A new matrix method for the Casimir operators of the Lie algebras $w\mathfrak{sp}(N,\mathbb{R})$ and $I\mathfrak{sp}(2N,\mathbb{R})$.

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Abstract. A method is given to determine the Casimir operators of the perfect Lie algebras $w\mathfrak{sp}(N,\mathbb{R}) = \mathfrak{sp}(2N,\mathbb{R}) \overrightarrow{\oplus}_{\Gamma_{\omega_1}\oplus\Gamma_0}\mathfrak{h}_N$ and the inhomogeneous Lie algebras $I\mathfrak{sp}(2N,\mathbb{R})$ in terms of polynomials associated to a parametrized $(2N+1)\times(2N+1)$-matrix. For the inhomogeneous symplectic algebras this matrix is shown to be associated to a faithful representation.
The method is extended to other classes of Lie algebras, and some applications to the missing label problem are given.

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1. Introduction

Symplectic Lie algebras constitute a quite interesting class of algebras for physical applications, due to their relation to some fundamental constructions. As known, the Hamiltonian of the most general system of linear oscillators is given by

$$H = \alpha^{ij} p_i p_j + \beta^{ij} p_i q_j + \gamma^{ij} q_i q_j,\ 1 \leq i, j \leq N, \tag{1}$$

where $q = \{q_i\}$ and $p = \{p_i\}$ are the usual configuration and momentum space variables. It is straightforward to verify that the observables of degree 2 in p and q generate the real Lie algebra $\mathfrak{sp}\,(2N, \mathbb{R})$, while those of degree ≤ 1 span the Heisenberg Lie algebra $\mathfrak{h}_N$. This constitutes an evidence that both algebras, as well as the semidirect product $w\mathfrak{sp}\,(N, \mathbb{R})$ of $\mathfrak{sp}\,(2N, \mathbb{R})$ and $\mathfrak{h}_N$ can be of interest for the study of internal symmetry schemes of particles. In this context, the case $N = 3$ has been shown to play a distinguished role in the theory of nuclear collective motions [1, 2]. On the other hand, we find that the unitary algebra $\mathfrak{u}\,(N)$ [and therefore the $\mathfrak{su}\,(N)$ algebra] is naturally embedded into $\mathfrak{sp}\,(2N, \mathbb{R})$ as the centralizer of the element

$$H' \frac{1}{2} \left(p^i p_i + q^i q_i \right). \tag{2}$$

Thus we have also a relation between symplectic groups and those applied to nuclear physics. Other important applications of the symplectic groups are for example the derivation of the dynamical noninvariance groups $SO\,(4, 2)$ for hydrogen-like atoms in 3 dimensions from the group $Sp\,(8, \mathbb{R})$, providing an additional approach to the classical method [3].

It is therefore natural that the Casimir operators of these algebras are relevant for the different problems analyzed, not only as an effective tool in the representation theory of these structures, but also for the obtention of quantum numbers and the labelling problems. In this work we develop a method that enables us to obtain the Casimir operators of the Lie algebras $w\mathfrak{sp}\,(N, \mathbb{R})$ and $I\mathfrak{sp}\,(2N, \mathbb{R})$ from a determinant associated to a parametrized matrix obtained from an extension of the generic matrix of the standard representation of the symplectic algebra $\mathfrak{sp}\,(2N, \mathbb{R})$. This method provides the invariants directly, without necessity of studying the corresponding enveloping algebras or taking contractions of Lie algebras, and is extremely easy to apply even for high values of N. For special cases we point out the relation of the matrix used and the existence of faithful representations of the algebra. As applications, it is shown that similar arguments hold for other Lie algebras such as the Poincaré algebra and some of its contractions. We finally give an application to the missing label problem, where the missing label operators for certain subalgebra chains can be obtained directly by means of determinants.

The most extended procedure to determine the (generalized) Casimir invariants of a Lie algebra $\mathfrak{g}$ is the analytical method, which turns out to be more practical than the traditional method of analyzing the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. This is particularly convenient in the study of completely integrable Hamiltonian systems, where Casimir operators in the classical sense do not have to exist, and where the transcendental invariant functions are not interpretable in terms of $\mathcal{U}(\mathfrak{g})$.

Given a basis $\{X_1, .., X_n\}$ of the Lie algebra $\mathfrak{g}$ and the structure tensor $\{C_{ij}^k\}$, $\mathfrak{g}$ can be realized in the space $C^\infty(\mathfrak{g}^*)$ by means of differential operators:

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{3}$$

where $[X_i, X_j] = C_{ij}^k X_k \quad (1 \leq i < j \leq n)$ and $\{x_1, .., x_n\}$ is a dual basis of $\{X_1, .., X_n\}$. In this context, an analytic function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system of PDEs:

$$\left\{\widehat{X}_i F = 0,\ 1 \leq i \leq n\right\}. \tag{4}$$

Polynomial solutions F correspond, after symmetrization, to classical Casimir operators, while nonpolynomial solutions of system (4) are usually called "generalized Casimir invariants". The cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions (in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis) is easily obtained from the classical criteria for PDEs:

$$\mathcal{N}(\mathfrak{g}) := \dim\, \mathfrak{g} - \text{rank}\left(C_{ij}^k x_k\right)_{1 \leq i < j \leq \dim \mathfrak{g}}, \tag{5}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the basis $\{X_1, .., X_n\}$. Evidently this quantity constitutes an invariant of the algebra. We remark that $\mathcal{N}(\mathfrak{g})$ can also be obtained from the Maurer-Cartan equations of the Lie group [4], which is of interest in the context of the missing label problem and the classification of subalgebras.

