QUANTUM RESONANCES WITHOUT ANALYTICITY

by

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Abstract. — We propose a definition for the resonances of Schrödinger operators with slowly decaying $\mathcal{C}^\infty$ potentials without any analyticity assumption. Our definition is based on almost analytic extensions for these potentials, and we describe a systematic way to build such an extension that coincide with the function itself whenever it is analytic. That way, if the potential is dilation analytic, our resonances are the usual ones. We show that our resonances with negative real part are exactly the eigenvalues of the operator. We also prove that our definition coincides with the usual ones in the case of smooth exponentially decaying potentials.

Then we consider semiclassical results. We show that, if the trapped set for some energy E is empty, there is no resonance in any complex vicinity of E of size $O(h\log(1/h))$. Finally, we investigate the semiclassical shape resonances and generalize some results of Helffer and Sjöstrand.

1. Introduction

The mathematical theory of quantum resonances has a rather long history, and the first rigorous definition is probably the one given in [**1**] in the case of one-body dilation analytic potentials, immediately generalized to many-body problems in [**2**]. Of course, the physical notion of quantum resonances is much older, and probably goes back to the early years of quantum mechanics itself.

This first mathematical definition was based on the notion of complex scaling and was very satisfactory from the point of view of abstract spectral theory

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as well as from the point of view of physical intuition. In particular, the resonances obtained in this way coincide with the poles (defined in a convenient and sufficiently convincing way) of the resolvent of the quantum Hamiltonian. However, somehow the rigidity of the definition did not permit to obtain more concrete results (such as the precise location of resonances) and has made necessary to find more flexible tools.

A first attempt has been made in [**20**] with the notion of exterior complex scaling, but one has had to wait the notion of analytic distortion introduced in [**11**] to really have a flexible way of defining and studying resonances. In the same year appeared the work [**8**], where this flexibility has even been pushed further, allowing very general complex deformations in phase space, and permitting in this way to obtain very precise results on the location of resonances (in the semiclassical limit) in relation with the geometry of the underlying classical flow.

However, all these theories of resonances required the analyticity of the potential, at least outside an arbitrarily large compact region of the position space. The only case where the potential is not assumed to be analytic is when it is exponentially decaying at infinity, that is its Fourier transform is analytic in a strip. In that case, working with a complex deformation in the momentum space only, it is still possible to construct a theory of resonances that involves the poles of the resolvent of the Schrödinger operator: See [**19, 4, 18, 17**].

Now, the point is that, from the physical point of view, resonances (at least those close enough to the real line) seem to exist even without assumption of analyticity or exponential decay at infinity on the potential. Indeed, physicist relate their existence to the behavior of various spectral quantities such as the phase shift of the scattering matrix (the definition of which only requires polynomial decay at infinity on the potential). Moreover, the absence of resonances near some energy level is traditionally related to the absence of trapped trajectories of the underlying classical flow at this level: Although this has been rigorously proved for analytic potentials only, it is clear that the condition on the classical flow has nothing to do with such an analyticity property.

Therefore, it has seemed necessary to us to investigate a possible definition of resonances without analyticity assumptions on the potential or its Fourier transform. Obviously, such a definition will necessarily contain some arbitrary aspect, since there is no way in general to extend the resolvent or the scattering matrix to complex values of the energy (and thus to be able to involve poles of such quantities). From the mathematical point of view, this arbitrary aspect will appear in the choice of an extension of the potential in the complex domain, that we will try to choose as "holomorphic" as possible by taking a so-called almost-analytic extension (in the sense of [**10**]). Such an extension is defined up to $O(\theta^\infty)$ in a complex sector of angle θ around $\mathbb{R}^n$, and is therefore subject to modifications that can affect the resonances themselves (defined by using the point of view of analytic distortion as in [**11**]). However, in order to minimize the effects of such possible modifications, we have taken care of two things:

i) Finding a systematic way of constructing the almost analytic extension, so that it becomes the holomorphic extension when the potential is analytic;
ii) Performing the distortion at a larger and larger distance R from the origin, and taking the limit $R \to +\infty$.

With these two cares, our resonances coincide with the usual ones whenever they exist (dilation analytic or exponentially decaying potentials), at least if one considers only those in a complex sector of angle $\theta = O(h \ln 1/h)$ around the real line.

Concerning the location of resonances in the semiclassical regime ($h \to 0$), we have investigated the cases of non-trapping energy level and shape resonances. In these two cases, we have been able to generalize the results proved in the analytic case, at least if $R = R(h) \to +\infty$ in a suitable way as $h \to 0$.

Surprinsingly, the latter case shows that our resonances are indeed defined with a precision that may be much better than $O(h^\infty)$, since the accuracy is then of order $e^{-2S_0/h}$, $S_0 > 0$ denoting the Agmon distance between the potential well and the non-trapping region.

Of course, it would be very interesting to link our resonances with the possible peaks of the scattering phase shift, as in the case of analytic potentials (where one has Breit-Wigner formulas at disposal). This is probably a difficult problem, and a positive answer would give an additional justification to our definition.

To end this introduction, let us indicate that a previous attempt has been made in [**7**] to define resonances in the semiclassical context for smooth potentials. In contrast with our approach, the one of [**7**] relies very much on the semiclassical aspect, and not at all on the behavior at infinity. As an advantage, their definition permits to study the time evolution in the vicinity of a resonance, in the limit $h \to 0$. However, it is not clear to us in which sense such a semiclassical definition can be related to the standard notion of resonances.

The paper is organized as follows. In the next section we give our assumptions, and we state the main results. We describe our almost analytic extension procedure in Section 3. Section 5 is devoted to the proof of Theorem 2.4, and Section 5 to that of Theorem 2.5. Then we consider the case of exponentially decaying potentials in Section 6. Section 7 consists in the proof of Theorem 2.6, and eventually, we consider the case of shape resonances in Section 8.

2. Assumptions and Main Results

We propose a definition for the resonances of Schrödinger operators

$$P(h) = -h^2\Delta + V, \tag{2.1}$$

with decaying $\mathcal{C}^\infty$ potentials V, without further analyticity assumption. Instead, we suppose that

(A1) The real-valued function V belongs to $L^1_{loc}(\mathbb{R}^n)$, and $V(\Delta + i)^{-1}$ is a compact operator.

(A2) There exists a real number $R_0 = R_0(V) > 0$ such that V is $\mathcal{C}^\infty$ in a neighborhood of $W(R_0) = \{x \in \mathbb{R}^n, |x| \geq R_0\}$.

(A3) For any $\alpha \in \mathbb{N}^n$, as $|x| \to \infty$,

$$|x^\alpha \partial_x^\alpha V(x)| = o(1).$$

Notice that, since V is supposed to be Δ-compact, it is Δ-bounded with relative bound 0, and Kato-Rellich theorem shows that the operator $P(h)$ on $L^2(\mathbb{R}^n)$ is self-adjoint with domain $H^2(\mathbb{R}^n)$.

Our definition is based on the notion of almost analytic extension. We recall that a function $\tilde{f}$ is called an almost analytic extension of a $\mathcal{C}^\infty$ function f on $\mathbb{R}^n$, when $\tilde{f}$ is a $\mathcal{C}^\infty$ function defined in a complex neighborhood of $\mathbb{R}^n \times \{0\}$ such that $\tilde{f}_{|\mathbb{R}^n} = f$ and, for any $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$|\partial_{\bar{z}}\tilde{f}(x+iy)| \leq C_N |y|^N. \tag{2.2}$$

There are many ways to obtain such an extension. This notion has been introduced by Hörmander, whose idea was to adapt the Borel lemma, and set (for a $\mathcal{C}^\infty$ function on $\mathbb{R}$):

$$\tilde{f}(x+iy) = \sum_k \frac{f^k(x)}{k!}(iy)^k \chi(\lambda_k y), \tag{2.3}$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ is such that $\chi = 1$ in a neighborhood of 0, and (λ_k) is an unbounded, quickly enough increasing sequence of real numbers. Another way to obtain an almost analytic extension is to use the Fourier transform, and set

$$\tilde{f}(x+iy) = \psi(x) \int e^{i(x+iy)\xi} \chi(y\xi) \hat{f}(\xi) \frac{d\xi}{2\pi}, \tag{2.4}$$

where $\psi \in \mathcal{C}_0^\infty$ is 1 in a neighborhood of $\operatorname{supp} f$, and χ is as above (see e.g. [**5**], Chapter 9 for details and references).

In the sequel, we describe a procedure that gives an almost analytic extension of V in a complex sector around $\mathbb{R}^n$, in such a way that, if V is dilation-analytic, the extension is nothing else than V itself (notice that it is not the case with the two extensions given above). From now on, we denote by $\tilde{V}$ such an analytic extension of V.

We choose a non decreasing function $\chi \in \mathcal{C}^\infty(\mathbb{R}^+)$ such that $\chi(x) = 0$ for $0 \leq x \leq 1/2$ and $\chi(x) = 1$ for $x \geq 1$. Then we set

$$\chi_R(x) = \chi(\frac{|x|}{R}). \tag{2.5}$$

Notice that the family $(\chi_R)_{R>2R_0}$ of $\mathcal{C}^\infty$ functions depends continuously on R, and that, for any $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha \chi_R = O(R^{-|\alpha|}). \tag{2.6}$$

Then, for some small enough μ, we denote by $\Phi_{R,\mu}$ the $\mathcal{C}^\infty$-diffeomorphism defined on $\mathbb{R}^n$ by

$$\Phi_{R,\mu}(x) = (1 + \mu\chi_R(x))x, \tag{2.7}$$

and we also define the (unitary) distortion operator $U_{R,\mu} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$U_{R,\mu}u(x) = |\det d\Phi_{R,\mu}(x)|^{1/2} u(\Phi_{R,\mu}(x)), \tag{2.8}$$

It is now classical that the family of operators $(U_{R,\mu} P U_{R,\mu}^{-1})_\mu$ can be extended as an analytic family of unbounded operators on $L^2(\mathbb{R}^n)$ for small enough complex μ, and we set, for $R > 2R_0(V)$,

$$P_{R,\theta} = U_{R,i\theta}(-h^2\Delta + \tilde{V})U_{R,i\theta}^{-1} = U_{R,i\theta}(-h^2\Delta)U_{R,i\theta}^{-1} + \tilde{V}(\Phi_{R,i\theta}(x)). \tag{2.9}$$

Sometimes, it will be convenient to say that $P_{R,\theta}$ is the operator $-h^2\Delta + \tilde{V}$ acting on $L^2(\Lambda_{R,\theta})$, where

$$\Lambda_{R,\theta} = \Phi_{R,i\theta}(\mathbb{R}^n). \tag{2.10}$$

It is also well-known that the essential spectrum of $P_{R,\theta}$ is $(1+i\theta)^{-2}\mathbb{R}^+$. In particular $P_{R,\theta}$ have discrete spectrum in the domain $\mathbb{C}\backslash(1+i\theta)^{-2}\mathbb{R}^+$. Moreover, as a one parameter continuous family, the eigenvalues of the operators $P_{R,\theta}$ (with θ fixed) can be enumerated in such a way that each depends continuously on R (see e.g. [**12**], Chap. II, Theorem 5.2). We fix such a numbering, and we denote

$$\Gamma_R(\theta) = \{\rho_j(R,\theta)\ ;\ j \in \mathbb{N}\}, \tag{2.11}$$

where the $\rho_j(R,\theta)$'s are the eigenvalues of $P_{R,\theta}$. We shall call $\Gamma_R(\theta)$ the set of (R,θ)-resonances.

Now we introduce a more satisfactory definition of resonances, getting rid in a natural way of the R-dependency.

