\lim_{r \to 1^-} B_\omega(r, \theta) = \int_{\mathbb{T}} \frac{d\mu_\omega(\alpha)}{4\sin^2((\alpha - \theta)/2)} \equiv B_\omega(\theta) \quad \text{exists for all } (\omega, \theta) \in \Omega \times \mathbb{T}. \tag{5.11}$$

Moreover, for (ω,θ) fixed, $B_\omega(r,\theta)$ is monotone non-decreasing in r as well. Now, Theorem 2.1 says for $0 < s < 1$,

$$\mathbb{E}(|\langle j|H_\omega(z)0\rangle|^s) \leq K(s)e^{-\gamma(s)|j|}, \quad \text{uniformly in } z. \tag{5.12}$$

Together with

$$\left(\sum_j |\langle j|H_\omega(z)0\rangle|^2\right)^{\tilde{s}} \leq \sum_j |\langle j|H_\omega(z)0\rangle|^s, \quad \text{for } \tilde{s} = s/2 < 1, \tag{5.13}$$

this implies

$$\mathbb{E}\left(\left(\|H_\omega(z)0\rangle\|^2\right)^{\tilde{s}}\right) \leq \sum_j K(s)e^{-\gamma(s)|j|} = \tilde{K}(s) < \infty. \tag{5.14}$$

Thus, we can apply the Monotone Convergence Theorem again to (5.11) for the measure $d\theta \times d\mathbb{P}(\omega)$ to get from (5.8) that,

$$\begin{aligned}\int_{\mathbb{T}} d\theta\ \mathbb{E}((B_\omega(\theta))^{\tilde{s}}) &= \lim_{r\to 1^-}\int_{\mathbb{T}} d\theta\ \mathbb{E}((B_\omega(r,\theta))^{\tilde{s}}) &(5.15)\\ &\leq \lim_{r\to 1^-}\int_{\mathbb{T}} d\theta\ \mathbb{E}\left(\left(B_\omega(r,\theta) + P[d\mu_\omega](re^{i\theta})\right)^{\tilde{s}}\right) \\ &= \lim_{r\to 1^-}\int_{\mathbb{T}} d\theta\ \mathbb{E}\left(\left(\|H_\omega(re^{i\theta})0\rangle\|^2\right)^{\tilde{s}}\right) \leq 2\pi\tilde{K}(s). &(5.16)\end{aligned}$$

Therefore, $B_\omega(\theta)$ is finite for almost all $(\theta,\omega) \in \mathbb{T}\times\Omega$, w.r.t. $d\theta \times d\mathbb{P}(\omega)$. By Fubini, this implies

Proposition 5.1 *Under the hypotheses of Theorem 2.1, there exists $\tilde{\Omega}_0 \subset \Omega$ of probability one and $J_\omega \in \mathbb{T}$ of full measure such that*

$$B_\omega(\theta) < \infty \quad \textit{if} \quad \omega \in \tilde{\Omega}_0 \quad \textit{and} \quad \theta \in J_\omega. \tag{5.17}$$

We are now in a position to apply the unitary version of [C] of the Simon-Wolff criterion for localization. Consider

$$\hat{H}_\omega(z) = \hat{U}_\omega(\hat{U}_\omega - z)^{-1} \quad \text{corresponding to (3.2)} \tag{5.18}$$

and $d\hat{\mu}_\omega$ the corresponding spectral measure associated with the vector $|0\rangle$. The relation (3.6) for $j = k = 0$ is equivalent to

$$\langle 0|H_\omega(z)0\rangle = \frac{\langle 0|\hat{H}_\omega(z)0\rangle}{\langle 0|\hat{H}_\omega(z)0\rangle(1-e^{i\theta_0(\omega)}) + e^{i\theta_0(\omega)}}. \tag{5.19}$$

The properties of the perturbed spectral measure $d\mu_\omega$, i.e. with $\theta_0(\omega)$ arbitrary, can be read from those of the unperturbed spectral measure $d\hat{\mu}_\omega$, i.e. with $\theta_0(\omega) = 0$, by means of the unitary analog of the Aronszajn-Donoghue characterization of supports of the Lebesgue decomposition of the spectral measure $d\mu_\omega$. We recall this characterization for completeness, changing slightly notations with respect to [C]: Combescure uses the resolvent rather than $H_\omega(z) = 1 + z(U_\omega - z)^{-1}$. Let $\hat{B}_\omega(\theta)$ be defined by (5.11) for $\hat{d}\mu_\omega$ in place of $d\mu_\omega$.