As commented above, real symplectic Lie algebras can easily be realized in terms of creation and annihilation operators [5]: consider the linear operators $a_i, a_j^\dagger \ (i, j = 1..N)$ satisfying the commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \tag{6}$$

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0 \tag{7}$$

Considering the operators $\left\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\right\}$, we generate the real symplectic Lie algebra $\mathfrak{sp}(2N, \mathbb{R})$. The brackets are easily obtained from $(6) - (7)$. For practical purposes, we label the basis in the following form

$$X_{i,j} = a_i^\dagger a_j,\ 1 \leq i, j \leq N \tag{8}$$

$$X_{-i,j} = a_i^\dagger a_j^\dagger \tag{9}$$

$$X_{i,-j} = a_i a_j \tag{10}$$

The brackets of $\mathfrak{sp}(2N, \mathbb{R})$ can then be comprised in a unique expression:

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} + \varepsilon_i \varepsilon_j \delta_{j,-l} X_{k,-i} - \varepsilon_i \varepsilon_j \delta_{i,-k} X_{-j,l}, \tag{11}$$

where $-N \leq i, j, k, l \leq N$, $\varepsilon_i = sgn\,(i)$ and

$$X_{i,j} + \varepsilon_i \varepsilon_j X_{-j,-i} = 0 \tag{12}$$

While this last basis is useful for the study of realisations of symplectic Lie algebras [6], the boson basis is more convenient for studying the semidirect products with Heisenberg algebras and their contractions [7, 8, 9, 10]. In fact, the operators $a_i, a_i^\dagger$ transform as follows by the generators $\left\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\right\}$ of $\mathfrak{sp}(2N,\mathbb{R})$:

$$\left[a_i^\dagger a_j, a_k^\dagger\right] = \delta_{jk} a_i^\dagger \tag{13}$$

$$\left[a_i^\dagger a_j, a_k\right] = -\delta_{ik} a_j \tag{14}$$

$$\left[a_i^\dagger a_j^\dagger, a_k\right] = -\delta_{jk} a_i^\dagger - \delta_{ik} a_j^\dagger \tag{15}$$

$$\left[a_i a_j, a_k^\dagger\right] = \delta_{ki} a_j + \delta_{kj} a_i. \tag{16}$$

With the labelling $P_i = a_i^\dagger$, $Q_i = a_i$ for $i = 1..N$, we immediately see that (13)-(16) is nothing but the standard $2n$-dimensional representation Γ_{ω_1} of $\mathfrak{sp}(2N,\mathbb{R})$. Using the variable Z for the identity operator $\mathbb{I}$, equations (6)-(7) and (11)-(16) are the brackets for the semidirect product $w\mathfrak{sp}(N,\mathbb{R})$ of the symplectic algebra $\mathfrak{sp}(2N,\mathbb{R})$ with the (2N+1)-dimensional Heisenberg-Weyl Lie algebra $\mathfrak{h}_N$ [7]. This type of construction for semidirect products is typical for the study of shift operator contractions and coherent state realisations of Lie algebras [7, 11].

Explicit expressions for the Casimir operators of semisimple Lie algebras have been proposed by many authors [12, 13], and even for nonsemisimple algebras there exist various procedures [7, 14, 15, 16, 17]. We recall in this section a quite economical method to determine the invariants of $w\mathfrak{sp}(2N,\mathbb{R})$ over the boson basis, and based on the classical matrix methods.

Proposition 1 *Let $N \geq 2$. Then the Casimir operators C_{2k} of $\mathfrak{sp}\,(2N,\mathbb{R})$ are given by the coefficients of the characteristic polynomial*

$$|A - T.id_{2N}| = T^{2N} + \sum_{k=1}^{N} C_{2k} T^{2N-2k}, \tag{17}$$

where

$$A = \begin{pmatrix} x_{1,1} & .. & x_{1,N} & -x_{-1,1} & .. & -x_{-1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{N,1} & .. & x_{N,N} & -x_{-1,N} & .. & -x_{-N,N} \\ x_{1,-1} & .. & x_{1,-N} & -x_{1,1} & .. & -x_{N,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{1,-N} & .. & x_{N,-N} & -x_{1,N} & .. & -x_{N,N} \end{pmatrix} \tag{18}$$

Moreover $\deg C_{2k} = 2k$ *for k=1..N.*

The proof follows easily from the classical formulae of Perelomov and Popov [18], or using the trace method introduced by Gruber and O'Raifeartaigh in [13]. Observe that in fact the matrix A can be rewritten as

$$A = \sum_{i=1}^{N} x_{i,j} \Gamma_{\omega_1}(X_{i,j}), \tag{19}$$

where $\Gamma_{\omega_1}(X_{i,j})$ is the matrix corresponding to the generator $X_{i,j}$ by the standard representation Γ_{ω_1} of $\mathfrak{sp}(2N,\mathbb{R})$.

2. The Lie algebras $w\mathfrak{sp}\,(N,\mathbb{R})$

As we have seen, the Lie algebras $w\mathfrak{sp}\,(N,\mathbb{R}) = \mathfrak{sp}\,(2N,\mathbb{R})\,\overrightarrow{\oplus}_{\Gamma_{\omega_1}\oplus\Gamma_0}\mathfrak{h}_N$ follow naturally from the boson realisation of $\mathfrak{sp}(2N,\mathbb{R})$. This fact has important consequences for the applications of these algebras and their irreducible representations, such as the theory of nuclear collective motions [2].

Among the various methods to obtain the Casimir operators of the semidirect products $w\mathfrak{sp}\,(N,\mathbb{R})$, C. Quesne introduced in [7] a quite practical procedure, which has been shown recently to hold also for excepcional Lie algebras [8]. The main idea is to obtain a semisimple Lie algebra $\mathfrak{g}'$ isomorphic to $\mathfrak{sp}\,(2N,\mathbb{R})$ in the enveloping algebra $\mathfrak{U}$ of $w\mathfrak{sp}\,(N,\mathbb{R})$ such that its generators commute with the Heisenberg algebra. The insertion of these new generators into the formulae for the Casimir operators of $\mathfrak{sp}(2N,\mathbb{R})$ gives the searched invariants.

Starting from this method, and combining it with proposition 1, we obtain a more direct matrix method for the computation of the invariants of $w\mathfrak{sp}(2N,\mathbb{R})$ from a certain polynomial associated to a $(2N+1)\times(2N+1)$-matrix obtained from the standard representation. Before proving the general case, we illustrate the procedure with the Lie algebra $w\mathfrak{sp}\,(2,\mathbb{R}) = \mathfrak{sp}\,(4,\mathbb{R})\,\overrightarrow{\oplus}_{\Gamma_{w_1}\oplus\Gamma_0}\mathfrak{h}_2$. This algebra clearly has three invariants, one of them corresponding to the generator Z of the centre. Using the insertion method, the operators $X'_{i,j} = X_{i,j} + P_iQ_j,\ X'_{-i,j} = X_{-i,j} + P_iP_j,\ X'_{i,-j} = X_{i,-j} + Q_iQ_j$ $(i,j=1,2)$ generate‡ a copy of $\mathfrak{sp}\,(4,\mathbb{R})$ in the enveloping algebra of $w\mathfrak{sp}\,(2,\mathbb{R})$. Since the generators P_i, Q_i correspond to the operators $a_i^\dagger$ and a_i, respectively, the boson formalism shows at once that the copy of $\mathfrak{sp}\,(4,\mathbb{R})$ in $\mathfrak{U}\,(w\mathfrak{sp}\,(2,\mathbb{R}))$ is realized by the operators $\left\{2a_i^\dagger a_i, 2a_i^\dagger a_j^\dagger, 2a_ia_j\right\}$. The corresponding (noncentral) Casimir operators are obtained from the characteristic polynomial of the matrix

