Simplicity of singular spectrum in Anderson type Hamiltonians

Vojkan Jakšić[1] and Yoram Last[2]

[1]Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, QC, H3A 2K6, Canada

[2]Institute of Mathematics
The Hebrew University, Givat Ram
91904 Jerusalem, Israel

August 6, 2004

Abstract

We study self adjoint operators of the form $H_\omega = H_0 + \sum \omega(n)(\delta_n|\cdot)\delta_n$, where the δ_n's are a family of orthonormal vectors and the $\omega(n)$'s are independent random variables with absolutely continuous probability distributions. We prove a general structural theorem which provides in this setting a natural decomposition of the Hilbert space as a direct sum of mutually orthogonal closed subspaces that are almost surely invariant under H_ω and which is helpful for the spectral analysis of such operators. We then use this decomposition to prove that the singular spectrum of H_ω is almost surely simple.

1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space and H_0 a bounded self adjoint operator on $\mathcal{H}$. Let $\{\delta_n\}_{n\in\mathcal{N}}$ be a set of orthonormal vectors in $\mathcal{H}$, where $\mathcal{N}$ is either finite or a countable infinite set. Let $\{p_n\}_{n\in\mathcal{N}}$ be absolutely continuous (w.r.t. Lebesgue measure) Borel probability measures on $\mathbb{R}$ and consider the probability space $(\Omega, \mathrm{d}P)$, where $\Omega = \mathbb{R}^{\mathcal{N}}$ and $\mathrm{d}P = \bigotimes_{\mathcal{N}} \mathrm{d}p_n$. For each $\omega \in \Omega$, we define

$$V_\omega := \sum_{n\in\mathcal{N}} \omega(n)(\delta_n|\cdot)\delta_n, \qquad H_\omega := H_0 + V_\omega. \tag{1.1}$$

We call families of self adjoint operators of the form $\{H_\omega\}_{\omega\in\Omega}$, Anderson type Hamiltonians [JL1]. They are a conceptually and technically convenient generalization of many specific models of discrete random Schrödinger operators discussed in the literature, including the standard Anderson model on $\mathbb{Z}^d$, models with decaying, sparse, or surface random potentials, models on Bethe lattice, etc.

We denote by $\mathcal{H}_{\omega,\psi}$, the cyclic subspace generated by H_ω and a vector $\psi \in \mathcal{H}$. This subspace is the closure of the linear span of the set of vectors $\{(H_\omega - z)^{-1}\psi : z \in \mathbb{C}\setminus\mathbb{R}\}$. By the spectral theorem, the operator $H_\omega \restriction \mathcal{H}_{\omega,\psi}$ is unitarily equivalent to the operator of multiplication by the parameter on $L^2(\mathbb{R}, \mathrm{d}\mu_{\omega,\psi})$, where $\mu_{\omega,\psi}$ is the spectral measure for H_ω and ψ. In the sequel, $\mu_{\omega,\psi,\mathrm{ac}}$ and $\mu_{\omega,\psi,\mathrm{sing}}$ denote, respectively, the absolutely continuous and the singular parts of $\mu_{\omega,\psi}$ (w.r.t. Lebesgue measure). For notational simplicity, we write $\mathcal{H}_{\omega,n}$ for $\mathcal{H}_{\omega,\delta_n}$, $\mu_{\omega,n}$ for μ_{ω,δ_n}, etc.

For any subset $\mathcal{M} \subseteq \mathcal{N}$, we let $\mathcal{H}_{\omega,\mathcal{M}}$ denote the cyclic subspace generated by H_ω and the family of vectors $\{\delta_n\}_{n\in\mathcal{M}}$. This subspace is the closure of the linear span of the set $\cup_{n\in\mathcal{M}}\mathcal{H}_{\omega,n}$. For every pair $n, m \in \mathcal{N}$, we define

$$E_{n,m} := \{\, \omega : \mathcal{H}_{\omega,n} \not\perp \mathcal{H}_{\omega,m}\}.$$

Let

$$\mathcal{S} := \{n \in \mathcal{N} : \text{for } P\text{-a.e. } \omega,\ \mu_{n,\omega} \text{ is not equivalent to the Lebesgue measure on } \mathbb{R}\},$$

where we say that two Borel measures are equivalent if they have the same sets of zero measure. Our first result is the following general structural theorem for Anderson type Hamiltonians.

Theorem 1.1 (1) *For every $\omega \in \Omega$, $\mathcal{H}_{\omega,\mathcal{N}}$ is equal to the cyclic subspace generated by H_0 and $\{\delta_n\}_{n\in\mathcal{N}}$, and is thus completely independent of ω.*
(2) *For every pair $n, m \in \mathcal{N}$, $P(E_{n,m}) \in \{0, 1\}$.*
(3) *The relation $n \sim m$ iff $P(E_{n,m}) = 1$ is an equivalence relation on $\mathcal{S}$.*
(4) *Let $\mathcal{S}_0 \equiv \mathcal{N} \setminus \mathcal{S}$ and let $\mathcal{S}_k$, $k = 1, 2, \ldots$ be the equivalence classes generated by $\sim$ within $\mathcal{S}$. Then for every $k \geq 0$, there exists a closed subspace $\mathcal{H}_{\mathcal{S}_k} \subseteq \mathcal{H}$, such that for P-a.e. $\omega \in \Omega$, $\mathcal{H}_{\omega,\mathcal{S}_k} = \mathcal{H}_{\mathcal{S}_k}$.*
(5) *For every $k, m \geq 0$, $k \neq m$, for P-a.e. $\omega \in \Omega$, $\mathcal{H}_{\omega,\mathcal{S}_k} \perp \mathcal{H}_{\omega,\mathcal{S}_m}$.*

We note that in many cases, $\mathcal{S} = \mathcal{N}$ (in particular, this is clearly the case whenever the spectrum of H_ω is almost surely not equal to $\mathbb{R}$). Moreover, as we discuss further in Section 3 below, there are many natural cases, including all of the above mentioned discrete random Schrödinger operators, where $\sim$ is an equivalence relation on $\mathcal{N}$, even if $\mathcal{S} \neq \mathcal{N}$. In such cases, one can prove a slightly simpler variant of Theorem 1.1, in which $\sim$ is an equivalence relation on $\mathcal{N}$ and where all of the $\mathcal{S}_k$'s are equivalence classes generated by $\sim$. In the event that $\mathcal{N} \setminus \mathcal{S} \neq \emptyset$ and that $\sim$ is not an equivalence relation on all of $\mathcal{N}$, one can still view the set $\mathcal{N} \setminus \mathcal{S}$ as a special equivalence class along side the equivalence classes generated by $\sim$ within $\mathcal{S}$. We thus see that, in either case, Theorem 1.1 provides a division of the set $\mathcal{N}$ into ω-independent equivalence classes. Each such equivalence class has an associated subspace of $\mathcal{H}_{\omega,\mathcal{N}}$, which is the cyclic subspace generated by H_ω and the set of δ_n vectors corresponding to this class. (4) implies that these subspaces coincide, P-a.s., with some ω-independent subspaces. (4) and (5) together then imply that these ω-independent subspaces are mutually orthogonal and that they are P-a.s. invariant under H_ω. Thus, Theorem 1.1 gives an ω-independent decomposition of $\mathcal{H}$ as a direct sum of mutually orthogonal closed subspaces which are P-a.s. invariant under H_ω.

Recall that the main results of [JL1] were obtained for Anderson type Hamiltonians of the form (1.1), under the additional assumption that $P(E_{n,m}) = 1$ for every pair $n, m \in \mathcal{N}$. The current paper is, in fact, a natural continuation of [JL1]. Part of the importance of Theorem 1.1 is that it allows one to apply the results of [JL1] to the more general setting considered here by essentially applying them independently to the restrictions of H_ω to the different invariant subspaces associated with the equivalence classes generated by $\sim$.

We note that by item (1) of Theorem 1.1, the invariant subspace $\mathcal{H}_{\omega,\mathcal{N}}$ is non-random and that if $\mathcal{H}_{\omega,\mathcal{N}} \neq \mathcal{H}$, then $H_\omega \restriction \mathcal{H}_{\omega,\mathcal{N}}^\perp = H_0 \restriction \mathcal{H}_{\omega,\mathcal{N}}^\perp$. Thus, our interest here is only in $H_\omega \restriction \mathcal{H}_{\omega,\mathcal{N}}$. Therefore, in what follows we identify H_ω with $H_\omega \restriction \mathcal{H}_{\omega,\mathcal{N}}$, namely, we assume that $\{\delta_n\}_{n\in\mathcal{N}}$ is a cyclic family for H_ω. This involves no real loss of generality. We denote by $\mathcal{H}_{\omega,\mathrm{sing}}$ and $\mathcal{H}_{\omega,\psi,\mathrm{sing}}$ the subspaces associated, correspondingly, with the singular spectra of H_ω and $H_\omega \restriction \mathcal{H}_{\omega,\psi}$. Recall that $\mathcal{S}_k$, $k = 1, 2, \ldots$, denote the equivalence classes generated by $\sim$ within $\mathcal{S}$. Our main result is the following:

Theorem 1.2 *Let* $\psi = \sum_{k\geq 1} a_k \delta_{n_k}$, *where* $n_k \in \mathcal{S}_k$, $a_k \neq 0$, *and* $\sum_k a_k^2 < \infty$. *Then for* P*-a.e.* ω,

$$\mathcal{H}_{\omega,\mathrm{sing}} = \mathcal{H}_{\omega,\psi,\mathrm{sing}}.$$

Remark 1. This says that, with probability one, the singular spectrum of H_ω is simple and ψ is a cyclic vector for $H_\omega \restriction \mathcal{H}_{\omega,\mathrm{sing}}$.