Definition 2.1. — *Let $\theta > 0$ be a fixed (small enough) real number. We define the set $\Gamma_\infty(\theta)$ of θ-resonances for $P = -h^2\Delta + V$ as the set of bounded, connected components in $\mathbb{C}\backslash(1+i\theta)^{-2}\mathbb{R}^+$ of*

$$\bigcap_{R'>2R_0(V)} \overline{\bigcup_{R>R'} \Gamma_R(\theta)}.$$

We call multiplicity of the θ-resonance $E \in \Gamma_\infty(\theta)$ the integer number

$$\liminf_{\varepsilon>0}\ \limsup_{R\to+\infty} N(R,\varepsilon),$$

where

$$N(R,\varepsilon) = \#\Gamma_R(\theta) \cap \{z \in \mathbb{C}, \mathrm{dist}(z,E) \leq \epsilon\}.$$

Remark 2.2. — *Instead of this definition, we could have consider as θ-resonances the limit sets $\rho_j^\infty(\theta)$ of the $\rho_j(R,\theta)$'s, given by*

$$\rho_j^\infty(\theta) := \bigcap_{R'>2R_0} \overline{\{\rho_j(R,\theta)\ ;\ R > R'\}}, \tag{2.12}$$

and say that the multiplicity of a resonance E is the integer number

$$\#\{j \in \mathbb{N}\ ;\ \rho_j^\infty(\theta) = E\}. \tag{2.13}$$

Then, as the intersection of a decreasing family of closed connected bounded subsets of $\mathbb{C}$, each resonance would automaticaly be a closed connected bounded subset of $\mathbb{C}$. But it is easy to think of configurations where the set of $\rho_j^\infty(\theta)$'s would depend on the choice of the numbering, and that's why we choose the somewhat more abstract definiton above. Nevertheless, since $E \in \Gamma_\infty(\theta)$ if and only if for any $\rho \in E$, there exist two sequences $(R_j)_j$ and

$(k_j)_j$ such that $R_j \to \infty$ and $\rho_{k_j}(R_j, \theta) \to \rho$ as $j \to +\infty$, it is clear that if, for example, the $\rho_j(R, \theta)$'s are separated uniformly for R large enough, these two points of view lead to the same set of θ-resonances.

In the case of analytic-dilation potentials, it can be shown that $\Gamma(\theta) \subset \Gamma(\theta')$ for any $\theta < \theta' < \theta_0$, where $\Gamma(\theta)$ denotes the usual set of resonances, and θ_0 is the angle of the maximum sector in the complex plane where V extends holomorphically. We do not have such a property here, and the sets $\Gamma_R(\theta)$ and $\Gamma_\infty(\theta)$ also depends on our choice of an almost analytic extension for V, as well as that of the distorsion field $\Phi_{R,\theta}$.

However, since the almost analytic extension of V that we build coincides with V when it is dilation-analytic, the $\rho_j(R, \theta)$'s do not depend on R in that case, and as a first result we have the

Theorem 2.3. — *Suppose that V satisfies assumption (A1) to (A3). Suppose moreover that V is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$, which extends as a holomorphic function in a sector*

$$\mathcal{S} = \{x \in \mathbb{C}\ ;\ |\operatorname{Re} x| > R_0, |\operatorname{Im} x| \leq \delta |\operatorname{Re} x|\},$$

and that (A3) holds in $\mathcal{S}$. Then, for $0 < \theta < \delta/2$ and $R > 2R_0$ large enough, one has $\Gamma_\infty(\theta) = \Gamma_R(\theta) = \Gamma(\theta)$, that is our set of resonances coincides with the usual one in this sector. This is also true for the notion of multiplicity of a resonance.

First, we investigate the a priori location of θ-resonances in the complex plane. As in the dilation-analytic case, we have the

Theorem 2.4. — *Suppose assumptions (A1) to (A3) hold, and let $\theta > 0$ be a small enough fixed number. Then $\Gamma_\infty(\theta) \cap \{z \in \mathbb{C}, |\operatorname{Im} z| > 0\} = \emptyset$.*

The next result follows essentially from the decay properties of the eigenfunctions associated to isolated eigenvalues:

Theorem 2.5. — *Suppose assumptions (A1) to (A3) hold, and let $\theta > 0$ be a small enough fixed number. Then $E \in \Gamma_\infty(\theta)$ is such that $E \cap \{z \in \mathbb{C}, \operatorname{Re} z < 0\} \neq \emptyset$ if and only if E is reduced to a single point of $\sigma_{pp}(P)$. Moreover the multiplicity of E as an eigenvalue of P and that of E as a resonance coincide.*

Then we address the case of $\mathcal{C}^\infty$, exponentially decaying potentials, already studied by S. Nakamura in [**18**] (see also [**19, 4**]). We prove in Section 6 that, in this case, the elements of $\Gamma_\infty(\theta)$ are reduced to a single point, and that as a set of points, $\Gamma_\infty(\theta)$ is exactly the set of resonances defined in [**18**], with same multiplicity.

Last, we turn to semiclassical results, and we consider the sets $\Gamma_R(\theta)$, where $R = R(h) > 1$ is such that

$$\frac{R(h)}{\ln(1/h)} \to 0 \text{ as } h \to 0. \tag{2.14}$$

Notice that, in this way, $R(h)$ may tend to $+\infty$ (rather slowly, though).

We investigate the existence of resonance-free domains. In the dilation analytic case, the general theory of Helffer and Sjöstrand [**8**] gives almost for free that, if, for some energy E, the trapped set $K(E)$ is empty, then there is a fixed (i.e. independent of h) complex neighborhood Ω of E such that, for any $h > 0$ small enough, the semiclassical Schrödinger operator $P(h)$ given in (2.1) has no resonance in Ω. Let us recall that $K(E)$ is defined as

$$K(E) = \{(x,\xi) \in p^{-1}(E), \exp(tH_p)(x,\xi) \not\to \infty, t \to \pm\infty\}, \tag{2.15}$$

where $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical principal symbol of the operator $P(h)$. The existence of a resonance-free domain around the real axis has been extended in [**15**] to the case of Schrödinger operators with potentials that are analytic out of a compact subset of $\mathbb{R}^n$.

Here, we adapt the arguments of [**15**], Section 7 to the present context. Since we need some pseudodifferential calculus, we have to replace the assumptions (A1)-(A3) by the stronger

(A4) V is $\mathcal{C}^\infty$ everywhere on $\mathbb{R}^n$, and, for any $\alpha \in \mathbb{N}^n$, as $|x| \to \infty$,

$$|x^\alpha \partial_x^\alpha V(x)| = o(1).$$

Then we prove the

Theorem 2.6. — *Suppose that V satisfies (A4), and that, for some $E > 0$,*

(A5) *The trapped set $K(E)$ is empty.*

Then, there exist $\eta > 0$ and $\nu > 0$ such that, for any $C > 0$, there exists $h_0 > 0$ such that, for any $0 < h \leq h_0$, and any $R = R(h) > 0$ verifying (2.14), the operator $P(h) = -h^2\Delta + V$ has no $(R(h),\theta)$-resonance in the domain $[E-\eta, E+\eta] - i[-\infty, \nu\theta]$ for $\theta = ChR(h)^{-1}\ln\frac{1}{h}$.

In the same framework, we consider the case of shape resonances, and we generalize the results of [**8**]. In particular, under standard geometrical assumptions (see Section 8), we show that the widths of the corresponding (R,θ)-resonances, with R and θ as in Theorem 2.6, are $O(e^{2(S_0-\epsilon)/h})$. Here S_0 is the

Agmon distance between the potential well and the non-trapping region, and $\epsilon > 0$ is any arbitrarily small number.

3. An Almost Analytic Extension

We describe a procedure that gives an almost analytic extension of a smooth function, in such a way that, if the function is analytic, the extension is nothing else than the function itself. We shall consider the following class of functions.

Definition 3.1. — *A function f on $\mathbb{R}^n$ will be said to belong to $\mathcal{C}^\infty_d(\mathbb{R}^n)$ if there exists a real number $R_0 = R_0(f) > 0$ such that*

i) $f \in L^1_{loc}(\mathbb{R}^n)$,
ii) f *is* $\mathcal{C}^\infty$ *in a neighborhood of* $W(R_0) = \{x \in \mathbb{R}^n, |x| \geq R_0\}$,
iii) For any $\alpha \in \mathbb{N}^n$, *there exists a constant* $C_\alpha > 0$ *such that, for any* $x \in W_{R_0}$,

$$|x^\alpha \partial_x^\alpha f(x)| \leq C_\alpha. \tag{3.1}$$

Let us recall the following representation of the identity (see e.g. [**21**], Section 6).

Lemma 3.2. — *Let $f \in \mathcal{C}^\infty_d(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ such that $|x| > R_0(f)$, and any t close enough to 1, it holds that,*

$$f(tx) = \iint e^{i(t-s)\sigma - \langle\sigma\rangle(t-s)^2/2} a(t-s,\sigma) f(sx) ds \frac{d\sigma}{2\pi}, \tag{3.2}$$

where the function a is defined by $a(t,\sigma) = 1 + \frac{i}{2} t \frac{\sigma}{\langle\sigma\rangle}$, and $\langle\sigma\rangle = (1+\sigma^2)^{1/2}$.

Proof. — Starting from the identity

$$\delta_{t=s} = \int e^{i(t-s)\sigma} \frac{d\sigma}{2\pi}, \tag{3.3}$$

and replacing the contour $\mathbb{R}_\sigma$ by the $(t-s)$-dependent complex contour $\mathbb{R} \ni \sigma \mapsto \sigma + \frac{i}{2}(t-s)\langle\sigma\rangle$, we obtain the (well-known) formula

$$\delta_{t=s} = \int e^{i(t-s)\sigma - \langle\sigma\rangle(t-s)^2/2} a(t-s,\sigma) \frac{d\sigma}{2\pi}\cdot \tag{3.4}$$

The result follows, applying this identity to the smooth function $t \mapsto f(tx)$ (with $|x| > R_0(f)$ fixed, and t in a neighborhood of 1). □

Let $f \in \mathcal{C}_d^\infty(\mathbb{R}^n)$, and $\delta > 0$ small enough such that f is $\mathcal{C}^\infty$ in $W((1-3\delta)R_0)$ (here we use the notation of Definition 3.1). We choose $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \subset [1-3\delta, 1+3\delta]$, and $\chi = 1$ on $[1-2\delta, 1+2\delta]$.