$$M_2 = \begin{pmatrix} zx_{1,1}+p_1q_1 & zx_{1,2}+p_1q_2 & -zx_{-1,1}-p_1^2 & -zx_{-1,2}-p_1p_2 \\ zx_{2,1}+p_2q_1 & zx_{2,2}+p_2q_2 & -zx_{-1,2}-p_1p_2 & -zx_{-2,2}-p_2^2 \\ zx_{1,-1}+q_1^2 & zx_{1,-2}+q_1q_2 & -zx_{1,1}-p_1q_1 & -zx_{2,1}-p_2q_1 \\ zx_{1,-2}+q_1q_2 & zx_{2,-2}+q_2^2 & -zx_{1,2}-p_1q_2 & -zx_{2,2}-p_2q_2 \end{pmatrix} \qquad (20)$$

The determinant $|M_2 - T.id_4|$ can be decomposed into a sum of 16 determinants, 11 of which are zero because the second summand in each column of the matrix (20) is a multiple of the column vector $(p_1,p_2,q_1,q_2)^t$. With this reduction, we obtain:

$|M_2 - T.id_4| =$

‡ Depending on the basis chosen, the expression for the new generators also changes. The basis used here differs slightly from that employed in [7]

$$\begin{vmatrix} p_1q_1 & zx_{1,2} & -zx_{-1,1} & -zx_{-1,2} \\ p_2q_1 & zx_{2,2}-T & -zx_{-1,2} & -zx_{-2,2} \\ q_1^2 & zx_{1,-2} & -zx_{1,1}-T & -zx_{2,1} \\ q_1q_2 & zx_{2,-2} & -zx_{1,2} & -zx_{2,2}-T \end{vmatrix} + \begin{vmatrix} zx_{1,1}-T & zx_{1,2} & -zx_{-1,1} & -p_1p_2 \\ zx_{2,1} & zx_{2,2}-T & -zx_{-1,2} & -p_2^2 \\ zx_{1,-1} & zx_{1,-2} & -zx_{1,1}-T & -p_2q_1 \\ zx_{1,-2} & zx_{2,-2} & -zx_{1,2} & -p_2q_2 \end{vmatrix} +$$

$$\begin{vmatrix} zx_{1,1}-T & zx_{1,2} & -p_1^2 & -zx_{-1,2} \\ zx_{2,1} & zx_{2,2}-T & -p_1p_2 & -zx_{-2,2} \\ zx_{1,-1} & zx_{1,-2} & -p_1q_1 & -zx_{2,1} \\ zx_{1,-2} & zx_{2,-2} & -p_1q_2 & -zx_{2,2}-T \end{vmatrix} + \begin{vmatrix} zx_{1,1}-T & p_1q_2 & -zx_{-1,1} & -zx_{-1,2} \\ zx_{2,1} & p_2q_2 & -zx_{-1,2} & -zx_{-2,2} \\ zx_{1,-1} & q_1q_2 & -zx_{1,1}-T & -zx_{2,1} \\ zx_{1,-2} & q_2^2 & -zx_{1,2} & -zx_{2,2}-T \end{vmatrix} +$$

$$+ \begin{vmatrix} zx_{1,1}-T & zx_{1,2} & -zx_{-1,1} & -zx_{-1,2} \\ zx_{2,1} & zx_{2,2}-T & -zx_{-1,2} & -zx_{-2,2} \\ zx_{1,-1} & zx_{1,-2} & -zx_{1,1}-T & -zx_{2,1} \\ zx_{1,-2} & zx_{2,-2} & -zx_{1,2} & -zx_{2,2}-T \end{vmatrix}. \tag{21}$$

A short calculation shows that this sum can be comprised in a unique determinant, namely

$$\frac{1}{T}\begin{vmatrix} zx_{1,1}-T & zx_{1,2} & -zx_{-1,1} & -zx_{-1,2} & p_1T \\ zx_{2,1} & zx_{2,2}-T & -zx_{-1,2} & -zx_{-2,2} & p_2T \\ zx_{1,-1} & zx_{1,-2} & -zx_{1,1}-T & -zx_{2,1} & q_1T \\ zx_{1,-2} & zx_{2,-2} & -zx_{1,2} & -zx_{2,2}-T & q_2T \\ q_1 & q_2 & -p_1 & -p_2 & -T \end{vmatrix}. \tag{22}$$

This determinant has the advantage of avoiding all zero summands of the decomposition of (21), and is therefore more practical for computation purposes. Factoring out the variable z of the centre from the remaining summands and symmetrizing, we obtain the Casimir operators of degree 3 and 5 of the Lie algebra.

Repeating the same argument for the general case $N \geq 2$ leads to a determinant which decomposes into a sum of 2^{2N} determinants of the same order, from which $2^{2N}-2N-1$ are identically zero. This shows that for large values of N the method of the copy of $\mathfrak{sp}(2N,\mathbb{R})$ in the enveloping algebra of the semidirect product is not the most economical method to compute the Casimir operators. The objetive of this section is to prove that the reduction leading to the unique determinant (22) can be extended to the general case.

Proposition 2 *Let* $N \geq 2$. *Then the noncentral Casimir operators* C_{2k+1} *of* $w\mathfrak{sp}\left(2N,\mathbb{R}\right)$ *are given by the coefficients of the polynomial*

$$|B - T.id_{2N+1}| = T^{2N+1} + \sum_{k=1}^{N} z^{2k-1} C_{2k+1} T^{2N+1-2k}, \tag{23}$$

where

$$B=\begin{pmatrix} zx_{1,1} & .. & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,N} & p_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{N,1} & .. & zx_{N,N} & -zx_{-1,N} & .. & -zx_{-N,N} & p_NT \\ zx_{1,-1} & .. & zx_{1,-N} & -zx_{1,1} & .. & -zx_{N,1} & q_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{N,N} & q_NT \\ q_1 & .. & q_N & -p_1 & .. & -p_N & 0 \end{pmatrix}. \tag{24}$$

