THE BAND-EDGE BEHAVIOR OF THE DENSITY OF SURFACIC STATES

WERNER KIRSCH AND FRÉDÉRIC KLOPP

Abstract. This paper is devoted to the asymptotics of the density of surfacic states near the spectral edges for a discrete surfacic Anderson model. Two types of spectral edges have to be considered : fluctuating edges and stable edges. Each type has its own type of asymptotics. In the case of fluctuating edges, one obtains Lifshitz tails the parameters of which are given by the initial operator suitably "reduced" to the surface. For stable edges, the surface density of states behaves like the surface density of states of a constant (equal to the expectation of the random potential) surface potential. Among the tools used to establish this are the asymptotics of the surface density of states for constant surface potentials.

0. Introduction

On $\mathbb{Z}^d$ ($d = d_1 + d_2$, $d_1 > 0$, $d_2 > 0$), we consider random Hamiltonians of the form

$$H_\omega = -\frac{1}{2}\Delta + V_\omega$$

where

- $-\Delta$ is the free Laplace operator, i.e., $-(\Delta u)(n) = \sum_{|m-n|=1} u(m)$;
- V_ω is a random potential concentrated on the sub-lattice $\mathbb{Z}^{d_1} \times \{0\} \subset \mathbb{Z}^d$ of the form

$$V_\omega(\gamma_1, \gamma_2) = \begin{cases} \omega_{\gamma_1} & \text{if } \gamma_2 = 0, \\ 0 & \text{if } \gamma_2 \neq 0. \end{cases}, \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} = \mathbb{Z}^d. \tag{0.1}$$

  and $(\omega_{\gamma_1})_{\gamma_1 \in \mathbb{Z}^{d_1}}$ is a family of i.i.d. bounded random variables. For the sake of simplicity, let us assume that the random variables are uniformly distributed in $[a, b]$ ($a < b$).

To keep the exposition as simple as possible in the introduction, we use these quite restrictive assumptions. We will deal with more general models in the next section.

The operator H_ω is bounded for almost every ω. It is ergodic with respect to shifts parallel to the surface. So we know there exists Σ the almost sure spectrum of H_ω (see e.g. [14, 23].

For H_ω, one defines the integrated density of surface states (the IDSS in the sequel), in the following way (see e.g. [8, 2, 3, 20]): for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we set

$$(\varphi, n_s) = \mathbb{E}(\mathrm{tr}(\Pi_1[\varphi(H_\omega) - \varphi(-\frac{1}{2}\Delta)]\Pi_1)) \tag{0.2}$$

where Π_1 is the orthogonal projector on the subspace $\mathbb{C}\delta_0 \otimes \ell^2(\mathbb{Z}^{d_2}) \subset \ell^2(\mathbb{Z}^d)$. Here, δ_0 denotes the vector with components $(\delta_{0j})_{j \in \mathbb{Z}^{d_1}}$.

Obviously, equation (0.2) defines the integrated density of surface states n_s only up to a constant. We choose this constant so that n_s vanishes below $\Sigma \cup \Sigma_0$ where Σ_0 is the spectrum of $-\frac{1}{2}\Delta$. We will see later on that, up to addition of a well controlled distribution, n_s is a positive measure.

One knows that $\Sigma = \sigma(-\frac{1}{2}\Delta) \cup \mathrm{supp}(dn_s)$ (see [8, 9, 2]. We will study the behavior of n_s at the edges of Σ. To simplify this set as much as possible, we will assume that the support of the random variables $(\omega_{\gamma_1})_{\gamma_1 \in \mathbb{Z}^{d_1}}$ is connected. Under this assumption, we know that

Lemma 0.1. *Σ is a compact interval given by*

$$\Sigma = \sigma(-\frac{1}{2}\Delta_{d_1}) + \bigcup_{\omega_0 \in [a,b]} \sigma(-\frac{1}{2}\Delta_{d_2} + \omega \Pi_0^2) \tag{0.3}$$

where Π_0^2 is the projector on the unit vector $\delta_0^2 \in \ell^2(\mathbb{Z}^{d_2})$.

This is a consequence of a standard characterization of Σ in terms of periodic potentials (see [14, 23]). The assumption that the random variables have connected support can be relaxed; more connected components for the support of the random variables will in general give rise to more spectral edges (as in the case of bulk randomness, see [16]). For the value of Σ, two different possibilities occur :

(1) $\Sigma = \sigma(-\frac{1}{2}\Delta) + [-\alpha, \beta] = [-d - \alpha, d + \beta]$ where $\alpha = \alpha(a)$, $\beta = \beta(b)$ and $\alpha + \beta > 0$; this occurs
  - if $d_2 \leq 2$ and either $a < 0$, in which case $\alpha(a) > 0$, or $b > 0$, in which case $\beta(b) > 0$,
  - if $d_2 \geq 3$ and $a > a_0$ or $b > b_0$, where, by (0.3), the thresholds a_0 and b_0 are uniquely determined by the family of operators $(-\frac{1}{2}\Delta_{d_2} + t\Pi_0^2)_{t\in\mathbb{R}}$.

   If $\alpha > 0$ (resp. $\beta > 0$), we say that the left (resp. right) edge is a "fluctuation edge" or "fluctuation boundary" (see [23]). If $\alpha = 0$ (resp. $\beta = 0$), we will speak of a "stable edge" or "stable boundary".

(2) $\Sigma = \sigma(-\frac{1}{2}\Delta)$; this occurs only in $d_2 \geq 3$ and if a is not too large, that is, if $a \in (0, a_0]$. In this case, both spectral edges are stable.

On the other hand, it is well known (see [24]) that,

- if $d_2 = 1, 2$, then, for $a > 0$, $\sigma(-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2) = [-d_2, d_2] \cup \{\lambda(a)\}$, and the spectrum in $[-d_2, d_2]$ is purely absolutely continuous and $\lambda(a)$ is a simple eigenvalue;
- if $d_2 \geq 3$, there exists $a^0 > 0$ such that
  - if $0 < a < a^0$, then, $\sigma(-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2) = [-d_2, d_2]$, and the spectrum is purely absolutely continuous;
  - if $a = a^0$, then
    * if $d_2 = 3, 4$, then $\sigma(-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2) = [-d_2, d_2]$, the spectrum is purely absolutely continuous, and $-d_2$ is a resonance for $-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2$;
    * if $d_2 \geq 5$, then $\sigma(-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2) = [-d_2, d_2]$, the spectrum is purely absolutely continuous in $[-d_2, d_2)$, and $-d_2$ is a simple eigenvalue for $-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2$;
  - if $a > a^0$, then, $\sigma(-\frac{1}{2}\Delta_{d_2} - a\Pi_0^2) = [-d_2, d_2] \cup \{\lambda(a)\}$, and the spectrum in $[-d_2, d_2]$ is purely absolutely continuous and $\lambda(a)$ is a simple eigenvalue;

For the operator $-\frac{1}{2}\Delta_{d_2} + b\Pi_0^2$, we have a symmetric situation.

Our aim is to study the density of surface states near the edges of Σ. In the present case, both edges are obviously symmetric. So we will only describe the lower edge. One has to distinguish between the case of fluctuation and stable edges. The behavior in the two cases are radically different.

0.1. **The stable edge.** As the discussion for lower and upper edge are symmetric, let us assume the lower edge is stable and work near that edge.

In the case of a stable edge, it is convenient to modify the normalization of the IDSS. Therefore, we introduce the operator

$$H_t = -\frac{1}{2}\Delta + t\mathbf{1} \otimes \Pi_0^2.$$

As above, let a be the infimum of the random variables $(\omega_j)_j$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, define

$$(\varphi, n_{s,\text{norm}}) = \mathbb{E}(\text{tr}(\Pi_1[\varphi(H_\omega) - \varphi(H_a)]\Pi_1))$$

The advantage of this renormalization is that the IDSS $n_{s,\text{norm}}$ is the distributional derivative of a positive measure. Indeed, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, define

$$(\varphi, dN_{s,\text{norm}}) = -\mathbb{E}(\text{tr}(\Pi_1[P(\varphi)(H_\omega) - P(\varphi)(H_a)]\Pi_1))$$

where

$$P(\varphi)(x) = \int_x^{+\infty} \varphi(t)dt.$$

Clearly, $dN_{s,\text{norm}}$ is independent of the anti-derivative of φ chosen to define it; it is a positive measure and

$$n_{s,\text{norm}} = -\frac{d}{dE} dN_{s,\text{norm}}.$$

Let n_s^t be the IDSS for H_t. As above, one can define a anti-derivative of n_s^t; denote it by $-dN_s^t$. Let $n_{s,\text{norm}}^t$ be the normalized version of n_s^t, i.e. $n_{s,\text{norm}}^t = n_s^t - n_s^a$. One has

$$n_{s,\text{norm}} + n_s = n_s^a. \tag{0.4}$$

One problem one encounters when studying n_s is that very little is known about its regularity for random surfacic models (see nevertheless [21]). Thanks to (0.4), we know that n_s is the difference of two distributions each of which is the derivative of a signed measure. So we can take the counting function of dN_s as $dN_s = dN_{s,\text{norm}} - dN_s^a$ is the difference of two measures. Thus, we define its counting function

$$N_s(E) = \int_{-d}^{E} dN_s(e). \tag{0.5}$$

An obvious consequence of (0.4) is the

Proposition 0.1. *One has*

$$N_s^a(E) \le N_s(E) \le N_s^b(E). \tag{0.6}$$

In section 5.1, we study the asymptotics for N_s^t. As a consequence of this study, we prove

Theorem 0.1. *Assume $d_2 = 1$ or 2. Then, one has*

$$N_s(E) \underset{\substack{E\to -d\\ E>-d}}{\sim} \begin{cases} \dfrac{Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}} \cdot (E+d)^{1+d_1/2} & \text{if } d_2 = 1,\\ \dfrac{2\,Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}} \cdot \dfrac{(E+d)^{1+d_1/2}}{|\log(E+d)|} & \text{if } d_2 = 2. \end{cases}$$

where $\mathbb{S}^{d_1-1}$ is the $d_1 - 1$ dimensional unit sphere.

If $a > 0$, this result is an immediate consequence of Proposition 0.1 and of Theorem 1.1 giving the asymptotics of the IDSS for constant surface potential (see also section 5.1). If $a = 0$, one needs to improve upon (0.6) as the left hand side of this inequality vanishes making it unusable. This is the purpose of Theorem 1.2.

When $d_2 \ge 3$, the situation becomes more complicated and we are only able to use Proposition 0.1 to get the two-sided estimate

$$C\frac{a(1+o(1))}{(1+aI)} \le \frac{(2\pi)^d}{s(E+d)(E+d)^{1+d_1/2}} \cdot N_s(E) \le C\frac{b(1+o(1))}{(1+bI)} \tag{0.7}$$

where C is a positive constant depending only on the dimensions d_1 and d_2 (see section 5.1) and

$$s(x) = \frac{1}{2}|x|^{\frac{d_2-2}{2}},$$

$$I = \frac{1}{2}\sup_{\theta_1\in\mathbb{T}^{d_1}} \int_{\theta_2\in\mathbb{T}^{d_2}} \left(d - \sum_{j=1}^{d_1}\cos(\theta_1^j) - \sum_{j=1}^{d_2}\cos(\theta_2^j)\right)^{-1} d\theta_2.$$

Here, and in the sequel, the measure $d\theta_\alpha$ ($\alpha \in \{1,2\}$) is the Haar measure on the torus $\mathbb{T}^{d_\alpha}$, i.e. the Lebesgue measure normalized to have total mass equal to one.

Let us note that, if $a < 0 < b$, the inequality (0.7) does not give much information of the actual behavior of $N_s(E)$ when $d_2 \ge 3$.

0.2. **The fluctuation edge.** Here, we assume that $E_0 = \inf \sigma(H_\omega)$ is strictly below $-d = \inf \sigma(-\frac{1}{2}\Delta)$. In this case, E_0 is a fluctuation edge of the spectrum.

Below the spectrum of $-\frac{1}{2}\Delta$, the density of surface states n_s is positive; hence, it is a Borel measure and the integrated density of surface states $N_s(E)$ can be defined as its distribution function, i.e. $N_s(E) = n_s((-\infty, E))$ for $E < -d$. We will prove Lifshitz type behavior for $N_s(E)$ for $E \searrow E_0$ which is characteristic for fluctuation edges. However, the Lifshitz exponent, in the homogeneous case typically equal to $-\frac{d}{2}$, is given by $-\frac{d_1}{2}$ in our case. More precisely, we will show

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\frac{d_1}{2}.$$

1. The main results

Let us now describe the general model we consider. Let H be a translational invariant Jacobi matrix with exponential off-diagonal decay that is $H = ((h_{\gamma-\gamma'}))_{\gamma,\gamma'\in\mathbb{Z}^d}$ such that,

(H0.a): $h_{-\gamma} = \overline{h_\gamma}$ for $\gamma \in \mathbb{Z}^d$ and for some $\gamma \neq 0$, $h_\gamma \neq 0$.

(H0.b): There exists $c > 0$ such that, for $\gamma \in \mathbb{Z}^d$,

$$|h_\gamma| \leq \frac{1}{c} e^{-c|\gamma|}.$$

The infinite matrix H defines a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. Using the Fourier transform, it is easily seen that H is unitarily equivalent to the multiplication by the function $\theta \mapsto h(\theta)$ defined by

$$h(\theta) = \sum_{\gamma\in\mathbb{Z}} h_\gamma e^{i\gamma\theta} \text{ where } \theta = (\theta_1, \dots, \theta_d),$$

acting as an operator on $L^2(\mathbb{T}^d)$ where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ (the Lebesgue measure on $\mathbb{T}^d$ is normalized so that the constant function 1 has norm 1). The function h is real analytic on $\mathbb{T}^d$. We normalize it so that it be non-negative and 0 be its minimum.

As both ends of the spectrum of our operator play symmetric parts, we only study what happens at a left edge, i.e. near the bottom of the spectrum. All our assumptions will reflect this fact.

1.1. The case of a constant surface potential.

We will start with a study of the density of surface states when the surfacic potential V is constant, i.e. $V = t\Pi_0^2$. We define the operator $H_t = H + t\mathbf{1} \otimes \Pi_0^2$. We prove two results on H_t. The first one is a criterion for the positivity of H_t and a description of its infimum when it is negative; the other result describes the density of the density of surface states near 0 when H_t is non-negative.

In the present section, we assume

(H1): the function $h : \mathbb{T}^d \to \mathbb{R}$ admits a unique minimum; it is quadratic non-degenerate.

If H is $-\frac{1}{2}\Delta$, then $h = h_0$ where

$$h_0(\theta) := \cos(\theta_1) + \cdots + \cos(\theta_d). \tag{1.1}$$

In this case, assumption (H1) is satisfied. Below, we give an example why considering more general Hamiltonians can be of interest.

For the sake of definiteness, we assume the minimum of h to be 0. This amounts to adding a constant to H.

We start with a characterization of the infimum of the spectrum of H_t. Therefore, write $h(\theta) = h(\theta_1, \theta_2)$ where $\theta = (\theta_1, \theta_2)$, $\theta_1 \in \mathbb{T}^{d_1}$, $\theta_2 \in \mathbb{T}^{d_2}$. Define

$$I(\theta_1, z) = \int_{\mathbb{T}^{d_2}} \frac{1}{h(\theta_1, \theta_2) - z} d\theta_2. \tag{1.2}$$

We recall that the measures $d\theta_2$ is normalized so that the measure of $\mathbb{T}^{d_2}$ be equal to 1.

We prove

Proposition 1.1. *Assume (H0) and (H1) are satisfied.*
H_t is non negative if and only if t satisfies

$$1 + tI_\infty \geq 0 \textit{ where } I_\infty := \sup_{\theta_1\in\mathbb{T}^{d_1}} \int_{\mathbb{T}^{d_2}} \frac{1}{h(\theta_1, \theta_2)} d\theta_2 \tag{1.3}$$

Assume now that $1 + tI_\infty < 0$. Then, there exists a unique $E_0 \in (-\infty, 0]$ such that

$$\forall \theta_1 \in \mathbb{T}^{d_1},\ 1 + tI(\theta_1, E_0) \geq 0 \textit{ and } \exists \theta_1 \in \mathbb{T}^{d_1},\ 1 + tI(\theta_1, E_0) = 0.$$

Moreover, E_0 is the infimum of the spectrum of H_t.

Proposition 1.1 is proved in section 5.

Criterion (1.3) immediately gives the obvious fact that if $t \geq 0$ then H_t is non-negative. As we assumed that h has only non degenerate minima, if $d_2 = 1, 2$ and $t < 0$, then H_t is not non-negative.

We now turn to our second result. It describes the asymptotics of N_s^t near 0 when (1.3) is satisfied. Recall that N_s^t is the density of surface states of H_t.