First of all, observe that the absolutely convergent integral

$$\iint e^{i(\tau-s)\sigma-(\tau-s)^2\langle\sigma\rangle/2} f(sx)a(\tau-s,\sigma)(1-\chi(s))ds\frac{d\sigma}{2\pi} \tag{3.5}$$

defines an analytic function of τ in $|\tau-1| < \delta$, and vanishes for τ real because of Lemma 3.2. It is therefore identically 0. Now, with $\tau = t + i\theta$ such that $|\tau - 1| < \delta$, and keeping in the argument of exponential only the terms that are either purely imaginary or real and negative, we set,

$$F(x,\tau,\sigma) = \int e^{i(t-s)\sigma-\langle\sigma\rangle(t-s)^2/2} e^{-i\theta(t-s)\langle\sigma\rangle} a(\tau-s,\sigma)f(sx)\chi(s)ds. \tag{3.6}$$

Integrating by parts with the operator $L(\sigma, D_s)$ defined as

$$L(\sigma, D_s) = (1+|\varphi|^2)^{-1}(1+\bar\varphi D_s), \tag{3.7}$$

where

$$\varphi = -\sigma - i(t-s)\langle\sigma\rangle + \theta\langle\sigma\rangle, \tag{3.8}$$

and using that

$$\partial_s^j(f(sx)) = \nabla^j f(sx)\cdot x^{\otimes j} = O(1), \tag{3.9}$$

we obtain for any $N \in \mathbb{N}$,

$$F(x,\tau,\sigma) = O_N(\langle\sigma\rangle^{-N}), \tag{3.10}$$

uniformly with respect to $|x| > R_0$ and $|\tau - 1| \le \delta$. We denote by $C_N > 0$ the smallest possible constant in (3.10), and we set, for $s \ge 1$,

$$\tilde L(s) = \inf_N C_N s^{-N}. \tag{3.11}$$

We denote $s_0 = \max\{(C_1)^2, e\}$, and for $s > s_0$,

$$L(s) = \exp\left(s \sup_{s_0<t<s}\left(\frac{\ln \tilde L(t)}{t}\right)\right). \tag{3.12}$$

Then of course, $s \mapsto \ln(L(s))/s$ is a non decreasing function, which is left continuous. Also notice that we have chosen s_0 such that

$$\log L(s) \le -\frac{1}{2}\log s, \tag{3.13}$$

since for $s_0 < t < s$, we have $\tilde L(t) \le C_1/t$ and

$$\frac{\ln\tilde L(t)}{t} \le \frac{1}{t}(\ln C_1 - \ln t) < -\frac{1}{2}\frac{\ln t}{t} < -\frac{1}{2}\frac{\ln s}{s}\cdot \tag{3.14}$$

Now for any $C \geq 2\delta s_0$, we define a function $[0, \delta) \ni \theta \mapsto \lambda_C(\theta)$ with values in $(0, +\infty]$ by

$$\lambda_C(\theta) = \sup\{\lambda > s_0 \; ; \; \forall s \in [s_0, \lambda), 2\theta s + \ln L(s) \leq C\}. \tag{3.15}$$

The function λ_C is clearly a non increasing function of θ, and since $\ln L(s) < 0$ by (3.13), we have

$$\lambda_C(\theta) \geq \frac{C}{2\theta}, \tag{3.16}$$

so that, in particular, $\lambda_C(0) = +\infty$.

Furthermore, if we suppose that f is an analytic function in a complex sector around $\mathbb{R}^n$, we also have $\lambda_C(\theta) = +\infty$ for all θ small enough. Indeed, in this case, (3.9) can be written more precisely as

$$|\partial_s^j(f(sx))| \leq C_0^{j+1} j^j, \tag{3.17}$$

and one can see that (with $C' \sim C_0 e$)

$$\frac{\ln(L(s))}{s} \leq -\frac{1}{C'} < 0\cdot \tag{3.18}$$

Thus $2\theta s + \ln L(s) \leq 0$ for any $\theta \in [0, \delta]$ provided δ is small enough.

Now, we build a regularized version of $\lambda_C(\theta)$. For convenience, we work with the function μ_C defined as

$$\mu_C(\theta) := \frac{1}{\lambda_C(\theta)}\cdot \tag{3.19}$$

Notice that we have

$$\mu_C(\theta) = \inf\left\{0 \leq \mu \leq \frac{1}{s_0}; \; \forall s \in [s_0, 1/\mu), \frac{C}{s} - \frac{\ln L(s)}{s} \geq 2\theta\right\}. \tag{3.20}$$

The function μ_C is a non decreasing function of θ with values in $[0, 1]$, and $\mu_C(0) = 0$. Now we show that μ_C is a Lipschitz function, namely that there exists a constant $\kappa > 0$ such that

$$\forall \theta' \leq \theta, \; \mu_C(\theta) \leq \mu_C(\theta') + \kappa(\theta - \theta'). \tag{3.21}$$

Indeed, for $s \in [s_0, 1/\mu_C(\theta')[$, we have, since $s \mapsto L(s)/s$ is non decreasing function,

$$\frac{C}{s} - \frac{\ln L(s)}{s} \geq \frac{C}{s} - \mu_C(\theta') \ln L(\frac{1}{\mu_C(\theta')}). \tag{3.22}$$

But since $s \mapsto \ln L(s)/s$ is left continuous we also have

$$-\mu_C(\theta') \ln L(\frac{1}{\mu_C(\theta')}) \geq 2\theta' - C\mu_C(\theta'), \tag{3.23}$$

so that for $s \in [s_0, 1/\mu_C(\theta')[$,

$$\frac{C}{s} - \frac{\ln L(s)}{s} \geq \frac{C}{s} + 2\theta' - C\mu_C(\theta'). \tag{3.24}$$

Thus it is sufficient to show that for $s \in [s_0, (\mu_C(\theta') + \kappa(\theta - \theta'))^{-1}]$, we have

$$\frac{C}{s} + 2\theta' - C\mu_C(\theta') \geq 2\theta, \tag{3.25}$$

and this is true as soon as $\kappa \geq 2/C$.

Now, it is not difficult to construct a non decreasing $\mathcal{C}^\infty$ function M_C on $[0, \delta/2)$ (with uniformly bounded derivative) such that

$$\mu_C(\theta) \leq M_C(\theta) \leq \mu_C(2\theta). \tag{3.26}$$

This can be made, e.g., by first approximating μ_C by a non decreasing function, linear on $[\frac{1}{k+1}, \frac{1}{k}]$ for any k large enough, and verifying (3.26), and then by regularizing this function. Then, we set,

$$\Lambda_C(\theta) := M_C(\theta)^{-1} \in (0, +\infty]. \tag{3.27}$$

The function $\Lambda_C(\theta)$ is non increasing on $[0, \delta/2)$, and satisfies $\Lambda_C(\theta) \geq \frac{C}{4\theta}$. Moreover

$$|\partial_\theta \Lambda_C(\theta)| = O\left(\Lambda_C(\theta)^2\right) \tag{3.28}$$

uniformly on $[0, \delta/2)$.

Proposition 3.3. — *Let $f \in \mathcal{C}_d^\infty(\mathbb{R}^n)$, and $\delta > 0$ such that f is $\mathcal{C}^\infty$ in $W((1-3\delta)R_0)$. Let also $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \in [1-3\delta, 1+3\delta]$, and $\chi = 1$ on $[1-2\delta, 1+2\delta]$. We set*

$$\mathcal{S}(\delta) = \{x \in \mathbb{C}^n, |\operatorname{Re} x| > R_0, |\operatorname{Im} x| \leq \delta(|\operatorname{Re} x|)\}, \tag{3.29}$$

and for any $x+iy \in \mathcal{S}(\delta)$, we have $x+iy = \tau x$, with $\tau = 1+i\theta$, $|\theta| < \delta$. Then, for $x+iy \in \mathcal{S}(\delta/2)$, we define

$$\tilde{f}(x,y) = \int_{|\sigma| \leq \Lambda_C(|\theta|)} e^{-\theta\sigma} e^{\theta^2\langle\sigma\rangle/2} F(x,\tau,\sigma) \frac{d\sigma}{2\pi} \tag{3.30}$$

where $C \geq 2\delta s_0$ is arbitrary, F is given by (3.6), and $\Lambda_C(|\theta|)$ is defined in (3.27). Then, the function $\tilde{f}$ is an almost analytic extension of f in $\mathcal{S}(\delta/2)$. Moreover, if f is analytic in $\mathcal{S}(\delta)$, then $\tilde{f} = f$ in $\mathcal{S}(\delta/2)$.

Remark 3.4. — *By slightly modifying our construction, it is clear that $\delta/2$ can be replaced by any $\delta' < \delta$.*

Proof. — Thanks to Lemma 3.2, the last property is obvious, as well as the fact that $(\tilde{f})_{|\mathbb{R}^n} = f$, since in those cases, we have $\Lambda_C(|\theta|) = +\infty$ for $|\theta| \leq \delta/2$. The fact that $\tilde{f}$ is a $\mathcal{C}^\infty$ function is also clear, since $\Lambda_C(\theta)$ is a smooth function at any point where it is not $+\infty$. Thus we are left with the proof that $\bar{\partial}\tilde{f}(x+iy) = O_N(|y|^N)$ for any $N \in \mathbb{N}$. Recall first that $G : \tau = (t+i\theta) \mapsto e^{-\theta\sigma}e^{\theta^2\langle\sigma\rangle/2}F(x,\tau,\sigma)$ is an analytic function, so that we only have to estimate

$$I(\tau) = \partial_\theta \Lambda_C(|\theta|)(G(x,\tau,\Lambda_C(|\theta|)) - G(x,\tau,-\Lambda_C(|\theta|))). \tag{3.31}$$

Now, by definition of Λ_C, we have, for any σ such that $s_0 \leq \langle\sigma\rangle \leq \Lambda_C(|\theta|)$,

$$\begin{aligned} -\theta\sigma + \frac{1}{2}\theta^2\langle\sigma\rangle + \frac{3}{4}\ln|F(x,\tau,\sigma)| &\leq (|\theta| + \frac{|\theta|^2}{2})\langle\sigma\rangle + \frac{3}{4}\ln L(\langle\sigma\rangle) \\ &\leq \frac{3}{2}|\theta|\langle\sigma\rangle + \frac{3}{4}\ln L(\langle\sigma\rangle) \leq \frac{3C}{4}, \end{aligned} \tag{3.32}$$

so that

$$|G(x,\tau,\Lambda_C(|\theta|))| \leq e^{3C/4}|F(x,\tau,\Lambda_C(|\theta|)|^{1/4}. \tag{3.33}$$

Therefore, thanks to (3.28) and (3.10), for any $N \in \mathbb{N}$ there exists a constant $M_N > 0$ such that we have,

$$|I(\tau)| \leq M_N \Lambda_C(|\theta|)^{2-N/4}. \tag{3.34}$$

Finally, since $\Lambda_C(|\theta|) \geq C/4|\theta|$, we obtain

$$\partial_{\bar{\tau}}\tilde{f}(x+iy) = O(|\theta|^\infty) = O(|y|^\infty). \tag{3.35}$$

where the last term is uniform with respect to $|x| \geq R_0$. □

4. Imaginary Part of the θ-Resonances

Here we prove Theorem 2.4. We show that for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, with $\|u\|_{L^2} = 1$, we have

$$-\operatorname{Im}\langle P_{R,\theta}u, u\rangle \geq e(R), \tag{4.1}$$

where $e(R) = o(1)$ as $R \to +\infty$. First of all, since $\tilde{V}\,|_{\mathbb{R}^n} = V\,|_{\mathbb{R}^n}$ is real-valued, and $\tilde{V}(x) \to 0$ as $|\operatorname{Re} x| \to \infty$, we have $\operatorname{Im}\tilde{V}(\Phi_{R,i\theta}(x)) = o(1)$ as $R \to +\infty$, and the estimate (4.1) reduces to the computation of $\operatorname{Im}\langle U_{R,i\theta}(-h^2\Delta)U_{R,i\theta}^{-1}u, u\rangle$.