Remark 2. If $P(E_{n,m}) = 1$ for every pair $n, m \in \mathcal{N}$, then one can take $\psi = \delta_n$ for any n. This is the case, for example, if $\mathcal{N}$ is the set of vertices of a connected graph of a bounded degree, $\mathcal{H} = \ell^2(\mathcal{N})$, the δ_n's are delta function vectors on $\mathcal{N}$, and H_0 is the associated discrete Laplacian ($(H_0\psi)(n) = \sum_{|n-m|=1} \psi(m)$, where $|n - m|$ is the distance on the graph between n and m). The special case $\mathcal{N} = \mathbb{Z}^d$ corresponds to the Anderson model. A natural case to have in mind where $P(E_{n,m}) = 0$ for some pairs is where $\mathcal{N}$ is the set of vertices of the union of two or more connected graphs which are disconnected from each other. In such a case, Theorem 1.2 says that the singular spectrum of H_ω is still almost surely simple, but one needs to take some linear combination of delta function vectors from the different disconnected components of $\mathcal{N}$ in order to get a cyclic vector for $H_\omega \restriction \mathcal{H}_{\omega,\mathrm{sing}}$.

Theorem 1.2 is motivated, in part, by the following result of [JL1]:

Theorem 1.3 (Theorem 1.1 of [JL1]) *Assume that $P(E_{n,m}) = 1$. Then the spectral measures $\mu_{\omega,n}$ and $\mu_{\omega,m}$ are P-a.s. equivalent and the operators $H_\omega \restriction \mathcal{H}_{\omega,n}$ and $H_\omega \restriction \mathcal{H}_{\omega,m}$ are thus P-a.s. unitarily equivalent.*

In this paper we prove

Theorem 1.4 *Assume that $P(E_{n,m}) = 1$. Then for P-a.e. ω,*

$$\mathcal{H}_{\omega,n,\mathrm{sing}} = \mathcal{H}_{\omega,m,\mathrm{sing}}.$$

As we shall see below, Theorem 1.2 follows fairly easily from Theorem 1.4. Similarly to [JL1], the proof of Theorem 1.4 can be naturally reduced to the special case $\mathcal{N} = \{1, 2\}$. We study this special case using some standard tools of rank one perturbation theory [Si2] and Poltoratskii's theorem concerning the ratios of Borel transforms of measures [Po, JL3]. We note that our proof of Theorem 1.4 relies heavily on the full strength of Poltoratskii's theorem.

For the case of pure point spectrum and a single class of equivalence, Theorem 1.2 has been essentially proven by Simon in [Si1]. Theorem 1.2 is thus an extension of Simon's result. In particular, it extends it to singular continuous spectrum. We note that while the random part (V_ω) of H_ω may be considered as the main generator of its spectral properties, it is also possible to consider Anderson type Hamiltonians where the random part is "very small." In particular, one can consider cases where V_ω is almost surely a trace class operator with an arbitrarily small trace norm. From this perspective, spectral properties which must hold for Anderson type Hamiltonians may be interpreted as "generic" properties of self adjoint operators, because any spectral property of a self adjoint operator that cannot occur with positive probability for Anderson type Hamiltonians must be very unstable—as it would be almost surely "removed" by a "tiny" random perturbation. From this point of view, Theorem 1.2 can be interpreted as saying that singular spectrum is "generically simple." It is thus connected with the many known results concerning the non-genericity of degenerate eigenvalues. As far as we know, Theorem 1.2 is the first result establishing (in some sense) the non-genericity of degenerate singular continuous spectrum.

We note that, since V_ω may be a trace class operator, H_ω and H_0 may have the same absolutely continuous spectrum with the same multiplicity. Thus, since H_0 may have absolutely continuous spectrum of arbitrary multiplicity, no general statement can be made regarding the multiplicity of the absolutely continuous spectrum of H_ω. The simplicity of the singular spectrum is thus the strongest possible statement that one can make regarding spectral multiplicity for general Anderson type Hamiltonians.

Another potentially interesting aspect of Theorem 1.2 is that it provides a new criterion for the existence of absolutely continuous spectrum for Anderson type Hamiltonians. Explicitly, in order to prove that an Anderson type Hamiltonian has some absolutely continuous spectrum (P-a.s.), it suffices to prove that its spectrum is not simple (with any positive probability). Moreover, for cases with a single class of equivalence, it suffices to show that for some n, δ_n is not a cyclic vector with positive probability. These criteria do not refer to any specific energy ranges (mobility edges) and could thus be potentially easier to establish than, e.g., the detailed picture of Anderson delocalization suggested by the physics literature for the Anderson model on $\mathbb{Z}^d$.

We note that (multidimensional) Anderson type Hamiltonians with singular continuous spectrum have been constructed recently by Last-Simon [LS]. Theorem 1.2 establishes the simplicity of the spectrum for these models.

We finish this discussion with some technical remarks. First, the assumption that the $\omega(n)$'s are completely independent can be relaxed. It suffices that for each n, the conditional probability distribution of $\omega(n)$, given any $\{\omega(m)\}_{m\neq n}$, is absolutely continuous, and (if $\mathcal{N}$ is infinite) that the tail σ-field of the sequence $\{\omega(n)\}_{n\in\mathcal{N}}$ is trivial (so that Kolmogorov's 0-1 law can be applied). Second, the assumption that H_0 is bounded can also be relaxed. With the single exception of item (1) of Theorem 1.1, our results (namely, items (2)–(5) of Theorem 1.1 and Theorems 1.2 and 1.4) hold with the same proofs also for an unbounded self adjoint H_0, as long as $H_\omega \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ is essentially self adjoint for P-a.e. ω. We note that this condition is non-other than the natural condition to ensure that H_ω makes sense as a self adjoint operator in this case. Regarding item (1) of Theorem 1.1, we note that if H_0 is unbounded, then a deterministic statement for *all* ω is not likely to be valid, since (if $\mathcal{N}$ is infinite) there will usually be some ω's in $\mathbb{R}^{\mathcal{N}}$ for which H_ω will not be self adjoint. However, our proof of this statement shows that $\mathcal{H}_{\omega,\mathcal{N}}$ is equal to the cyclic subspace generated by H_0 and $\{\delta_n\}_{n\in\mathcal{N}}$ for every ω for which both $H_\omega \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ and $H_0 \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ are essentially self adjoint. A natural case where this clearly holds is when V_ω is bounded. If $H_0 \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ is not essentially self adjoint, one can still use an argument similar to our proof of item (4) of Theorem 1.1 below (namely, using rank one perturbations and Kolmogorov's 0-1 law) to show that for P-a.e. $\omega \in \Omega$, $\mathcal{H}_{\omega,\mathcal{N}}$ coincides with some non-random subspace. However, we cannot exclude in this case that the almost sure $\mathcal{H}_{\omega,\mathcal{N}}$ is strictly contained in the cyclic subspace generated by H_0 and $\{\delta_n\}_{n\in\mathcal{N}}$. We also note that, similarly, if we do have that both $H_\omega \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ and $H_0 \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ are essentially self adjoint, then one can use an argument similar to our proof of item (1) of Theorem

1.1 below to prove a slightly stronger variant of item (4) of Theorem 1.1, namely, that for every $k \geq 0$, for P-a.e. $\omega \in \Omega$, $\mathcal{H}_{\omega,\mathcal{S}_k}$ coincides with the cyclic subspace generated by H_0 and $\{\delta_n\}_{n\in\mathcal{S}_k}$.

The rest of this paper is organized as follows. In Section 2 we give some preliminaries and in Section 3 we prove Theorem 1.1. In Section 4 we prove a theorem about rank one perturbations and in Section 5 we prove a theorem about rank two perturbations, which is a simple consequence of the theorem we prove in Section 4. Finally, in Section 6 we prove Theorems 1.4 and 1.2.

Acknowledgments
VJ's work was partially supported by NSERC. YL's work was partially supported by THE ISRAEL SCIENCE FOUNDATION (grant no. 188/02).