Theorem 1.1. *Assume t satisfies condition (1.3). Define*

$$I = \int_{\mathbb{T}^{d_2}} \frac{1}{h(0,\theta_2)} d\theta_2.$$

One has

- *if $d_2 = 1$:*

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} \frac{Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{Det(Q_1 - RQ_2^{-1}R^*)}} \cdot E^{1+d_1/2}$$

- *if $d_2 = 2$:*

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} \frac{2\,Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{Det(Q_1 - RQ_2^{-1}R^*)}} \frac{E^{1+d_1/2}}{|\log E|}$$

If $d_2 \geq 3$ and $1 + t\cdot I > 0$, then, one has

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} \frac{c(d_1,d_2)\,Vol(\mathbb{S}^{d_2-1})\,Vol(\mathbb{S}^{d_1-1})}{d(2\pi)^d\sqrt{Det Q}} \cdot \frac{t}{1+tI} \cdot s(E)E^{1+d_1/2}$$

If $d_2 \geq 3$ and $1 + t\cdot I = 0$, if we assume, moreover, that $\theta_1 \mapsto I(\theta_1, 0) := \int_{\mathbb{T}^{d_2}} (h(\theta_1,\theta_2))^{-1} d\theta_2$ has a local maximum for $\theta_1 = 0$, then one has

- *if $d_2 = 3$:*

$$\int_0^E dN_s^t(e)de \underset{E\to 0^+}{\sim} \frac{\int_{|\theta_1|\leq 1} Arg(-i|1-\theta_1^2|^{1/2} + \tilde{g}(\theta_1))d\theta_1}{d_1(d_1+2)\pi(2\pi)^{d_1}\sqrt{Det(Q_1 - RQ_2^{-1}R^*)}} \cdot E^{1+d_1/2}$$

- *if $d_2 = 4$:*

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} -\frac{2\,Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{Det(Q_1 - RQ_2^{-1}R^*)}} \frac{E^{1+d_1/2}}{|\log E|}$$

- *if $d_2 \geq 5$:*

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} \frac{c(d_1,d_2)\,Vol(\mathbb{S}^{d_2-1})\,Vol(\mathbb{S}^{d_1-1})}{d(2\pi)^d\sqrt{Det Q}} \cdot \frac{-1}{J} \cdot s(E)E^{d_1/2}$$

Here, we used the following notations:

- *$Arg(\cdot)$ denotes the principal determination of the argument of a complex number,*
- *for $n \in \{d_1, d_2\}$, $\mathbb{S}^{n-1}$ is the $n-1$ dimensional unit sphere,*
- *$\tilde{g}$ is a linear form defined below,*
- *the function s and the constants $c(d_1,d_2)$ and J are defined by*

$$s(x) = \frac{1}{2}|x|^{\frac{d_2-2}{2}}, \quad c(d_1,d_2) = \int_0^1 r^{d_1-1}(1-r^2)^{(d_2-2)/2}dr, \quad J = \int_{\mathbb{T}^{d_2}} \frac{1}{h^2(0,\theta_2)} d\theta_1$$

- *Q is the Hessian matrix of h at 0 that can be decomposed as $Q = \begin{pmatrix} Q_1 & R^* \\ R & Q_2 \end{pmatrix}$.*

About the function $\tilde{g}$, it is defined as follows. We assume $d_2 \geq 3$ and $1+tI=0$. Let $h_2(\theta_1) = \inf_{\theta_2\in\mathbb{T}^{d_2}} h(\theta_1,\theta_2)$. In section 5.1, we show that the function $\theta_1 \mapsto \int_{\mathbb{T}^{d_2}} (h(\theta_1,\theta_2)-h_2(\theta_1))^{-1} d\theta_2$ is real analytic in a neighborhood of 0. Using the Taylor expansion of this function near 0, one obtains

$$1 + t\int_{\mathbb{T}^{d_2}} \frac{1}{h(\theta_1,\theta_2)-h_2(\theta_1)} d\theta_2 = tg(\theta_1) + O(|\theta_1|^2).$$

This defines the linear form g uniquely. Then, $\tilde{g}$ is defined by

$$\tilde{g}(\theta) := (2\pi)^{d_2}\sqrt{\mathrm{Det}\,(Q_2)}\, g((Q_1 - RQ_2^{-1}R^*)^{-1/2}\theta_1).$$

If the variables (θ_1,θ_2) separate in h, i.e., if $h(\theta_1,\theta_2) = \tilde{h}_1(\theta_1) + \tilde{h}_2(\theta_2)$, the function $\tilde{g}$ is identically 0.

1.2. **The case of a random surface potential.** Let V_ω be a random potential concentrated on the sub-lattice $\mathbb{Z}^{d_1}\times\{0\} \subset \mathbb{Z}^d$ (d_1 is chosen as in section 0) of the form

$$V_\omega(\gamma_1,\gamma_2) = \begin{cases} \omega_{\gamma_1} & \text{if } \gamma_2 = 0, \\ 0 & \text{if } \gamma_2 \neq 0. \end{cases}, \gamma = (\gamma_1,\gamma_2) \in \mathbb{Z}^{d_1}\times\mathbb{Z}^{d_2} = \mathbb{Z}^d. \tag{1.4}$$

and $(\omega_{\gamma_1})_{\gamma_1\in\mathbb{Z}^{d_1}}$ is a family of i.i.d. bounded, non constant random variables.
Let $\omega_\pm$ be respectively the maximum and minimum of the random variables $(\omega_{\gamma_1})_{\gamma_1\in\mathbb{Z}^{d_1}}$, and let $\overline{\omega}$ be its expectation.

Finally, we define the random surfacic model by

$$H_\omega = H + V_\omega, \tag{1.5}$$

and its IDSS by

$$(n_s,\varphi) = \mathbb{E}(\mathrm{tr}(\Pi_1[\varphi(H_\omega)-\varphi(H)]\Pi_1))$$

Following section 0, one regularizes n_s into N_s as in (0.5).

Remark 1.1. An interesting case which can be brought back to a Hamiltonian of the form (1.5) with H and V_ω as above is the following.
Consider Γ, a sub-lattice of $\mathbb{Z}^d$ obtained in the following way $\Gamma = G(\{0\}\times\mathbb{Z}^{d_2})$ where G is a matrix in $\mathrm{GSL}_d(\mathbb{Z})$, the d-dimensional special linear group over $\mathbb{Z}$, i.e. the multiplicative group of invertible matrices with coefficients in $\mathbb{Z}$ and unit determinant. One easily shows that the random operator

$$H_\omega(\Gamma) = -\frac{1}{2}\Delta + \sum_{\gamma\in\Gamma}\omega_\gamma\Pi_\gamma$$

(where Π_γ is the projector onto the vector $\delta_\gamma \in \ell^2(\mathbb{Z}^d)$) is unitarily equivalent to $H+V_\omega$ where V_ω is defined in (1.4) and $h(\theta) = h_0(G'\cdot\theta)$; here, h_0 is defined in (1.1) and G' is the inverse of the transpose of G, i.e. $G' = {}^tG^{-1}$.

Definition 1.1. We say that E, an edge (or boundary) of the spectrum of H_ω, is *stable* if it is an edge of the spectrum of $H+tV_\omega$ for all $t\in[0,1]$. If an edge is not stable, we call it a *fluctuation* edge.

Note that in the case of the introduction, this definition is equivalent to that given there.
As in the introduction, one has to distinguish between

1. stable boundaries : at these boundaries, the IDSS is given by the IDSS of a model operator computed from the random model.
2. fluctuation boundaries: at these boundaries, one has standard Lifshitz tails.

To complete this section, let us give a very simple description of the spectrum of H_ω. One has

Proposition 1.2. *Let H_ω be defined as above. Then*

$$\sigma(H_\omega) = \bigcup_{t\in supp(P_0)} \sigma(H_t).$$

Here and in the following P_0 denotes the common distribution of the random variables $(\omega_{\gamma_2})_{\gamma_2}$.

1.3. **The stable boundaries.** The stable boundary we are studying is the lower boundary that we assumed to be 0. Let us first give a criterion for the lower edge of the spectrum of H (that we assume to be equal to 0) to be a stable edge. We prove

Proposition 1.3. *Write $h(\theta)=h(\theta_1,\theta_2)$ where $\theta=(\theta_1,\theta_2)$, $\theta_1\in\mathbb{T}^{d_1}$, $\theta_2\in\mathbb{T}^{d_2}$. Then, 0 is a stable spectral edge if and only if ω_- satisfies condition (1.3).*

Proposition 1.3 is an immediate consequence of Proposition 1.1 and Proposition 1.2. It gives the obvious fact that, if $\omega_-\geq 0$, then 0 is a stable edge. As we assumed that h has only non degenerate minima, we see that if $d_2=1,2$ and $\omega_-<0$, then 0 is never a stable edge. Actually, it need not be an edge of the spectrum of H_ω.

Using the same notations as above, we prove

Theorem 1.2. *Assume (H0) and (H1) are verified. Assume, moreover, that 0 is a stable spectral edge for H_ω. Then, one has*

$$\text{if } \overline{\omega}>0, \text{ then } \liminf_{E\to 0^+}\frac{N_s(E)}{N_s^{\overline{\omega}}(E)}\geq 1 \quad\text{and}\quad \text{if } \overline{\omega}<0, \text{ then } \limsup_{E\to 0^+}\frac{N_s(E)}{N_s^{\overline{\omega}}(E)}\leq 1 \tag{1.6}$$

where $N_s^{\overline{\omega}}$ is the IDSS of the operator with constant surface potential $\overline{\omega}$, the common expectation value of the random variables $(\omega_{\gamma_1})_{\gamma_1}$.

This result admits an immediate corollary

Theorem 1.3. *Assume (H0) and (H1) hold. Assume, moreover, that 0 is a stable spectral edge for H_ω. Then,*

- *if $d_2=1$:*

$$N_s(E)\underset{E\to 0^+}{\sim}\frac{Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{Det(Q_1-RQ_2^{-1}R^*)}}\cdot E^{1+d_1/2};$$

- *if $d_2=2$:*

$$N_s(E)\underset{E\to 0^+}{\sim}\frac{2\,Vol(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{Det(Q_1-RQ_2^{-1}R^*)}}\frac{E^{1+d_1/2}}{|\log E|}.$$

Theorem 1.3 is an immediate consequence of Theorem 1.2 and the bound

$$N_s^{\omega_-}(E)\leq N_s(E)\leq N_s^{\omega_+}(E).$$

As noted in the introduction, Theorem 1.2 is only necessary when $\omega_-=0$ (in which case $\overline{\omega}>0$). Moreover, one obtains the analogue of (0.7) in the present case for $d_2\geq 3$.

The above results may lead to the belief that

$$N_s(E)\underset{E\to 0}{\sim}N_s^{\overline{\omega}}(E)$$

for all dimensions d_2. Let us now explain why this result, if true, is not obtained for dimension $d_2\geq 3$. Therefore, we explain the heuristics behind the proof of Theorem 1.2; it is very similar to that of standard Lifshitz tails with one big difference when $d_2\geq 3$.

Restrict H_ω to some large cube. One wants to estimate the IDSS for H_ω; for this restriction, this comes up to estimating the differences between the integrated density of states (the usual one) of the operator H_ω and the integrated density of the operator H_{ω_-} (see Lemma 2.2). So we want to count the eigenvalues of H_ω below energy E, say, subtract the number of eigenvalues of H_{ω_-} below energy E, divide by the volume of the cube, and see how this behaves when E gets small. Assume φ is a normalized eigenfunction associated to an eigenvalue of H_ω below E. Then, one has $\langle(H+V_\omega)\varphi,\varphi\rangle\geq E$. Assume for a moment that V_ω is non negative. Then, we see that one must have both $\langle H\varphi,\varphi\rangle\geq E$ and $\langle V_\omega\varphi,\varphi\rangle\geq E$. The first of these conditions guarantees that φ is localized in momentum. So it has to be extended in space. If one plugs this information into the second condition, one sees that $\langle V_\omega\varphi,\varphi\rangle\sim\overline{\omega}$ with a large probability. So that, to φ, H_ω roughly looks like $H+\overline{\omega}\Pi_0^2$. There is one problem with this reasoning:
as V_ω only lives on a hyper-surface, and as φ is flat, it only sees a very small part of φ; a simple calculation

shows that $\|\Pi_0^2\varphi\| \sim E^{d_2/2}$; on the other hand, when one says that φ is roughly constant, one makes an error of size E^α (for some $0<\alpha<1$); hence, for dimension $d_2 \geq 3$, this error is much larger than the term we want to estimate, namely, $\langle V_\omega\varphi,\varphi\rangle$. In other words, because φ is very flat, we can modify it on the hyper-surface (e.g. localize the part of it living on the hyper-surface) with almost no change to the total energy of φ; hence, we cannot guarantee that φ is also flat on the hyper-surface, which implies that $\langle V_\omega\varphi,\varphi\rangle$ need not be close $\overline{\omega}$ with a large probability.

1.4. **The fluctuation boundaries.** In this section we assume that the infimum of Σ which we call E_0 is (strictly) below $\inf(\sigma(H))$, so that E_0 is a fluctuation edge. In this case, we consider a "reduced" operator $\tilde{H}$ which acts on $\ell^2(\mathbb{Z}^{d_1})$. In Fourier representation this operator is multiplication by the function $\tilde{h}$ given by:

$$\tilde{h}(\theta_1) = \left(\int_{\mathbb{T}^{d_2}} \frac{1}{h(\theta_1,\theta_2)-E_0}\,d\theta_2\right)^{-1} + E_0 \tag{1.7}$$

We will reduce the proof of Lifshitz tails for $H_\omega = H + V_\omega$ to a proof of Lifshitz tails for the reduced operator $\tilde{H}_\omega = \tilde{H} + \tilde{V}_\omega$ (where $\tilde{V}_\omega$ is a diagonal matrix with entries $(\omega_{\gamma_1})_{\gamma_1}$). To prove Lifshitz tail behavior for $\tilde{H}_\omega$ we have to impose a condition on the behavior of $\tilde{h}$ near its minimum. We either suppose:

(H2): the function $\tilde{h}:\ \mathbb{T}^{d_1}\to\mathbb{R}$ admits a unique quadratic minimum.

or we assume the weaker hypothesis:

(H2'): the function $\tilde{h}:\ \mathbb{T}^{d}\to\mathbb{R}$ is not constant.

Moreover, we always assume that the random variables ω_{γ_1} defining the potential (0.1) are independent with a common distribution P_0. We set $\omega_- = \inf(\mathrm{supp}(P_0))$ and assume:

(H3): P_0 is not concentrated in a single point and $P_0([\omega_-,\omega_-+\varepsilon)) \geq C\,\varepsilon^k$ for some k.

We will prove below:

Theorem 1.4. *If (H2) and (H3) are satisfied then*

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\frac{d_1}{2}.$$

We have an additional result for low dimension of the surface:

Theorem 1.5. *Assume (H2') and (H3) hold. If $d_1 = 1$ then*

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\lim_{E\searrow E_0}\frac{\ln(n(E))}{(E-E_0)}$$

where $n(E)$ is the integrated density of states for $\tilde{H}$.
If $d_2 = 2$, then

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\alpha$$

where the computation of α is explained below.

For the sake of simplicity, let us assume $E_0 = 0$. The Lifshitz exponent α will depend on the way $\tilde{h}$ vanishes at $\mathcal{S} = \{\theta_1|\tilde{h}=0\}$ and on the curvature of $\mathcal{S}$.

To describe it precisely, we need to introduce some objects from analytic geometry (see [19] for more details). If $\mathcal{E}$ is a set contained in the closed first quadrant in $\mathbb{R}^2$ then its *exterior convex hull* is the convex hull of the union of the rectangles $R_{xy} = [x,\infty)\times[y,\infty)$, where the union is taken over all $(x,y)\in\mathcal{E}$.

Pick $\theta_0 \in \mathcal{S}$ and consider the Newton diagram of $\tilde{h}$ at θ_0, i.e.,

(1) Express $\tilde{h}$ as a Taylor series at θ_0, $\tilde{h}(\theta^1,\theta^2) = \sum_{ij} a_{ij}(\theta^1-\theta_0^1)^i(\theta^2-\theta_0^2)^j$, $\theta = (\theta^1,\theta^2)$.
(2) Form the exterior convex hull of the points (i,j) with $a_{ij}\neq 0$. This is a convex polygon, called the *Newton polygon*.
(3) The boundary of the polygon is the *Newton diagram*.

The Newton decay exponent is then defined as follows. The Newton diagram consists of certain line segments. Extend each to a complete line and intersect it with the diagonal line $\theta^1 = \theta^2$. This gives a collection of points (a_k, a_k), one for each boundary segment. Take the reciprocal of the largest a_k and call this number $\tilde{\alpha}(\tilde{h}, \theta_0)$; it is the *Newton decay exponent*. Define $\alpha(\tilde{h}, \theta_0) = \min\{\tilde{\alpha}(\tilde{h} \circ T_0, \theta_0) : T_0(\cdot) = \theta_0 + T(\cdot - \theta_0),\ T \in SL(2, \mathbb{R})\}$.