Let us recall from Section 2 that, for μ small enough,

$$\begin{cases} U_{R,\mu}u(x) = J_{R,\mu}(x)u(\Phi_{R,\mu}(x)), \\ \Phi_{R,\mu}(x) = x + \mu\chi(\frac{|x|}{R})x, \ J_{R,\mu}(x) = |\det d\Phi_{R,\mu}(x)|^{1/2}. \end{cases} \tag{4.2}$$

We also write

$$\left\{\begin{array}{l} U_{R,\mu}^{-1}u(x) = K_{R,\mu}(x)u(\Psi_{R,\mu}(x)), \\ \Psi_{R,\mu}(x) = (\Phi_{R,\mu})^{-1}(x),\ K_{R,\mu}(x) = |\det d\Psi_{R,\mu}(x)|^{1/2}. \end{array}\right. \tag{4.3}$$

Now we compute, for $\mu \in \mathbb{R}$ small enough, and $j \in \{1, \dots, n\}$,

$$\langle -U_{R,\mu}h^2\Delta U_{R,\mu}^{-1}u, u\rangle_{L^2(\mathbb{R})} = \sum_{j=1}^{n} \|U_{R,\mu}h\partial_{x_j}U_{R,\mu}^{-1}u\|^2$$

$$= \sum_{j=1}^{n}\int\big[|hdu(x)h\partial_j\Psi_{R,\mu}(\Phi_{R,\mu}(x))|^2 + |J_{R,\mu}(x)h\partial_jK_{R,\mu}(\Phi_{R,\mu}(x))u(x)|^2$$

$$+2\operatorname{Re} hdu(x)\partial_j\Psi_{R,\mu}(\Phi_{R,\mu}(x))J_{R,\mu}(x)h\partial_jK_{R,\mu}(\Phi_{R,\mu}(x))\overline{u(x)}\big]\,dx. \tag{4.4}$$

We notice that

$$d\Phi_{R,\mu}(x) = (1+\mu\chi(\frac{|x|}{R}))I + \mu\chi'(\frac{|x|}{R})\frac{1}{R|x|}x^tx. \tag{4.5}$$

In particular, $d\Phi_{R,\mu}(x)$ is a rank one, symmetric perturbation of $(1+\mu\chi(\frac{|x|}{R}))I$, and we obtain

$$\det d\Phi_{R,\mu}(x) = \left(1+\mu\chi(\frac{|x|}{R})\right)^{n-1}\left(1+\mu\chi(\frac{|x|}{R}) + \mu\chi'(\frac{|x|}{R})\frac{|x|}{R}\right), \tag{4.6}$$

so that $J_{R,\mu}(x) = O(1)$ as $R \to +\infty$. Now we have

$$K_{R,\mu}(x) = |\det d\Phi_{R,\mu}(\Psi_{R,\mu}(x))|^{-1/2}, \tag{4.7}$$

and using the fact that the entries of $d\Phi_{R,\mu}(x)$ are $O(1)$ as $R \to +\infty$, we obtain with (4.6) that, for any j,

$$\partial_jK_{R,\mu}(\Phi_{R,\mu}(x)) = O(\frac{1}{R}). \tag{4.8}$$

Therefore, the two last terms in (4.4) are $o(1)$ as $R \to +\infty$, and we are left with the study of the first term, or

$$q_\mu(x) = \sum_{j=1}^{n}\int |hdu(x)h\partial_j\Psi_{R,\mu}(\Phi_{R,\mu}(x))|^2dx = h^2\int |(d\Phi_{R,\mu}(x))^{-1}\nabla u(x)|^2dx. \tag{4.9}$$

As we have seen, there exists a (smooth) family of orthogonal matrices $M(x)$ such that $d\Phi_{R,\mu}(x) = M(x)D_{R,\mu}(x)M(x)^{-1}$, where

$$D_{R,\mu}(x) = \operatorname{diag}\left(1+\mu\lambda_1(x), \dots, 1+\mu\lambda_n(x)\right), \tag{4.10}$$

and $((1+\mu\lambda_j(x))_{j=1,\dots,n}$ are the eigenvalues of $d\Phi_{R,\mu}(x)$. Notice that, as can be seen in (4.6), one has $\lambda_j(x) \geq 0$ for all $j \in \{1,\dots,n\}$. With these notations, we have

$$\begin{aligned} q_\mu(x) &= h^2 |D_{R,\mu}(x)^{-1} M(x)^{-1} \nabla u(x)|^2 \\ &= h^2 \sum_{j=1}^{n} \frac{1}{(1+\mu\lambda_j(x))^2} |(M(x)^{-1}\nabla u(x))_j|^2. \end{aligned} \tag{4.11}$$

Therefore, if we take the analytic extension with respect to μ of $q_\mu(x)$, and choose $\mu = i\theta$, we see that

$$\operatorname{Im} q_{i\theta}(x) \geq 0, \tag{4.12}$$

and this finishes the proof of Theorem 2.4.

5. Bounded States

Now we prove Theorem 2.5. We essentially follow [**8**], Section 9, although our result is not semiclassical. Indeed, here both θ and h are supposed to be fixed, and we omit them in the notations.

First we suppose that $E \in \sigma_{pp}(P)$, and we denote by Ω a complex neighborhood of E such that $\Omega \cap \sigma(P) = \{E\}$. Let $\{\phi_1,\dots,\phi_m\}$ be a set of normalized eigenfunctions in $L^2(\mathbb{R}^n)$ associated with the eigenvalue E, and K the corresponding eigenspace. We also denote by K' the orthogonal complement of K in $L^2(\mathbb{R}^n)$. We recall that, by Agmon-type estimates, we have for some $C > 0$ and for any $j \in \{0,\dots,m\}$,

$$\phi_j(x) = O(e^{-\langle x\rangle/C}). \tag{5.1}$$

Notice that, in the sequel, when we write error terms like $O(e^{-R/C})$, the constant C may change from line to line.

Let $R > 2R_0(V)$. For $z \in \Omega$, we investigate the possible invertibility of $P_R - z$. First, for any $z \in \Omega$, the operator $P'(z) = (P-z)_{|_{K'}} : K' \to K'$ is invertible.

We choose $\psi \in \mathcal{C}_0^\infty$ such that $\psi(x) = 1$ in B_{R-1}, $\psi(x) = 0$ out of B_R, and $\nabla\psi$ is uniformly bounded with respect to R. Here, and in what follows, for $r > 0$, we denote by B_r the open set

$$B_r = \{x \in \mathbb{R}^n \,;\, |x| < r\}. \tag{5.2}$$

Notice that if $u \in L^2(\Lambda_R)$ or $u \in L^2(\mathbb{R}^n)$, then $\psi u \in L^2(\mathbb{R}^n) \cap L^2(\Lambda_R)$.

Since $V \to 0$ at infinity and $E < 0$, one can find a real-valued smooth function W with compact support, such that

$$\inf_{x \in \mathbb{R}^n} V(x) + W(x) > E. \tag{5.3}$$

We can suppose that R is large enough, so that $\operatorname{supp}(W) \subset B_{R/4}$. Thus, if we set $Q_R = P_R + W$, we have

$$Q_R(1-\psi) = P_R(1-\psi). \tag{5.4}$$

Also we can arrange so that $Q_R - z$ and $P'(z)$ are invertible for $z \in \Omega$, and we have the following classical estimate (see e.g. [**9**]):

Lemma 5.1. — *Let $\psi_{1,R}, \psi_{2,R} \in \mathcal{C}^\infty(\mathbb{R}^n)$ be bounded functions on $\mathbb{R}^n$, uniformly with respect to R large enough, such that* $\operatorname{dist}(\operatorname{supp}\psi_{1,R}, \operatorname{supp}\psi_{2,R}) > R/C_0$ *for some constant $C_0 > 0$. There is a constant $C > 0$ such that, for any $z \in \Omega$,*

$$\|\psi_{1,R}(Q_R - z)^{-1}\psi_{2,R}\| \leq Ce^{-R/C},$$

and

$$\|\psi_{1,R}P'(z)^{-1}\psi_{2,R}\| \leq Ce^{-R/C},$$

uniformly for R large enough.

We choose $\tilde{\psi} \in \mathcal{C}_0^\infty(B_{R/2}, [0,1])$ such that $\tilde{\psi} = 1$ on $B_{(R-1)/2}$. We denote by $\mathcal{R}_- : \mathbb{C}^m \to L^2(\Lambda_R)$ and $\mathcal{R}_+ : L^2(\Lambda_R) \to \mathbb{C}^m$ the operators defined by

$$\mathcal{R}_- u_- = \psi \sum_{j=1}^m u_{-,j}\phi_j, \qquad \mathcal{R}_+ u = (\langle u, \tilde{\psi}\phi_j\rangle)_{j=1\dots m}, \tag{5.5}$$

where $\langle\ ,\ \rangle$ denotes the scalar product in $L^2(\Lambda_R)$. Thanks to (5.1), we have

$$\mathcal{R}_-\mathcal{R}_+ = \psi\pi_K\tilde{\psi}, \quad \mathcal{R}_+\mathcal{R}_- = I_m + O(e^{-R/C}), \tag{5.6}$$

where I_m is the identity matrix on $\mathbb{C}^m$.

Now we consider the operator $\mathcal{P}_R(z)$ on $L^2(\Lambda_R) \times \mathbb{C}^m$ defined by

$$\mathcal{P}_R(z) = \begin{pmatrix} P_R - z & \mathcal{R}_- \\ \mathcal{R}_+ & 0 \end{pmatrix}. \tag{5.7}$$

We claim that

$$\mathcal{P}_R(z)\mathcal{F}_R(z) = I + \mathcal{K}(z), \text{ with } \mathcal{K}(z) = O(e^{-R/C}), \tag{5.8}$$

where $\mathcal{F}_R(z)$ is the operator on $L^2(\Lambda_R) \times \mathbb{C}^m$ given by

$$\mathcal{F}_R(z) = \begin{pmatrix} \psi P'(z)^{-1}(1-\pi_K)\tilde{\psi} + (Q_R - z)^{-1}(1-\tilde{\psi}) & \mathcal{R}_- \\ \mathcal{R}_+ & (z-E)I_m \end{pmatrix}. \tag{5.9}$$

Indeed, if we set

$$\mathcal{P}_R(z)\mathcal{F}_R(z) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{5.10}$$

we have immediately

$$a_{22} = \mathcal{R}_+\mathcal{R}_- = I_m + O(e^{-R/C}), \tag{5.11}$$

and

$$a_{12} = (P_R-z)\mathcal{R}_- + (z-E)\mathcal{R}_- = (P-z)\mathcal{R}_- + (z-E)\mathcal{R}_- = O(e^{-R/C}). \tag{5.12}$$

Moreover, since $P'(z)^{-1}(1-\pi_K)$ is a bounded operator, and using (5.1), we have

$$\begin{aligned} \mathcal{R}_+\psi P'(z)^{-1}(1-\pi_K)\tilde{\psi} &= (\langle \psi P'(z)^{-1}(1-\pi_K)\tilde{\psi}\ .\ , \tilde{\psi}\phi_j\rangle)_{j=1\dots m} \\ &= (\langle P'(z)^{-1}(1-\pi_K)\tilde{\psi}\ .\ , \tilde{\psi}\phi_j\rangle)_{j=1\dots m} \\ &= -(\langle P'(z)^{-1}(1-\pi_K)\tilde{\psi}\ .\ , (1-\tilde{\psi})\phi_j\rangle)_{j=1\dots m} \\ &= O(e^{-R/C}). \end{aligned} \tag{5.13}$$

With (5.1) and Lemma 5.1, we also have

$$\mathcal{R}_+(Q_R-z)^{-1}(1-\tilde{\psi}) = (\langle (Q_R-z)^{-1}(1-\tilde{\psi})\ .\ , \tilde{\psi}\phi_j\rangle)_{j=1\dots m} = O(e^{-R/C}), \tag{5.14}$$

so that $a_{21} = O(e^{-R/C})$. Finally, we compute

$$a_{11} = (P_R-z)[\psi P'(z)(1-\pi_K)\tilde{\psi} + (Q_R-z)^{-1}(1-\tilde{\psi})] + \mathcal{R}_-\mathcal{R}_+. \tag{5.15}$$