2 Preliminaries

Throughout the paper we will use the shorthand $\mathbb{C}_\pm := \{z : \pm\mathrm{Im}\, z > 0\}$. The Borel transform of a complex valued measure μ on $\mathbb{R}$ is defined by

$$F_\mu(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x-z}, \qquad \mathrm{Im}\, z \neq 0.$$

If $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$, then $f\mu$ denotes the measure defined by $(f\mu)(S) = \int_S f\mathrm{d}\mu$. We will need the following celebrated result of Poltoratskii [Po, JL3]:

Theorem 2.1 *Let μ be a complex valued measure on $\mathbb{R}$ and $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$. Then*

$$\lim_{\epsilon\downarrow 0} \frac{F_{f\mu}(E+\mathrm{i}\epsilon)}{F_\mu(E+\mathrm{i}\epsilon)} = f(E)$$

for μ_{sing}-a.e. E.

We will also need the celebrated theorem of F. & M. Riesz [Ri]:

Theorem 2.2 *If μ is a non-vanishing complex valued measure on $\mathbb{R}$ and if $F_\mu(z)$ vanishes on $\mathbb{C}_+$, then $|\mu|$ is equivalent to the Lebesgue measure on $\mathbb{R}$.*

In the literature one can find many different proofs of Theorem 2.2. For example, three different proofs are given in [Ko]. To the best of our knowledge, however, it has not been previously noticed that the F. & M. Riesz theorem is an easy consequence of Poltoratskii's theorem. Since this fact is of some independent interest, we include the proof below.

Proof of Theorem 2.2. We first prove that $|\mu|$ is absolutely continuous w.r.t. Lebesgue measure. Write $\mu = h|\mu|$, where $|h(E)| = 1$ for all E. By Poltoratskii's theorem,

$$\lim_{\epsilon\downarrow 0} \frac{|F_\mu(E+\mathrm{i}\epsilon)|}{|F_{|\mu|}(E+\mathrm{i}\epsilon)|} = |h(E)| = 1,$$

for $|\mu|_{\rm sing}$-a.e. E. Since $\lim_{\epsilon\downarrow 0} |F_{|\mu|}(E+\mathrm{i}\epsilon)| = \infty$ for $|\mu|_{\rm sing}$-a.e. E, we must have that $\lim_{\epsilon\downarrow 0} |F_\mu(E+\mathrm{i}\epsilon)| = \infty$ for $|\mu|_{\rm sing}$-a.e. E. Thus, if $|\mu|_{\rm sing} \neq 0$, F_μ cannot vanish in $\mathbb{C}_+$, and we conclude that $|\mu|_{\rm sing} = 0$.

To prove that $|\mu|$ is equivalent to the Lebesgue measure on $\mathbb{R}$, we need to show that

$$\operatorname{Im} F_{|\mu|}(E+\mathrm{i}0) = \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}} \frac{\epsilon\,\mathrm{d}|\mu|(x)}{(x-E)^2+\epsilon^2} > 0 \tag{2.2}$$

for Lebesgue a.e. E. Since, for any measure ν,

$$F_\nu(E+\mathrm{i}\epsilon) = \int_{\mathbb{R}} \frac{(x-E)\,\mathrm{d}\nu(x)}{(x-E)^2+\epsilon^2} + \mathrm{i}\int_{\mathbb{R}} \frac{\epsilon\,\mathrm{d}\nu(x)}{(x-E)^2+\epsilon^2}, \tag{2.3}$$

we see that if $\operatorname{Im} F_{|\mu|}(E+\mathrm{i}0) = 0$ for some E, then $F_\mu(E+\mathrm{i}0) = F_\mu(E-\mathrm{i}0)$ (in the sense that any one of these limits exists if and only if the other exists and then they are equal). Since μ is non-vanishing and F_μ vanishes on $\mathbb{C}_+$, F_μ is non-vanishing on $\mathbb{C}_-$, and by well known results about boundary values of analytic functions [Ko], $F_\mu(E-\mathrm{i}0) \neq 0$ for Lebesgue a.e. E. Hence, if $\operatorname{Im} F_{|\mu|}(E+\mathrm{i}0) = 0$ on a set of positive Lebesgue measure, then $F_\mu(E+\mathrm{i}0) \neq 0$ on a set of positive Lebesgue measure and this contradicts the assumption that F_μ vanishes on $\mathbb{C}_+$. □

Another fact that we need is:

Lemma 2.3 *Let μ be a finite positive Borel measure on $\mathbb{R}$ and let $f \in L^2(\mathbb{R}, \mathrm{d}\mu)$. Then for Lebesgue a.e. $E \in \mathbb{R}$ for which $\operatorname{Im} F_\mu(E+\mathrm{i}0) = 0$, $F_{f\mu}(E+\mathrm{i}0) = F_{f\mu}(E-\mathrm{i}0)$.*

Proof. By the Cauchy-Schwartz inequality and (2.3), we have

$$\begin{aligned} \left| \int_{\mathbb{R}} \frac{\epsilon f\,\mathrm{d}\mu(x)}{(x-E)^2+\epsilon^2} \right| &\leq \sqrt{\int_{\mathbb{R}} \frac{\epsilon |f|^2\,\mathrm{d}\mu(x)}{(x-E)^2+\epsilon^2} \int_{\mathbb{R}} \frac{\epsilon\,\mathrm{d}\mu(x)}{(x-E)^2+\epsilon^2}} \\ &= \sqrt{\operatorname{Im} F_{|f|^2\mu}(E+\mathrm{i}\epsilon)\operatorname{Im} F_\mu(E+\mathrm{i}\epsilon)}\,. \end{aligned} \tag{2.4}$$

Since $\limsup_{\epsilon\to 0} \operatorname{Im} F_{|f|^2\mu}(E+\mathrm{i}\epsilon) < \infty$ for Lebesgue a.e. $E \in \mathbb{R}$, we see that the last expression in (2.4) goes to zero as $\epsilon \to 0$ for Lebesgue a.e. $E \in \mathbb{R}$ for which $\operatorname{Im} F_\mu(E+\mathrm{i}0) = 0$. Thus, by (2.3) again, we see that for an appropriate set of E's

$$\lim_{\epsilon\downarrow 0} F_{f\mu}(E+\mathrm{i}\epsilon) = \lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{(x-E) f\,\mathrm{d}\mu(x)}{(x-E)^2+\epsilon^2} = \lim_{\epsilon\downarrow 0} F_{f\mu}(E-\mathrm{i}\epsilon).$$

□

We also need:

Lemma 2.4 (the spectral averaging lemma) *Let μ be a Borel probability measure on $\mathbb{R}$, let A denote the operator of multiplication by the parameter on $L^2(\mathbb{R}, \mathrm{d}\mu)$, let $\mathbf{1}$ denote the constant function $\mathbf{1}(x) = 1 \ \forall x \in \mathbb{R}$, and for every $\lambda \in \mathbb{R}$, let μ_λ be the spectral measure for the vector $\mathbf{1}$ and the operator $A + \lambda(\mathbf{1}|\cdot)\mathbf{1}$ on $L^2(\mathbb{R}, \mathrm{d}\mu)$. Then for any Borel set $S \subseteq \mathbb{R}$,*

$$\int_{\mathbb{R}} \mu_\lambda(S)\,\mathrm{d}\lambda = |S|,$$

where $|\cdot|$ denotes Lebesgue measure.

Proof. This is an immediate consequence of Theorem I.8 of [Si2]. □

In our proofs we will deal with functions on the probability space and with subsets of it that are defined through the spectral theory of H_ω. It is not difficult to show that all of the functions and sets which appear in our paper are measurable. We give below two of the relevant measurability arguments, largely as an example of how these can be obtained. The others are left to the reader. For definitions and basic results concerning measurability for random self adjoint operators, we refer the reader to [CL].

Lemma 2.5 *Let $\mathbf{1}_{\omega,\psi}$ be the orthogonal projection onto $\mathcal{H}_{\omega,\psi}$. Then the map $\omega \to \mathbf{1}_{\omega,\psi}$ is measurable.*

Proof. Let $\{z_k\}$ be an ordering of points in $\mathbb{C} \setminus \mathbb{R}$ whose both coordinates are rational numbers. Let $\phi_{k,\omega} = (H_\omega - z_k)^{-1}\psi$. Obviously, $\phi_{k,\omega} \neq 0$, the linear span of the set $\{\phi_{k,\omega}\}$ is dense in $\mathcal{H}_{\omega,\psi}$ for all ω, and the functions $\omega \to \phi_{k,\omega} \in \mathcal{H}$ are measurable. Let $\{u_{k,\omega}\}$ be an orthonormal basis of $\mathcal{H}_{\omega,\psi}$ obtained from $\{\phi_{k,\omega}\}$ by the Gram-Schmidt procedure. Then the functions $\omega \to u_{k,\omega}$ are measurable. Since, for any $\phi \in \mathcal{H}$,

$$\mathbf{1}_{\omega,\psi}\phi = \sum_k (u_{k,\omega}|\phi)u_{k,\omega},$$

the function $\omega \to \mathbf{1}_{\omega,\psi}$ is measurable. □

Lemma 2.6 *Let $\psi, \varphi \in \mathcal{H}$. Then the sets*

$$\Omega_1 := \{\omega : \mathcal{H}_{\omega,\psi} \perp \mathcal{H}_{\omega,\varphi}\} \quad \textit{and} \quad \Omega_2 := \{\omega : \mathcal{H}_{\omega,\psi,\mathrm{sing}} = \mathcal{H}_{\omega,\varphi,\mathrm{sing}}\}$$

are measurable.