Similarly, define $\alpha(\tilde{h}, \theta)$ if θ is any other point in $\mathcal{S}$, the zero set of $\tilde{h}$. Then, the *Lifshitz exponent* α is defined by

$$\alpha = \min_{\theta \in \mathcal{S}} \alpha(\tilde{h}, \theta). \tag{1.8}$$

The Lifshitz exponent α is positive as $\theta \mapsto \alpha(\tilde{h}, \theta)$ is a positive, lower semi-continuous function and $\mathcal{S}$ is compact (see [19]).

Remark 1.2. Let us return to the example given in Remark 1.1. In the section 6, we check that (H.2') holds in this case; so for $d = d_1 + d_2 = 3$, Theorem 1.5 applies.

2. Approximating the IDSS

To approximate the IDSS, we use a method that has proved useful to approximate the density of states of random Schrödinger operators, the periodic approximations. We shall show that the IDSS is well approximated by the suitably normalized density of states of a well chosen periodic operator.

2.1. Periodic approximations.

Let $(\omega_{\gamma_1})_{\gamma_1 \in \mathbb{Z}^{d_1}}$ be a realization of the random variables defined above. Fix $N \in \mathbb{N}^*$. We define H^N_ω, a periodic operator acting on $\ell^2(\mathbb{Z}^d)$ by

$$H^N_\omega = H + V^N_\omega = H + \sum_{\gamma_1 \in \mathbb{Z}^{d_1}_{2N+1}} \omega_n \sum_{\substack{\beta_1 \in (2N+1)\mathbb{Z}^{d_1} \\ \beta_2 \in (2N+1)\mathbb{Z}^{d_2}}} |\delta_{\gamma_1+\beta_1} \otimes \delta_{\beta_2}\rangle\langle\delta_{\gamma_1+\beta_1} \otimes \delta_{\beta_2}|.$$

Here, $\mathbb{Z}^{\tilde{d}}_{2N+1} = \mathbb{Z}^{\tilde{d}}/(2N+1)\mathbb{Z}^{\tilde{d}}$, $\delta_l = (\delta_{jl})_{j \in \mathbb{Z}^{\tilde{d}}}$ is a vector in the canonical basis of $\ell^2(\mathbb{Z}^{\tilde{d}})$ where δ_{jl} is the Kronecker symbol and, $\tilde{d} = d_1$ or $\tilde{d} = d_2$, the choice being clear from the context. As usual, $|u\rangle\langle u|$ is the orthogonal projection on a unit vector u.
By definition, H^N_ω is periodic with respect to the (non degenerate) lattice $(2N+1)\mathbb{Z}^d$. We define the density of states denoted by n^N_ω as usual for periodic operators: for $\varphi \in \mathcal{C}^\infty_0(\mathbb{R})$,

$$(\varphi, dn^N_\omega) = \int_{\mathbb{R}} \varphi(x) dn^N_\omega(x) = \lim_{L \to +\infty} \frac{1}{(2L+1)^d} \sum_{\substack{\gamma \in \mathbb{Z}^d \\ |\gamma| \leq L}} \langle \delta_\gamma, \varphi(H^N_\omega)\delta_\gamma\rangle.$$

This limit exists (see e.g. [4, 23]). In a similar way, one can define the density of states of H; we denote it by dn_0. The operators $(H^N_\omega)_{\omega,N}$ are uniformly bounded; hence, their spectra are contained in a fixed compact set, say $\mathcal{C}$. This set also contains the spectrum of H_ω and H. We prove

Lemma 2.1. *Pick $\mathcal{U} \subset \mathbb{R}$ a relatively compact open set such that $\mathcal{C} \subset \mathcal{U}$. There exists $C > 1$ such that, for $\varphi \in \mathcal{C}^\infty_0(\mathbb{R})$, for $K \in \mathbb{N}$, $K \geq 1$, and $N \in \mathbb{N}^*$, we have*

$$\left|(\varphi, dn) - (2N+1)^{d_2}\mathbb{E}\{(\varphi, [dn^N_\omega - dn_0])\}\right| \leq \left(\frac{CK}{N}\right)^K \sup_{\substack{x \in \mathcal{U} \\ 0 \leq J \leq K+d+2}} \left|\frac{d^J\varphi}{d^Jx}(x)\right|. \tag{2.1}$$

Proof of Lemma 2.1 Fix $\varphi \in \mathcal{C}^\infty_0(\mathbb{R})$. As the spectra of the operators H^N_ω are contained in $\mathcal{U}$, we may restrict ourselves to φ supported in $\mathcal{U}$ which we do from now on. By the definition (0.2), one has

$$(\varphi, n_s) = \mathbb{E}\left(\sum_{\gamma \in \mathbb{Z}^{d_2}} \langle \delta_0 \otimes \delta_{\gamma_2}, [\varphi(H_\omega) - \varphi(H)]\delta_0 \otimes \delta_{\gamma_2}\rangle\right) = M_N(\varphi) + R_N(\varphi) \tag{2.2}$$

where

$$M_N(\varphi)=\mathbb{E}\left(\sum_{\substack{\gamma_2\in\mathbb{Z}^{d_2}\\|\gamma_2|\le N}}\langle\delta_0\otimes\delta_{\gamma_2},[\varphi(H_\omega)-\varphi(H)]\delta_0\otimes\delta_{\gamma_2}\rangle\right),$$

$$R_N(\varphi)=\mathbb{E}\left(\sum_{\substack{\gamma_2\in\mathbb{Z}^{d_2}\\|\gamma_2|> N}}\langle\delta_0\otimes\delta_{\gamma_2},[\varphi(H_\omega)-\varphi(H)]\delta_0\otimes\delta_{\gamma_2}\rangle\right).$$

Let us now show that

$$|R_N(\varphi)|\le\left(\frac{CK}{N}\right)^K\sup_{\substack{x\in\mathcal{U}\\0\le J\le K+d+2}}\left|\frac{d^J\varphi}{d^Jx}(x)\right|. \tag{2.3}$$

Therefore, we use some ideas from the proof of Lemma 1.1 in [17]. Helffer-Sjöstrand's formula ([10]) reads

$$\varphi(H_\omega)=\frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde\varphi}{\partial\bar z}(z)\cdot(z-H_\omega)^{-1}dz\wedge d\bar z.$$

where $\tilde\varphi$ is an almost analytic extension of φ (see [22]), i.e. a function satisfying

(1) for $z\in\mathbb{R}$, $\tilde\varphi(z)=\varphi(z)$;
(2) $\mathrm{supp}(\tilde\varphi)\subset\{z\in\mathbb{C};\ |\mathrm{Im}(z)|<1\}$;
(3) $\tilde\varphi\in\mathcal{S}(\{z\in\mathbb{C};\ |\mathrm{Im}(z)|<1\})$;
(4) the family of functions $x\mapsto\frac{\partial\tilde\varphi}{\partial\bar z}(x+iy)\cdot|y|^{-n}$ (for $0<|y|<1$) is bounded in $\mathcal{S}(\mathbb{R})$ for any $n\in\mathbb{N}$; more precisely, there exists $C>1$ such that, for all $p,q,r\in\mathbb{N}$, there exists $C_{p,q}>0$ such that

$$\sup_{0<|y|\le1}\sup_{x\in\mathbb{R}}\left|x^p\frac{\partial^q}{\partial x^q}\left(|y|^{-r}\cdot\frac{\partial\tilde\varphi}{\partial\bar z}(x+iy)\right)\right|\le C^rC_{p,q}\sup_{\substack{q'\le r+q+2\\p'\le p}}\sup_{x\in\mathbb{R}}\left|x^{p'}\frac{\partial^{q'}\varphi}{\partial x^{q'}}(x)\right|. \tag{2.4}$$

As we are working with φ with compact support in $\mathcal{U}$, its almost analytic extension can be taken to have support in $(\mathcal{U}+[-1,1])+i[-1,1]$ (see e.g. [6]).
We estimate $\mathbb{E}(|\langle\delta_0\otimes\delta_{\gamma_2},[\varphi(H_\omega)-\varphi(H)]\delta_0\otimes\delta_{\gamma_2}\rangle|)$ for $|\gamma_2|>N$. Using the fact that the random variables $(\omega_{\gamma_2})_{\gamma_2}$ are bounded, we get

$$\begin{aligned}&\mathbb{E}(|\langle\delta_0\otimes\delta_{\gamma_2},[\varphi(H_\omega)-\varphi(H)]\delta_0\otimes\delta_{\gamma_2}\rangle|)\\&\le\frac{1}{4\pi}\mathbb{E}\left(\int_{\mathbb{C}}\left|\frac{\partial\tilde\varphi}{\partial\bar z}(z)\right||\langle\delta_0\otimes\delta_{\gamma_2},\left((z-H_\omega^N)^{-1}-(z-H)^{-1}\right)\delta_0\otimes\delta_{\gamma_2}\rangle|dxdy\right)\\&\le C\sum_{\gamma_1\in\mathbb{Z}^{d_1}}\int_{\mathbb{C}}\left|\frac{\partial\tilde\varphi}{\partial\bar z}(z)\right|\cdot\mathbb{E}\left(|\langle\delta_0\otimes\delta_{\gamma_2},(z-H_\omega^N)^{-1}\delta_{\gamma_1}\otimes\delta_0\rangle|\cdot|\langle\delta_{\gamma_1}\otimes\delta_0,(z-H)^{-1}\delta_0\otimes\delta_{\gamma_2}\rangle|\right)dxdy\end{aligned}$$

where $z=x+iy$.
By a Combes-Thomas argument (see e.g. [18]), we know that there exists $C>1$ such that, uniformly in $(\omega_\gamma)_\gamma$, $\gamma_1\in\mathbb{Z}^{d_1}$ and $N\ge1$, we have, for $\mathrm{Im}(z)\neq0$,

$$\begin{aligned}&|\langle\delta_{\gamma_1}\otimes\delta_{\gamma_2},(z-H_\omega^N)^{-1}\delta_{\gamma_1'}\otimes\delta_{\gamma_2'}\rangle|+|\langle\delta_{\gamma_1}\otimes\delta_{\gamma_2},(z-H)^{-1}\delta_{\gamma_1'}\otimes\delta_{\gamma_2'}\rangle|\le\\&\qquad\le\frac{C}{|\mathrm{Im}(z)|}e^{-|\mathrm{Im}(z)|(|\gamma_1-\gamma_1'|+|\gamma_2-\gamma_2'|)/C}\end{aligned} \tag{2.5}$$

Hence, for some $C > 1$,

$$|R_N(\varphi)| \leq C \sum_{\gamma_1 \in \mathbb{Z}^{d_1}} \int_{\mathbb{C}} \left| \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) \right| \cdot \frac{1}{|\mathrm{Im}(z)|^2} e^{-|\mathrm{Im}(z)(|\gamma_1|+|\gamma_2|)|/C} dxdy$$
$$\leq C \int_{\mathbb{C}} \left| \frac{\partial \tilde{\varphi}}{\partial \overline{z}}(z) \right| \frac{1}{|\mathrm{Im}(z)|^{d+2}} e^{-|\mathrm{Im}(z) N|/C} dxdy.$$

Taking into account the properties of almost analytic extensions (2.4), for some $C > 1$, for $K \geq 1$ and $N \geq 1$, we get

$$|R_N(\varphi)| \leq C^{K+1} \int_{(\mathcal{U}+[-1,1])+i[-1,1]} |y|^K e^{-|yN|/C} dxdy \sup_{\substack{x \in \mathcal{U} \\ 0 \leq J \leq K+d+2}} \left| \frac{d^J \varphi}{d^J x}(x) \right|$$
$$\leq \left(\frac{CK}{N} \right)^K \sup_{\substack{x \in \mathcal{U} \\ 0 \leq J \leq K+d+2}} \left| \frac{d^J \varphi}{d^J x}(x) \right|.$$

This completes the proof of (2.3).

We now compare $M_N(\varphi)$ to $(2N+1)^{d_2} \mathbb{E}\{(\varphi, [dn_\omega^N - dn_0])\}$. Therefore, we rewrite this last term as follows. Using the $(2N+1)\mathbb{Z}^d$ periodicity of H_ω^N and H, we get

$$\sum_{\substack{\gamma \in \mathbb{Z}^d \\ |\gamma| \leq N + L(2N+1)}} \langle \delta_\gamma, \varphi(H_\omega^N) \delta_\gamma \rangle = (2L+1)^d \sum_{\substack{\gamma \in \mathbb{Z}^d \\ |\gamma| \leq N}} \langle \delta_\gamma, \varphi(H_\omega^N) \delta_\gamma \rangle.$$

This gives

$$(2N+1)^d (\varphi, dn_\omega^N) = \mathbb{E} \left(\sum_{\substack{\gamma \in \mathbb{Z}^d \\ |\gamma| \leq N}} \langle \delta_\gamma, \varphi(H_\omega^N) \delta_\gamma \rangle \right). \tag{2.6}$$

On the other hand, as the random variables $(\omega_{\gamma_2})_{\gamma_2}$ are i.i.d. and as H is $\mathbb{Z}^d$-periodic, as in [18], one computes

$$\mathbb{E} \left(\sum_{\substack{\gamma \in \mathbb{Z}^d \\ |\gamma| \leq N}} \langle \delta_\gamma, \varphi(H_\omega^N) \delta_\gamma \rangle \right) = \mathbb{E} \left(\sum_{\substack{\gamma_1 \in \mathbb{Z}^{d_1},\ |\gamma_1| \leq N \\ \gamma_2 \in \mathbb{Z}^{d_2},\ |\gamma_2| \leq N}} \langle \delta_{\gamma_1} \otimes \delta_{\gamma_2}, \varphi(H_\omega^N) \delta_{\gamma_1} \otimes \delta_{\gamma_2} \rangle \right)$$
$$= (2N+1)^{d_1} \mathbb{E} \left(\sum_{\substack{\gamma_2 \in \mathbb{Z}^{d_2} \\ |\gamma_2| \leq N}} \langle \delta_0 \otimes \delta_{\gamma_2}, \varphi(H_\omega^N) \delta_0 \otimes \delta_{\gamma_2} \rangle \right)$$

Combining this with (2.6), we get

$$(2N+1)^{d_2} \mathbb{E}[(\varphi, dn_\omega^N)] = \mathbb{E} \left(\sum_{\substack{\gamma_2 \in \mathbb{Z}^{d_2} \\ |\gamma_2| \leq N}} \langle \delta_0 \otimes \delta_{\gamma_2}, \varphi(H_\omega^N) \delta_0 \otimes \delta_{\gamma_2} \rangle \right)$$

Of course, such a formula also holds when H^N_ω is replaced with H. In view of (0.2), (2.3) and (2.2), to complete the proof of Lemma 2.1, we need only to prove

$$(2.7)\qquad \mathbb{E}\left|\sum_{\substack{\gamma_2\in\mathbb{Z}^{d_2}\\ |\gamma_2|\leq N}}\langle\delta_0\otimes\delta_{\gamma_2},[\varphi(H^N_\omega)-\varphi(H_\omega)]\delta_0\otimes\delta_{\gamma_2}\rangle\right| \leq \left(\frac{CK}{N}\right)^K \sup_{\substack{x\in\mathcal{U}\\ 0\leq J\leq K+d+2}}\left|\frac{d^J\varphi}{d^Jx}(x)\right|.$$

for φ, K J and N as in Lemma 2.1.
Proceeding as above, for $\gamma_2\in\mathbb{Z}^{d_2}$, $|\gamma_2|\leq N$, we estimate

$$\begin{aligned}
&|\langle\delta_0\otimes\delta_{\gamma_2},[\varphi(H^N_\omega)-\varphi(H_\omega)]\delta_0\otimes\delta_{\gamma_2}\rangle|\\
&\leq C\left[\sum_{\substack{\gamma_1'\in\mathbb{Z}^{d_1}\\ \gamma_2'\in((2N+1)\mathbb{Z}^{d_2})^*}}+\sum_{\substack{\gamma_1'\in\mathbb{Z}^{d_1},\ |\gamma_1'|>N\\ \gamma_2'=0}}\right]\int_{\mathbb{C}}\left|\frac{\partial\tilde\varphi}{\partial\bar z}(z)\right|dxdy\cdot\\
&\qquad\mathbb{E}\Big(|\langle\delta_0\otimes\delta_{\gamma_2},(z-H^N_\omega)^{-1}\delta_{\gamma_1'}\otimes\delta_{\gamma_2'}\rangle|\cdot\\
&\qquad\qquad|\langle\delta_{\gamma_1'}\otimes\delta_{\gamma_2'},(z-H_\omega)^{-1}\delta_0\otimes\delta_{\gamma_2}\rangle|\Big).
\end{aligned}$$

Here we used the fact that the operators H_ω and H^N_ω coincide in the cube $\{|\gamma|\leq N\}$.
As H_ω satisfies the same Combes-Thomas estimate (2.5) as H^N_ω, doing the same computations as in the estimate for $R_N(\varphi)$, we obtain (2.7). This completes the proof of Lemma 2.1. □

Obviously, one has an analogue of (2.1) for $n_{s,\mathrm{norm}}$, n^t_s or $n^t_{s,\mathrm{norm}}$. One needs to replace H^N_ω and H with their obvious counterparts i.e., choose the random variables $(\omega_{\gamma_2})_{\gamma_2}$ to be the appropriate constant.