Moreover $\deg C_{2k+1}=2k+1$ *for k=0..N.*

Proof. Since the Casimir operators of $w\mathfrak{sp}(2N,\mathbb{R})$ are obtained by replacing the generators of $\mathfrak{sp}(2N,\mathbb{R})$ by new generators spanning a copy of the symplectic algebra in the enveloping algebra into the matrix (18), the invariants are given by the following determinant:

$$\Delta=\begin{vmatrix} zx_{1,1}+p_1q_1-T & .. & zx_{1,N}+p_1q_N & -zx_{-1,1}-p_1^2 & .. & -zx_{-1,N}-p_1p_N \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1}+p_Nq_1 & .. & zx_{N,N}+p_Nq_N-T & -zx_{-1,N}-p_1q_1 & .. & -zx_{-N,N}-p_N^2 \\ zx_{1,-1}+q_1^2 & .. & zx_{1,-N}+q_1q_N & -zx_{1,1}-p_1q_1-T & .. & -zx_{N,1}-p_Nq_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N}+q_1q_N & .. & zx_{N,-N}+q_N^2 & -zx_{1,N}-p_1q_N & .. & -zx_{N,N}-p_Nq_N-T \end{vmatrix} \tag{25}$$

Now this can be simplified using the elementary rules for determinants, and taking into account that the second summand in each column of (25) is a multiple of $(p_1,..p_N,q_1,..,q_N)^t$, Δ reduces to:

$$\Delta=\begin{vmatrix} zx_{1,1}-T & .. & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} & .. & zx_{N,N}-T & -zx_{-1,N} & .. & -zx_{-N,N} \\ zx_{1,-1} & .. & zx_{1,-N} & -zx_{1,1}-T & .. & -zx_{N,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{N,N}-T \end{vmatrix}+$$

$$\sum_{j=1}^{N}\begin{vmatrix} zx_{1,1}-T & .. & zx_{1,j-1} & p_1q_j & zx_{1,j+1} & .. & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} & .. & zx_{N,j-1} & p_Nq_j & zx_{N,j+1} & .. & zx_{N,N}-T & -zx_{-1,N} & .. & -zx_{-N,N} \\ zx_{1,-1} & .. & zx_{1,-j+1} & q_1q_j & zx_{1,-j-1} & .. & zx_{1,-N} & -zx_{1,1}-T & .. & -zx_{N,1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} & .. & zx_{j-1,-N} & q_Nq_j & zx_{j+1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{N,N}-T \end{vmatrix}+$$

$$\sum_{j=1}^{N}\begin{vmatrix} zx_{1,1}-T & .. & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,j-1} & -p_1p_j & -zx_{-1,j+1} & .. & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & & & \vdots & & & \vdots \\ zx_{N,1} & .. & zx_{N,N}-T & -zx_{-1,N} & .. & -zx_{-N,j-1} & -p_Np_j & -zx_{-N,j+1} & .. & -zx_{-N,N} \\ zx_{1,-1} & .. & zx_{1,-N} & -zx_{1,1}-T & .. & -zx_{j-1,1} & -q_1p_j & -zx_{j+1,1} & .. & -zx_{N,1} \\ \vdots & & \vdots & \vdots & & & & & & \vdots \\ zx_{1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{j-1,N} & -q_Np_j & -zx_{j+1,N} & .. & -zx_{N,N}-T \end{vmatrix} \tag{26}$$

The structure of these determinants suggest that they can be obtained as minors of some another determinant. Let us now consider the matrix B. The polynomial $\Delta' = |B - T.id_{2n+1}|$ is given by the determinant

$$\Delta' = \begin{vmatrix} zx_{1,1}-T & .. & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,N} & p_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{N,1} & .. & zx_{N,N}-T & -zx_{-1,N} & .. & -zx_{-N,N} & p_NT \\ zx_{1,-1} & .. & zx_{1,-N} & -zx_{1,1}-T & .. & -zx_{N,1} & q_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{N,N}-T & q_NT \\ q_1 & .. & q_N & -p_1 & .. & -p_N & -T \end{vmatrix} \tag{27}$$

Solving it by the elements of the last row, we decompose the determinant into:

$$\Delta' = -T\left(|(B-Tid_{2N+1})_{2N+1,2N+1}| + \sum_{j=1}^{N}(-1)^{2N+1+j}\, q_j\, |(B-Tid_{2N+1})_{2N+1,j}|\right)$$
$$+ (\sum_{j=1}^{N}(-1)^{3N+1+j}\, p_j\, |(B-Tid_{2N+1})_{2N+1,N+j}|)T, \tag{28}$$

where $(B-Tid_{2N+1})_{i,j}$ is the minor of $B-Tid_{2N+1}$ obtained deleting the i^{th} row and j^{th} column. Inserting the variable q_j (respectively p_j) in the minor $(B-Tid_{2N+1})_{2N+1,j}$ (respectively $(B-Tid_{2N+1})_{2N+1,N+j}$), we recover the summands of (26). Comparison of the determinants of (35) and (37) shows that they are related as follows:

$$\Delta T + \Delta' = 0. \tag{29}$$

■

It is important to realise that the matrix B used depends on the variable T taken to evaluate the determinant $|B - T.id_{2N+1}|$. Therefore we cannot speak formally of the characteristic polynomial of B, but of a polynomial closely related to it.

3. The inhomogeneous algebras $I\mathfrak{sp}\,(2N,\mathbb{R})$

The previous formula (27) constitutes a simplification of the determinant (25) used to determine the Casimir operators of $w\mathfrak{sp}(N,\mathbb{R})$, but its real interest lies in its application to the computation of the invariants of the inhomogeneous Lie algebras $I\mathfrak{sp}(2N,\mathbb{R})$, by virtue of the simple Inönü-Wigner contraction§

$$w\mathfrak{sp}(N,\mathbb{R}) \rightsquigarrow I\mathfrak{sp}(2N,\mathbb{R}) \oplus \mathbb{R} \tag{30}$$

determined by the change of basis

$$X'_{i,j} = X_{i,j}, i,j = -N.. -1, 1, ..N; \quad Z' = Z \tag{31}$$

$$P'_i = \frac{1}{\sqrt{t}} P_i, i = 1..N; \quad Q'_i = \frac{1}{\sqrt{t}} Q_i, i = 1..N \tag{32}$$

for $t \to \infty$, but this process rapidly becomes tedious with large N. It is therefore convenient to develop a direct method to obtain the Casimir operators, independently of the limiting process of (31)-(32). These algebras have, as known, N independent Casimir operators [20].