Proof. Let $\{\phi_i\}$ be a countable dense set in $\mathcal{H}$. Then

$$\Omega_1 = \{\omega \,:\, (\mathbf{1}_{\omega,\psi}\phi_i|\mathbf{1}_{\omega,\varphi}\phi_j) = 0, \ \ \forall i, j\},$$

and the set on the r.h.s. is measurable by the previous lemma.

Let $\mathbf{1}_{\mathrm{sing}}(H_\omega)$ be the projection on $\mathcal{H}_{\omega,\mathrm{sing}}$. The map $\omega \to \mathbf{1}_{\mathrm{sing}}(H_\omega)$ is measurable (see [CL]). Let $\mathbf{1}_{\omega,\psi,\mathrm{sing}}$ be the projection on $\mathcal{H}_{\omega,\psi,\mathrm{sing}}$. Then, $\mathbf{1}_{\omega,\psi,\mathrm{sing}} = \mathbf{1}_{\mathrm{sing}}(H_\omega)\mathbf{1}_{\omega,\psi}$ and so the function $\omega \to \mathbf{1}_{\omega,\psi,\mathrm{sing}}$ is measurable. Finally, the measurability of Ω_2 follows from

$$\Omega_2 = \{\omega : (\phi_i|\mathbf{1}_{\omega,\psi,\mathrm{sing}}\phi_j) = (\phi_i|\mathbf{1}_{\omega,\varphi,\mathrm{sing}}\phi_j), \ \ \forall i, j\}.$$

□

3 The relation $\sim$

We recall the following well known result (see, e.g., the proof of Corollary 1.1.3 in [JL1]):

Proposition 3.1 *Assume that for P-a.e. ω, $\mu_{\omega,n,\mathrm{ac}}$ is equivalent to the Lebesgue measure on $\mathbb{R}$. Then $\mu_{\omega,n,\mathrm{sing}} = 0$ for P-a.e. ω, namely, $n \notin \mathcal{S}$.*

Proof of Theorem 1.1. (1) We assume here that both $H_\omega \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ and $H_0 \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ are essentially self adjoint. (This clearly holds if H_0 is bounded.) Let $\mathcal{H}_{0,\mathcal{N}}$ denote the cyclic subspace generated by H_0 and $\{\delta_n\}_{n\in\mathcal{N}}$. Without loss of generality, we assume that $\mathcal{N} = \{1, 2, \dots\}$, and for every $n \in \mathcal{N}$, we define

$$H_\omega^{(n)} = H_0 + \sum_{1\le p\le n} \omega(p)(\delta_p|\,\cdot\,)\delta_p,$$

$$\tilde{H}_\omega^{(n)} = H_0 + \sum_{p>n} \omega(p)(\delta_p|\,\cdot\,)\delta_p.$$

For all $z \in \mathbb{C} \setminus \mathbb{R}$ and $n, k \in \mathcal{N}$, we have

$$(H_\omega^{(n)} - z)^{-1}\delta_k = (H_0 - z)^{-1}\delta_k - \sum_{1\le p\le n} \omega(p)(\delta_p|(H_\omega - z)^{-1}\delta_k)(H_0 - z)^{-1}\delta_p,$$

$$(\tilde{H}_\omega^{(n)} - z)^{-1}\delta_k = (H_\omega - z)^{-1}\delta_k + \sum_{1\le p\le n} \omega(p)(\delta_p|(\tilde{H}_\omega^{(n)} - z)^{-1}\delta_k)(H_\omega - z)^{-1}\delta_p,$$

and thus

$$(H_\omega^{(n)} - z)^{-1}\delta_k \in \mathcal{H}_{0,\mathcal{N}}, \qquad (\tilde{H}_\omega^{(n)} - z)^{-1}\delta_k \in \mathcal{H}_{\omega,\mathcal{N}}.$$

If $\mathcal{N}$ is finite, then there is an n for which $H_\omega^{(n)} = H_\omega$ and $\tilde{H}_\omega^{(n)} = H_0$, and so we see that $\mathcal{H}_{0,\mathcal{N}} = \mathcal{H}_{\omega,\mathcal{N}}$. Otherwise, our assumption that both $H_\omega \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ and $H_0 \restriction \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}\,(V_\omega)$ are essentially self adjoint implies that $\lim_{n\to\infty} H_\omega^{(n)} = H_\omega$ and $\lim_{n\to\infty} \tilde{H}_\omega^{(n)} = H_0$, where both limits are in the strong resolvent sense. Thus, we have

$$(H_\omega - z)^{-1}\delta_k = \lim_{n\to\infty} (H_\omega^{(n)} - z)^{-1}\delta_k \in \mathcal{H}_{0,\mathcal{N}},$$

$$(H_0 - z)^{-1}\delta_k = \lim_{n\to\infty} (\tilde{H}_\omega^{(n)} - z)^{-1}\delta_k \in \mathcal{H}_{\omega,\mathcal{N}},$$

and so $\mathcal{H}_{0,\mathcal{N}} = \mathcal{H}_{\omega,\mathcal{N}}$.

(2) First, note that $\mathcal{H}_{\omega,n} \not\perp \mathcal{H}_{\omega,m}$ iff there exists $z \in \mathbb{C}\setminus\mathbb{R}$ such that $(\delta_n|(H_\omega - z)^{-1}\delta_m) \neq 0$. Let $\omega \in E_{n,m}$ and let z be such that $(\delta_n|(H_\omega - z)^{-1}\delta_m) \neq 0$. Let $p \in \mathcal{N}$ and

$$H_{\omega,\lambda} = H_\omega + \lambda(\delta_p|\,\cdot\,)\delta_p. \tag{3.5}$$

Then

$$
\begin{aligned}
(\delta_n|(H_{\omega,\lambda}-z)^{-1}\delta_m) &= (\delta_n|H_\omega - z)^{-1}\delta_m) - \lambda(\delta_n|(H_\omega - z)^{-1}\delta_p)(\delta_p|(H_{\omega,\lambda}-z)^{-1}\delta_m) \\
&= (\delta_n|H_\omega - z)^{-1}\delta_m) - \lambda\frac{(\delta_n|(H_\omega - z)^{-1}\delta_p)(\delta_p|(H_\omega - z)^{-1}\delta_m)}{1+\lambda(\delta_p|(H_\omega - z)^{-1}\delta_p)},
\end{aligned}
\tag{3.6}
$$

where the second equality in (3.6) is obtained by applying the first equality to the case $n = p$ and then using the resulting expression for $(\delta_p|(H_{\omega,\lambda}-z)^{-1}\delta_m)$. Hence, there exists at most one λ such that $(\delta_n|(H_{\omega,\lambda}-z)^{-1}\delta_m) = 0$. Since the random variable $\omega(p)$ has density, the event $E_{n,m}$ is independent of $\omega(p)$ for all $p \in \mathcal{N}$. If $\mathcal{N}$ is finite, this yields the statement. If $\mathcal{N}$ is infinite, then $E_{n,m}$ is measurable w.r.t. the tail σ-field of the sequence $\{\omega(p)\}$ and Kolmogorov's 0-1 law yields that $P(E_{n,m}) \in \{0, 1\}$.

(3) Let

$$R_{n,m} := \{\omega \,:\, (\delta_n|(H_\omega - z)^{-1}\delta_m) \neq 0 \,\text{for some}\; z \in \mathbb{C}_+\}.$$

Obviously, $R_{n,m} \subseteq E_{n,m}$. By the same argument as in the proof of (2), we see that $P(R_{n,m}) \in \{0, 1\}$. If $P(E_{n,m}) = 1$ and $P(R_{n,m}) = 0$, then, by Theorem 2.2, $\mu_{n,\omega,\mathrm{ac}}$ and $\mu_{m,\omega,\mathrm{ac}}$ are equivalent to the Lebesgue measure on $\mathbb{R}$ for P-a.e. ω, and thus, by Proposition 3.1, $n, m \in \mathcal{N} \setminus \mathcal{S}$. Hence, for $n, m \in \mathcal{S}$, $P(E_{n,m}) = 1$ iff $P(R_{n,m}) = 1$.

Consider now $n, m, p \in \mathcal{S}$ and assume that $n \sim p$ and $p \sim m$. Let $\omega \in R_{n,p} \cap R_{p,m}$. Let $z \in \mathbb{C}_+$ be such that $(\delta_n|(H_\omega - z)^{-1}\delta_p)(\delta_p|(H_\omega - z)^{-1}\delta_m) \neq 0$ (such z must exist, since the the product of two non-vanishing analytic functions is a non-vanishing analytic function). Then (3.6) again yields that there exists at most one λ such that $(\delta_n|(H_{\omega,\lambda}-z)^{-1}\delta_m) = 0$. Let $\delta^{(p)} \in \Omega = \mathbb{R}^{\mathcal{N}}$ be the sequence with $\delta^{(p)}(p) = 1$ and $\delta^{(p)}(n) = 0$ for $n \neq p$. Then for every $\omega \in R_{n,p} \cap R_{p,m}$, $\omega + \lambda\delta^{(p)} \in R_{n,m}$ for Lebesgue a.e. λ. Since the random variable $\omega(p)$ has density, the Fubini theorem yields that $P((R_{n,p} \cap R_{p,m}) \setminus R_{n,m}) = 0$. Since $P(R_{n,p} \cap R_{p,m}) = 1$, we have $P(R_{n,m}) = 1$ and so $n \sim m$. This shows that $\sim$ is transitive. Since it is obviously symmetric, we see that it is an equivalence relation.