This enables us to prove

Lemma 2.2. *Fix I, a compact interval. Pick $\alpha>0$. There exists $\nu_0>0$ and $\rho>0$ such that, for $\gamma\in[0,1]$, $E\in I$, $\nu\in(0,\nu_0)$ and $N\geq\nu^{-\rho}$, one has*

$$(2.8)\qquad \mathbb{E}(N^N_\omega(E-\nu))-e^{-\nu^{-\alpha}}\leq N_s(E)\leq\mathbb{E}(N^N_\omega(E+\nu))+e^{-\nu^{-\alpha}}$$

where $N^N_{norm,\omega}=N^N_\omega-N^N_{\omega_-}$, and N^N_ω (resp. $N^N_{\omega_-}$) is the integrated density of states of H^N_ω (resp $H^N_{\omega_-}$, i.e. H^N_ω where $\omega_\gamma=\omega_-$ for all γ).

Let us note here that one can prove a similar result for the approximation of $N_{s,\mathrm{norm}}$ by $N^N_{\mathrm{norm},\omega}$ or that of N^t_s by $N^{t,N}_s$.
Proof Let us now prove Lemma 2.2. Pick φ a Gevrey class function of Gevrey exponent $\alpha>1$ (see [11]); assume, moreover, that φ has support in $(-1,1)$, that $0\leq\varphi\leq1$ and that $\varphi\equiv1$ on $(-1/2,1/2]$. Let $E\in I$ and $\nu\in(0,1)$,and set

$$\varphi_{E,\nu}(\cdot)=\mathbf{1}_{[0,E]}*\varphi\left(\frac{\cdot}{\nu}\right).$$

Then, by Lemma 2.1 and the Gevrey estimates on the derivatives of φ, there exist $C>1$ such that, for $N\geq1$, $k\geq1$ and $0<\nu<1$, we have

$$(2.9)\qquad |\mathbb{E}((\varphi_{E,\nu},dN^N_\omega))-(\varphi_{E,\nu},dN_s)|\leq C(N\nu)^3\left(\frac{Ck^{1+\alpha}}{N\nu}\right)^k.$$

We optimize the right hand side of (2.9) in k and get that, there exist $C>1$ such that, for $N\geq1$ and $0<\nu<1$, we have

$$|\mathbb{E}((\varphi_{E,\nu},dN^N_\omega))-(\varphi_{E,\nu},dN_s)|\leq C(N+\nu^{-1})^3e^{-(N\nu/C)^{1/(1+\alpha)}+C(N\nu/C)^{-1/(1+\alpha)}}$$

Now, there exist $\nu_0>0$ such that, for $0<\nu<\nu_0$ and $N\geq\nu^{-1-\eta}$, we have

$$(2.10)\qquad |\mathbb{E}((\varphi_{E,\nu},dN^N_\omega))-(\varphi_{E,\nu},dN_s)|\leq e^{-\nu^{-\eta/(2\alpha)}}.$$

By definition, $\varphi_{E,\nu} \equiv 1$ on $[0,E]$, and $\varphi_{E,\nu}$ has support in $[-\nu, E+\nu]$ and is bounded by 1. As dN_ω^N and dN_s are positive measures, we have

$$\mathbb{E}(N_{s,\omega}^N(E)) \le \mathbb{E}((\varphi_{E,\nu}, dN_\omega^N)) \le \mathbb{E}(N_{s,\omega}^N(E+\nu)). \tag{2.11}$$

Hence, by (2.10) and (2.11), we obtain

$$\begin{aligned} N_s(E) \le (\varphi_{E,\nu}, dN_s) &= \mathbb{E}[(\varphi_{E,\nu}, dN_\omega^N)] + \left[(\varphi_{E,\nu}, dN_s) - \mathbb{E}((\varphi_{E,\nu}, dN_\omega^N))\right] \\ &\le \mathbb{E}(N_\omega^N(E+\nu)) + e^{-\nu^{-\eta/(2\alpha)}} \end{aligned}$$

and

$$\begin{aligned} N_s(E) \ge (\varphi_{E-\nu,\nu}, dN_s) &= \mathbb{E}[(\varphi_{E-\nu,\nu}, dN_\omega^N)] + \left[(\varphi_{E-\nu,\nu}, dN_s) - \mathbb{E}((\varphi_{E-\nu,\nu}, dN_\omega^N))\right] \\ &\ge \mathbb{E}(N_\omega^N(E-\nu)) - e^{-\nu^{-\eta/(2\alpha)}} \end{aligned}$$

This completes the proof of Lemma 2.2. □

2.2. **Some Floquet theory.** To analyze the spectrum of H_ω^N, we use some Floquet theory that we develop now. We denote by $\mathcal{F}: L^2([-\pi,\pi]^d) \to \ell^2(\mathbb{Z}^d)$ the standard Fourier series transform. Then, we have, for $u \in L^2([-\pi,\pi]^d)$,

$$(\hat{H}_\omega u)(\theta) = (\mathcal{F}^* H_\omega \mathcal{F} u)(\theta) = h(\theta)u(\theta) + \sum_{\beta\in\mathbb{Z}^d} \omega_\beta (\Pi_\beta u)(\theta)$$
$$\text{where } (\Pi_\beta u)(\theta) = \frac{1}{(2\pi)^d} e^{i\beta\theta} \int_{[-\pi,\pi]^d} e^{-i\beta\theta} u(\theta) d\theta.$$

Define the unitary equivalence

$$\begin{aligned} U : L^2([-\pi,\pi]^d) &\to L^2([-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d) \otimes \ell^2(\mathbb{Z}_{2N+1}^d) \\ u &\mapsto (Uu)(\theta) = (u_\gamma(\theta))_{\gamma\in\mathbb{Z}_{2N+1}^d} \end{aligned}$$

where the $(u_\gamma(\theta))_{\gamma\in\mathbb{Z}_{2N+1}^d}$ are defined by

$$u(\theta) = \sum_{\gamma\in\mathbb{Z}_{2N+1}^d} e^{i\gamma\theta} u_\gamma(\theta) \text{ where the functions } (\theta \mapsto u_\gamma(\theta))_{\gamma\in\mathbb{Z}_{2N+1}^d} \text{ are } \frac{2\pi}{2N+1}\mathbb{Z}^d\text{-periodic.} \tag{2.12}$$

The functions $(u_\gamma)_{\gamma\in\mathbb{Z}_{2N+1}^d}$ are computed easily; if the Fourier coefficients of u are denoted by $(\hat{u}_\gamma)_{\gamma\in\mathbb{Z}^d}$, then, one gets

$$u_\gamma(\theta) = \sum_{\beta\in\mathbb{Z}^d} \hat{u}_{\gamma+(2N+1)\beta} e^{i(2N+1)l\beta\theta}. \tag{2.13}$$

The operator $U\mathcal{F}^* H_\omega^N \mathcal{F} U^*$ acts on $L^2([-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d) \otimes \ell^2(\mathbb{Z}_{2N+1}^d)$; it is the multiplication by the matrix

$$M_\omega^N(\theta) = H^N(\theta) + V_\omega^N \tag{2.14}$$

where

$$H^N(\theta) = ((h_{\beta-\beta'}(\theta)))_{(\beta,\beta')\in(\mathbb{Z}_{2N+1}^d)^2} \text{ and } V_\omega^N = ((\omega_{\beta_1}\delta_{\beta_1\beta_1'}\delta_{\beta_2 0}\delta_{\beta_2' 0}))_{\substack{(\beta_1,\beta_1')\in(\mathbb{Z}_{2N+1}^{d_1})^2 \\ (\beta_2,\beta_2')\in(\mathbb{Z}_{2N+1}^{d_2})^2}}. \tag{2.15}$$

Here, the functions $(h_\gamma)_{\gamma\in\mathbb{Z}_{2N+1}^d}$ are the components of h decomposed according to (2.12). The $(2N+1)^d \times (2N+1)^d$-matrices $H^N(\theta)$ and V_ω^N are non-negative matrices.

This immediately tells us that the Floquet eigenvalues and eigenvectors of H_ω^N with Floquet quasi-momentum θ (i.e. the vectors, $u = (u_\beta)_{\beta\in\mathbb{Z}^d}$), solution to the problem

$$\begin{cases} H_\omega^N u &= \lambda u, \\ u_{\beta+\gamma} &= e^{-i\gamma\theta} u_\beta \text{ for } \beta\in\mathbb{Z}^d,\ \gamma\in(2N+1)\mathbb{Z}^d) \end{cases}$$

are the eigenvalues and eigenvectors (once extended quasi-periodically) of the $(2N+1)^d \times (2N+1)^d$ matrix $M_\omega^N(\theta)$. For $E \in \mathbb{R}$, one has

$$\mathcal{N}_\omega^N(E) = \int_0^E dn_\omega^N(E) = \int_{[-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d} \sharp\{\text{eigenvalues of } M_\omega^N(\theta) \text{ in } [0,E]\} d\theta.$$

Considering H as $(2N+1)\mathbb{Z}^d$-periodic on $\mathbb{Z}^d$, we see that the Floquet eigenvalues of H (for the quasi-momentum θ) are $(h(\theta + \frac{2\pi\gamma}{2N+1}))_{\gamma \in \mathbb{Z}^d_{2N+1}}$; the Floquet eigenvalue $h(\theta + \frac{2\pi\gamma}{2N+1})$ is associated to the Floquet eigenvector $u_\gamma(\theta)$, $\gamma \in \mathbb{Z}^d_{2N+1}$ defined by

$$u_\gamma(\theta) = \frac{1}{(2N+1)^{d/2}} (e^{-i(\theta + \frac{2\pi\gamma}{2N+1})\beta})_{\beta \in \mathbb{Z}^d_{2N+1}}.$$

In the sequel, the vectors in $\ell^2(\mathbb{Z}^d_{2N+1})$ are given by their components in the orthonormal basis $(u_\gamma(\theta))_{\gamma \in \mathbb{Z}^d_{2N+1}}$. The vectors of the canonical basis denoted by $(v_l(\theta))_{l \in \mathbb{Z}^d_{2N+1}}$ have the following components in this basis

$$v_l(\theta) = \frac{1}{(2N+1)^{d/2}} (e^{i(\theta + \frac{2\pi\gamma}{2N+1})l})_{\gamma \in \mathbb{Z}^d_{2N+1}}.$$

We define the vectors $(v_l)_{l \in \mathbb{Z}^d_{2N+1}}$ by

$$v_l = e^{-il\theta} v_l(\theta) = \frac{1}{(2N+1)^{d/2}} (e^{i\frac{2\pi\gamma l}{2N+1}})_{\gamma \in \mathbb{Z}^d_{2N+1}}.$$

3. The proof of Theorem 1.2

To prove Theorem 1.2, we will use Lemma 2.2 and the Floquet theory developed in 2.2. We will start with

3.1. The Floquet theory for constant surface potential.

We consider the operator $H_t^N = H_\omega^N$ where $\omega = (t)_{\gamma_1 \in \mathbb{Z}^{d_1}_{2N+1}}$ is the constant vector and $t \neq 0$. The matrix $M_t^N(\theta)$ defined by (2.14) for H_t^N takes the form (2.14) where

$$V_t^N = t((\delta_{\beta_1\beta_1'}\delta_{\beta_2 0}\delta_{\beta_2' 0}))_{\substack{(\beta_1,\beta_1') \in (\mathbb{Z}^{d_1}_{2N+1})^2 \\ (\beta_2,\beta_2') \in (\mathbb{Z}^{d_2}_{2N+1})^2}}. \tag{3.1}$$

Our goal is to describe the eigenvalues and eigenfunctions of $M_t^N(\theta)$. As usual, we write $\theta = (\theta_1, \theta_2)$. By definition, the operator H_t^N is $\mathbb{Z}^{d_1} \times (2N+1)\mathbb{Z}^{d_2}$-periodic. It can be seen as acting on $\ell^2(\mathbb{Z}^{d_1}, \ell^2(\mathbb{Z}^{d_2}))$; as such, we can perform a Floquet analysis in the θ_1-variable as in section 2.2 (in this case, just a discrete Fourier transform in θ_1) to obtain that H_t^N is unitarily equivalent to the direct sum over θ_1 in $\mathbb{T}^{d_1}$ of the $2N+1$-periodic operator $H_t^N(\theta_1)$ acting on $\ell^2(\mathbb{Z}^{d_2})$ defined by the matrix

$$H_t^N(\theta_1) = ((h(\theta_1; \beta_2 - \beta_2') + t \sum_{\gamma_2 \in (2N+1)\mathbb{Z}^{d_2}} \delta_{\beta_2\gamma_2}\delta_{\beta_2'\gamma_2}))_{(\beta_2,\beta_2') \in (\mathbb{Z}^{d_2})^2}.$$

Here $h(\theta_1; \beta_2)$ is the partial Fourier transform of $h(\theta_1, \theta_2)$ in the θ_2-variable.
For each θ_1, we now perform a Floquet reduction for $H_t^N(\theta_1)$ to obtain that $H_t^N(\theta_1)$ is unitarily equivalent to the multiplication by the matrix

$$\tilde{M}_t^N(\theta_1, \theta_2) = ((h(\theta_1, \theta_2; \beta_2 - \beta_2') + t\delta_{\beta_2 0}\delta_{\beta_2' 0}))_{(\beta_2,\beta_2') \in (\mathbb{Z}^{d_2}_{2N+1})^2}$$

The matrix-valued function $(\theta_1, \theta_2) \mapsto \tilde{M}_t^N(\theta_1, \theta_2)$ is $2\pi\mathbb{Z}^{d_1}$-periodic in θ_1 and $\frac{2\pi}{2N+1}\mathbb{Z}^{d_2}$-periodic in θ_2. It is a rank one perturbation of the matrix $\tilde{M}_0^N(\theta_1, \theta_2)$; the eigenvalues of this matrix are the values $h\left(\theta_1, \theta_2 + \dfrac{2\pi\gamma_2}{2N+1}\right)$. Let us for a while order these values increasingly and call them $(E_n^N(\theta_1, \theta_2, t))_{1 \le n \le n_N}$ where $n_N \le (2N+1)^{d_2}$ (we do not repeat the eigenvalues according to multiplicity). Assume $t > 0$. The standard theory of rank one perturbations [24] yields

Lemma 3.1. *For $1 \le n \le n_N$, if $E_n^N(\theta_1, \theta_2, 0)$ is an eigenvalue of multiplicity k, then*

- *either it is an eigenvalue of multiplicity k for $\tilde{M}_t^N(\theta_1)$;*
- *or it is an eigenvalue of multiplicity $k-1$ for $\tilde{M}_t^N(\theta_1)$ and the interval $(E_n^N(\theta_1,0),E_{n+1}^N(\theta_1,\theta_2,0))$ contains exactly one simple eigenvalue; this eigenvalue is given by the condition*

$$t\langle\delta_0,(E-\tilde{M}_0^N(\theta_1))^{-1}\delta_0\rangle=1;$$

Here, we took the convention $E_{n_N+1}^N(\theta_1,\theta_2,0)=+\infty$. One has a symmetric statement for $t<0$.

For $1\le n\le n_N$, let $(\varphi_{n,j}^N(\theta_1,\theta_2,t))_{1\le j\le j_n}$ denote orthonormalized eigenvectors associated to the eigenvalue $E_n^N(\theta_1,\theta_2,t)$ where j_n denotes its multiplicity.
In the sequel, it will be convenient to reindex the eigenvalues and eigenfunctions of the matrix $\tilde{M}_t^N(\theta_1,\theta_2)$ as $(E_{\gamma_2}^N(\theta_1,\theta_2,t))_{\gamma_2\in\mathbb{Z}_{2N+1}^{d_2}}$ and $(\varphi_{\gamma_2}^N(\theta_1,\theta_2,t))_{\gamma_2\in\mathbb{Z}_{2N+1}^{d_2}}$. Clearly, the functions $(\theta_1,\theta_2)\mapsto E_{\gamma_2}^N(\theta_1,\theta_2,t)$ and $(\theta_1,\theta_2)\mapsto\varphi_{\gamma_2}^N(\theta_1,\theta_2,t)$ can be chosen to be $2\pi\mathbb{Z}^{d_1}$-periodic in θ_1 and $\frac{2\pi}{2N+1}\mathbb{Z}^{d_2}$-periodic in θ_2.
Let us now show the

Lemma 3.2. *The eigenvalues of $M_t^N(\theta)$ are the values $\{E_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t);\ \gamma_1\in\mathbb{Z}_{2N+1}^{d_1},\ \gamma_2\in\mathbb{Z}_{2N+1}^{d_2}\}$ where*

$$E_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t)=E_{\gamma_2}^N\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)\tag{3.2}$$

A normalized eigenfunction associated to the eigenvalue $E_{\gamma_2}^N\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)$ is the vector

$$v_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t):=(2N+1)^{-d_1/2}\left(e^{-i\beta_1(\theta_1+\frac{2\pi\gamma_1}{2N+1})}\varphi_{\gamma_2}^N\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)\right)_{\beta_1\in\mathbb{Z}_{2N+1}^{d_1}},\tag{3.3}$$

i.e. the vector of components

$$(2N+1)^{-d_1/2}\left(e^{-i\beta_1(\theta_1+\frac{2\pi\gamma_1}{2N+1})}c_{\gamma_2}^{\beta_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1}\right)\right)_{\substack{\beta_1\in\mathbb{Z}_{2N+1}^{d_1}\\ \beta_2\in\mathbb{Z}_{2N+1}^{d_2}}}\tag{3.4}$$

if $\varphi_{\gamma_2}^N(\theta_1,t)$ has components $(c_{\gamma_2}^{\beta_2}(\theta_1))_{\beta_2\in\mathbb{Z}_{2N+1}^{d_2}}$.
The vectors $(v_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t))_{\substack{\gamma_1\in\mathbb{Z}_{2N+1}^{d_1}\\ \gamma_2\in\mathbb{Z}_{2N+1}^{d_2}}}$ form an orthonormal basis of $\ell^2(\mathbb{Z}_{2N+1}^{d_1}\times\mathbb{Z}_{2N+1}^{d_2})$.