Proposition 3 *Let $N \geq 2$. Then the Casimir operators C_{2k} of $I\mathfrak{sp}\,(2N,\mathbb{R})$ are given by the coefficients of the polynomial*

$$|C - T.id_{2N+1}| - |A - T.id_{2N}|\, T = T^{2N+1} + \sum_{k=1}^{N} C_{2k+1} T^{2N+1-2k}, \tag{33}$$

where

$$C = \begin{pmatrix} x_{1,1} & .. & x_{1,N} & -x_{-1,1} & .. & -x_{-1,N} & p_1 T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{N,1} & .. & x_{N,N} & -x_{-1,N} & .. & -x_{-N,N} & p_N T \\ x_{1,-1} & .. & x_{1,-N} & -x_{1,1} & .. & -x_{N,1} & q_1 T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{1,-N} & .. & x_{N,-N} & -x_{1,N} & .. & -x_{N,N} & q_N T \\ q_1 & .. & q_N & -p_1 & .. & -p_N & 0 \end{pmatrix} \tag{34}$$

Moreover $\deg C_{2k+1} = 2k+1$.

Proof.

The proof is esentially the same as that of proposition 2. By the contraction, the invariants of the inhomogeneous algebras are obtained from the limit for $t \to \infty$ of the invariants of $w\mathfrak{sp}(N,\mathbb{R})$, and are given by:

$$\lim_{t\to\infty} \frac{1}{t} \begin{vmatrix} zx_{1,1} + tp_1q_1 - T & .. & zx_{1,N} + tp_1q_N & -zx_{-1,1} - tp_1^2 & .. & -zx_{-1,N} - tp_1p_N \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} + tp_Nq_1 & .. & zx_{N,N} + tp_Nq_N - T & -zx_{-1,N} - tp_1q_1 & .. & -zx_{-N,N} - tp_N^2 \\ zx_{1,-1} + tq_1^2 & .. & zx_{1,-N} + tq_1q_N & -zx_{1,1} - tp_1q_1 - T & .. & -zx_{N,1} - tp_Nq_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} + tq_1q_N & .. & zx_{N,-N} + tq_N^2 & -zx_{1,N} - tp_1q_N & .. & -zx_{N,N} - tp_Nq_N - T \end{vmatrix} \tag{35}$$

§ For the definition of simple IW-contraction used here see e.g. reference [19].

Reducing the determinant by standard methods and taking the limit, we obtain the following sum:

$$\sum_{j=1}^{N}\begin{vmatrix} zx_{1,1}-T & .. & zx_{1,j-1} & p_1q_j & zx_{1,j+1} & .. & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} & .. & zx_{N,j-1} & p_Nq_j & zx_{N,j+1} & .. & zx_{N,N}-T & -zx_{-1,N} & .. & -zx_{-N,N} \\ zx_{1,-1} & .. & zx_{1,-j+1} & q_1q_j & zx_{1,-j-1} & .. & zx_{1,-N} & -zx_{1,1}-T & .. & -zx_{N,1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} & .. & zx_{j-1,-N} & q_Nq_j & zx_{j+1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{N,N}-T \end{vmatrix}+$$

$$\sum_{j=1}^{N}\begin{vmatrix} zx_{1,1}-T & ... & zx_{1,N} & -zx_{-1,1} & .. & -zx_{-1,j-1} & -p_1p_j & -zx_{-1,j+1} & .. & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & & & \vdots & & & \vdots \\ zx_{N,1} & .. & zx_{N,N}-T & -zx_{-1,N} & .. & -zx_{-N,j-1} & -p_Np_j & -zx_{-N,j+1} & .. & -zx_{-N,N} \\ zx_{1,-1} & .. & zx_{1,-N} & -zx_{1,1}-T & .. & -zx_{j-1,1} & -p_1q_j & -zx_{j+1,1} & .. & -zx_{N,1} \\ \vdots & & \vdots & \vdots & & & & & & \vdots \\ zx_{1,-N} & .. & zx_{N,-N} & -zx_{1,N} & .. & -zx_{j-1,N} & -p_Nq_j & -zx_{j+1,N} & .. & -zx_{N,N}-T \end{vmatrix} \tag{36}$$

This sum is very similar to that of (26), up to the fact that here all involved determinants have a column whose entries are products of the variables p_i and q_j associated to the standard representation Γ_{ω_1} of $\mathfrak{sp}(2N,\mathbb{R})$. Expanding the sum, we obtain a polynomial:

$$\sum_{k=1}^{N} z^{2k-1}C_{2k+1}T^{2N-2k}, \tag{37}$$

where the C_{2k+1} are homogeneous polynomials of degree $2k+1$. The invariants of $I\mathfrak{sp}(2N,\mathbb{R})$ are given by the C_{2k+1}, while z is the invariant of the direct summand $\mathbb{R}$ of the contraction.

If we now expand the determinant $|C-T.id_{2n+1}|$ (compare with (27)), we get the decomposition

$$|C-T.id_{2n+1}| = -T\left(|(C-Tid_{2N+1})_{2N+1,2N+1}| + \sum_{j=1}^{N}(-1)^{2N+1+j}\,q_j\,|(C-Tid_{2N+1})_{2N+1,j}|\right)$$
$$+(\sum_{j=1}^{N}(-1)^{3N+1+j}\,p_j\,|(C-Tid_{2N+1})_{2N+1,N+j}|)T. \tag{38}$$

Proceeding like before, it is not difficult to see that the sum

$$\sum_{j=1}^{N}\left((-1)^{2N+1+j}\,q_j\,|(C-Tid_{2N+1})_{2N+1,j}| + (-1)^{3N+1+j}\,p_j\,|(C-Tid_{2N+1})_{2N+1,N+j}|\right)T \tag{39}$$

coincides with the sum (36) when we set $z=1$. The remaining summand $|(C-Tid_{2N+1})_{2N+1,2N+1}|$ is nothing but the characteristic polynomial of the matrix

A of $\mathfrak{sp}(2N,\mathbb{R})$ associated to the standard representation (see (18)) multiplied by T. Therefore the difference

$$|C - T.id_{2N+1}| - |A - T.id_{2N}|\,T = T^{2N+1} + \sum_{k=1}^{N} C_{2k+1} T^{2N+1-2k},$$

gives the Casimir operators of the inhomogeneous algebra. ■

The advantage of this determinantal procedure for the invariants of $\mathfrak{sp}(2N,\mathbb{R})$ in comparison with the contraction or embedding methods usually applied in the literature is remarkable, since we are only using the structure tensor of the algebra, and not specific realisations of it.

4. Relation of C to a faithful representation of $I\mathfrak{sp}(2N,\mathbb{R})$

The formula (33) for the invariants of the Lie algebras $I\mathfrak{sp}\,(2N,\mathbb{R})$ is of special interest, not only because of its simplicity, but also because, in a certain sense, the matrix C of (34) used for its computation is related to a faithful representation of the inhomogeneous algebras.