(4) Consider any $k \geq 0$. We first show that for any $n \in \mathcal{S}_k$ and $m \in \mathcal{N} \setminus \mathcal{S}_k$, $P(E_{n,m}) = 0$. For $n, m \in \mathcal{S}$, this is immediate from the definition of $\sim$. If $k = 0$ and $P(E_{n,m}) = 1$, then, by Theorem 1.3, $\mu_{\omega,n}$ and $\mu_{\omega,m}$ are P-a.s. equivalent and so $\mu_{\omega,m}$ must be equivalent to the Lebesgue measure on $\mathbb{R}$ with positive probability. Thus, $m \notin \mathcal{S}$ and it follows that $m \in \mathcal{S}_0 = \mathcal{S}_k$. Similarly, $m \in \mathcal{S}_0$ and $P(E_{n,m}) = 1$ imply $k = 0$. Thus, we see that in either case, $n \in \mathcal{S}_k$ and $m \in \mathcal{N} \setminus \mathcal{S}_k$ imply $P(E_{n,m}) = 0$.

For every $\omega \in \Omega$, $p \in \mathcal{N}$, and $\lambda \in \mathbb{R}$, let $H_{\omega,\lambda}$ be given by (3.5) and let $\mathcal{H}_{\omega,\lambda,\mathcal{S}_k}$ be the cyclic subspace generated by $H_{\omega,\lambda}$ and $\{\delta_n\}_{n\in\mathcal{S}_k}$. Similarly to 3.6, we have for any $n \in \mathcal{N}$,

$$(H_{\omega,\lambda}-z)^{-1}\delta_n = (H_\omega - z)^{-1}\delta_n - \lambda\frac{(\delta_p|(H_\omega - z)^{-1}\delta_n)}{1+\lambda(\delta_p|(H_\omega - z)^{-1}\delta_p)}(H_\omega - z)^{-1}\delta_p. \tag{3.7}$$

Consider now $n \in \mathcal{S}_k$. If $p \in \mathcal{S}_k$, we see from (3.7) that $\mathcal{H}_{\omega,\lambda,\mathcal{S}_k} \subseteq \mathcal{H}_{\omega,\mathcal{S}_k}$ for every $\omega \in \Omega$ and $\lambda \in \mathbb{R}$. By considering $\upsilon = \omega + \lambda\delta^{(p)}$, this also implies $\mathcal{H}_{\omega,\mathcal{S}_k} = \mathcal{H}_{\upsilon,-\lambda,\mathcal{S}_k} \subseteq \mathcal{H}_{\upsilon,\mathcal{S}_k} = \mathcal{H}_{\omega,\lambda,\mathcal{S}_k}$, and so we see that for $p \in \mathcal{S}_k$, $\mathcal{H}_{\omega,\lambda,\mathcal{S}_k} = \mathcal{H}_{\omega,\mathcal{S}_k}$ for every $\omega \in \Omega$ and $\lambda \in \mathbb{R}$. If $p \notin \mathcal{S}_k$, then $P(E_{n,p}) = 0$, and so we have $(\delta_p|(H_\omega - z)^{-1}\delta_n) = 0$ for P-a.e. ω, for every $z \in \mathbb{C}\setminus\mathbb{R}$. Thus, we have in this case that $(H_{\omega,\lambda} - z)^{-1}\delta_n = (H_\omega - z)^{-1}\delta_n$ for P-a.e. ω, for every $z \in \mathbb{C}\setminus\mathbb{R}$, and so we conclude that $\mathcal{H}_{\omega,\lambda,\mathcal{S}_k} = \mathcal{H}_{\omega,\mathcal{S}_k}$ for P-a.e. ω, for every $\lambda \in \mathbb{R}$. Therefore, for any $p \in \mathcal{N}$, we have that $\mathcal{H}_{\omega,\lambda,\mathcal{S}_k} = \mathcal{H}_{\omega,\mathcal{S}_k}$ for P-a.e. ω, for every $\lambda \in \mathbb{R}$. That is, the projection on $\mathcal{H}_{\omega,\mathcal{S}_k}$ is a measurable function of ω which is P-a.s. independent of $\omega(p)$ for any $p \in \mathcal{N}$. Thus, Kolmogorov's 0-1 law yields the required statement.

(5) Consider $k, m \geq 0$, $k \neq m$. For every $n \in \mathcal{S}_k$ and $p \in \mathcal{S}_m$, we have $P(E_{n,p}) = 0$, and so for P-a.e. ω, $\mathcal{H}_{\omega,n} \perp \mathcal{H}_{\omega,p}$. Thus, we clearly obtain that for P-a.e. ω, $\mathcal{H}_{\omega,\mathcal{S}_k} \perp \mathcal{H}_{\omega,\mathcal{S}_m}$. □

We note that the only thing which is (possibly) preventing $\sim$ from being an equivalence relation on all of $\mathcal{N}$ is the possibility that in some cases where $\mathcal{H}_{\omega,n} \not\perp \mathcal{H}_{\omega,m}$, we would nevertheless have $(\delta_n|(H_\omega - z)^{-1}\delta_m) = 0$ for all $z \in \mathbb{C}_+$ (in which case we must have that $(\delta_m|(H_\omega - z)^{-1}\delta_n)$ doesn't vanish on $\mathbb{C}_+$, and so $P(R_{n,m})$ need not be symmetric in n and m). This can prevent the relation $\sim$ from being transitive. Indeed, it is not difficult to construct examples (albeit with an unbounded H_0) with $\mathcal{N} = \{1, 2, 3\}$, where $1 \sim 2$ and $2 \sim 3$, but $1 \not\sim 3$. Fortunately, the F. & M. Riesz theorem, Theorem 2.2, along with Proposition 3.1, assure us that this kind of "anomaly" can only happen in cases where $\mu_{\omega,n}$ and $\mu_{\omega,m}$ are both equivalent to the Lebesgue measure on $\mathbb{R}$. Thus, one can simply consider the class $\mathcal{S}_0 \equiv \mathcal{N} \setminus \mathcal{S}$ as a special equivalence class. This class is different from the other classes $\mathcal{S}_k$, $k = 1, 2, \ldots$, in that for $k \geq 1$, we have $P(E_{n,m}) = 1$ for every $n, m \in \mathcal{S}_k$, whereas we may have $P(E_{n,m}) = 0$ for $n, m \in \mathcal{S}_0$, so that $\mathcal{H}_{\omega,\mathcal{S}_0}$ may consist of "disconnected" components and $H_\omega \restriction \mathcal{H}_{\omega,\mathcal{S}_0}$ may be a more complex object than $H_\omega \restriction \mathcal{H}_{\omega,\mathcal{S}_k}$ for $k \geq 1$. Since, however, for $n \in \mathcal{S}_0$, $\mu_{\omega,n}$ must be almost surely equivalent to the Lebesgue measure on $\mathbb{R}$, the spectral properties of $H_\omega \restriction \mathcal{H}_{\omega,\mathcal{S}_0}$ are essentially known (in particular, it must have purely absolutely continuous spectrum, P-a.s., and it thus plays no role for the singular spectrum of H_ω) and so, from the perspective of spectral theory, there is not much loss here.

We further note that $\sim$ is, in fact, an equivalence relation on all of $\mathcal{N}$ in the following cases (even if $\mathcal{N} \setminus \mathcal{S} \neq \emptyset$):
(a) The spectral measure for H_0 and the pair of vectors δ_n,δ_m is real-valued (namely, a signed measure) for every pair $n, m \in \mathcal{N}$.
(b) $\delta_n \in \mathrm{Dom}\,(H_\omega^j)$ for all $n \in \mathcal{N}$, $j \geq 0$ and P-a.e. ω.
The case (a) is seen immediately from equation (2.3), since one sees that in this case, $(\delta_n|(H_\omega - \overline{z})^{-1}\delta_m) = \overline{(\delta_n|(H_\omega - z)^{-1}\delta_m)}$, where $\overline{\cdot}$ denotes complex conjugation. For a proof of (b), see Lemma 5.10 in [JL2]. In any of these cases, one clearly has a slightly simpler variant of Theorem 1.1, in which $\sim$ is an equivalence relation on $\mathcal{N}$ and where all of the $\mathcal{S}_k$'s are equivalence classes generated by $\sim$.