Proof. Orthonormality is easily checked using the fact that the vectors $(\varphi_{\gamma_2}^N(\theta_1,\theta_2,t))_{\gamma_2\in\mathbb{Z}_{2N+1}^{d_2}}$ form an orthonormal basis.
Let us now check that $v_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t)$ satisfies the eigenvalue equation for $M_t^N(\theta)$ and $E_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t)$ given in (3.2). Therefore, first note that the matrix $M_t^N(\theta)$ is nothing but the multiplication operator by the matrix-valued function $\tilde{M}_t^N(\theta_1)$ to which one has applied the Floquet reduction of in the θ_1-variable. Hence, by (2.13), the matrix elements of $M_t^N(\theta)$ given by (2.15) satisfy, for $\beta_1\in\mathbb{Z}_{2N+1}^{d_1}$,

$$\tilde{M}_t^N\left(\theta_1+\frac{2\pi\gamma_1}{2N+1}\right)e^{-i\beta_1(\theta_1+\frac{2\pi\gamma_1}{2N+1})}=\sum_{\beta_1'\in\mathbb{Z}_{2N+1}^{d_1}}m_{\beta_1-\beta_1'}(\theta_1)e^{-i\beta_1'(\theta_1+\frac{2\pi\gamma_1}{2N+1})}\tag{3.5}$$

Both sides in this equality are matrices acting on $\ell^2(\mathbb{Z}_{2N+1}^{d_2})$, the matrices $m_{\beta_1-\beta_1'}(\theta)$ being defined as

$$m_{\beta_1-\beta_1'}(\theta)=((h_{\beta_1-\beta_1',\beta_2-\beta_2'}(\theta)))_{(\beta_2,\beta_2')\in(\mathbb{Z}_{2N+1}^{d_2})^2}.$$

If we now apply both sides of equation (3.5) to the vector $\varphi^N_{\gamma_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},t\right)$, we obtain, for $\beta_1\in\mathbb{Z}^{d_1}_{2N+1}$,

$$\begin{aligned}\sum_{\beta_1'\in\mathbb{Z}^{d_1}_{2N+1}} & h_{\beta_1-\beta_1'}(\theta_1)e^{-i\beta_1'(\theta_1+\frac{2\pi\gamma_1}{2N+1})}\varphi^N_{\gamma_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)\\ &=\tilde M^N_t\left(\theta_1+\frac{2\pi\gamma_1}{2N+1}\right)e^{-i\beta_1(\theta_1+\frac{2\pi\gamma_1}{2N+1})}\varphi^N_{\gamma_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},t\right)\\ &=E^N_{\gamma_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)e^{-i\beta_1(\theta_1+\frac{2\pi\gamma_1}{2N+1})}\varphi^N_{\gamma_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)\end{aligned}$$

This is nothing but to write

$$M^N_t(\theta)v_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t)=E^N_{\gamma_2}\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2,t\right)v_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t).$$

This completes the proof of Lemma 3.2. □

In the course of the proof of Theorem 1.2, we will use the

Lemma 3.3. *Fix t such that $t>0$ if $d_2=1,2$ and $1+tI_\infty>0$ if $d_2\geq 3$. Then, for $\rho>2$, there exists $C>0$ such that, for $N\geq E^{-\rho}$ and E sufficiently small, the eigenvalues of M^N_t satisfy*

$$E_{\gamma_1,\gamma_2}(\theta_1,\theta_2,t)\leq E\Longrightarrow\left(\frac{1+|\gamma_1|}{2N+1}\right)^2\leq CE \tag{3.6}$$

Proof. When t is positive, (3.6) is clear by Lemmas 3.2 and 3.1, that is, by the intertwining of the eigenvalues of $M^N_0(\theta)$ and $M^N_t(\theta)$, and as the eigenvalues of $M^N_0(\theta)$ are the values $h\left(\theta_1+\frac{2\pi\gamma_1}{2N+1},\theta_2+\frac{2\pi\gamma_2}{2N+1}\right)$ which satisfy (3.6) as $h(\theta)\geq C|\theta|^2$.

Assume now that $d_2\geq 3$ and t satisfies $1+tI_\infty>0$. To complete the proof of Lemma 3.3, by Lemma 3.2, it is then enough to prove that, there exists $C>0$ such that for, one has

$$|\theta_1|^2>CE\Longrightarrow\forall\gamma_2,\ E^N_{\gamma_2}(\theta_1,\theta_2,t)>E.$$

By the intertwining properties and the properties of h, this is clear except for the lowest of the $(E^N_{\gamma_2})_{\gamma_2}$. Assume now that $|\theta_1|^2\geq E$. Then, by our assumptions on the behavior of h near its minimum, for some $C>0$, one has that $(\theta_1,e)\mapsto I(\theta_1,e)$ is real analytic in $\{|\theta_1|^2\geq E\}\times\{|e|\leq E/C\}$. Hence, using a standard estimate for Riemann sums, we get that, for $|\theta_1|^2\geq E$ and $|e|\leq E/C$,

$$1+t\langle\delta_0,(\tilde M^N_0(\theta_1)-e)^{-1}\delta_0\rangle=1+tI(\theta_1,e)+O(E^{-2}E^\rho)$$

So that, as $1+tI_\infty>0$, for E sufficiently small, the equation $1+t\langle\delta_0,(\tilde M^N_0(\theta_1)-e)^{-1}\delta_0\rangle=0$ has no solution for $|\theta_1|^2\geq E$ and $|e|\leq E/C$. By the above discussion, this implies that, all the $E^N_{\gamma_2}(\theta_1,\theta_2,t)$ lie above E/C. This completes the proof of Lemma 3.3. □

3.2. **The proof of Theorem 1.2.** We now have all the tools necessary to prove Theorem 1.2. Notice that, as $\overline{\omega}>\omega_-$, as $1+\omega_-I_\infty\geq 0$, we know that $1+\overline{\omega}I_\infty>0$. So that the asymptotics for $N^{\overline{\omega}}_s(E)$ are given by

$$N^{\overline{\omega}}_s(E)\underset{E\to0^+}{\sim}C(\overline{\omega})\cdot f(E).$$

The precise value of the constant $C(\overline{\omega})$ and of the function $f(E)$ are given in Theorem 1.1. The constant $C(\overline{\omega})$ is a continuous function of $\overline{\omega}$; and, for any $c\in\mathbb{R}$, the function $f(E)$ satisfies $f(E+cE^2)\sim f(E)$ when $E\to 0$; moreover, f is at most polynomially small in E. All these facts will be useful.

We start with the proof of (1.6). We will use Lemma 2.2. As above, fix N large but not too large, say $N\sim E^{-\rho}$ for some large ρ. Fix $\delta>0$ small. Consider the matrix $M^N_{\overline{\omega}+\delta}(\theta)$ obtained by the Floquet reduction of $H^N+(\overline{\omega}+\delta)\Pi^2_0$. Let $\mathcal{H}^N_\delta(E,\theta)$ be the spectral space of $M^N_{\overline{\omega}+\delta}(\theta)$ associated the eigenvalues less that E. Then, we prove that

Lemma 3.4. *Fix $\delta > 0$, $\rho > 2$ and $\alpha \in (0, 1/2)$. For $N \sim E^{-\rho}$ and E sufficiently small, with a probability at least $1 - e^{E^{-\alpha}}$, for all θ and all $\varphi \in \mathcal{H}_\delta^N(E,\theta)$, one has*

$$\langle M_\omega^N(\theta)\varphi, \varphi\rangle \leq E\|\varphi\|^2.$$

This lemma immediately implies the desired lower bound. Indeed, it implies that, for $N \sim E^{-\rho}$, with a probability at least $1 - e^{E^{-\alpha}}$, one has

$$\begin{aligned} N_{\overline{\omega}+\delta}^N(E) &= \int_{[-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d} \sharp\{\text{eigenvalues of } M_{\overline{\omega}+\delta}^N(\theta) \text{ in } [0,E]\} d\theta \\ &\leq \int_{[-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d} \sharp\{\text{eigenvalues of } M_\omega^N(\theta) \text{ in } [0,E]\} d\theta \\ &= N_\omega^N(E) \end{aligned}$$

Taking the expectation of both side, and using (2.8) (and the fact that the number of eigenvalues of $M_\omega^n(\theta)$ and $M_{\overline{\omega}+\delta}^N(\theta)$ are bounded by $(2N+1)^d$), we obtain

$$N_s^{\overline{\omega}+\delta}(E - E^2) - CE^{d\rho}e^{-E^{-\alpha}} \leq N_s(E)$$

Considering the remarks made above, we obtain

$$C(\overline{\omega}) \leq \liminf_{E \to 0^+} \frac{N_s(E)}{f(E)}.$$

As $C(\overline{\omega})$ has the same sign as $\overline{\omega}$, this completes the proof of (1.6).

Proof of Lemma 3.4. Pick E small and $\varphi \in \mathcal{H}_\delta^N(E,\theta)$. Then, by Lemma 3.3, φ can be expanded as

$$\varphi = \sum_{\substack{|\gamma_1| \leq CE^{1/2}N \\ \gamma_2 \in \mathbb{Z}_{2N+1}^{d_2}}} a_{\gamma_1,\gamma_2} v_{\gamma_1,\gamma_2}(\theta, t)$$

where the vectors $(v_\gamma(\theta))_\gamma$ are given by (3.3) and (3.4). Using these equations, we compute

$$\langle V_\omega^N \varphi, \varphi\rangle = \sum_{\beta_1 \in \mathbb{Z}_{2N+1}^{d_1}} \omega_{\beta_1} |A_{\beta_1}|^2 \tag{3.7}$$

where

$$A_{\beta_1} = \frac{1}{(2N+1)^{d_1/2}} \sum_{|\gamma_1| \leq CE^{1/2}N} e^{i\frac{2\pi\beta_1\cdot\gamma_1}{2N+1}} c_{\gamma_1} \text{ and } c_{\gamma_1} = \sum_{\gamma_2 \in \mathbb{Z}_{2N+1}^{d_2}} a_{\gamma_1,\gamma_2} \left\langle \delta_0, \varphi_{\gamma_2}^N\left(\theta_1 + \frac{2\pi\gamma_1}{2N+1}, \theta_2, t\right)\right\rangle \tag{3.8}$$

So the vector $(A_{\beta_1})_{\beta_1}$ is the discrete Fourier transform of the vector $c = (c_{\gamma_1})_{\gamma_1}$ supported in a ball of radius $CE^{1/2}N$. To estimate this Fourier transform, we used the following result

Lemma 3.5 ([18])**.** *Assume N, L, K, K' L' are positive integers such that*

- $2N+1 = (2K+1)(2L+1) = (2K'+1)(2L'+1)$
- *$K < K'$ and $L' < L$.*

Pick $a = (a_n)_{n\in\mathbb{Z}_{2N+1}^d} \in \ell^2(\mathbb{Z}_{2N+1}^d)$ such that,

$$\text{for } |n| > K,\ a_n = 0.$$

Then, there exists $\tilde{a} \in \ell^2(\mathbb{Z}_{2N+1}^d)$ such that

1. *$\|a - \tilde{a}\|_{\ell^2(\mathbb{Z}_{2N+1}^d)} \leq C_{K,K'}\|a\|_{\ell^2(\mathbb{Z}_{2N+1}^d)}$ where $C_{K,K'} \asymp_{K/K'\to 0} K/K'$.*
2. *write $\tilde{a} = (\tilde{a}_j)_{j\in\mathbb{Z}_{2L+1}^d}$; for $l' \in \mathbb{Z}_{2L'+1}^d$ and $k' \in \mathbb{Z}_{2K'+1}^d$, we have*

$$\sum_{j\in\mathbb{Z}_{2L+1}^d} \tilde{a}_j e^{i\frac{2\pi j\cdot(l'+k'(2L'+1))}{2N+1}} = \sum_{j\in\mathbb{Z}_{2L+1}^d} \tilde{a}_j e^{i\frac{2\pi j\cdot k'}{2K'+1}}.$$

(3) $\|a\|_{\ell^2(\mathbb{Z}^d_{2N+1})} = \|\tilde{a}\|_{\ell^2(\mathbb{Z}^d_{2N+1})}$.

This lemma is a quantitative version of the Uncertainty Principle; it says that, if a vector is localized in a small neighborhood of 0 (here, of size K/N), up to a small error δ, its Fourier transform is constant over cube of size $N/(\delta K)$.

To apply Lemma 3.5, we pick N such that $(2N+1) = (2K+1)(2L'+1)(2M+1)$ where $K \geq CE^{1/2}N$; this is possible as $N \sim E^{-\rho}$ with ρ large; we pick for example, $L' \sim CE^{-(1-\nu)/2}$ and $L' \sim CE^{-\nu/2}$ (for some fixed $0 < \nu < 1$).

We apply Lemma 3.5 to the vector $c = (c_{\gamma_1})_{\gamma_1}$ defined in (3.8); by Lemma 3.5, there exists $\tilde{c} = (\tilde{c}_{\gamma_1})_{\gamma_1}$ so that, if we set

$$\tilde{A}_{\beta_1} = \frac{1}{(2N+1)^{d_1/2}} \sum_{\gamma_1 \in \mathbb{Z}^{d_1}_{2N+1}} e^{i\frac{2\pi\beta_1\cdot\gamma_1}{2N+1}} c_{\gamma_1}$$

then, for $\gamma_1' \in \mathbb{Z}^d_{2L'+1}$ and $\beta_1' \in \mathbb{Z}^d_{2K'+1}$, we have

$$\tilde{A}_{\gamma_1'+\beta_1'(2L'+1)} = \tilde{A}_{\beta_1'(2L'+1)}. \tag{3.9}$$

Fix $\eta > 0$ small to be chosen later. We replace A by $\tilde{A}$ in (3.7) and use the boundedness of the random variables to obtain

$$\langle V^N_\omega \varphi, \varphi\rangle \leq (1+\eta) \sum_{\beta_1 \in \mathbb{Z}^{d_1}_{2N+1}} \omega_{\beta_1} |\tilde{A}_{\beta_1}|^2 + \frac{C}{\eta}\|A - \tilde{A}\|^2$$

Using (3.9) and points (1) and (3) of Lemma 3.5, we get that

$$\langle V^N_\omega \varphi, \varphi\rangle \leq \sum_{\beta_1' \in \mathbb{Z}^{d_1}_{2K'+1}} \left[\frac{C}{\eta}E^{\nu/2} + \frac{1}{(2L'+1)^{d_1}}\left(\sum_{\gamma_1' \in \mathbb{Z}^{d_1}_{2L'+1}} (1+\eta)\omega_{\gamma_1'+\beta_1'(2L'+1)}\right)\right](2L'+1)^{d_1}|\tilde{A}_{\beta_1'}(2L'+1)|^2$$

Pick η such that $\eta \cdot \omega_+ < \delta/4$ and E sufficiently small that $\frac{C}{\eta}E^{\nu/2} < \delta/4$. We then obtain

$$\langle V^N_\omega \varphi, \varphi\rangle \leq \sum_{\beta_1' \in \mathbb{Z}^{d_1}_{2K'+1}} \left[\delta/2 + \frac{1}{(2L'+1)^{d_1}}\left(\sum_{\gamma_1' \in \mathbb{Z}^{d_1}_{2L'+1}} \omega_{\gamma_1'+\beta_1'(2L'+1)}\right)\right](2L'+1)^{d_1}|\tilde{A}_{\beta_1'}(2L'+1)|^2 \tag{3.10}$$

Now, if ω satisfies

$$\forall \beta_1' \in \mathbb{Z}^{d_1}_{2K'+1},\ \frac{1}{(2L'+1)^{d_1}} \sum_{\gamma_1' \in \mathbb{Z}^{d_1}_{2L'+1}} \omega_{\gamma_1'+\beta_1'(2L'+1)} \leq \overline{\omega} + \delta/2$$

then, (3.10) gives

$$\langle V^N_\omega \varphi, \varphi\rangle \leq (\overline{\omega}+\delta) \sum_{\beta_1' \in \mathbb{Z}^{d_1}_{2K'+1}} (2L'+1)^{d_1}|\tilde{A}_{\beta_1'}(2L'+1)|^2 = \langle V^N_{\overline{\omega}+\delta}\varphi, \varphi\rangle$$

where V^N_t is defined in (3.1). Here, we have used the points (2) and (3) of Lemma 3.5, and the definition (3.8) of the vector $c = (c_{\gamma_1})_{\gamma_1}$.