Proposition 4 *For $N \geq 2$ the matrix C of* (34) *decomposes as*

$$C = M_N \begin{pmatrix} id_{2N} & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & .. & 0 & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & .. & 0 & 0 \\ q_1 & .. & q_N & -p_1 & .. & -p_N & 0 \end{pmatrix}, \tag{40}$$

where

$$M_N := \begin{pmatrix} x_{1,1} & .. & x_{1,N} & -x_{-1,1} & .. & -x_{-1,N} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{N,1} & .. & x_{N,N} & -x_{-1,N} & .. & -x_{-N,N} & p_N \\ x_{1,-1} & .. & x_{1,-N} & -x_{1,1} & .. & -x_{N,1} & q_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{1,-N} & .. & x_{N,-N} & -x_{1,N} & .. & -x_{N,N} & q_N \\ 0 & .. & 0 & 0 & .. & 0 & 0 \end{pmatrix}. \tag{41}$$

Moreover, M_N defines a $(2N+1)$-dimensional faithful representation of $I\mathfrak{sp}\,(2N,\mathbb{R})$.

Proof. The proof of the decomposition follows at once. Let $I_{i,j}$ be the elementary matrix whose entry is 1 in the ith row and j^{th} column, and zero elsewhere. For the generators $(8)-(10)$ define the mapping‖

‖ According to the contraction (32), the brackets for the generators P_i' and Q_i' of $I\mathfrak{sp}(2N,\mathbb{R})$ are given by

$$[P_i', Q_i'] = \lim_{t\to\infty} [P_i', Q_i'] = 0.$$

$$\begin{array}{ll} \Phi\left(X_{i,j}\right)=I_{ij}-I_{j+N,i+N} & 1\leq i\leq j\leq N \\ \Phi\left(X_{-i,i}\right)=I_{i,N+i};\ \Phi\left(X_{i,-i}\right)=I_{N+i,i}; & 1\leq i\leq N \\ \Phi\left(X_{-i,j}\right)=-\left(I_{i,N+j}+I_{j,N+i}\right); & 1\leq i<j\leq N \\ \Phi\left(X_{i,-j}\right)=\left(I_{N+i,j}+I_{N+j,i}\right); & 1\leq i<j\leq N \\ \Phi\left(P_i'\right)=I_{i,2N+1};\ \Phi\left(Q_i'\right)=I_{N+i,2N+1}; & 1<i\leq N \end{array} \tag{42}$$

It is straightforward to verify that the matrix commutator satisfies the relations (11)-(16), thus define a representation of $I\mathfrak{sp}\left(2N,\mathbb{R}\right)$. Since no element is mapped onto the zero matrix, Φ is faithful. ■

We thus recover the usual standard representation of $I\mathfrak{sp}(2N,\mathbb{R})$. Equation (40) can be seen as the adaptation of the classical Gel'fand method to the computation of invariants of inhomogeneous algebras.

5. Applications to other inhomogeneous Lie algebras

The preceding sections show how the Casimir operators of the semidirect products $w\mathfrak{sp}(N,\mathbb{R})$ and the inhomogeneous Lie algebras $I\mathfrak{sp}(2N,\mathbb{R})$ can be obtained easily by evaluation of certain determinants. In this section we show that the matrix method can be extended to other algebras which are neither semidirect products with a Heisenberg algebra nor contractions of such products. This constitutes an evidence that the procedure is not dependent on the special case of the algebras $w\mathfrak{sp}(N,\mathbb{R})$ treated, but holds for a wide class of semidirect products. Recall that the main idea in the $w\mathfrak{sp}(2N,\mathbb{R})$ case is the existence of a copy of the Levi part $\mathfrak{sp}(2N,\mathbb{R})$ in the enveloping algebra of $w\mathfrak{sp}(N,\mathbb{R})$, while in the second we used the fact that $I\mathfrak{sp}(2N,\mathbb{R})$ is obtained as a direct summand of a certain Inönü-Wigner contraction of $w\mathfrak{sp}(N,\mathbb{R})$.

Now let us consider the kinematical algebras in (3+1)-dimensions [21]. Over the generators $\{J_i,K_i,P_i,H\}_{1\leq i\leq 3}$ the nonzero brackets of the Poincaré Lie algebra $I\mathfrak{so}(3,1)$ are given by:

$$\begin{array}{lll} [J_i,J_j]=\varepsilon_{ijk}J_k; & [J_i,P_j]=\varepsilon_{ijk}P_k; & [J_i,K_j]=\varepsilon_{ijk}K_k; \\ [H,K_i]=P_i; & [K_i,K_j]=-\varepsilon_{ijk}J_k; & [P_i,K_i]=H. \end{array} \tag{43}$$

In particular, $\{J_i,K_i\}$ generate the Lorentz algebra, whose Casimir operators are easily obtained using the Gel'fand method [15, 22]. If

$$\widehat{A}=\begin{pmatrix} 0 & j_3 & j_2 & -k_1 \\ -j_3 & 0 & j_1 & k_2 \\ -j_2 & -j_1 & 0 & -k_3 \\ -k_1 & k_2 & -k_3 & 0 \end{pmatrix}, \tag{44}$$

then

$$\left|\widehat{A}-Tid_4\right|=T^4+\left(j^\alpha j_\alpha-k^\alpha k_\alpha\right)T^2-\left(j^\alpha k_\alpha\right)^2 \tag{45}$$

Since the direct sum of the Poincaré algebra and $\mathbb{R}$ cannot be obtained from an eleven dimensional perfect Lie algebra with Heisenberg radical, the method of the enveloping algebra combined with contractions is not applicable to the inhomogeneous Lorentz group. However, the Casimir operators can still be obtained directly using the preceding matrix method. Considering the matrix

$$D=\begin{pmatrix} 0 & j_3 & j_2 & -k_1 & p_1T \\ -j_3 & 0 & j_1 & k_2 & -p_2T \\ -j_2 & -j_1 & 0 & -k_3 & p_3T \\ -k_1 & k_2 & -k_3 & 0 & hT \\ p_1 & -p_2 & p_3 & -h & 0 \end{pmatrix} \tag{46}$$