We also need the following fact:

Proposition 3.2 *Let $n, m \in \mathcal{S}$ and suppose that $n \not\sim m$. Then for P-a.e. ω, the measures*

$\mu_{\omega,n,\text{sing}}$ *and* $\mu_{\omega,m,\text{sing}}$ *are mutually singular.*

Proof. We decompose the probability space Ω along the n-th coordinate, $\Omega = \mathbb{R} \times \tilde{\Omega}$, $\tilde{\Omega} = \mathbb{R}^{\mathcal{N}\setminus\{n\}}$, $\mathrm{d}P = \mathrm{d}p_n \otimes \mathrm{d}\tilde{P}$, $\mathrm{d}\tilde{P} = \bigotimes_{\mathcal{N}\setminus\{n\}} \mathrm{d}p_j$, and we write $\omega = (\lambda, \tilde{\omega})$, where $\lambda \in \mathbb{R}$ and $\tilde{\omega} \in \tilde{\Omega}$. Similarly to equation (3.6), we have for every $p \in \mathcal{N}$ and $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(\delta_p|(H_{(\lambda,\tilde{\omega})} - z)^{-1}\delta_m) = (\delta_p|H_{(0,\tilde{\omega})} - z)^{-1}\delta_m) - \lambda \frac{(\delta_p|(H_{(0,\tilde{\omega})} - z)^{-1}\delta_n)(\delta_n|(H_{(0,\tilde{\omega})} - z)^{-1}\delta_m)}{1 + \lambda(\delta_n|(H_{(0,\tilde{\omega})} - z)^{-1}\delta_n)}. \tag{3.8}$$

Since $P(E_{n,m}) = 0$, we must have $(\delta_n|(H_\omega - z)^{-1}\delta_m) = 0$ for P-a.e. ω, for every $z \in \mathbb{C}\setminus\mathbb{R}$, and so by setting $p = n$ in (3.8), we see that we must also have $(\delta_n|(H_{(0,\tilde{\omega})} - z)^{-1}\delta_m) = 0$ for $\tilde{P}$-a.e. $\tilde{\omega}$, for every $z \in \mathbb{C} \setminus \mathbb{R}$ (otherwise, the r.h.s. of (3.8) cannot vanish for more than one value of λ). Setting now $p = m$ in (3.8), we conclude that for $\tilde{P}$-a.e. $\tilde{\omega}$, for every $z \in \mathbb{C} \setminus \mathbb{R}$, $(\delta_m|(H_{(\lambda,\tilde{\omega})} - z)^{-1}\delta_m)$ is independent of λ. Now let

$$S_{\omega,m} \equiv \{E \in \mathbb{R} \,:\, \limsup_{\epsilon\downarrow 0} |(\delta_m|(H_\omega - E - \mathrm{i}\epsilon)^{-1}\delta_m)| = \infty\},$$

then for every ω, $S_{\omega,m}$ is a Borel set of zero Lebesgue measure which supports $\mu_{\omega,m,\text{sing}}$. Since, for $\tilde{P}$-a.e. $\tilde{\omega}$, $(\delta_m|(H_{(\lambda,\tilde{\omega})} - z)^{-1}\delta_m)$ is independent of λ, we see that $S_{(\lambda,\tilde{\omega}),m}$ is also independent of λ, namely, for $\tilde{P}$-a.e. $\tilde{\omega}$, $S_{(\lambda,\tilde{\omega}),m} = S_{(0,\tilde{\omega}),m}$ for every $\lambda \in \mathbb{R}$.

By Fubini's theorem,

$$\int_\Omega \mu_{\omega,n,\text{sing}}(S_{\omega,m})\,\mathrm{d}P(\omega) = \int_{\tilde{\Omega}} \left[\int_{\mathbb{R}} \mu_{(\lambda,\tilde{\omega}),n,\text{sing}}(S_{(0,\tilde{\omega}),m})\,\mathrm{d}p_n(\lambda)\right] \mathrm{d}\tilde{P}(\tilde{\omega}).$$

Noting that $\mu_{(\lambda,\tilde{\omega}),n}$ is the spectral measure for $H_{(0,\tilde{\omega})} + \lambda(\delta_n|\cdot)\delta_n$ and δ_n, and that the measure p_n is absolutely continuous w.r.t. the Lebesgue measure, we have, by Lemma 2.4,

$$\int_{\mathbb{R}} \mu_{(\lambda,\tilde{\omega}),n,\text{sing}}(S_{(0,\tilde{\omega}),m})\,\mathrm{d}p_n(\lambda) = 0$$

for all $\tilde{\omega} \in \tilde{\Omega}$. Hence, $\int_\Omega \mu_{\omega,n,\text{sing}}(S_{\omega,m})\,\mathrm{d}P(\omega) = 0$, and it follows that $\mu_{\omega,n,\text{sing}} \perp \mu_{\omega,m,\text{sing}}$ for P-a.e. ω. □

4 Rank one perturbations

The starting point of this and the next section is a self adjoint operator A on a separable Hilbert space $\mathcal{H}$ and two orthonormal vectors $\psi, \varphi \in \mathcal{H}$, which are a cyclic family for A. Namely, we assume that the linear span of the set of vectors

$$\{(A - z)^{-1}\psi \,:\, z \in \mathbb{C} \setminus \mathbb{R}\} \cup \{(A - z)^{-1}\varphi \,:\, z \in \mathbb{C} \setminus \mathbb{R}\}$$

is dense in $\mathcal{H}$.

In this section we consider the family of operators

$$A_\lambda = A + \lambda(\psi|\cdot)\psi, \qquad \lambda \in \mathbb{R}.$$

The vectors ψ, φ are a cyclic family for A_λ, for all $\lambda \in \mathbb{R}$. We denote by $\mathcal{H}_{\lambda,\psi}$ and $\mathcal{H}_{\lambda,\varphi}$ the cyclic subspaces generated by A_λ and, correspondingly, ψ and φ. These subspaces are not orthogonal for all λ iff they are not orthogonal for $\lambda = 0$. Note, also, that $\mathcal{H}_{\lambda,\psi} = \mathcal{H}_\psi$ for all λ.

Recall the following result of [JL1]:

Theorem 4.1 (Theorem 2.4 of [JL1]) *Let $\mu_{\lambda,\psi}$ and $\mu_{\lambda,\varphi}$ be the spectral measures for A_λ and, correspondingly, ψ and φ. Assume that the cyclic subspaces $\mathcal{H}_{\lambda,\psi}$ and $\mathcal{H}_{\lambda,\varphi}$ are not orthogonal. Then for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mu_{\lambda,\psi}$ is absolutely continuous w.r.t. $\mu_{\lambda,\varphi}$.*

Remark. We note that there is a minor error in the proof of Theorem 4.1 in [JL1]. Explicitly, Proposition 2.1 of [JL1] is not correct as stated. It should have been formulated with the additional condition that the total variation of the complex spectral measure $\mu_{\varphi,\psi}$ of the two vectors is not equivalent to the Lebesgue measure on $\mathbb{R}$. This means that, in essence, Theorem 4.1 is proven in [JL1] under the assumption that $(\psi|(A-z)^{-1}\varphi)$ and $(\psi|(A-z)^{-1}\varphi)$ do not vanish identically in $\mathbb{C}_+$, which is stronger than assuming just $\mathcal{H}_{\lambda,\psi} \not\perp \mathcal{H}_{\lambda,\varphi}$. However, if either $(\psi|(A-z)^{-1}\varphi)$ or $(\psi|(A-z)^{-1}\varphi)$ vanishes identically in $\mathbb{C}_+$ and if $\mathcal{H}_{\lambda,\psi} \not\perp \mathcal{H}_{\lambda,\varphi}$, then by Theorem 2.2, $\mu_{\lambda,\psi,\mathrm{ac}}$ and $\mu_{\lambda,\varphi,\mathrm{ac}}$ are both equivalent to the Lebesgue measure on $\mathbb{R}$, and so, by Proposition 3.1, $\mu_{\lambda,\psi}$ is equivalent to the Lebesgue measure on $\mathbb{R}$ for Lebesgue a.e. λ, and we see that Theorem 4.1 holds. Thus, Theorem 4.1, as well as all of the other results of [JL1] (which are mainly its consequences), are correct as stated.

Let $\mathcal{H}_{\lambda,\mathrm{sing}}$ and $\mathcal{H}_{\lambda,\psi,\mathrm{sing}}$ be the subspaces associated, correspondingly, with the singular spectra of A_λ and $A_\lambda \restriction \mathcal{H}_{\lambda,\psi}$. Our goal in this section is to prove:

Theorem 4.2 *Assume that the subspaces $\mathcal{H}_{\lambda,\psi}$ and $\mathcal{H}_{\lambda,\varphi}$ are not orthogonal and that for Lebesgue a.e. $\lambda \in \mathbb{R}$, the measures $\mu_{\lambda,\psi}$ and $\mu_{\lambda,\varphi}$ are equivalent. Then, for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mathcal{H}_{\lambda,\mathrm{sing}} = \mathcal{H}_{\lambda,\psi,\mathrm{sing}}$.*

Proof. Let $\mathbf{1}_{\mathrm{sing}}(A_\lambda)$ be the projection on $\mathcal{H}_{\lambda,\mathrm{sing}}$. To prove the statement, we need to show that for Lebesgue a.e. λ, $\mathbf{1}_{\mathrm{sing}}(A_\lambda)\varphi \in \mathcal{H}_{\lambda,\psi,\mathrm{sing}}$. We use similar notations to those in the proof of Theorem 2.4 of [JL1]. In particular, for $\phi_1, \phi_2 \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we write $G_\lambda(\phi_1, \phi_2, z) \equiv (\phi_1|(A_\lambda - z)^{-1}\phi_2)$.