Summing all this up, we have proved

Lemma 3.6. *Pick $0 < \nu < 1$. Pick N as described above. For E sufficiently small, the probability that, for all θ and all $\varphi \in \mathcal{H}^N_\delta(E,\theta)$, one has*

$$\langle M^N_\omega(\theta)\varphi, \varphi\rangle \leq E\|\varphi\|^2.$$

is larger than the probability of the set

$$\left\{\omega;\ \forall \beta_1' \in \mathbb{Z}^{d_1}_{2K'+1},\ \frac{1}{(2L'+1)^{d_1}} \sum_{\gamma_1' \in \mathbb{Z}^{d_1}_{2L'+1}} \omega_{\gamma_1'+\beta_1'(2L'+1)} \leq \overline{\omega} + \delta/2\right\}$$

The probability of this event is estimated by the usual large deviation estimates (see e.g. [7, 5]). This completes the proof of Lemma 3.4. ☐

4. The fluctuating edges

In this section, we investigate the behavior of the density of surface states $N_s(E)$ at the bottom E_0 of the spectrum of H_ω in the case when $E_0 < \inf \sigma(H) = 0$. As we saw in Section 0.1, this is always the case for dimension $d_2 = 1$ or $d_2 = 2$ and it holds in arbitrary dimensions if the support of common distribution P_0 of the ω_{γ_1} has a sufficiently negative part. Thus, we are looking at a fluctuation edge as described in Section 0.2. Due to the symmetry of the problem we may, of course, consider the top of the spectrum in an analogous way.

4.1. **A reduced Hamiltonian.** In the present situation it is convenient to think of the Hilbert space $\ell^2(\mathbb{Z}^{d_1+d_2})$ as a direct some of $\ell^2(\mathbb{Z}^{d_1} \times \{0\}) =: \mathcal{H}_b$ and $\ell^2(\mathbb{Z}^{d_1+d_2} \setminus \mathbb{Z}^{d_1} \times \{0\}) =: \mathcal{H}_s$, the indices referring to "bulk" and "surface" respectively (see [13] whose notations we follow). According to the decomposition $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_b$ we can write any operator A on $\mathcal{H}$ as a matrix

$$A = \begin{bmatrix} A_{ss} & A_{sb} \\ A_{bs} & A_{bb} \end{bmatrix}$$

where A_{ss} and A_{bb} act on $\mathcal{H}_s$ and $\mathcal{H}_b$ respectively and $A_{sb} : \mathcal{H}_b \to \mathcal{H}s$, $A_{bs} : \mathcal{H}_s \to \mathcal{H}_b$ "connect" the two Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_b$. The bounded operator A is symmetric if $A^*_{ss} = A_{ss}$, $A^*_{bb} = A_{bb}$ and $A^*_{sb} = A_{bs}$. In the case of our random Hamiltonian H_ω we have: $(H_\omega)_{ss} = (H_0)_{ss} + V_\omega$ while $(H_\omega)_{bb} = (H_0)_{bb}$ and H_{sb} as well as H_{bs} are independent of the randomness. Moreover, by assumption, $(H_\omega)_{bb} \geq 0$, while $\inf \sigma(H_\omega) < 0$. Consequently, the operator $((H_0)_{bb} - E\mathbb{1}_{bb})^{-1}$ exists for all $E < 0$ and the operator

$$G_s(E) := (H_0)_{ss} + V_\omega - H_{sb} \left((H_0)_{bb} - E\mathbb{1}_{bb}\right)^{-1} H_{bs} - E\mathbb{1}_{ss}$$

the so called resonance function is well defined. The operator $G_s(E)$ is a sort of a reduced Hamiltonian. Its inverse plays the role of a resolvent. It is not hard to show that the set $\mathcal{R}(H_\omega) = \{E \in]-\infty, 0[; 0 \in \sigma(G_s(E))\}$ (the resonant spectrum) agrees with the negative part of $\sigma(H_\omega)$. See Prop.1.2 in [13] for details. For later reference, we state this as a lemma:

Lemma 4.1. *For $E < 0$, E is an eigenvalue of H_ω if and only if 0 is an eigenvalue at $G_s(E)$. Moreover the multiplicities agree.*

In fact a little linear algebra proves that, for block matrices, we have

$$\left(\begin{array}{cc} A & B \\ C & D \end{array} \right)^{-1} = \left(\begin{array}{cc} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A + BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{array} \right) \tag{4.1}$$

when all the terms make sense.
We denote by $N(A, E)$ the number of eigenvalues (counted according to multiplicity) of the operator A below E. For $\Lambda_L = [-L, L]^d$ we set $(H_{\omega,L})_{ij} = (H_\omega)_{ij}$ if $i, j \in \Lambda_L$ and $(H_{\omega,L})_{ij} = 0$ otherwise. For energies E below zero the integrated density of surface states of H_ω is given by

$$N_s(E) = \lim_{L\to\infty} \frac{1}{(2L+1)^{d_1}} N(H_{\omega,L}, E).$$

Defining

$$G_s^L(E) = (H_{\omega,L})_{ss} - (H_L)_{sb}((H_L)_{bb} - E\mathbb{1}_{bb})^{-1}(H_L)_{bs} - E\mathbb{1}_{ss}.$$

We have, as above, that $E < 0$ is an eigenvalue of $H_{\omega,L}$ if and only if 0 is an eigenvalue of $G_s^L(E)$.
In the following, we will express the density of surface states $N_s(E)$ (for $E < 0$) in terms of the operators $G_s^L(E)$.

Lemma 4.2. *The eigenvalues $\rho_n(E)$ of $G_s^L(E)$ are continuous and decreasing functions of E (for $E < 0$).*

Proof of Lemma 4.2. Continuity is obvious from the explicit form of the entries of the (finite-dimensional) matrix $G^L_S(E)$. In the following calculations we omit the superscript L. Let $0 > E_2 > E_1$, then

$$\begin{aligned} G_s(E_1) - G_s(E_2) &= -H_{sb}((H_{bb} - E_1)^{-1} - (H_{bb} - E_2)^{-1})H_{bs} - (E_1 - E_2) \\ &= (E_2 - E_1)H_{sb}((H_{bb} - E_1)^{-1}(H_{bb} - E_2)^{-1})H_{bs} + (E_2 - E_1). \end{aligned}$$

Since $E_1, E_2 < 0$ the operator $(H_{bb} - E_1)^{-1}(H_{bb} - E_2)^{-1}$ is positive, the operator $G_s(E_1) - G_s(E_2)$ is positive. □

Proposition 4.1. *For $E < 0$:*

$$N(H_{\omega,L}, E) = N(G^L_s(E), 0).$$

Proof of Proposition 4.1. For E sufficiently negative, $G_s(E)$ is a positive operator. Let us now increase E (toward $E = 0$). Then, E is an eigenvalue of $H_{\omega,L}$ if one of the eigenvalues of $G^L_s(E)$ passes through zero and becomes negative. □

It follows from this proposition that (for $E < 0$)

$$N_s(E) = \lim_{L\to\infty} N(G^L_s(E), 0)$$

$G^L_s(E)$ depends on E in a rather complicated way through the resonance function. We will therefore approximate $G^L_s(E)$ by an operator with much simpler dependence on E in the following way:
let $E_0 = \inf \sigma(H_\omega)$ then we set:

$$\tilde{G}^L_s(E) = (H_{\omega,L})_{ss} - (H_L)_{sb}((H_L)_{bb} - E_0)^{-1}(H_L)_{bs} - E$$

This operator should give a good estimate for the eigenvalues of H_ω near E_0, in fact:

Lemma 4.3. *For $E_0 < E < 0$:*

$$N(\tilde{G}_s(E), 0) \leq N(G_s(E), 0)$$

Proof of Lemma 4.3.

$$\tilde{G}_s(E) - G_s(E) = (E - E_0)H_{sb}\left((H_{bb} - E)^{-1}(H_{bb} - E_0)^{-1}\right)H_{bs}.$$

So

$$\tilde{G^L_s}(E) \geq G^L_s(E).$$

□

For a bound in the other direction we observe that:

Lemma 4.4. *For $E_0 \leq E \leq E_1 < 0$ we have*

$$\tilde{G}_s(E) - G_s(E) \leq C(E - E_0)$$

Remark: The constant C in the above estimate depends on E_0 and E_1.
Proof of Lemma 4.4.

$$\begin{aligned} \tilde{G}_s(E) - G_s(E) &= (E - E_0)H_{sb}((H_{bb} - E)^{-1}(H_{bb} - E_0)^{-1})H_{bs} \\ &\leq (E - E_0)H_{sb}((H_{bb} - E_1)^{-1}(H_{bb} - E_0)^{-1})H_{bs} \\ &\leq C(E - E_0). \end{aligned}$$

Here, we used that

$$(H_{bb} - E)^{-1} \leq (H_{bb} - E_1)^{-1}.$$

□

Summarizing, we have got:

Proposition 4.2. *There is a constant C, such that for $E_0 \leq E \leq E_0/2 < 0$*

$$N(\tilde{G}_s(E), 0) \leq N(H_\omega, E) \leq N(\tilde{G}_s(E) - C(E - E_0), 0).$$

The advantage of having $\tilde{G}_s(E)$ rather than $G_s(E)$ lies in the fact that $\tilde{G}_s(E)$ depends linearly on E, in fact:

$$\begin{aligned} G_s(E) &= H_{ss} - H_{sb}(H_{bb} - E_0)^{-1} H_{bs} + V_\omega - E \\ &= \tilde{H} + V_\omega - E \end{aligned}$$

where $\tilde{H}$ is the operator

$$\tilde{H} = H_{ss} - H_{sb}(H_{bb} - E_0)^{-1} H_{bs}.$$

This operator is of a similar form as the Hamiltonian H, however it acts on $\ell^2(\mathbb{Z}^{d_1})$, i.e. on the surface only where the random potential V_ω lives. The price to pay is the complicated looking "bulk term" $H_{sb}(H_{bb} - E_0)^{-1} H_{bs}$.

Nevertheless, $\tilde{H}$ is still a Toeplitz operator and it is not too hard to compute its symbol, i.e. its Fourier representation.

In fact, a look at formula (4.1) shows that

$$\tilde{H} = \left[\left((H - E_0)^{-1} \right)_{ss} \right]^{-1} + E_0. \tag{4.2}$$

Consequently the symbol of $\tilde{H}$ is given by:

$$\tilde{h}(\theta_1) = \left(\int \frac{1}{h(\theta_1, \theta_2) - E_0} \, d\theta_2 \right)^{-1} + E_0.$$

We summarize these results in a theorem:

Theorem 4.1. *Let $H_\omega = H + V_\omega$ as in (1.5) satisfying assumption (H1). Assume moreover, that $E_0 = \inf \sigma(H_\omega) \leq 0$. Define $\tilde{H}_\omega = \tilde{H} + \tilde{V}_\omega$ as in (4.2) and let $N_s(H_\omega, E)$ be the integrated density of surface states of H_ω and $N(\tilde{H}_\omega, E)$ the integrated density of states for $\tilde{H}_\omega$. Then*

$$\lim_{E \searrow E_0} \frac{\ln |\ln N_s(H_\omega, E)|}{\ln(E - E_0)} = \lim_{E \searrow E_0} \frac{\ln |\ln N(\tilde{H}_\omega, E)|}{\ln(E - E_0)}$$

where the equality should be interpreted in the following way: if one of the sides exists so does the other one and they agree.

In other words, the Lifshitz exponent for the density of *surface* states of H_ω and and the Lifshitz exponent for the density of states for $\tilde{H}_\omega$ agree.

4.2. **Lifshitz tails.** In this section we investigate the integrated density of surface states $N_s(E)$ for the operator $H_\omega = H + V_\omega$ acting on $\ell^2(\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2})$. We assume throughout that $E_0 = \inf \sigma(H_\omega)$ is (strictly) negative and $E < 0$.

By the previous section the investigation of $N_s(E)$ for E near E_0 can be reduced to estimates for the integrated density of states $N(E)$ of the operator $\tilde{H}_\omega = \tilde{H} + \tilde{V}_\omega$ which acts on $\ell^2(\mathbb{Z}^{d_1})$. Hence the problem of surface Lifshitz tails boils down to ordinary Lifshitz tails in a lower dimensional configuration space. However the (free) operator is somewhat more complicated, in fact in Fourier representation it is multiplication by

$$\tilde{h}(\theta_1) = \left(\int \frac{1}{h(\theta_1, \theta_2) - E_0} \; d\theta_2 \right)^{-1} + E_0.$$

We remind the reader that $\tilde{V}_\omega(\gamma_1) = \omega_{\gamma_1}$ for $\gamma_1 \in \mathbb{Z}^{d_1}$ and $(\omega_{\gamma_1})_{\gamma_1 \in \mathbb{Z}^{d_1}}$ is a family of independent random variables with a common distribution P_0.

Throughout this section we assume that $\operatorname{supp}(P_0)$ is a compact set. Moreover, if we set $\omega_- = \inf(\operatorname{supp}(P_0))$ we suppose that $P_0([\omega_-, \omega_- + \varepsilon) \geq C\, \varepsilon^k)$ for some k.

Theorem 4.2. *If $\tilde{h}$ has a unique quadratic minimum then*

$$\lim_{E \searrow E_0} \frac{\ln |\ln(N_s(E))|}{\ln(E - E_0)} = -\frac{d_1}{2}.$$

Proof. The theorem follows from [16, 19] and the considerations above. □

For dimensions $d_1 = 1$ and $d_1 = 2$ we have the following result:

Theorem 4.3. *Assume that $\tilde{h}$ is not constant. If $d_1 = 1$ then*

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\lim_{E\searrow E_0} \frac{\ln(n(E))}{(E-E_0)}$$

where $n(E)$ is the integrated density of states for $\tilde{H}$.
If $d_2 = 2$, then

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\alpha$$

where α is defined in (1.8).

Note that $n(E) \sim (E-E_0)^\rho$ for some $\rho > 0$. See [16, 19] for details.

To conclude this section we consider some examples that fulfill the assumptions of the previous theorems. Let us first assume that H is separable, i.e. that

$$h(\theta_1, \theta_2) = h_1(\theta_1) + h_2(\theta_2).$$

This is satisfied for example by the discrete Laplacian where h is equal to h_0 given in (1.1). The function h has a unique quadratic minimum if and only if both h_1 and h_2 have unique quadratic minima (which we may assume to be attained at $\theta_1 = \theta_2 = 0$).
We will show in the following that the function

$$\tilde{h}(\theta_1) = \left(\int \frac{1}{h_1(\theta_1) + h_2(\theta_2) - E_0}\, d\theta_2\right)^{-1} + E_0$$

has a unique quadratic minimum in this case as well. Differentiating the function

$$\rho(\theta_1) = \int \frac{1}{h_1(\theta_1) + h_2(\theta_2) - E_0}\, d\theta_2$$

we obtain:

$$\nabla\rho(\theta_1) = -\int \frac{\nabla h_1(\theta_1)}{(h_1(\theta_1) + h_2(\theta_2) - E_0)^2}\, d\theta_2$$

so the (possible) maximum of ρ is at $\theta_1 = 0$.
The second derivative at $\theta_1 = 0$ is given by:

$$\nabla\nabla\rho(0) = -\nabla\nabla h_1(0) \int \frac{1}{(h_1(0) + h_2(\theta_2) - E_0)^2}\, d\theta_2$$

which obviously gives a negative definite Hessian.
We remark that no assumptions on h_2 were needed; in fact, the above arguments work for $h_2 = const$ as well.
The same reasoning also shows that $\tilde{h}$ is not constant as long as h_1 is not constant.
So we have proved:

Theorem 4.4. *Suppose $h(\theta_1, \theta_2) = h_1(\theta_1) + h_2(\theta_2)$ then*

(1) *If h_1 has a unique quadratic minimum, then*

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\frac{d_1}{2}.$$

(2) *If $d_1 = 1$ and $\tilde{h}$ is not constant then*

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\lim_{E\searrow E_0} \frac{\ln(n(E))}{(E-E_0)}.$$

where $n(E)$ is the integrated density of states for $\tilde{H}$.

(3) *If $d_2 = 2$, then*

$$\lim_{E\searrow E_0} \frac{\ln|\ln(N_s(E))|}{\ln(E-E_0)} = -\alpha$$

where α is defined in (1.8).

5. The density of surface states for a constant surfacic potential

In this section, we prove some useful results on the density of surface states for a constant surface potential. In some cases, this density may even be computed explicitly (see e.g. [2]).