First we have, using again Lemma 5.1,

$$\begin{aligned} (P_R-z)\psi P'(z)^{-1}(1-\pi_K)\tilde{\psi} &= (P-z)\psi P'(z)^{-1}(1-\pi_K)\tilde{\psi} \\ &= \psi(1-\pi_K)\tilde{\psi} + [P,\psi]P'(z)^{-1}(1-\pi_K)\tilde{\psi} \\ &= \psi(1-\pi_K)\tilde{\psi} + O(e^{-R/C}). \end{aligned} \tag{5.16}$$

On the other hand, we have

$$(P_R-z)(Q_R-z)^{-1}(1-\tilde{\psi}) = (1-\tilde{\psi}) - W(Q_R-z)^{-1}(1-\tilde{\psi}), \tag{5.17}$$

so that, with Lemma 5.1, using also (5.6),

$$a_{11} = \psi(1-\pi_K)\tilde{\psi} + (1-\tilde{\psi}) + \psi\pi_K\tilde{\psi} + O(e^{-R/C}) = I + O(e^{-R/C}). \tag{5.18}$$

In the same way, if we denote by $\mathcal{G}_R$ the operator on $L^2(\Lambda_R)\times\mathbb{C}^m$ defined by

$$\mathcal{G}_R(z) = \begin{pmatrix} \tilde{\psi}P'(z)^{-1}(1-\pi_K)\psi + (1-\tilde{\psi})(Q_R-z)^{-1} & \mathcal{R}_- \\ \mathcal{R}_+ & (z-E)I_m \end{pmatrix}, \tag{5.19}$$

we can prove,

$$\mathcal{G}_R(z)\mathcal{P}_R(z) = I + O(e^{-R/C}). \tag{5.20}$$

Thus $\mathcal{P}_R(z)$ is invertible, and its inverse $\mathcal{E}_R(z)$ is a holomorphic function of z, since $\mathcal{F}_R(z)$ is holomorphic with respect to $z \in \Omega$, and

$$\mathcal{E}_R(z) = \mathcal{F}_R(z)(I + \mathcal{K}(z))^{-1}. \tag{5.21}$$

Moreover $\mathcal{E}_R(z) = \mathcal{F}_R(z) + O(e^{-R/C})$. Writing

$$\mathcal{E}_R(z) = \begin{pmatrix} E(z) & E_-(z) \\ E_+(z) & E_{-+}(z) \end{pmatrix}, \tag{5.22}$$

we can easily see that $P_R - z$ is invertible if and only if $E_{-+}(z)$ is invertible, and that we have the so-called Schur complement formula:

$$(P_R - z)^{-1} = E(z) - E_-(z)E_{-+}(z)^{-1}E_+(z). \tag{5.23}$$

Now we have $E_{-+}(z) = (z - E)I_m + O(e^{-R/C})$, so that, by Rouché's Theorem, $\sigma_{pp}(P_R) \cap \Omega$ consists exactly in m complex numbers $\rho_1(R), \dots, \rho_m(R)$ with, for each $j \in \{1, \dots, m\}$,

$$\rho_j(R) - E = O(e^{-R/C}), \tag{5.24}$$

This proves that $\{E\} \in \Gamma_\infty(\theta)$, and that the multiplicity of E as a resonance of P is m.

Now we suppose that $E \in \Gamma_\infty(\theta)$, with multiplicity m, and $E \cap \{z \in \mathbb{C}, \operatorname{Re} z < 0\} \neq \emptyset$. We choose $\rho \in E \cap \{z \in \mathbb{C}, \operatorname{Re} z < 0\}$, and we prove by contradiction that $\rho \in \sigma_{pp}(P)$.

Indeed, if $\rho \notin \sigma_{pp}(P)$, then $P - z$ is invertible for any z close to ρ with bounded inverse. Then we set as before $Q_R = P_R + W$, where W is now such that

$$\inf_{x \in \mathbb{R}^n} V(x) + W(x) > \operatorname{Re}(\rho), \tag{5.25}$$

and we can see, as above, that the operators F_1 and G_1, defined as

$$\begin{cases} F_1 = \psi(P - z)^{-1}\tilde{\psi} + (Q_R - z)^{-1}(1 - \tilde{\psi}), \\ G_1 = \tilde{\psi}(P - z)^{-1}\psi + (1 - \tilde{\psi})(Q_R - z)^{-1}, \end{cases} \tag{5.26}$$

satisfy

$$\begin{cases} (P_R - z)F_1 = I + O(e^{-R/C}), \\ G_1(P_R - z) = I + O(e^{-R/C}). \end{cases} \tag{5.27}$$

As a consequence, $P_R - z$ is invertible with bounded inverse, uniformly for R large enough. Therefore there exists a fixed (i.e. independent of R) neighborhood of ρ that does not intersect $\sigma(P_R)$. In particular ρ cannot belong to the resonance E.

Since E is connected, and $\sigma_{pp}(P)$ is discrete, E is necessarily reduced to a point in $\sigma_{pp}(P)$, and the first part of the proof shows that its multiplicity is also m.

Remark 5.2. — *In fact we have proved a result on $\Gamma_R(\theta)$, namely that, for R large enough, the elements of $\Gamma_R(\theta)$ differs from those in $\sigma_{pp}(P)$ by $O(e^{-R/C})$.*

6. Exponentially Decaying Potentials

This section is devoted to a case already studied by S. Nakamura in [**18**] (see also [**19, 4**]), namely the case where V is exponentially decaying at infinity. Actually, [**18**] addresses the more general case of a sum of a dilation-analytic potential and an exponentially decaying potential. As we shall see in Remark 6.3 below, that case can also be handled, but for now we concentrate on the simpler situation, and we assume that

(A6) V is $\mathcal{C}^\infty$ on $\mathbb{R}^n$ and there exists $\delta > 0$ such that, for any $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha V(x) = O(e^{-\delta\langle x\rangle})$$

uniformly on $\mathbb{R}^n$.

Notice that **(A6)** also implies **(A1)-(A3)**, so that the set $\Gamma_\infty(\theta)$ is well-defined. On the other hand, fixing $E > 0$, one can define the resonances of P near E by a complex deformation in the momentum space in the following way (see, e.g., [**17**]):

We choose $v \in \mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $v(\xi) = \xi$ if $|\xi| \le E - \inf V + 1$, and $v(\xi)\cdot\xi \ge 0$ for any $\xi \in \mathbb{R}^n$, and we define the distortion $\tilde{U}_\mu$ by

$$\tilde{U}_\mu\phi = \mathcal{F}^{-1}(\tilde{J}_\mu(\xi)^{1/2}(\mathcal{F}\phi)(\xi - \mu v(\xi)) \tag{6.1}$$

where $\mathcal{F}$ stands for the semiclassical Fourier transform,

$$\mathcal{F}\phi(\xi) = (2\pi h)^{-n/2}\int e^{-ix\xi/h}\phi(x)dx, \tag{6.2}$$

and $\tilde{J}_\mu(x)$ is the Jacobian of the application $T_\mu : \xi \mapsto \xi - \mu v(\xi)$. Then, (see [**18**]), the family of operators

$$P^{\mu v} := \tilde{U}_\mu P \tilde{U}_\mu^{-1},\ \mu \in \mathbb{R}, \tag{6.3}$$

can be extended to an analytic family of operators for $\mu \in \mathbb{C}$, $|\mu| < \delta_1 h$, where $\delta_1 > 0$ is a small enough constant. Moreover, it is easy to see that

$$\mathcal{F}\tilde{U}_\mu hD_x\tilde{U}_\mu^{-1}\mathcal{F}^{-1} = T_\mu(\xi), \tag{6.4}$$

and that $\sigma_{ess}(P^{\mu v}) = \{(\xi - \mu v(\xi))^2 \,;\, \xi \in \mathbb{R}^n\}$. For $\mu \in i\mathbb{R}_+$, one has

$$|T_\mu(\xi)^2 - E|^2 \geq |\xi^2 - E|^2 + |\mu|^4\xi^4 + 2E|\mu|^2 v(\xi)^2 + 2|\mu|^2(v(\xi).\xi)^2 \tag{6.5}$$

so that

$$|T_\mu(\xi)^2 - E| \geq \frac{|\mu|}{C} \tag{6.6}$$

for some constant $C > 0$. In particular $\sigma(P^{\mu v})$, the spectrum of $P^{\mu v}$, is discrete in a neighborhood of E of size $\sim |\mu|$ when $\mu \in i\mathbb{R}_+$, and its eigenvalues are shown to be the poles of the meromorphic continuation of the resolvent of P starting from $\{z \in \mathbb{C}\,;\, \mathrm{Re}\, z > 0\,,\, \mathrm{Im}\, z > 0\}$ ([**18**], Corollary 1.5).

Now we fix $\mu = i\delta_1' h$ with $\delta_1' < \delta_1$, and $0 < \delta_2 \leq \delta_1'$ such that the spectrum of $P^{\mu v}$ is discrete in

$$\Omega_E(h) = \{z \in \mathbb{C}, |z - E| < \delta_2 h\}. \tag{6.7}$$

We want to prove that, for any fixed small enough h, in $\Omega_E(h)$, the set $\Gamma_\infty(\theta)$ coincides exactly with the spectrum of $P^{\mu v}$ for any $\theta > \delta_1' h$. We shall mimic the proof of Theorem 2.5, working with the pair $(P^{\mu v}, P_{R,\theta})$ instead of $(P, P_{R,\theta})$.

As a matter of fact, we shall compare the operators $P^{\mu v}$ and $(P_{R,\theta})^{\mu v}$, where $(P_{R,\theta})^{\mu v}$ stands for $\tilde{U}_\mu P_{R,\theta}\tilde{U}_\mu^{-1}$, and first of all, we give a meaning to that expression for complex values of μ. We observe that, when **(A6)** is satisfied, our construction of an analytic extension $\tilde{V}$ of V verifies, for all $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha \tilde{V}(x) = O(e^{-\delta'\langle x\rangle}) \text{ in } \{|\,\mathrm{Im}\, x| \leq \theta\langle \mathrm{Re}\, x\rangle\}, \tag{6.8}$$

for any $\delta' \in (0, \delta)$. This can be seen directly on the formulas (3.6) and (3.30). In particular, for θ small enough, the results of [**18**] show that $\tilde{U}_\mu\tilde{V}(\Phi_{R,\theta}(x))\tilde{U}_\mu^{-1}$ can be extended to an analytic family of Δ-compact operators for $\mu \in \mathbb{C}$, $|\mu| < \delta_1'' h$, with $\delta_1'' > \delta_1'$.

Moreover, we can write (see (4.4)),

$$U_{R,i\theta} hD_x U_{R,i\theta}^{-1} = (1 + i\theta)^{-1} hD_x + F_{R,\theta}(x) hD_x + G_{R,\theta}(x), \tag{6.9}$$

where $F_{R,\theta}$ is a $n\times n$-matrix-valued compactly supported function, and $G_{R,\theta}(x)$ is a smooth compactly supported function with values in $\mathbb{R}^n$, as can be seen from (4.6). Therefore we have,

$$\tilde{U}_\mu \left(U_{R,i\theta} hD_x U_{R,i\theta}^{-1}\right)\tilde{U}_\mu^{-1} =$$
$$(1 + i\theta)^{-1}\tilde{U}_\mu hD_x\tilde{U}_\mu^{-1} + \tilde{U}_\mu F_{R,\theta}(x)\tilde{U}_\mu^{-1}\tilde{U}_\mu hD_x\tilde{U}_\mu^{-1} + \tilde{U}_\mu G_{R,\theta}(x)\tilde{U}_\mu^{-1}, \tag{6.10}$$

and $\tilde{U}_\mu h D_x \tilde{U}_\mu^{-1}$, $\tilde{U}_\mu F_{R,\theta}(x)\tilde{U}_\mu^{-1}$ and $\tilde{U}_\mu G_{R,\theta}(x)\tilde{U}_\mu^{-1}$ can be extended analytically (see (6.4)), as well as $\tilde{U}_\mu \left((d\Phi_{R,\theta}(x))^{-1} h D_x\right)^2 \tilde{U}_\mu^{-1}$ for $\mu \in \mathbb{C}$, $|\mu| < \delta_1 h$.