From the operator identity $A^{-1} - B^{-1} = B^{-1}(B-A)A^{-1}$, we have

$$G_\lambda(\phi_1, \phi_2, z) = G_0(\phi_1, \phi_2, z) - \lambda G_0(\phi_1, \psi, z) G_\lambda(\psi, \phi_2, z). \tag{4.9}$$

By setting $\phi_1 = \phi_2 = \psi$ in (4.9), we get

$$G_\lambda(\psi,\psi,z) = \frac{G_0(\psi,\psi,z)}{1+\lambda G_0(\psi,\psi,z)}, \tag{4.10}$$

and similarly, by setting $\phi_1 = \psi$, $\phi_2 = \varphi$, we get

$$G_\lambda(\psi,\varphi,z) = \frac{G_0(\psi,\varphi,z)}{1+\lambda G_0(\psi,\psi,z)}. \tag{4.11}$$

By setting in (4.9) $\phi_1 = \phi_2 = \varphi$, and then using both (4.10) and (4.11), we obtain (see [JL1])

$$G_\lambda(\varphi,\varphi,z) = G_0(\varphi,\varphi,z) - \lambda \frac{G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)}{G_0(\psi,\psi,z)} G_\lambda(\psi,\psi,z). \tag{4.12}$$

Without loss of generality, we assume that $G_0(\varphi,\psi,z)$ and $G_0(\psi,\varphi,z)$ do not vanish identically in $\mathbb{C}_+$ (otherwise, Theorem 2.2 and Proposition 3.1 would imply that for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mu_{\lambda,\psi}$ and $\mu_{\lambda,\varphi}$ are both equivalent to the Lebesgue measure on $\mathbb{R}$, and so $\mathcal{H}_{\lambda,\mathrm{sing}} = \mathcal{H}_{\lambda,\psi,\mathrm{sing}} = \emptyset$ for Lebesgue a.e. $\lambda \in \mathbb{R}$). Let S_0 be the set of all $E \in \mathbb{R}$ for which the limits $G_0(\psi,\psi,E+\mathrm{i}0)$, $G_0(\psi,\varphi,E+\mathrm{i}0)$, $G_0(\varphi,\psi,E+\mathrm{i}0)$, and $G_0(\varphi,\varphi,E+\mathrm{i}0)$ exist, are finite, and $G_0(\psi,\psi,E+\mathrm{i}0) \neq 0$, $G_0(\varphi,\varphi,E+\mathrm{i}0) \neq 0$. By well-known results about boundary values of analytic functions [Ko], the set $\mathbb{R} \setminus S_0$ has Lebesgue measure zero.

By Lemma 2.3, we see that for Lebesgue a.e. E where $G_0(\psi,\psi,E+\mathrm{i}0) \in \mathbb{R}$, we must also have $G_0(\psi,\varphi,E+\mathrm{i}0) = G_0(\psi,\varphi,E-\mathrm{i}0)$. We define S_1 to be the subset of S_0 where either $G_0(\psi,\psi,E+\mathrm{i}0) \notin \mathbb{R}$ or else $G_0(\psi,\varphi,E+\mathrm{i}0) = G_0(\psi,\varphi,E-\mathrm{i}0)$. Then the set $\mathbb{R}\setminus S_1$ has Lebesgue measure zero.

Recall that $\mu_{\lambda,\psi,\mathrm{sing}}$ is supported on the set

$$S_2 := \{E \,:\, G_0(\psi,\psi,E+\mathrm{i}0) = -\lambda^{-1}\},$$

a fact which can be easily seen from (4.10) (or see, e.g., [Si2]). The set S_2 need not be contained in S_1. However, by Lemma 2.4,

$$\int_{\mathbb{R}} \mu_{\lambda,\psi,\mathrm{sing}}(\mathbb{R}\setminus S_1)\,\mathrm{d}\lambda = 0,$$

and so for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mu_{\lambda,\psi,\mathrm{sing}}$ is supported on $S_2 \cap S_1$. Hence, by using (4.10) and (4.11), we see that for Lebesgue a.e. $\lambda \in \mathbb{R}$, for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$,

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \frac{G_\lambda(\psi,\varphi,E+\mathrm{i}\epsilon)}{G_\lambda(\psi,\psi,E+\mathrm{i}\epsilon)} &= \lim_{\epsilon\downarrow 0} \frac{G_0(\psi,\varphi,E+\mathrm{i}\epsilon)}{G_0(\psi,\psi,E+\mathrm{i}\epsilon)} \\ &= \frac{G_0(\psi,\varphi,E+\mathrm{i}0)}{G_0(\psi,\psi,E+\mathrm{i}0)} = -\lambda G_0(\psi,\varphi,E+\mathrm{i}0). \end{aligned} \tag{4.13}$$

In the sequel, ϕ_ψ denotes the projection of a vector ϕ on the cyclic subspace $\mathcal{H}_\psi = \mathcal{H}_{\lambda,\psi}$. We identify $\mathcal{H}_\psi = L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, $\psi = 1$. Then, by Theorem 2.1, for every $\lambda \in \mathbb{R}$,

$$\lim_{\epsilon\downarrow 0} \frac{G_\lambda(\psi, \varphi, E + \mathrm{i}\epsilon)}{G_\lambda(\psi, \psi, E + \mathrm{i}\epsilon)} = (\mathbf{1}_{\mathrm{sing}}(A_\lambda)\varphi)_\psi(E)$$

for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$. This and (4.13) yield that for Lebesgue a.e. $\lambda \in \mathbb{R}$, for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$,

$$\left(\mathbf{1}_{\mathrm{sing}}(A_\lambda)\varphi\right)_\psi(E) = -\lambda G_0(\psi, \varphi, E + \mathrm{i}0). \tag{4.14}$$

Let

$$\mathrm{d}\mu_{\lambda,\varphi} = f_{\lambda,\varphi,\psi}\, \mathrm{d}\mu_{\lambda,\psi},$$

be the Lebesgue decomposition of $\mu_{\lambda,\varphi}$ w.r.t. $\mu_{\lambda,\psi}$ (which is known to exist for Lebesgue a.e. $\lambda \in \mathbb{R}$ by our assumption that $\mu_{\lambda,\varphi}$ and $\mu_{\lambda,\psi}$ are equivalent for Lebesgue a.e. $\lambda \in \mathbb{R}$). Theorem 2.1 yields that for Lebesgue a.e. $\lambda \in \mathbb{R}$, for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$,

$$\lim_{\epsilon\downarrow 0} \frac{G_\lambda(\varphi, \varphi, E + \mathrm{i}\epsilon)}{G_\lambda(\psi, \psi, E + \mathrm{i}\epsilon)} = f_{\lambda,\varphi,\psi}(E).$$

Since $\lim_{\epsilon\to 0} |G_\lambda(\psi, \psi, E + \mathrm{i}\epsilon)| = \infty$ for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$, it follows from (4.12) that for Lebesgue a.e. $\lambda \in \mathbb{R}$,

$$\begin{aligned}\lim_{\epsilon\downarrow 0} \frac{G_\lambda(\varphi, \varphi, E + \mathrm{i}\epsilon)}{G_\lambda(\psi, \psi, E + \mathrm{i}\epsilon)} &= -\lambda \frac{G_0(\varphi, \psi, E + \mathrm{i}0) G_0(\psi, \varphi, E + \mathrm{i}0)}{G_0(\psi, \psi, E + \mathrm{i}0)} \\ &= \lambda^2 G_0(\varphi, \psi, E + \mathrm{i}0) G_0(\psi, \varphi, E + \mathrm{i}0),\end{aligned}$$

for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$. It thus follows that for Lebesgue a.e. $\lambda \in \mathbb{R}$, for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$,

$$f_{\lambda,\varphi,\psi}(E) = \lambda^2 G_0(\varphi, \psi, E + \mathrm{i}0) G_0(\psi, \varphi, E + \mathrm{i}0).$$

Since $G_0(\psi, \psi, E + \mathrm{i}0)$ is real for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$ (in fact, equal to $-\lambda^{-1}$), we have

$$G_0(\varphi, \psi, E + \mathrm{i}0) = \overline{G_0(\psi, \varphi, E - \mathrm{i}0)} = \overline{G_0(\psi, \varphi, E + \mathrm{i}0)}$$

for Lebesgue a.e. $\lambda \in \mathbb{R}$, for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$. Thus, by using (4.14), we see that for Lebesgue a.e. $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mathrm{d}\mu_{\lambda,\varphi,\mathrm{sing}}(E) &= \lambda^2 |G_0(\psi, \varphi, E + \mathrm{i}0)|^2\, \mathrm{d}\mu_{\lambda,\psi,\mathrm{sing}}(E) \\ &= |(\mathbf{1}_{\mathrm{sing}}(A_\lambda)\varphi)_\psi(E)|^2\, \mathrm{d}\mu_{\lambda,\psi.\mathrm{sing}}(E).\end{aligned}$$

Hence, for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mathbf{1}_{\mathrm{sing}}(A_\lambda)\varphi \in \mathcal{H}_{\lambda,\psi,\mathrm{sing}}$. $\square$

5 Rank two Perturbations

Let A, $\mathcal{H}$, ψ, and φ be as in the previous section. For $(\lambda, \eta) \in \mathbb{R}^2$, we define

$$A_{\lambda,\eta} = A + \lambda(\psi|\cdot)\psi + \eta(\varphi|\cdot)\varphi.$$

The vectors ψ, φ are a cyclic family for $A_{\lambda,\eta}$, for all $(\lambda, \eta) \in \mathbb{R}^2$. We denote by $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$ the cyclic subspaces generated by $A_{\lambda,\eta}$ and, correspondingly, ψ and φ. These subspaces are not orthogonal for all $(\lambda, \eta) \in \mathbb{R}^2$ iff they are not orthogonal for $\lambda = \eta = 0$.