Proposition 5.2 *With the notations above, a support of the singular continuous part of $d\mu_\omega$ is*

$$S_\omega = \left\{\theta \in \mathbb{T} \mid \lim_{r\to 1^-} \hat{H}_\omega(re^{i\theta}) = \frac{e^{i\theta_0}}{e^{i\theta_0}-1} \text{ and } \hat{B}_\omega(\theta) = \infty\right\}, \tag{5.20}$$

the set of atoms of $d\mu_\omega$ is

$$P_\omega = \left\{\theta \in \mathbb{T} \mid \lim_{r\to 1^-} \hat{H}_\omega(re^{i\theta}) = \frac{e^{i\theta_0}}{e^{i\theta_0}-1} \text{ and } \hat{B}_\omega(\theta) < \infty\right\}, \tag{5.21}$$

whereas a support of the absolutely continuous part of $d\mu_\omega$ is

$$A_\omega = \left\{\theta \in \mathbb{T} \mid \lim_{r\to 1^-} P[d\mu_\omega](re^{i\theta}) = \frac{d\mu_\omega(\theta)}{d\theta} \in (0,\infty)\right\}. \tag{5.22}$$

These sets are mutually disjoint.

The key proposition from [C] regarding the properties of $d\mu_\omega$ in our setting is the following unitary version of the Simon-Wolff criterion:

Proposition 5.3 *Let $d\hat{\mu}_\omega$ and $d\mu_\omega$ be related by (5.19).*

$$\hat{B}_\omega(\theta) < \infty \text{ for a.e } \theta \in \mathbb{T} \Longleftrightarrow d\mu_\omega \text{ is purely atomic for a.e. } \theta_0 \in \mathbb{T}. \tag{5.23}$$

Indeed, considering $\hat{U}_\omega := e^{i\theta_0(\omega)}U_\omega$ instead of U_ω, we deduce from Proposition 5.1 and the criterion above that for any $\omega \in \tilde{\Omega}_0$, the spectral measure for $|0\rangle$ of

$$\tilde{U}_\omega = e^{-i\beta|0\rangle\langle 0|}U_\omega = \text{diag}(e^{-i\theta_j^\beta(\omega)})S, \quad \text{where} \quad \theta_j^\beta(\omega) = \theta_j(\omega) + \beta\delta_{j,0}, \tag{5.24}$$

is purely atomic for almost all $\beta \in \mathbb{T}$. But, as the distribution of phases is absolutely continuous, this means that the spectral measure $d\mu_\omega(\cdot) = \langle 0|dE(\cdot)0\rangle$ of U_ω is purely atomic for $\omega \in \Omega_0$, a set of probability one. Repeating the argument for the spectral measures $\langle j|E(\cdot)j\rangle$, $j \in \mathbb{Z}^d$, this yields the same result for $\omega \in \Omega_j$, where Ω_j is a set of probability one. Therefore, U_ω is pure point for $\omega \in \cap_{j\in\mathbb{Z}^d}\Omega_j$, a set of probability one.

■

6 Appendix

6.1 Proof of the Decoupling Lemma

Let us start with the second part of the Lemma. For any $\lambda > 0$,

$$\begin{aligned}\int_{\mathbb{T}} \frac{d\nu(\theta)}{|e^{i\theta}-\beta|^s} &\leq \lambda \int_{\{|e^{i\theta}-\beta|^{-s}\leq\lambda\}} d\nu(\theta) + \int_{\{|e^{i\theta}-\beta|^{-s}\geq\lambda\}} \frac{d\nu(\theta)}{|e^{i\theta}-\beta|^s} \\ &\leq \lambda + \int_\lambda^\infty \nu\{|e^{i\theta}-\beta|^{-s} \geq \lambda'\}d\lambda',\end{aligned} \tag{6.1}$$

where

$$\nu\{|e^{i\theta}-\beta|^{-s} \geq \lambda\} \leq \|\tau\|_\infty \int_{\{|e^{i\theta}-\beta|\leq 1/\lambda^{1/s}\}} d\theta. \tag{6.2}$$

In the last integral, we can assume without loss that $\beta \geq 0$ and it yields the arclength of the intersection of the unit circle with a circle of radius $1/\lambda^{1/s}$, centered at β. We first note that if $\lambda^{-1/s} \geq 1$, i.e. $\lambda \leq 1$, the integral takes its maximal value 2π, obtained with $\beta = 0$. If $\lambda > 1$, the integral is maximized by the choice $\beta = \beta(\lambda) = \sqrt{1-\lambda^{-2/s}}$ to give

$$\int_{\{|e^{i\theta}-\beta|\leq 1/\lambda^{1/s}\}} d\theta = 2\arcsin(1/\lambda^{1/s}), \quad \text{if} \quad 1/\lambda^{1/s} < 1. \tag{6.3}$$