Now, by standard results on such analytic families of operators, the discrete spectrum of $\tilde{U}_\mu P_{R,\theta} \tilde{U}_\mu^{-1}$ does not depend on μ, and, in particular for $\mu = i\delta_1' h$, we have

$$\sigma_{disc}(\tilde{U}_\mu P_{R,\theta}\tilde{U}_\mu^{-1}) = \sigma_{disc}(P_{R,\theta}). \tag{6.11}$$

Therefore $\Gamma_\infty(\theta)$ is indeed obtained by taking the limit as $R \to +\infty$ of the discrete eigenvalues of $(P_{R,\theta})^{\mu v} := \tilde{U}_\mu P_{R,\theta}\tilde{U}_\mu^{-1}$, and we shall prove our result comparing $(P_{R,\theta})^{\mu v}$ and $P^{\mu v}$.

The next step consists in proving some decay properties of the eigenfunctions and of cut-off resolvent of these operators for energies close to E, as in Lemma 5.1.

Proposition 6.1. — *Let ϕ be an eigenfunction of $P^{\mu v}$ for some eigenvalue $z \in \Omega_E(h)$. For any $N \geq 1$, one has,*

$$\phi(x) = O(\langle x\rangle^{-N}).$$

Moreover, if $\psi_{1,R}, \psi_{2,R} \in \mathcal{C}^\infty(\mathbb{R}^n)$ are bounded functions on $\mathbb{R}^n$, uniformly with respect to R large enough, such that $\operatorname{dist}(\operatorname{supp}\psi_{1,R}, \operatorname{supp}\psi_{2,R}) > C_0 R$ *for some constant $C_0 > 0$, then one has for $z \in \Omega_E(h)$ and $z \notin \sigma(P^{\mu v})$,*

$$\|\psi_{1,R}(P^{\mu v} - z)^{-1}\psi_{2,R}\| = O(R^{-N}),$$

and for $z \notin \sigma((P_{R,\theta})^{\mu v})$,

$$\|\psi_{1,R}((P_{R,\theta})^{\mu v} - z)^{-1}\psi_{2,R}\| = O(R^{-N}),$$

uniformly for R large enough.

Proof. — Let χ be a real-valued $\mathcal{C}^\infty$ function with support out of some large enough ball B_{R_1}. For any $z \in \Omega_E(h)$, we define $Q_\chi(z)$ as the selfadjoint operator

$$Q_\chi(z) := \chi(P^{\mu v} - z)^*(P^{\mu v} - z)\chi. \tag{6.12}$$

For any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ $(H^2(\mathbb{R}^n))$, we have

$$\langle Q_\chi(z)u, u\rangle^{1/2} = \|\mathcal{F}(P^{\mu v} - z)\mathcal{F}^{-1}\mathcal{F}\chi u\| = \|(T_\mu(\xi)^2 - z)\mathcal{F}\chi u + B\mathcal{F}\chi u\|, \tag{6.13}$$

where $B = \mathcal{F}U_\mu V U_\mu^{-1}\mathcal{F}^{-1}$. The kernel of B can be explicitely computed:

$$b(\xi,\eta) = \frac{1}{(2\pi h)^n} J_\mu(\xi)^{1/2} J_\mu(\eta)^{1/2} (\mathcal{F}V)(T_\mu(\xi) - T_\mu(\eta)), \tag{6.14}$$

and using (6.8), we obtain that, for any $N > 0$, $\|B\mathcal{F}\chi u\| = \|\chi u\| O(R_1^{-N})$. Finally, recalling (6.6), provided R_1 is large enough (recall that h is held fixed) we have

$$\langle Q_\chi(z)u, u\rangle^{1/2} \geq \frac{|\mu|}{C}\|\chi u\|. \tag{6.15}$$

Now let f be a real-valued function with uniformly bounded derivatives of all positive orders, and $t > 0$ a constant to be chosen small enough. For $z \in \sigma_E(h)$, we set

(6.16)

$$Q_{\chi,f}(z) := \chi(x)(\langle f(tx)\rangle^N P^{\mu v}\langle f(tx)\rangle^{-N} - z)^*(\langle f(tx)\rangle^N P^{\mu v}\langle f(tx)\rangle^{-N} - z)\chi(x),$$

and we want to show that the an estimate like (6.15) holds also for $Q_{\chi,f}(z)$. We have

$$\langle Q_{\chi,f}(z)u, u\rangle = \|\langle f(tx)\rangle^N (P^{\mu v} - z)\langle f(tx)\rangle^{-N}\chi(x)u(x)\|^2, \tag{6.17}$$

and, writing

$$\begin{aligned} &\langle f(tx)\rangle^N (P^{\mu v} - z)\langle f(tx)\rangle^{-N} = \\ &(P^{\mu v} - z) + \left[\langle f(tx)\rangle^N, \tilde{U}_\mu h^2 \Delta \tilde{U}_\mu^{-1}\right]\langle f(tx)\rangle^{-N}, \end{aligned} \tag{6.18}$$

we are left with an estimate for $A := \left[\langle f(tx)\rangle^N, \tilde{U}_\mu h^2 \Delta \tilde{U}_\mu^{-1}\right]\langle f(tx)\rangle^{-N}$. Writing this operator explicitely (using (6.4)), we see that it is $O(th)$. Therefore, if $t < \delta_2$, there exists a constant $C > 0$ such that

$$\langle Q_{\chi,f}(z)u, u\rangle^{1/2} \geq \frac{|\mu|}{C}\|\chi u\|. \tag{6.19}$$

At last, using (6.9), we see that this estimate holds also for $Q_{\chi,f}^{R,\theta}(z)$, defined as in (6.16), but with $(P_{R,\theta})^{\mu v}$ instead of $P^{\mu v}$.

We apply the energy estimate (6.19) with $u = \langle f(tx)\rangle^N \phi(x)$, where ϕ is a normalized eigenfunction of $P^{\mu v}$ for the eigenvalue $z \in \sigma_E(h)$, and with $f(x) = x$. Then we get

$$\|\langle f(tx)\rangle^N (P^{\mu v} - z)\chi\phi\| = \langle Q_{\chi,f}u, u\rangle^{1/2} \geq \frac{|\mu|}{C}\|\chi\langle tx\rangle^N \phi\|. \tag{6.20}$$

Writing the L.H.S. as $\|\langle f(tx)\rangle^N [P^{\mu v}, \chi]\phi\|$, we obtain that, for some constant $C > 0$,

$$\|\chi\langle tx\rangle^N \phi\| \leq \frac{C}{|\mu|}(tR_1)^N. \tag{6.21}$$

Since $\|(1-\chi)\langle tx\rangle^N \phi\|$ is also $O((tR_1)^N)$, we obtain

$$\|\langle x\rangle^N \phi\|_{L^2(\mathbb{R}^n)} = O(1), \tag{6.22}$$

and the first estimate of the proposition then follows from standard arguments of elliptic regularity (see e.g. [**14**], Section 3).

Now we prove the first resolvent estimate. We suppose that $z \in \Omega_E(h)$, $z \notin \sigma(P^{\mu v})$, and we apply (6.19) with $u = \langle f(tx)\rangle^N (P^{\mu v} - z)^{-1}\psi_2(x)v(x)$, with $v \in L^2(\mathbb{R}^n)$. Here, we choose $\chi = \psi_1$, and the function f is a suitable regularization of $dist(\,\cdot\,, \operatorname{supp}\psi_2)$. We obtain

$$\begin{aligned} &\|\langle f(tx)\rangle^N (P^{\mu v} - z)\psi_1 (P^{\mu v} - z)^{-1}\psi_2(x)v(x)\| \geq \\ &\qquad \frac{|\mu|}{C}\|\langle f(tx)\rangle^N \psi_1(x)(P^{\mu v} - z)^{-1}\psi_2(x)v(x)\|, \end{aligned} \tag{6.23}$$

or, since $\psi_1\psi_2 = 0$,

$$\begin{aligned} &\|\langle f(tx)\rangle^N [P^{\mu v}, \psi_1](P^{\mu v} - z)^{-1}\psi_2(x)v(x)\| \geq \\ &\qquad \frac{|\mu|}{C}\|\langle f(tx)\rangle^N \psi_1(x)(P^{\mu v} - z)^{-1}\psi_2(x)v(x)\|. \end{aligned} \tag{6.24}$$

Since $z \notin \sigma(P^{\mu v})$, we have $\|(P^{\mu v} - z)^{-1}\psi_2(x)v(x)\| \leq C\|v\|$, and we get for some $C > 0$ depending on μ, t, R and N,

$$\|\psi_1(x)(P^{\mu v} - z)^{-1}\psi_2(x)v(x)\| \leq C dist(\operatorname{supp}\psi_1, \operatorname{supp}\psi_2)^{-N}\|v\|. \tag{6.25}$$

The other estimates can be obtained along the same lines. □

Finally, proceeding as in Section 5, observing also that $V - \tilde{V}_R = O(e^{-\delta' R})$ uniformly (as well as $(V^{\mu v} - (\tilde{V}_R)^{\mu v})$), we obtain the

Theorem 6.2. — *Let us fix $\theta > \delta_1' h$ small enough. Then, $Z \in \Gamma_\infty(\theta)$ is such that $Z \cap \Omega_E(h) \neq \emptyset$ if and only if Z is reduced to a single point of $\Gamma_E(h)$.*

Remark 6.3. — *Potentials that are sum of a dilation-analytic potential and an exponentially decaying potential: Instead of (A6), we assume now as in* [**17**], *that*

(A6') *$V = V_1 + V_2$ where V_1 and V_2 are such that,*

- *V_1 is holomorphic on a complex sector of the form $S_a := \{x \in \mathbb{C}^n\,;\, |\operatorname{Im} x| < a\langle \operatorname{Re} x\rangle\}$ with $a > 0$, and V_1 tends to 0 as $|\operatorname{Re} x| \to +\infty$ ($x \in S_a$);*
- *V_2 is $\mathcal{C}^\infty$ on $\mathbb{R}^n$, and there exists $\delta > 0$ such that, for any $\alpha \in \mathbb{N}^n$,*

$$\partial^\alpha V_2(x) = O(e^{-\delta\langle x\rangle})$$

uniformly on $\mathbb{R}^n$.

In that case, we have to slightly modify our definition of resonances in the following way: Instead of constructing an almost analytic extension of V as in Section 3, we prefer to do it for V_2 only, and keep V_1 as it is. Then, observing *that* $V_1(\Phi_{R,\theta}(x)) = V_1((1+i\theta)x) + V_3(x)$ *with* $V_3 \in \mathcal{C}_0^\infty$, *we see that we can define again* $(P_{R,\theta})^{\mu v}$, *and the previous arguments show that its eigenvalues near* E *coincide with both those of* $P^{\mu v}$ *and those of* $P_{R,\theta}$. *That, is, our resonances that are in a neighborhood of* E *of size* $\sim h$ *coincide with those defined in* [**18**].

7. Resonance-Free Domains

Here we prove Theorem 2.6, and we assume that the potential V satisfies assumptions (A4) and (A5). As in [**13, 15**], the key point for the proof is a weighted microlocal estimate that we recall now.