We denote by $\mathcal{H}_{\lambda,\eta,\mathrm{sing}}$ the subspace associated with the singular spectrum of $A_{\lambda,\eta}$. Similarly, we denote by $\mathcal{H}_{\lambda,\eta,\psi,\mathrm{sing}}$ and $\mathcal{H}_{\lambda,\eta,\varphi,\mathrm{sing}}$ the subspaces associated, correspondingly, with the singular spectra of $A_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\psi}$ and $A_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\varphi}$.

Recall the following theorem of [JL1], which is an easy consequence of Theorem 4.1.

Theorem 5.1 (Theorem 2.5 of [JL1]) *Let $\mu_{\lambda,\eta,\psi}$ and $\mu_{\lambda,\eta,\varphi}$ be the spectral measures for $A_{\lambda,\eta}$ and, correspondingly, ψ and φ. Suppose that the cyclic subspaces $\mathcal{H}_{\lambda,\eta,\varphi}$ and $\mathcal{H}_{\lambda,\eta,\psi}$ are not orthogonal. Then for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$, the measures $\mu_{\lambda,\eta,\psi}$ and $\mu_{\lambda,\eta,\varphi}$ are equivalent.*

In this section we prove

Theorem 5.2 *Assume that the subspaces $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$ are not orthogonal. Then for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$,*

$$\mathcal{H}_{\lambda,\eta,\mathrm{sing}} = \mathcal{H}_{\lambda,\eta,\psi,\mathrm{sing}} = \mathcal{H}_{\lambda,\eta,\varphi,\mathrm{sing}}.$$

Proof. Let

$$F_1 := \{(\lambda, \eta) : \mathcal{H}_{\lambda,\eta,\mathrm{sing}} = \mathcal{H}_{\lambda,\eta,\psi,\mathrm{sing}}\}, \qquad F_2 := \{(\lambda, \eta) : \mathcal{H}_{\lambda,\eta,\mathrm{sing}} = \mathcal{H}_{\lambda,\eta,\varphi,\mathrm{sing}}\}.$$

By Theorem 5.1, for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$, the measures $\mu_{\lambda,\eta,\varphi}$ and $\mu_{\lambda,\eta,\psi}$ are equivalent. Fubini's theorem and Theorem 4.2 thus yield that for Lebesgue a.e. η, the relation

$$\mathcal{H}_{\lambda,\eta,\mathrm{sing}} = \mathcal{H}_{\lambda,\eta,\psi,\mathrm{sing}},$$

holds for Lebesgue a.e. λ. Thus, by Fubini's theorem again, the set F_1 has full Lebesgue measure in $\mathbb{R}^2$. A similar argument shows that F_2 has full Lebesgue measure in $\mathbb{R}^2$ and so we conclude that $F_1 \cap F_2$ has full Lebesgue measure in $\mathbb{R}^2$. □

6 Proofs of the main theorems

Proof of Theorem 1.4. Without loss of generality, we may assume that $\mathcal{N} = \{n \in \mathbb{N} : n < N\}$, where N is either finite or ∞, and that $n = 1$, $m = 2$. We will use the decompositions $\Omega = \mathbb{R}^2 \times \tilde{\Omega}$, $\omega = (\lambda, \eta, \tilde{\omega})$,

$$\mathrm{d}P(\omega) = (\mathrm{d}p_1(\lambda) \otimes \mathrm{d}p_2(\eta)) \otimes \mathrm{d}\tilde{P}(\tilde{\omega}).$$

Let Ω_0 be the set of all ω's such that $\mathcal{H}_{\omega,1}$ and $\mathcal{H}_{\omega,2}$ are not orthogonal. Then, by Fubini's theorem, there exists a set $\tilde{\Omega}_0 \subset \tilde{\Omega}$ such that $\tilde{P}(\tilde{\Omega}_0) = 1$ and for every $\tilde{\omega} \in \tilde{\Omega}_0$, the subspaces $\mathcal{H}_{(\lambda,\eta,\tilde{\omega}),1}$ and $\mathcal{H}_{(\lambda,\eta,\tilde{\omega}),2}$ are not orthogonal for $p_1 \otimes p_2$-a.e. $(\lambda, \eta) \in \mathbb{R}^2$ (and hence for all $(\lambda, \eta) \in \mathbb{R}^2$). Theorem 5.2 yields that for every $\tilde{\omega} \in \tilde{\Omega}_0$, for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$,

$$\mathcal{H}_{(\lambda,\eta,\tilde{\omega}),1,\mathrm{sing}} = \mathcal{H}_{(\lambda,\eta,\tilde{\omega}),2,\mathrm{sing}}. \tag{6.15}$$

Since $p_1 \otimes p_2$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^2$, (6.15) holds for every $\tilde{\omega} \in \tilde{\Omega}_0$, for $p_1 \otimes p_2$-a.e. $(\lambda, \eta) \in \mathbb{R}^2$. By Fubini's theorem, again, the relation

$$\mathcal{H}_{\omega,1,\mathrm{sing}} = \mathcal{H}_{\omega,2,\mathrm{sing}}$$

holds for P-a.e. ω. □

Proof of Theorem 1.2. The linear span of the set $\cup_{n \in \mathcal{S}} \mathcal{H}_{\omega,n,\mathrm{sing}}$ is dense in $\mathcal{H}_{\omega,\mathrm{sing}}$ for P-a.e. ω. Since $\mathcal{H}_{\omega,n,\mathrm{sing}} = \mathcal{H}_{\omega,m,\mathrm{sing}}$ P-a.s. if $n \sim m$, and $\mathcal{H}_{\omega,n,\mathrm{sing}} \perp \mathcal{H}_{\omega,m,\mathrm{sing}}$ P-a.s. if $n \not\sim m$, we have, for any choice of $n_k \in \mathcal{S}_k$,

$$\mathcal{H}_{\omega,\mathrm{sing}} = \bigoplus_k \mathcal{H}_{\omega,n_k,\mathrm{sing}}$$

for P-a.e. ω. By Proposition 3.2, if $k \neq j$, then the measures $\mu_{\omega,n_k,\mathrm{sing}}$ and $\mu_{\omega,n_j,\mathrm{sing}}$ are mutually singular for P-a.e. ω. This implies that for P-a.e. ω,

$$\mathcal{H}_{\omega,\psi,\mathrm{sing}} = \bigoplus_k \mathcal{H}_{\omega,n_k,\mathrm{sing}},$$

and so Theorem 1.2 follows. □

References

[CL] R. Carmona, J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhauser, Boston 1990

[Ko] P. Koosis, *Introduction to H_p spaces*, Second Ed., Cambridge Tracts in Mathematics **115**, Cambridge, Cambridge University Press 1998

[JL1] V. Jakšić, Y. Last, *Spectral structure of Anderson type Hamiltonians*, Invent. Math. **141** (2000), 561–577

[JL2] V. Jakšić, Y. Last, *Corrugated surfaces and a.c. spectrum*, Rev. Math. Phys. **12** (2000), 1465–1503

[JL3] V. Jakšić, Y. Last, *A new proof of Poltoratskii's theorem*, J. Funct. Anal., in press

[LS] Y. Last, B. Simon, in preparation

[Po] A.G. Poltoratskii, *The boundary behavior of pseudocontinuable functions*, St. Petersburg Math. J. **5** (1994), 389–406

[Ri] F. Riesz, M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrieme Congrès des Math. Scand., 1916

[Si1] B. Simon, *Cyclic vectors in the Anderson Model*, Rev. Math. Phys. **6** (1994), 1183–1185

[Si2] B. Simon, *Spectral analysis of rank one perturbations and applications*, CRM Lecture Notes Vol. 8 (J. Feldman, R. Froese, L. Rosen, eds), pp. 109-149, AMS, Providence, RI, 1995