and the determinant $|D-Tid_5|$, it can be easily verified that the polynomial

$$P=|D-Tid_5|-T\left|\widehat{A}-Tid_4\right|=T^4\left(h^2-p_\alpha p^\alpha\right)+\left\{p_\alpha p^\alpha\left(k_\beta k^\beta+k_\gamma k^\gamma-j_\alpha j^\alpha\right)\right\}T$$
$$+T\left\{-2\left(j_\alpha p^a j_\beta p^\beta+p_\alpha k^\alpha p_\beta k^\beta\right)+h^2 j_\alpha j^\alpha+2h\begin{vmatrix} k_1 & k_2 & k_3 \\ p_1 & p_2 & p_3 \\ j_1 & j_2 & j_3 \end{vmatrix}\right\}, \tag{47}$$

where $\alpha\neq\beta\neq\gamma$, allows us to recover the familiar Casimir operators m^2 and $W^2=W_\mu W^\mu$ determining the mass and spin of a particle, where W_μ is the Pauli-Lubanski spin operator. Again, we have that the matrix D decomposes as

$$D=\begin{pmatrix} 0 & j_3 & j_2 & -k_1 & p_1 \\ -j_3 & 0 & j_1 & k_2 & -p_2 \\ -j_2 & -j_1 & 0 & -k_3 & p_3 \\ -k_1 & k_2 & -k_3 & 0 & h \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & T \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ p_1 & -p_2 & p_3 & -h & 0 \end{pmatrix}, \tag{48}$$

where the first matrix on the right side defines a faithful representation of the Poincaré algebra. This example shows that the matrix method can be applied to a more wide class of algebras, and does not constitute merely a reformulation of the method developed in [7] for the invariants of semidirect products of simple and Heisenberg Lie algebras. Obviously equation (45) could also be used to determine the invariants of the Galilei algebra $G(2)$ via the contraction

$$I\mathfrak{so}\,(3,1)\rightsquigarrow G(2) \tag{49}$$

determined by the change of basis

$$K_i=\frac{1}{\sqrt{t}}K_i,\;P_i=\frac{P_i}{\sqrt{t}},\;i=1..3. \tag{50}$$

However, the limiting procedure can again be avoided, and the Casimir operators result from the following determinants:

$$P(k_\alpha,p_\beta):=\begin{vmatrix} -T & 0 & 0 & -k_1 & p_1T \\ 0 & -T & 0 & k_2 & -p_2T \\ 0 & 0 & -T & -k_3 & p_3T \\ -k_1 & k_2 & -k_3 & -T & hT \\ p_1 & -p_2 & p_3 & -h & -T \end{vmatrix}-\begin{vmatrix} -T & 0 & 0 & -k_1 \\ 0 & -T & 0 & k_2 \\ 0 & 0 & -T & -k_3 \\ -k_1 & k_2 & -k_3 & -T \end{vmatrix}=$$
$$=-T^2\left(T^2p_\alpha p^\alpha+2p_\alpha k^\alpha p_\beta k^\beta+k_\alpha k^a p_\beta p^\beta\right). \tag{51}$$

The independence from the variables j_α spanning the $\mathfrak{so}(3)$-Levi part follows at once from the space isotropy [4]. This implies moreover that a matrix decomposition similar to that of (48) is not associated to a faithful representation of the Galilei algebra, due to the absence of the variables corresponding to the rotation generators J_α.

Conclusions

The main purpose has been to point out a formal matrix method to determine the Casimir operators of the Lie algebras $w\mathfrak{sp}(N,\mathbb{R})$ and $I\mathfrak{sp}(2N,\mathbb{R})$. Traditionally the invariants of the semidirect product $w\mathfrak{sp}(N,\mathbb{R})$ are computed by exhibiting a copy of its Levi part in the enveloping algebra, to which the classical formulae for $\mathfrak{sp}(2N,\mathbb{R})$-invariants is applied. The method has been refined and simplified by constructing a $(2N+1)\times(2N+1)$-matrix B depending on a parameter T whose determinant $|B-T-id_{2N+1}|$ gives the searched invariants. This procedure has the advantage of avoiding the step involving the new generators generating the copy in the enveloping algebra, and seems more practical for obtaining explicit expressions of the invariants. Taking into account the contraction $w\mathfrak{sp}(N,\mathbb{R})\rightsquigarrow I\mathfrak{sp}(2N,\mathbb{R})\oplus\mathbb{R}$, we have obtained a similar matrix for the inhomogeneous symplectic algebras $I\mathfrak{sp}(2N,\mathbb{R})$. With this matrix, the corresponding Casimir operators can also be computed directly. The important fact is the parameter used in matrices also appears in the determinant, so that we cannot speak strictly of characteristic polynomials. This is due to the non-semisimplicity of the analyzed algebras. However, for the inhomogeneous algebras $I\mathfrak{sp}(2N,\mathbb{R})$, the method has an interesting consequence, namely its relation to a faithful representation. The factorization (40) can thus be interpreted as a kind of generalization of the traditional matrix methods for the study of semisimple Lie algebras.

Although the method arises primarily from the analysis of semidirect products of symplectic algebras with Heisenberg algebras, it can also be applied to cases where the intrinsic procedure of [7] is no more valid, such as the kinematical algebras in (3+1) dimensions. Under some circumstances we can still find a faithful representation of the Lie algebra associated to the matrix giving the invariants. However, the existence of such a representation can only be deduced when the Casimir operators are dependent on all generators of the algebra, as shows the example with the Galilei algebra.

The matrix method developed here can not only be used to analyze different inhomogeneous algebras and their contractions, but has potential interest in the analysis of missing label operators [23]. To this extent, consider the symplectic algebra $\mathfrak{sp}(4,\mathbb{R})$ generated by the operators $\left\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\right\}$ for $i,j=1,2$ and the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ generated by $\left\{a_1^\dagger a_1, a_1^\dagger a_1^\dagger, a_1 a_1\right\}$. According to [24], the number of missing labels for the chain $\mathfrak{sl}(2,\mathbb{R})\hookrightarrow\mathfrak{sp}(4,\mathbb{R})$ is given by

$$n=\frac{1}{2}\left(\dim\mathfrak{sp}(4,\mathbb{R})-\mathcal{N}(\mathfrak{sp}(4,\mathbb{R}))-\dim\mathfrak{sl}(2,\mathbb{R})-\mathcal{N}(\mathfrak{sl}(2,\mathbb{R}))\right)=2, \tag{52}$$

there are thus 4 available missing label operators. Using the basis $(8)-(10)$, these operators are obtained from the subsystem formed by the following equations:

$$2x_{-1,1}\frac{\partial}{\partial x_{-1,1}}-2x_{1,-1}\frac{\partial}{\partial x_{1,-1}}-x_{-1,2}\frac{\partial}{\partial x_{-1,2}}-x_{1,-2}\frac{\partial}{\partial x_{1,-2}}+x_{1,2}\frac{\partial}{\partial x_{1,2}}-x_{2,1}\frac{\partial}{\partial x_{2,1}}=0 \quad (53)$$

$$-2x_{-1,1}\frac{\partial}{\partial x_{1,1}}-4x_{1,1}\frac{\partial}{\partial x_{1,-1}}-2x_{1,2}\frac{\partial}{\partial x_{1,-2}}-2x_{-1,2}\frac{\partial}{\partial x_{2,1}}=0 \quad (54)$$

$$2x_{1,-1}\frac{\partial}{\partial x_{1,1}}+4x_{1,1}\frac{\partial}{\partial x_{-1,1}}+2x_{2,1}\frac{\partial}{\partial x_{-1,2}}+2x_{1,-2}\frac{\partial}{\partial x_{1,2}}=0. \quad (55)$$