We denote by $\mathcal{T}$ the F.B.I. (or Bargmann) transform, given by

$$\mathcal{T}u(x,\xi,h) = 2^{-n/2}(\pi h)^{-3n/4}\int_{\mathbb{R}^n} e^{i(x-y)\xi/h-(x-y)^2/2h}u(y)dy, \tag{7.1}$$

which is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. First, we recall Corollary 3.5.3 of [**16**]:

Proposition 7.1. — *Let* $p \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$ *be a function that extends holomorphicaly in a strip* $\{|\operatorname{Im} x| < a\}$ *for some* $a > 0$. *We suppose that, uniformly in that strip,* $\partial^\alpha_{x,\xi} p(x,\xi) = O_\alpha(\langle\xi\rangle^\ell)$, *for some* $\ell \geq 0$ *and any* $\alpha \in \mathbb{N}^{2n}$.

Let also $\psi \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$ *be a real-valued function. We suppose that* $\partial^\alpha\psi(x,\xi) = O_\alpha(1)$ *for all* $\alpha \in \mathbb{N}^{2n}$, *and that* $\sup_{(x,\xi)}|\nabla\psi(x,\xi)| < a$.

Then there exist $h_0 > 0$, *a function* $q : T^*\mathbb{R}^n \times]0,h_0] \to \mathbb{C}$ *and a bounded operator* $\mathcal{R}(h)$ *on* $L^2(\mathbb{R}^{2n})$ *such that*

i) $q(x,\xi,h) \sim \sum_{j\geq 0} q_j(x,\xi)h^j$ *as* $h \to 0$, *with*

$$q_0(x,\xi) = p(x - \partial_x\psi - i\partial_\xi\psi, \xi - \partial_\xi\psi + i\partial_x\psi),$$

ii) $\|\mathcal{R}(h)\|_{\mathcal{L}(L^2(\mathbb{R}^{2n}))} = O(h^\infty)$,

iii) for all $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\langle e^{\psi/h}\mathcal{T}\operatorname{Op}_h^w(p)u, e^{\psi/h}\mathcal{T}v\rangle = \langle (q(x,\xi,h) + \mathcal{R}(h)\langle\xi\rangle^\ell)e^{\psi/h}\mathcal{T}u, e^{\psi/h}\mathcal{T}v\rangle,$$

where $\langle\ ,\ \rangle$ *stands for the usual scalar product on* $L^2(\mathbb{R}^{2n})$.

The proof of Proposition 7.1 has been adapted in [**15**] to the case of a $\mathcal{C}^\infty$ symbol p defined on some distorted space Λ_{tG}, using an almost analytic extension. We give here another version, staying on $\mathbb{R}^{2n}$. It is also very close to Theorem 3 in [**3**], but we have here to take into account some dependency on R. In order to make the statements below clearer, we denote by $\mathcal{S}_u(\langle\xi\rangle^l)$

the space of functions $a(x,\xi,t,\theta,R,h)$ that are $\mathcal{C}^\infty$ with respect to (x,ξ), and such that, for any $\alpha\in\mathbb{N}^{2n}$,

$$\partial^\alpha_{(x,\xi)}a(x,\xi,t,\theta,R,h)=O(\langle\xi\rangle^l), \tag{7.2}$$

uniformly with respect to $(x,\xi)\in T^*\mathbb{R}^n$, t and θ small, R large, and $h\in[0,h_0]$. As usual, we shall also say that $a\in\mathcal{S}_u^{cl}(\langle\xi\rangle^l)$ when there exists a sequence $(a_j(x,\xi,t,\theta,R))_j$ of functions in $\mathcal{S}_u(\langle\xi\rangle^l)$ such that, for any $N\in\mathbb{N}$.

$$a(x,\xi,t,\theta,R,h)-\sum_{j=0}^N a_j(x,\xi,t,\theta,R)h^j\in h^{N+1}\mathcal{S}_u(\langle\xi\rangle^l). \tag{7.3}$$

We denote $\tilde p_{R,\theta}$ the Weyl semiclassical symbol of the operator $P_{R,\theta}$ that we have defined in (2.9). As for (6.8), we can easily see that, by construction, and thanks to assumption (A4), we have, as $|x|\to\infty$ in $\{|\operatorname{Im}x|\le\delta\langle\operatorname{Re}x\rangle\}$,

$$x^\alpha\partial_x^\alpha\tilde V(x)=o(1). \tag{7.4}$$

Moreover, since $d\Phi_{R,\theta}(x)$ is $O(1)$ uniformly with respect to R (see (2.7)), we see that $\tilde p_{R,\theta}\in\mathcal{S}_u^{cl}(\langle\xi\rangle^2)$, and that

$$\begin{aligned}\tilde p_{R,\theta}(x,\xi)&=\tilde p(\Phi_{R,\theta}(x),(d\Phi_{R,\theta}(x))^{-1}\xi)+h\mathcal{S}_u^{cl}(\langle\xi\rangle^2)\\&=((d\Phi_{R,\theta}(x))^{-1}\xi)^2+\tilde V(\Phi_{R,\theta}(x))+h\mathcal{S}_u^{cl}(\langle\xi\rangle^2).\end{aligned} \tag{7.5}$$

Then, following line by line the proof of Theorem 3 in [**3**], we obtain the following

Proposition 7.2. — *Let $(\psi_R)_{R>R_0}$ be a family of functions in $\mathcal{C}_0^\infty(T^*\mathbb{R}^n)$, such that, for any $\alpha,\beta\in\mathbb{N}^n$,*

$$\partial_\xi^\beta\partial_x^\alpha\psi_R=O(R^{(1-|\alpha|)_+}). \tag{7.6}$$

Then, for any $t\in\mathbb{R}$ small enough, and for all $u,v\in\mathcal{C}_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}&\langle e^{t\psi_R/h}\mathcal{T}P_{R,\theta}u,e^{t\psi_R/h}\mathcal{T}v\rangle_{L^2(\mathbb{R}^{2n})}=\\&\langle(q(x,\xi,t,\theta,R,h)+\mathcal{R}_{R,\theta}(t,h)\langle\xi\rangle^2)e^{t\psi_R/h}\mathcal{T}u,e^{t\psi_R/h}\mathcal{T}v\rangle_{L^2(\mathbb{R}^{2n})},\end{aligned} \tag{7.7}$$

where $q(x,\xi,t,\theta,R,h)=\sum_j q_j(x,\xi,t,\theta,R)h^j\in\mathcal{S}_u^{cl}(\langle\xi\rangle^2)$ with

$$q_0(x,\xi,t,\theta,R)=\tilde p_{R,\theta}(x-t\partial_x\psi_R-it\partial_\xi\psi_R,\xi-t\partial_\xi\psi_R+it\partial_x\psi_R), \tag{7.8}$$

modulo $h\mathcal{S}_u^{cl}(\langle\xi\rangle^2)$, and $\mathcal{R}_{R,\theta}(t,h)$ is a bounded operator on $L^2(\mathbb{R}^{2n})$ such that, for some constant $C_0>0$, uniformly for θ small enough and $R>1$,

$$\mathcal{R}_{R,\theta}(t,h)=O(h^\infty)+O(h^{-3n/2}|t|^\infty e^{C_0R|t|/h}). \tag{7.9}$$

Remark 7.3. — *As follows from Lemma 6.1 in* [**15**], *the last error term in (7.9) is* $O(h^{-3n/2}|t|^\infty e^{C_0|t|\sup|\psi_R|/h})$, *so we cannot have a better estimate with respect to* R *if we only assume (7.6).*

Now we shall prove that if $\rho \in \mathbb{C}$ is such that $\rho \in [E-\eta, E+\eta]-i[0, \nu Ch \ln \frac{1}{h}]$, ($\nu > 0$ small enough) then, for any R large enough, and with $\theta = Ch \ln \frac{1}{h}$, the kernel of $P_{R,\theta} - \rho$ is reduced to 0. We need a so-called escape function ψ_R, chosen so that q_0 becomes invertible. Indeed, as in [**3**] Lemma 3.3 or [**15**], Lemma 4.1, we have

$$
\begin{aligned}
q_0(x,\xi,t,\theta,R) &= p(x,\xi) - t\nabla p.\nabla\psi_R(x,\xi) \\
&\quad -iH_p(\theta\chi_R(x)x.\xi - t\psi_R)(x,\xi) + O(\theta^2+t^2)\mathcal{S}_u^{cl}(\langle\xi\rangle^2),
\end{aligned}
\tag{7.10}
$$

and we would like to find ψ_R such that $-\operatorname{Im}(q_0-\rho) > C > 0$ for some constant $C \in \mathbb{R}$.

To begin with, we fix $R_0 > 0$ and we proceed as in [**15**]. We can fix $\eta > 0$ such that for $|x| \geq R_0$ and $(x,\xi) \in p^{-1}(]E-\eta, E+\eta[)$, we have

$$
H_p(x.\xi) = 2\xi^2 - x\nabla V(x) \geq E. \tag{7.11}
$$

Then it has been shown in [**6**], that there exists a real-valued function $f \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, bounded as well as all its derivatives, such that $H_p(f) \geq 0$ everywhere, $H_p(f) = 1$ for $|x| \leq R_0$ and (x,ξ) close enough to $p^{-1}(E)$, and $H_p(f) = 0$ for $|x| \geq R_0 + 1$. We set

$$
\psi_0(x,\xi) = -\chi_1(x)\chi_2(\xi)f(x,\xi), \tag{7.12}
$$

where $\chi_1, \chi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n, [0,1])$ are such that $\chi_1(x) = 1$ for $|x| \leq R_0 + 1$, and $\chi_2(\xi) = 1$ for $|\xi|^2 \leq E + \sup|V| + \eta$. Then we can show as in [**15**], Lemma 4.1 that, for $(x,\xi) \in p^{-1}(]E-\eta, E+\eta[)$,

$$
H_p(\theta F_{R_0}(x).\xi - t\psi_0(x,\xi)) \geq \theta E\chi_{R_0}(x) + (t - C_0\theta)H_p f(x,\xi) - \mu t, \tag{7.13}
$$

where $\mu = \sup|fH_p\chi_1|$, and $C_0 = \sup_{p^{-1}([E-\eta,E+\eta])}|(x.\xi)H_p\chi_{R_0}|$.

Now we set

$$
\psi_R(x,\xi) = \psi_0(x,\xi) - \frac{1}{2C_0}\chi_2(\xi)(\chi_{R_0}(x)x.\xi - \chi_R(x)x.\xi), \tag{7.14}
$$

and we notice that $\psi_R \in \mathcal{C}_0^\infty(\mathbb{R}^{2n})$, since $\chi_{R_0}(x) = \chi_R(x)$ for $|x| \geq R$. Moreover, still for $(x,\xi) \in p^{-1}(]E-\eta, E+\eta[)$, we have $\chi_2 = 1$ and

$$
\begin{aligned}
&H_p(\theta\chi_R(x)x.\xi - t\psi_R(x,\xi)) = \\
&\quad (\theta - \frac{t}{2C_0})H_p(\chi_R(x)x.\xi) - tH_p\psi_0 + \frac{t}{2C_0}H_p(\chi_{R_0}(x)x.\xi).
\end{aligned}
\tag{7.15}
$$

Therefore, if we choose

$$t = 2C_0\theta, \tag{7.16}$$

we obtain, for any $R > R_0$,

$$H_p(\theta\chi_R(x)x.\xi - t\psi_R(x,\xi)) \geq \theta E\chi_{R_0}(x) + C_0\theta H_p f(x,\xi) - 2C_0\mu\theta. \tag{7.17}$$

But μ can be made arbitrarily small since χ_1 can be chosen arbitrarily flat, and we can suppose that

$$\mu \leq \frac{1}{4C_0}\min\{C_0, E\}. \tag{7.18}$$

Then, noticing that $H_p(f)+\chi_{R_0} \geq 1$, we obtain, for $(x,\xi) \in p^{-1}(]E-\eta, E+\eta[)$,

$$H_p(\theta\chi_R(x)x.\xi + t\psi_R(x,\xi)) \geq \frac{\theta}{2}\min\{C_0, E\}. \tag{7.19}$$

Observe that the constant in the R.H.S. of (7.19) does not depend on R.