The model we consider is the model introduced in Proposition 1.2 namely $H_t = H + t\mathbf{1}\otimes\Pi_0^2$ where H is chosen as in section 1 and t is a real coupling constant. The proof of all the results we now state is based on rank one perturbation theory (see e.g. [24]). The main formula that we will use is the following: for $z\notin\mathbb{R}$, one has

$$(H_t - z)^{-1} - (H-z)^{-1} = -t\frac{1}{1+tI(z)\otimes Id_{\theta_2}}(H-z)^{-1}\mathbf{1}\otimes\Pi_0^2(H-z)^{-1} \tag{5.1}$$

where $I(z)$ is the operator acting on $\ell^2(\mathbb{Z}^{d_1})$ that, in Fourier representation, is the multiplication by the function $I(\theta_1, z)$ defined in (1.2).

Formula (5.1) is easily proved if one makes a partial Fourier transform in the (γ_1,θ_1) variable of H and H_t. If one does so, one obtains a direct integral representation for both H and H_t namely

$$H = \int_{\mathbb{T}^{d_1}} H(\theta_1)d\theta_1 \text{ and } H_t = \int_{\mathbb{T}^{d_1}} H_t(\theta_1)d\theta_1$$

where $H(\theta_1)$ and $H_t(\theta_1)$ (both acting on $\ell^2(\mathbb{Z}^{d_2})$) differ only by a rank one operator, namely,

$$H_t(\theta_1) - H(\theta_1) = t\Pi_0^2.$$

Formulae (5.1) and (1.2) then follow immediately from the well known resolvent formula for rank one perturbations that can be found e.g. in [24].

Proposition 1.3 follows immediately from Proposition 1.2 and formulae (5.1) and (1.2). Indeed, by formula (5.1) and the special form of the operator $I(z)$, z is a point in $\sigma(H_t)\setminus\sigma(H)$ if and only if, for some θ_1, one has

$$1 + tI(\theta_1, z) = 0.$$

If we pick $z\in\mathbb{R}$ below 0 (recall that $0 = \inf(\sigma(H)) = \inf(h(\mathbb{R}^d))$, we see that $z\in\sigma(H_t)$ if and only if $tI(\theta_1,z) = -1$ for some θ_1. As, for $z<0$, $I(\theta_1,z)$ is a negative decreasing function of z that tends to 0 when $z\to-\infty$, we see that this can happen if an only if $tI(\theta_1, 0) < -1$ for some θ_1. This is the first statement of Proposition 1.1. Indeed, the function $\theta_1\mapsto tI(\theta_1,0)$ is continuous of $\mathbb{T}^{d_1}$ except, possibly, at the points where h assumes its minimum, and it takes its minimal value exactly at one of those points.

As, for the second statement, let $I(z) := \max_{\theta_1\in\mathbb{T}^{d_1}} I(\theta_1, z)$ and consider the function $f:\ z\mapsto 1+tI(z)$. This function is clearly continuous and strictly decreasing on $]-\infty,0[$ and by assumption, it is negative near 0 (as $1+tI_\infty<0$) and $f(z)\to1$ as $z\to-\infty$. So, the function f admits a unique zero that we denote by E_0. The analysis given above immediately shows that E_0 is the infimum of H_t: as $\theta_1\mapsto I(\theta_1,z)$ is continuous on $\mathbb{T}^{d_1}$ that is compact, for some θ_1, one has $1+tI(\theta_1,E_0)=0$. So that E_0 belongs to $\sigma(H_t)$; on the other hand, for $E<E_0$, for any θ_1, one has $1+tI(\theta_1,E)\geq 1+tI(E)>0$, hence, $E\notin\sigma(H_t)$. This completes the proof of Proposition 1.1.

5.1. Asymptotics of the density of surface states.

The starting point for this computation is again formula (5.1). This enables us to get a very simple formula for the Stieltjes-Hilbert transform of the density of surface states n_s^t for the pair (H_t, H). Using the Fourier representation and Parseval's formula, one computes

$$\begin{aligned}
&\operatorname{tr}(\Pi_1[(H_t-z)^{-1} - (H-z)^{-1}]\Pi_1)\\
&= \sum_{\gamma_2\in\mathbb{Z}^{d_2}}\int_{\mathbb{T}^{d_1}}\frac{-t}{1+tI(\theta_1,z)}\int_{\mathbb{T}^{d_2}}\frac{e^{i\gamma_2\theta_2}d\theta_2}{h(\theta_1,\theta_2)-z}\int_{\mathbb{T}^{d_2}}\frac{e^{i\gamma_2\theta_2}d\theta_2}{h(\theta_1,\theta_2)-z}d\theta_1\\
&= \sum_{\gamma_2\in\mathbb{Z}^{d_2}}\int_{\mathbb{T}^{d_1}}\frac{-t}{1+tI(\theta_1,z)}\overline{\int_{\mathbb{T}^{d_2}}\frac{e^{-i\gamma_2\theta_2}d\theta_2}{h(\theta_1,\theta_2)-z}}\int_{\mathbb{T}^{d_2}}\frac{e^{i\gamma_2\theta_2}d\theta_2}{h(\theta_1,\theta_2)-z}d\theta_1\\
&= \int_{\mathbb{T}^{d_1}}\frac{-t}{1+tI(\theta_1,z)}\int_{\mathbb{T}^{d_2}}\frac{d\theta_2}{(h(\theta_1,\theta_2)-z)^2}d\theta_1
\end{aligned}$$

One then notices that

$$\int_{\mathbb{T}^{d_1}} \frac{-t}{1+tI(\theta_1,z)} \int_{\mathbb{T}^{d_2}} \frac{d\theta_2}{(h(\theta_1,\theta_2)-z)^2} d\theta_1 = -\frac{d}{dz}\int_{\mathbb{T}^{d_1}} \log(1+tI(\theta_1,z))d\theta_1.$$

Here, and in the sequel, log denotes the principal determination of the logarithm.
This immediately yields that the Stieltjes-Hilbert transform of N_s^t is given by

$$\left\langle \frac{1}{\cdot - z}, dN_s^t \right\rangle = \int_{\mathbb{T}^{d_1}} \log(1+tI(\theta_1,z))d\theta_1$$

where I is defined by (1.2).
It is well known that one can invert the Stieltjes-Hilbert transform to recover the signed measure dN_s^t (see e.g. the appendix of [23]). By the Stieltjes-Perron inversion formula, one has

$$\begin{aligned}\int_0^E dN_s^t(e) &= \lim_{\varepsilon\to 0^+} \frac{1}{2i\pi}\int_0^E \left(\left\langle \frac{1}{\cdot - e - i\varepsilon}, dN_s^t\right\rangle - \left\langle \frac{1}{\cdot - e + i\varepsilon}, dN_s^t\right\rangle \right) de \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2i\pi}\int_0^E \int_{\mathbb{T}^{d_1}} [\log(1+tI(\theta_1,e+i\varepsilon)) - \log(1+tI(\theta_1,e-i\varepsilon))]d\theta_1 de.\end{aligned} \tag{5.2}$$

Notice that, for e real,

$$\mathrm{Im}\,(1+tI(\theta_1,e+i\varepsilon)) = t\varepsilon \int_{\mathbb{T}^{d_2}} \frac{1}{(h(\theta_1,\theta_2)-e)^2+\varepsilon^2} d\theta_2;$$

hence, this imaginary part keeps a fixed sign. So, for $\theta_1 \in \mathbb{T}^{d_1}$, one has

$$\log(1+tI(\theta_1,e+i\varepsilon)) - \log(1+tI(\theta_1,e-i\varepsilon)) = \log\left(\frac{1+tI(\theta_1,e+i\varepsilon)}{1+tI(\theta_1,e-i\varepsilon)}\right)$$

For $e\in\mathbb{R}$, one has $|1+tI(\theta_1,e+i\varepsilon)| = |1+tI(\theta_1,e-i\varepsilon)|$. As moreover the imaginary part of $1+tI(\theta_1,e+i\varepsilon)$ keeps a fixed sign, one has

$$|\log(1+tI(\theta_1,e+i\varepsilon)) - \log(1+tI(\theta_1,e-i\varepsilon))| \leq 2\pi.$$

As $\mathbb{T}^{d_1}$ and $[0,E]$ are compact, one can apply Lebesgue's dominated convergence Theorem to (5.2) and thus obtain

$$\int_0^E dN_s^t(e) = \int_0^E \int_{\mathbb{T}^{d_1}} f(\theta_1,e)d\theta_1 de. \tag{5.3}$$

where

$$f(\theta_1,e) = \lim_{\varepsilon\to 0^+}\frac{1}{2i\pi}\log\left(\frac{1+tI(\theta_1,e+i\varepsilon)}{1+tI(\theta_1,e-i\varepsilon)}\right) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\mathrm{Arg}(1+tI(\theta_1,e+i\varepsilon)) \tag{5.4}$$

where Arg is the principal determination of the argument of a complex number. Notice here that this formula is the analogue of the well-known Birman-Kreĭn formula (see e.g. [1, 25]) for surface perturbations.

We will now compute the asymptotics of $f(\theta_1,e)$ for e small. First, let us notice that we need only to compute these for θ_1 small, i.e. close to 0. Indeed, we have assumed that h takes its minimum only at 0. Therefore, as $\mathbb{T}^d$, is compact, if $|\theta_1|\geq\delta$, we know that, for some $\delta'>0$, for all θ_2, one has $h(\theta_1,\theta_2)\geq\delta'$. Hence, if $|\theta_1|\geq\delta$, the function $I(\theta_1,z)$ is analytic in a neighborhood of 0, so that $f(\theta_1,e)=0$ for e sufficiently small (independent of θ_1). So, we now assume that $|\theta_1|<\delta$ for some $\delta>0$ to be chosen later on.

We now study $I(\theta_1,z)$ for $|z|$ small. Pick χ a smooth cut-off function in θ_2, i.e. such that $\chi(\theta_2)=1$ if $|\theta_2|\leq\delta_\chi$ and $\chi(\theta_2)=0$ if $|\theta_2|\geq 2\delta_\chi$. Write

$$I(\theta_1,z) = \int_{\mathbb{T}^{d_2}} \frac{\chi(\theta_2)}{h(\theta_1,\theta_2)-z}d\theta_2 + \int_{\mathbb{T}^{d_2}} \frac{1-\chi(\theta_2)}{h(\theta_1,\theta_2)-z}d\theta_2. \tag{5.5}$$

For the same reason as above, the second integral in the right hand side term is analytic for $|z|$ small for all θ_1. We only need to study the integral

$$J(\theta_1,z) = \int_{\mathbb{T}^{d_2}} \frac{\chi(\theta_2)}{h(\theta_1,\theta_2)-z}d\theta_2. \tag{5.6}$$

Therefore, we use the assumptions that 0 is the unique minimum of h and that it is quadratic non-degenerate. This implies, that for $\delta > 0$ sufficiently small, for $|\theta_1| < \delta$, the function $\theta_2 \mapsto h(\theta_1, \theta_2)$ has a unique minimum, say $\theta_2(\theta_1)$, that this minimum is quadratic non-degenerate. Let $h_2(\theta_1)$ be the minimal value, i.e. $h_2(\theta_1) = h(\theta_1, \theta_2(\theta_1))$. Then, the functions $\theta_1 \mapsto \theta_2(\theta_1)$ and $\theta_1 \mapsto h(\theta_1)$ are real analytic in $|\theta_1| < \delta$.
All these statements are immediate consequences of the analytic Implicit Function Theorem applied to the system of equations $\nabla_{\theta_2} h(\theta_1, \theta_2) = 0$.
So, for $|\theta| < \delta$, one can write

$$h(\theta_1, \theta_2) = h_2(\theta_1) + \langle (\theta_2 - \theta_2(\theta_1)), Q_2(\theta_1)(\theta_2 - \theta_2(\theta_1)) \rangle + O(|\theta_2 - \theta_2(\theta_1)|^3)$$

where $Q_2(\theta_1)$ is the Hessian matrix of $h(\theta_1, \theta_2)$ at the point $\theta_2(\theta_1)$.
We can now use the analytic Morse Lemma (see e.g. [12]) uniformly in the parameter θ_1. That is, for some $\delta_0 > 0$ small, there exists $B_2(0, \delta_0) \subset U$ (the ball of center 0 and radius δ_0 in $\mathbb{T}^{d_2}$) and $\psi(\theta_1): \ \theta_2 \in U \to \psi(\theta_1, \theta_2) \in B_2(\theta_2(\theta_1), 2\delta_0)$, a real analytic diffeomorphism so that, for $\theta \in U$,

$$h(\theta_1, \psi(\theta_1, \theta_2)) = h_2(\theta_1) + (\theta_2, Q_2(\theta_1)\theta_2). \tag{5.7}$$

Moreover, the Jacobian matrix of ψ at $\theta_2(\theta_1)$ is the identity matrix, and the mapping $\theta_1 \mapsto \psi(\theta_1)$ is real analytic (here, we take the norm in the Banach space of real analytic function in a neighborhood of 0).
Before we return to the analysis of J, let us describe $h_2(\cdot)$ and $\theta_2(\cdot)$ more precisely. Let Q be the Hessian matrix of h at 0. As h has a quadratic non degenerate minimum at 0, Q is definite positive. We can write this $d \times d$-matrix in the form

$$Q = \begin{pmatrix} Q_1 & R^* \\ R & Q_2 \end{pmatrix} \tag{5.8}$$

where $Q_{1,2}$ is the restriction of Q to $\mathbb{R}^{d_1, d_2}$ when one decomposes $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Both Q_1 and Q_2 are positive definite; actually, the positive definiteness of Q ensures that the matrices $Q_1 - R^* Q_2^{-1} R$ and $Q_2 - R Q_1^{-1} R^*$ are positive definite. Using the Taylor expansion of h near 0, one computes

$$\begin{aligned} \theta_2(\theta_1) &= -Q_2^{-1} R \theta_1 + O(|\theta_1|^2), \quad Q_2(\theta_1) = Q_2 + O(|\theta_1|), \\ h_2(\theta_1) &= ([Q_1 - R^* Q_2^{-1} R]\theta_1, \theta_1) + O(|\theta_1|^3). \end{aligned} \tag{5.9}$$

Let us also note here that

$$\mathrm{Det}\, Q = \mathrm{Det}\, Q_1 \cdot \mathrm{Det}\, (Q_2 - R^* Q_1^{-1} R) = \mathrm{Det}\, Q_2 \cdot \mathrm{Det}\, (Q_1 - R Q_2^{-1} R^*) \tag{5.10}$$

We now return to J. Performing the change of variables $\theta \to \psi(\theta)$ in $J(\theta_1, z)$, we get

$$J(\theta_1, z) = \int_{\mathbb{T}^{d_2}} \frac{\tilde{\chi}(\theta_1, \theta_2)}{(\theta_2, Q_2(\theta_1)\theta_2) + h_2(\theta_1) - z} d\theta_2 \text{ where } \tilde{\chi}(\theta_1, \theta_2) := \chi(\psi(\theta_1, \theta_2)) \mathrm{Det}\, (\nabla_{\theta_2} \psi(\theta_1, \theta_2)). \tag{5.11}$$

Choosing δ sufficiently small with respect to δ_χ (defining χ), we see that $\chi(\psi(\theta_1, \theta_2)) = 1$ for all $|\theta_1| < \delta$ and $|\theta_2| < \delta$. Hence, the function $\tilde{\chi}(\theta_1, \theta_2)$ is real analytic in a neighborhood of $(0, 0)$.

To compute the integral in the right hand side of (5.11), we change to polar coordinates (recall that $\tilde{\chi}$ is supported near 0) to obtain

$$J(\theta_1, z) = \mathrm{Det}\, (Q_2(\theta_1))^{-1/2} \int_0^{+\infty} \frac{\hat{\chi}(\theta_1, r) r^{d_2 - 1}}{r^2 + h_2(\theta_1) - z} dr \tag{5.12}$$

where

$$\check{\chi}(\theta_1, r) := \frac{1}{(2\pi)^{d_2}} \int_{\mathbb{S}^{d_2 - 1}} \tilde{\chi}(\theta_1, r\xi) d\xi. \tag{5.13}$$

The factor $(2\pi)^{-d_2}$ in the last integral comes from the fact that $d\theta_2$ denotes the normalized Haar measure on $\mathbb{T}^{d_2}$, i.e. the Lebesgue measure divided by $(2\pi)^{d_2}$. Note again that $(\theta_1, r) \mapsto \check{\chi}(\theta_1, r)$ is real analytic in a neighborhood of 0, and

$$\check{\chi}(\theta_1, 0) = \frac{1}{(2\pi)^{d_2}} \mathrm{Det}\, (\nabla_{\theta_2} \psi(\theta_1, \theta_2(\theta_1))) \cdot \mathrm{Vol}(\mathbb{S}^{d_2 - 1}).$$

Moreover, as $\int_{\mathbb{S}^{d_2}} \xi^k d\xi = 0$ if k is multi-index of odd length, we known that the Taylor expansion of $\check{\chi}(\theta_1, r)$ contains only even powers of r, i.e. there exists a function $\hat{\chi}(\theta_1, r)$ analytic in a neighborhood of $(0,0)$ such that $\check{\chi}(\theta_1, r) = \hat{\chi}(\theta_1, r^2)$.