Instead of integrating it (although this case is extremely simple), we observe that the variables $x_{2,2}, x_{-2,2}$ and $x_{2,-2}$ do not appear in the differentials $\frac{\partial}{\partial x_{i,j}}$¶, thus can be taken as solutions of (53) − (55). The fourth independent solution is obtained from the characteristic polynomial of the matrix

$$\mathcal{M}=\begin{pmatrix} x_{1,1} & x_{1,2} & -x_{-1,1} & -x_{-1,2} \\ x_{2,1} & 0 & -x_{-1,2} & 0 \\ x_{1,-1} & x_{1,-2} & -x_{1,1} & -x_{2,1} \\ x_{1,-2} & 0 & -x_{1,2} & 0 \end{pmatrix}. \quad (56)$$

Observe that $\mathcal{M}$ is nothing but the matrix A of (18), but where the entries corresponding to $x_{2,2}, x_{-2,2}$ and $x_{2,-2}$ have been replaced by zero. Now

$$|M-Tid_4|=T^4+\left(2\left(x_{1,-2}x_{-1,2}-x_{1,2}x_{2,1}\right)+\left(x_{1,-1}x_{-1,1}-x_{1,1}^2\right)\right)T^2+\left(x_{1,-2}x_{-1,2}-x_{1,2}x_{2,1}\right)^2. \quad (57)$$

Taking $I_1=(x_{1,-2}x_{-1,2}-x_{1,2}x_{2,1})$, we obtain a fourth independent missing label operator. Observe further that the coefficient of T^2 is nothing but

$$\left(2\left(x_{1,-2}x_{-1,2}-x_{1,2}x_{2,1}\right)+\left(x_{1,-1}x_{-1,1}-x_{1,1}^2\right)\right)=2I_1+C_2, \quad (58)$$

where C_2 is the quadratic Casimir operator of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra. A more interesting example is the chain $\mathfrak{sp}(4,\mathbb{R})\hookrightarrow\mathfrak{sp}(6,\mathbb{R})$. In this case, there are $m=6$ available missing operators, three of which can be taken to be the variables $x_{3,3}, x_{-3,3}$ and $x_{3,-3}$. The other three operators can be obtained from the characteristic polynomial of the matrix

$$\mathcal{M}_1=\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & -x_{-1,1} & -x_{-1,2} & -x_{-1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} & -x_{-1,2} & -x_{-2,2} & -x_{-2,3} \\ x_{3,1} & x_{3,2} & 0 & -x_{-1,3} & -x_{-2,3} & 0 \\ x_{1,-1} & x_{1,-2} & x_{1,-3} & -x_{1,1} & -x_{2,1} & -x_{3,1} \\ x_{1,-2} & x_{2,-2} & x_{2,-3} & -x_{1,2} & -x_{2,2} & -x_{3,2} \\ x_{1,-3} & x_{2,-3} & 0 & -x_{1,3} & -x_{2,3} & 0 \end{pmatrix}. \quad (59)$$

We obtain

$$|\mathcal{M}_4-Tid_6|=T^6+C_2T^4+C_4T^2+C_6, \quad (60)$$

where C_{2i} is a polynomial of degree $2i$ for $i=1,2,3$. Further, for example,

$$C_2=2\left(x_{1,-3}x_{-1,3}+x_{2,-3}x_{-2,3}-x_{3,1}x_{1,3}-x_{2,3}x_{3,2}\right)+P_2, \quad (61)$$

¶ This is obvious, since the operators $X_{2,2}=a_2^{\dagger}a_2$, $X_{-2,2}=a_2^{\dagger}a_2^{\dagger}$ and $X_{2,-2}=a_2a_2$ generate an independent copy of $\mathfrak{sl}(2,\mathbb{R})$.

where P_2 is the quadratic Casimir operator of $\mathfrak{sp}(4,\mathbb{R})$ over the given basis. The first nontrivial missing label operator can thus be taken as $L_1 = C_2 - P_2$. Symplifying C_4 and C_6 in analogous manner, we obtain two other independent missing label operators of degrees four and six, respectively. In general, such operators can be obtained whenever we consider a chains $\mathfrak{k} \hookrightarrow \mathfrak{g}$ where the subsystem of (4) corresponding to the generators of $\mathfrak{k}$ are not dependent on all variables associated to the generators of the algebra $\mathfrak{g}$ and there exist polynomial solutions$^+$. However, it must be observed that not all available missing label operators must arise by this technique. This can be easily illustrated. Take again $\mathfrak{sp}(4,\mathbb{R})$ and the two dimensional subalgebra $\mathfrak{g}$ generated by $X_{1,1}$ and $X_{-1,1}$. Then the available missing label operators are $m = 6$. They are obtained from the equations (53) and (54). Again the equations do not depend on $x_{2,2}, x_{-2,2}$ and $x_{2,-2}$, so we can again use matrix (56) . We thus obtain the polynomial of (57) :

$$T^4 + \left(2\left(x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1}\right) + \left(x_{1,-1}x_{-1,1} - x_{1,1}^2\right)\right)T^2 + \left(x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1}\right)^2 . \quad (62)$$

Taking $I_1 = (x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1})$, we obtain an independent missing label operator. Since $\mathfrak{k}$ has none invariants, the coefficient of T^2 in (62) provides another independent solution of the system, namely $I_2 = x_{1,-1}x_{-1,1} - x_{1,1}^2$, which could be taken as the fifth missing label operator. But there is no possibility of obtaining a sixth independent operator by this method.

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$^+$ Taking the chain $\mathfrak{u}(N) \hookrightarrow sp(2N,\mathbb{R})$, we get no missing label operators, since the equations associated to the generators of $\mathfrak{u}(N)$ involve all generators of the symplectic algebra

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