From now on, we suppose that $\rho \in \mathbb{C}$ is such that,

$$\rho \in [E-\eta, E+\eta] - i]-\infty, \nu\theta[, \tag{7.20}$$

with $\nu := \frac{1}{4}\min\{C_0, E\}$, and we fix $t = 2C_0\theta$ as in (7.16). We denote by Σ the subset of $T^*\mathbb{R}^n$ defined by

$$\Sigma = \{(x,\xi) \in T^*\mathbb{R}^n, |\operatorname{Re} q(x,\xi,t,\theta,R,h) - \rho| \leq \frac{\langle\xi\rangle^2}{a}\}, \tag{7.21}$$

where $a > 1$ is a fixed constant. Notice that, recalling (7.10), $|\xi|$ is bounded on Σ, since $p = \xi^2 + V$ and ψ_R has compact support. Also with (7.19), we have on Σ, for some constant $C > 0$,

$$-\operatorname{Im}(q(x,\xi,t,\theta,R,h) - \rho) \geq \frac{\theta}{C} - Ch. \tag{7.22}$$

Now we suppose that $\theta = ChR^{-1}\ln(1/h)$ and $t = 2C_0\theta$. Then, if $R = R(h)$ verifies (2.14), we have $h = o(\theta)$, and, on the other hand, we can apply Proposition 7.2 where the remainder term $\mathcal{R}_{R,\theta}(t,h)$ becomes $O(h^\infty)$. Therefore we obtain, for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\operatorname{Im}\langle e^{t\psi_R/h}\mathcal{T}(P_{R,\theta}-\rho)u, e^{t\psi_R/h}\mathcal{T}u\rangle &= \langle \operatorname{Im}(q_0-\rho)e^{t\psi_R/h}\mathcal{T}u, e^{t\psi_R/h}\mathcal{T}u\rangle \\ &+O(h)\|\langle\xi\rangle^2 e^{t\psi_R/h}\mathcal{T}u\|\,\|e^{t\psi_R/h}\mathcal{T}u\|,\end{aligned} \tag{7.23}$$

where the scalar products and norms are taken in $L^2(\mathbb{R}^{2n})$.

Therefore we have, for some constant $C > 0$ that may differ from the previous one,

$$\begin{aligned} &|\operatorname{Im}\langle e^{t\psi_R/h}\mathcal{T}(P_{R,\theta}-\rho)u, e^{t\psi_R/h}\mathcal{T}u\rangle| \\ &\geq \frac{\theta}{C}\|e^{t\psi_R/h}\mathcal{T}u\|_{\Sigma}^2 - C\|\langle\xi\rangle^2 e^{t\psi_R/h}\mathcal{T}u\|_{\Sigma^c}^2, \end{aligned} \tag{7.24}$$

where we also used the fact that $p \in \mathcal{S}_u(\langle\xi\rangle^2)$ in particular on Σ^c.

Applying Proposition (7.2) with $v = (P_{R,\theta} - \rho)u$, one can easily obtain as in [**15**], Corollary 3.2, that, for some constant $C' > 0$,

$$\|e^{t\psi_R/h}\mathcal{T}(P_{R,\theta}-\rho)u\|^2 \geq \frac{1}{C'}\|\langle\xi\rangle^2 e^{t\psi_R/h}\mathcal{T}u\|_{\Sigma^c}^2 - C'h\|e^{t\psi_R/h}\mathcal{T}u\|_{\Sigma}^2. \tag{7.25}$$

From (7.24) and (7.25), it follows that, for h small enough,

$$\|\langle\xi\rangle^2 e^{t\psi_R/h}\mathcal{T}u\| \leq \frac{CR}{h\ln\frac{1}{h}}\|e^{t\psi_R/h}\mathcal{T}(P_{R,\theta}-\rho)u\|, \tag{7.26}$$

where C is indeed independent of R. Since, by density, this estimate holds also for any u in the domain of $P_{R,\theta}$, we have proved Theorem 2.6.

Remark 7.4. — *We can also consider the case where V belongs to some Gevrey class, as in* [**15**]. *Indeed if V is Gevrey with index $s > 1$, Proposition 7.2 still holds, with the better estimate*

$$\mathcal{R}_{R,\theta}(t,h) = O(h^\infty) + O(h^{-3n/2}e^{-\delta t^{1-1/s}}e^{C_0 R|t|/h}), \tag{7.27}$$

for some $\delta > 0$. Therefore, choosing $t = \delta' R^{-1}h^{1-1/s}$ with $0 < \delta' < \delta$, we obtain the same result if we suppose, instead of (2.14), that

$$R(h)h^{1/s} \to 0 \text{ as } h \to 0. \tag{7.28}$$

8. Shape Resonances

In this section, we investigate the resonances of P in the case where V presents the well-known structure of a "well in an island", as described e.g. in [**8**]. More precisely, we assume (A4), and that, for some energy level $E > 0$,

(A7) There exists a connected open subset $\ddot{O}$ of $\mathbb{R}^n$ and a compact $U \subset \ddot{O}$ such that

$$V(x) \leq E \text{ on } U,\ V > E \text{ on } \ddot{O}\backslash U,\ V = E \text{ on } \partial\ddot{O}.$$

(A8) The diameter of U for the Agmon distance d_{V-E} is 0.

Let us recall that the Agmon distance d_{V-E} is the pseudo-distance associated to the degenerate metric $\max(V-E,0)dx^2$.

We also assume that there is no trapped trajectory of energy E above the sea $\ddot{O}^C := \mathbb{R}^n \backslash \ddot{O}$, that is, with the notation of (2.15),

(A9) $K(E) \cap \ddot{O}^C \times \mathbb{R}^n = \emptyset$.

Then, the construction made in [**6**] gives the existence, for any $R_0 > 0$, of a real-valued smooth function f supported in a neighborhood of $\ddot{O}^C$, such that $H_p(f) \geq 0$ everywhere, $H_p(f) = 1$ in $\{|x| \leq R_0\} \cup \{x \in \ddot{O}^C\} \cup p^{-1}(E)$, and $H_p(f) = 0$ for $|x| \geq R_0 + 1$. As in the previous section, we can deduce for any $R > 0$ large enough, the existence of a function $\psi_R \in C_0^\infty(\mathbb{R}^{2n})$ such that, for $(x,\xi) \in p^{-1}(E-\eta, E+\eta) \cap \{x \in \ddot{O}^C\}$,

$$H_p(\theta F_R(x).\xi - t\psi_R(x,\xi)) \geq \frac{\theta}{C_1}. \tag{8.1}$$

where C_1 and η are positive constants, and $t = 2C_0\theta$ as in (7.16) ($\theta > 0$ small enough). We also set

$$S_0 := d_{V-E}(U, \ddot{O}^C) \tag{8.2}$$

and, following [**8**], for any $\eta > 0$ small enough, we denote by P_η the Dirichlet realization of P on $M_\eta := \overline{B(U, S_0 - \eta)}$, where $B(U, S_0 - \eta) = \{x\,; d_{V-E}(U,x) < S_0 - \eta\}$.

Finally, we assume the existence of two positive functions $a(h)$ and $b(h)$ such that,

(A10) $a(h) + b(h) \to 0$ as $h \to 0^+$;

(A11) There exists $C' > 0$ and, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $\delta_\varepsilon e^{-\varepsilon/h} \leq a(h) \leq C' h \ln 1/h$ for all h small enough;

(A12) $\sigma(P_\eta) \cap \{\lambda \in \mathbb{R}\,; b(h) < |\lambda - E| \leq b(h) + 2a(h)\} = \emptyset$.

Observe that, by the results of [**9**], if this assumption is satisfied for some small enough value of η, then it is also satisfied for any smaller value of η. Moreover, such an assumtion is always satisfied in the case of a non degenerate point-well, i.e. when U is reduced to a single point where the Hessian of V is positive definite.

Now, following [**8**] Section 9, we choose $W \in \mathcal{C}_0^\infty(B(U,\eta); \mathbb{R}_+)$ such that $V + W > E$ on U. We can see as in the previous section that, for $\theta = ChR^{-1} ln 1/h$ and $\rho \in [E-\eta, E+\eta] - i[-\infty, \theta/2C_1]$, (8.1) implies the invertibility of $Q_{R,\theta} - \rho$, where $Q_{R,\theta}$ is the operator defined by

$$Q_{R,\theta} := P_{R,\theta} + W \,:\, L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{8.3}$$

Moreover, the estimate (7.26) is valid with $P_{R,\theta}$ replaced by $Q_{R,\theta}$, and this means that, if we endow $L^2(\mathbb{R}^n)$ with the norm:

$$\|u\|_{L^2_{\psi_R}} := \|e^{t\psi_R/h}\mathcal{T}u\|_{L^2(\mathbb{R}^{2n})}, \tag{8.4}$$

then, the norm of $(Q_{R,\theta} - \rho)^{-1}$ is $O(R/(h\ln 1/h))$ uniformly with respect to R and h.

At this point, one can readily follow the arguments of [**8**] Section 9, and conclude that there exists a bijection

$$\begin{aligned}\beta : \sigma(P_\eta) \cap [E - b(h), E + b(h)] \\ \to \tilde{\Gamma}(\theta) \cap [E - b(h) - a(h), E + b(h) + a(h)] - i[-\infty, \theta/2C_1]\end{aligned} \tag{8.5}$$

with the property,

$$\beta(\lambda) - \lambda = O\left(e^{-(2S_0 - \varepsilon(\eta))/h}\right) \tag{8.6}$$

where $\varepsilon(\eta) \to 0$ as $\eta \to 0^+$, and the O is uniform with respect to R and h.

Summing up, we have proved the

Theorem 8.1. — *Assume (A4) and (A7)-(A12). Then, for any $R = R(h) > 1$ verifying (2.14), any $C > 0$, and with $\theta = ChR(h)^{-1}\ln(1/h)$, we have that any (R,θ)-resonance ρ of P inside $[E-b(h)-a(h), E+b(h)+a(h)]-i[-\infty, \frac{\theta}{2C_1}]$ verifies, for any $\varepsilon > 0$,*

$$|\operatorname{Im}\rho| = O\left(e^{-(2S_0-\varepsilon)/h}\right).$$

Moreover, the number of these resonances is the same as the number of eigenvalues of P_η in $[E - b(h), E + b(h)]$ and there exists a bijection between these two sets as in (8.5) with the property (8.6).

Remark 8.2. — *Since the almost-analytic extension of V is not uniquely defined, but only up to $O(\theta^\infty)$ in the sector $\mathcal{S}(\theta)$, in the case $\theta = ChR(h)^{-1}\ln 1/h$ one would expect that our resonances are defined up to $O(h^\infty)$ only. However, one can see on the previous result that this is actually not true: In that case, any almost-analytic extension of V gives the same resonances up to $O\left(e^{-(2S_0-\varepsilon)/h}\right)$, since the eigenvalues of P_η do not depend on such an extension.*

Remark 8.3. — *By the results of [**9**] applied to P_η, in the case of a non degenerate point-well, Theorem 8.1 also permits to obtain a semiclassical asymptotic expansion, in powers of $h^{1/2}$, of the (R,θ)-resonances of P that are in a domain of the type $[E, E + Ch] - i[-\infty, \frac{\theta}{2C_1}]$ with $C > 0$ arbitrarily large.*

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