As $\lambda \to \infty$, this integral behaves as $1/\lambda^{1/s}$, which is integrable for $0 < s < 1$. At this point we optimize our upper bound (6.1) on λ by choosing λ such that

$$1 - \|\tau\|_\infty 2\arcsin(1/\lambda^{1/s}) = 0. \tag{6.4}$$

Since $\|\tau\|_\infty \geq 2\pi$ to ensure normalization, the minimizer is

$$\lambda = [\sin(1/(2\|\tau\|_\infty)))]^{-s} > 1. \tag{6.5}$$

Therefore we have proven the existence of a constant $C_\nu^{(2)}(s)$ depending on s and ν only such that (4.10) holds.

The first part of the Lemma is proven along the lines of [AG]. It is shown in the appendix C of this paper that for $0 < s < 1$ and for any $u, v, \alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}\frac{1}{|v-\beta|^s} + \frac{1}{|u-\beta|^s} \quad &\leq \quad \frac{|v-\alpha|^s}{|v-\beta|^s}\left(\frac{1}{|u-\alpha|^s} + \frac{1}{|u-\beta|^s}\right) \\ &+ \quad \frac{|u-\alpha|^s}{|u-\beta|^s}\left(\frac{1}{|v-\alpha|^s} + \frac{1}{|v-\beta|^s}\right).\end{aligned} \tag{6.6}$$

Then, replacing v and u by $e^{i\theta}$ and $e^{i\theta'}$ respectively, and integrating over $d\nu(\theta)d\nu(\theta')$, we get

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{T}} d\nu(\theta)d\nu(\theta')\left(\frac{1}{|e^{i\theta}-\beta|^s} + \frac{1}{|e^{i\theta'}-\beta|^s}\right) = 2\int_{\mathbb{T}} d\nu(\theta)\frac{1}{|e^{i\theta}-\beta|^s} \\ \leq \quad &\int_{\mathbb{T}} d\nu(\theta)\frac{|e^{i\theta}-\alpha|^s}{|e^{i\theta}-\beta|^s}\int_{\mathbb{T}} d\nu(\theta')\left(\frac{1}{|e^{i\theta'}-\alpha|^s} + \frac{1}{|e^{i\theta'}-\beta|^s}\right) + (\theta' \leftrightarrow \theta)\end{aligned} \tag{6.7}$$

where $(\theta' \leftrightarrow \theta)$ means the same expression with θ and θ' exchanged. We finally get (4.9) with $C_\nu^{(1)}(s) = 1/(2C_\nu^{(2)}(s))$ by applying the bound (4.10). ■

6.2 Proof of Lemma 4.2

We first observe that if $\varphi = \{\varphi(k)\}_{k\in\mathbb{Z}^d} \in L^\infty(\mathbb{Z}^d)$ is real valued, such that $\varphi(j) \leq 0$ and satisfies

$$C\varphi(k) \leq (\sigma\varphi)(k), \quad \forall k \neq j, \tag{6.8}$$

then $\varphi(k) \leq 0$, for any k. Indeed, if it were not the case, $M \equiv \sup_k \varphi(k)$ would be strictly positive. But that would imply

$$C\varphi(k) \leq \sum_{l\neq k} \sigma(k,l)\varphi(l) \leq NM \ \Rightarrow CM \leq NM, \tag{6.9}$$

which contradicts $N < C$. Then one applies the above to

$$\varphi(k) = f(k) - f(j)e^{-\gamma|k-j|}, \quad \text{s. t.} \quad \varphi(j) = 0. \tag{6.10}$$

Since

$$(\sigma e^{-\gamma|\cdot-j|})(k) = \sum_{l\neq k}\sigma(k,l)e^{-\gamma(|l-j|-|k-j|)}e^{-\gamma|k-j|} \leq \sum_{l\neq k}\sigma(k,l)e^{\gamma|l-k|}e^{-\gamma|k-j|}$$
$$\leq Ce^{-\gamma|k-j|}, \quad (6.11)$$

by hypothesis, we get, using $f(j) \leq 0$,

$$(\sigma\varphi)(k) = (\sigma f)(k) - f(j)(\sigma e^{-\gamma|\cdot-j|})(k) \geq C(f(k) - f(j)e^{-\gamma|k-j|}) = C\varphi(k), \quad (6.12)$$

hence $f(k) \leq f(j)e^{-\gamma|k-j|}$. ■

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