We now use the

Lemma 5.1. *Let $\hat{\chi}$ be a smooth compactly supported function such that $\hat{\chi}$ be real analytic is a neighborhood of 0. Define the integral $J_{\hat{\chi}}(z)$ to be*

$$J_{\hat{\chi}}(z) = \int_0^{+\infty} \frac{\hat{\chi}(r^2) r^{n-1}}{r^2 + z} dr.$$

Then, one has

$$J_{\hat{\chi}}(z) = S(z) \cdot H(z) + G(z) \tag{5.14}$$

where

1. *G and H are real analytic in a neighborhood of 0;*
2. *they satisfy $H(0) = \hat{\chi}(0)$ and $G(0) > 0$ if $\hat{\chi}(0) > 0$ and $\hat{\chi} \geq 0$;*
3. *the function S is defined by*
    - *if n is even, then $S(z) = \frac{1}{2} \cdot (-1)^{\frac{n}{2}} z^{\frac{n-2}{2}} \cdot \log z$;*
    - *if n is odd, then $S(z) = \frac{\pi}{2} \cdot (-1)^{\frac{n-1}{2}} z^{\frac{n-1}{2}} \frac{1}{\sqrt{z}}$.*

    Here, $\sqrt{z}$ and $\log z$ denote respectively the principal determination of the square root and of the logarithm.

The proof of this result is elementary; after a cut-off near zero, one expands $\hat{\chi}$ in a Taylor series near 0, and computes the resulting integrals term by term essentially explicitly (see [15] for more details).

Putting (5.5), (5.6), (5.11), (5.12) and (5.14) together, we obtain that

$$I(\theta_1, z) = S(h_2(\theta_1) - z) \cdot H(\theta_1, h_2(\theta_1) - z) + G(\theta_1, h_2(\theta_1) - z) \tag{5.15}$$

where

- S is described in point (3) of Lemma 5.1;
- $(\theta_1, z) \mapsto H(\theta_1, z)$ and $(\theta_1, z) \mapsto G(\theta_1, z)$ are real analytic in θ_1 and z in a neighborhood of 0;
- one has

  $$H(\theta_1, 0) = \frac{1}{(2\pi)^{d_2}} \mathrm{Det}\,(Q_2(\theta_1))^{-1/2} \cdot \mathrm{Det}\,(\nabla_{\theta_2} \psi(\theta_1, \theta_2)) \mathrm{Vol}(\mathbb{S}^{d_2 - 1})$$

  and $G(0,0)$ is positive.

The last point here is obtained combining point (2) of Lemma 5.1, (5.12) and (5.13), and using the decomposition (5.5).

The first immediate consequence of (5.15) is that, if $e \in \mathbb{R}$ and $h_2(\theta_1) > e$, then

$$I(\theta_1, e + i\varepsilon) - I(\theta_1, e - i\varepsilon) \to 0 \text{ when } \varepsilon \to 0^+.$$

This implies that, if $h_2(\theta_1) > e$, one has

$$f(\theta_1, e) = 0.$$

Assume now that $h_2(\theta_1) \leq e$. As $0 \leq h_2(\theta_1)$, $-e \leq h_2(\theta_1) - e \leq 0$. We now need to distinguish different cases according to the dimension d_2. Consider the case

- $d_2 = 1$: by (5.15), as H and G are analytic, one has

  $$\lim_{\varepsilon \to 0^+} I(\theta_1, e + i\varepsilon) = -\frac{\pi}{2} \frac{i}{\sqrt{|h_2(\theta_1) - e|}} H(\theta_1, h_2(\theta_1) - e) + G(\theta_1, h_2(\theta_1) - e).$$

  Using again the fact that H and G are analytic and that $H(\theta_1, 0)$ does not vanish for θ_1 small, we get

  $$\lim_{\varepsilon \to 0^+} \frac{1 + tI(\theta_1, e + i\varepsilon)}{1 + tI(\theta_1, e - i\varepsilon)} = -1 + i \frac{2G(0,0)\sqrt{|h_2(\theta_1) - e|}}{tH(0,0)} + o\left(\sqrt{|h_2(\theta_1) - e|}\right).$$

As $G(0,0)$, $H(0,0)$ and t are also positive, one finally obtains

$$f(\theta_1,e)=\frac{1}{2}\left[1+O\left(\sqrt{|h_2(\theta_1)-e|}\right)\right]\cdot\mathbf{1}_{\{h_2(\theta_1)\le e\}}.$$

- $d_2=2$: in this case, one computes

$$\lim_{\varepsilon\to0^+}I(\theta_1,e+i\varepsilon)=\frac{1}{2}(|\log|h_2(\theta_1)-e||+i\pi)H(\theta_1,h_2(\theta_1)-e)+G(\theta_1,h_2(\theta_1)-e).$$

  Using again the fact that H and G are analytic, we get

$$\lim_{\varepsilon\to0^+}\frac{1+tI(\theta_1,e+i\varepsilon)}{1+tI(\theta_1,e-i\varepsilon)}=\left(1+\frac{2i\pi}{|\log|h_2(\theta_1)-e||}\right)\cdot(1+O[(\log|h_2(\theta_1)-e|)^{-1}]).$$

  So that finally, one has

$$f(\theta_1,e)=\frac{1}{|\log|h_2(\theta_1)-e||}(1+O[(\log|h_2(\theta_1)-e|)^{-1}])\cdot\mathbf{1}_{\{h_2(\theta_1)\le e\}}.$$

- $d_2\ge3$: in this case, one has to distinguish two cases whether $1+tI(0,0)=0$ or not, as well as the case of even and odd dimensions.
  Let us first assume:
  - that $1+tI(0,0)>0$: as H and G are analytic, one has

$$\lim_{\varepsilon\to0^+}I(\theta_1,e+i\varepsilon)=\left(\lim_{\varepsilon\to0^+}S(h_2(\theta_1)-e-i\varepsilon)\right)\cdot H(\theta_1,h_2(\theta_1)-e)+G(\theta_1,h_2(\theta_1)-e).$$

    As G is analytic and as $S(0)=0$, one has $G(\theta_1,h_2(\theta_1))=I(\theta_1,0)$. So, for θ_1 small, we know that $1+tG(\theta_1,0)\neq0$. Here, we used the continuity of G and the fact that $h_2(\theta_1)$ is of size $|\theta_1|^2$ hence small. This gives

$$\lim_{\varepsilon\to0^+}\frac{1+tI(\theta_1,e+i\varepsilon)}{1+tI(\theta_1,e-i\varepsilon)}=1+\frac{t\cdot s(h_2(\theta_1)-e)\cdot H(\theta_1,0)}{1+t\cdot G(\theta_1,0)}\cdot(1+R)$$

    where

$$s(x)=\lim_{\varepsilon\to0^+}[S(x-i\varepsilon)-S(x+i\varepsilon)],R=O((h_2(\theta_1)-e)\cdot|S(h_2(\theta_1)-e)|,(h_2(\theta_1)-e)).$$

    So that finally, for e small, one has

$$f(\theta_1,e)=\frac{t\cdot s(h_2(\theta_1)-e)\cdot H(\theta_1,0)}{1+tG(\theta_1,0)}\mathbf{1}_{\{h_2(\theta_1)\le e\}}(1+R).\tag{5.16}$$

    where

$$s(x)=\frac{1}{2}|x|^{\frac{d_2-2}{2}}\tag{5.17}$$

    and R is given above.

From these asymptotics and from (5.3), integrating f in (5.3), using (5.9) and (5.10), one gets that

- if $d_2=1$:

$$\int_0^E dN_s^t(e)\underset{E\to0^+}{\sim}\frac{\mathrm{Vol}(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{\mathrm{Det}\,(Q_1-RQ_2^{-1}R^*)}}\cdot E^{1+d_1/2}\tag{5.18}$$

- if $d_2=2$:

$$\int_0^E dN_s^t(e)\underset{E\to0^+}{\sim}\frac{2\mathrm{Vol}(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{\mathrm{Det}\,(Q_1-RQ_2^{-1}R^*)}}\frac{E^{1+d_1/2}}{|\log E|}$$

- if $d_2\ge3$ and $1+tI(0,0)>0$:

$$\int_0^E dN_s^t(e)\underset{E\to0^+}{\sim}\frac{c(d_1,d_2)\mathrm{Vol}(\mathbb{S}^{d_2-1})\mathrm{Vol}(\mathbb{S}^{d_1-1})}{d(2\pi)^d\sqrt{\mathrm{Det}\,Q}}\cdot\frac{t}{1+tI(0,0)}\cdot s(E)E^{1+d_1/2}\tag{5.19}$$

where $\mathbb{S}^{d_{1,2}-1}$ are respectively the $d_{1,2}-1$ dimensional unit spheres, and s is given by (5.17). Here, $c(d_1, d_2)$ is the integral

$$c(d_1,d_2) = \int_0^1 r^{d_1-1}(1-r^2)^{(d_2-2)/2}dr. \tag{5.20}$$

5.1.1. *The borderline case.* Though it will not find direct applications in this paper, let us now turn to the case when $d_2 \geq 3$ and $1 + tI(0,0) = 0$. Notice that this assumption implies $t < 0$. When $1 + tI(0,0) = 0$, one has to take a closer look at the vanishing of $1 + tG(\theta_1, 0)$ when $\theta_1 \to 0$. We will now assume that

(H): $I(\theta_1, 0)$ has a local maximum for $\theta_1 = 0$.

Remark 5.1. Notice that this assumption was also necessary for fluctuating edges. Actually, in that setting, we even required that the maximum be non-degenerate if $d_1 \geq 3$. This seems quite natural as the case $1 + tI(0,0) = 0$ is exactly the border line between the fluctuating edges and stable edges.

Let us recall that as above, we need to compute the asymptotic when $e \to 0^+$ of the integral

$$\int_{\mathbb{T}^{d_1}} f(\theta_1, e)d\theta_1 = \int_{\{h_2(\theta_1)\leq e\}} f(\theta_1, e)d\theta_1$$

where f is defined by (5.4). Using (5.9) we can find a analytic change of variable $\theta_1 \mapsto \psi(\theta_1)$ such that $h_2(\psi^{-1}(\theta_1)) = \langle \tilde{Q}_1\theta_1, \theta_1\rangle =: q_2(\theta_1)$ where $\tilde{Q}_1 = Q_1 - R^*Q_2^{-1}R$ (the matrices $Q_{1,2}$ and R are defined in (5.8)) and $\psi(\theta_1) = \theta_1 + O(|\theta_1|^2)$. So, we want to study

$$\int_{\{q_2(\theta_1)\leq e\}} f(\psi(\theta_1), e)|\mathrm{Det}\nabla_{\theta_1}\psi(\theta_1)|d\theta_1$$

Let us perform one more change of variable in the integral above, namely $\theta_1 \leftrightarrow \sqrt{e}\theta_1$; hence, we need to study

$$\int_{\{q_2(\theta_1)\leq 1\}} f(\psi(\sqrt{e}\theta_1), e)|\mathrm{Det}\nabla_{\theta_1}\psi(\sqrt{e}\theta_1)|d\theta_1$$

Notice that, for e small, on $\{\langle \tilde{Q}_1\theta_1, \theta_1\rangle \leq 1\}$, one has

$$|\mathrm{Det}\nabla_{\theta_1}\psi(\sqrt{e}\theta_1)| = 1 + O(\sqrt{e}).$$

We now study $f(\psi(\sqrt{e}\theta_1), e)$ for e small and $\{q_2(\theta_1) \leq 1\}$.
Using the analyticity of G and H, for $\varepsilon > 0$, we start with rewriting (5.15) in the following way

$$\begin{aligned} 1 + tI(\psi(\sqrt{e}\theta_1), e + i\varepsilon) &= 1 + tG(\psi(\sqrt{e}\theta_1), 0) + te\partial_z G(0,0)(q_2(\theta_1) - 1) + \\ &+ tS(e\cdot(q_2(\theta_1)-1) - i\varepsilon))H(\psi(\sqrt{e}\theta_1), 0) + O(\varepsilon + e^2 + |e\cdot S(e)|). \end{aligned} \tag{5.21}$$

Let us now distinguish between the different dimensions, i.e. between the cases $d_2 = 3$, $d_2 = 4$ and $d_2 \geq 5$. Substituting the asymptotics for S given in Lemma 5.1 and using the analyticity of G and H, one obtains the following:

- If $d_2 = 3$: define $F_\pm(\theta_1, e) = \lim_{\varepsilon\to 0^+} 1 + tI(\psi(\sqrt{e}\theta_1), e \pm i\varepsilon)$. For $q_2(\theta_1) < 1$, one has

$$F_\pm(\theta_1, e) = \sqrt{e}\left(\mp it(2\pi)^{-d_2}|1 - q_2(\theta_1)|^{1/2}\mathrm{Det}\,(Q_2)^{-1/2} + t\cdot g(\theta_1) + o(\sqrt{e})\right)$$

  where

$$t\cdot g(\theta_1) = \lim_{e\to 0^+} \frac{1}{\sqrt{e}}[1 + tG(\psi(\sqrt{e}\theta_1), 0)]. \tag{5.22}$$

  This gives, for $q_2(\theta_1) < 1$,

$$f(\psi(\sqrt{e}\theta_1), e) \underset{e\to 0^+}{\sim} \frac{1}{\pi}\mathrm{Arg}(-i(2\pi)^{-d_2}\mathrm{Det}\,(Q_2)^{-1/2}|1 - q_2(\theta_1)|^{1/2} + g(\theta_1)).$$

We notice that this last argument is non positive. As a result we obtain that

$$\int_0^E dN_s^t(e)de \underset{E\to 0^+}{\sim} \frac{\displaystyle\int_{|\theta_1|\leq 1} \mathrm{Arg}(-i|1-\theta_1^2|^{1/2}+\tilde{g}(\theta_1))d\theta_1}{d_1(d_1+2)\pi(2\pi)^{d_1}\sqrt{\mathrm{Det}\,(Q_1-RQ_2^{-1}R^*)}} \cdot E^{1+d_1/2} \tag{5.23}$$

where

$$\tilde{g}(\theta_1) = (2\pi)^{d_2}\sqrt{\mathrm{Det}\,(Q_2)}g((Q_1-RQ_2^{-1}R^*)^{-1/2}\theta_1)$$

and g is defined by (5.22).

Remark 5.2. In some cases, $\tilde{g}$ and g are identically vanishing. This happens for example if h is a "separate variable" function, i.e. if $h(\theta_1,\theta_2) = \tilde{h}_1(\theta_1) + \tilde{h}_2(\theta_2)$. Indeed, in this case, $h_2(\theta_1) = \tilde{h}_1(\theta_1)$ and $I(\theta_1, h_2(\theta_1)) = I(0,0)$, hence, G does not depend on θ_1, i.e. $G(\theta_1, z) = G(z)$.
When $\tilde{g}$ vanishes identically, formula (5.23) becomes (5.18) except for the sign which changes to $-$. The integral $\displaystyle\int_{|\theta_1|\leq 1} \mathrm{Arg}(-it|1-\theta_1^2|^{1/2}+\tilde{g}(\theta_1))d\theta_1$ is negative. Hence, comparing (5.23) to (5.19), we see that, asymptotically when $E \to 0^+$, $\int_0^E dN_s^t(e)de$ is larger when $1+tI(0,0)=0$ than when $1+tI(0,0)>0$. This is explained by the fact that, when $1+tI(0,0)=0$, a zero energy resonance (or eigenvalue if $d_2 \geq 5$) is created. This resonance (eigenvalue) carries more weight. Of course, the same phenomenon happens for the spectral shift function.
To conclude the case $d_2 = 3$, let us notice that we did not use assumption (H).

- If $d_2 = 4$: let us start with computing $\partial_{\theta_1} G(0,0)$. Therefore, we use $G(\theta_1, 0) = I(\theta_1, h_2(\theta_1))$ and compute

$$\begin{aligned}\partial_{\theta_1} G(0,0) = \partial_{\theta_1}[I(\theta_1,h_2(\theta_1))]_{|\theta_1=0} &= -\left(\int_{\mathbb{T}^{d_2}} \frac{\partial_{\theta_1}(h(\theta_1,\theta_2)-h_2(\theta_1))}{(h(\theta_1,\theta_2)-h_2(\theta_1))^2}d\theta_1\right)_{|\theta_1=0} \\ &= -\int_{\mathbb{T}^{d_2}} \frac{\partial_{\theta_1} h(0,\theta_2)}{(h(0,\theta_2))^2}d\theta_1 = 0\end{aligned}$$

as 0 is a local maximum of $I(\theta_1, 0)$. This computation immediately gives that $1+tG(\psi(\sqrt{e}\theta_1),0) = O(e)$. Hence, equation (5.21) gives

$$F_\pm(\theta_1,e) = te(q_2(\theta_1)-1)(\log e + \log|q_2(\theta_1)-1| + h(e)) \cdot \left(1+\frac{\mp i\pi}{\log e + \log|q_2(\theta_1)-1| + h(e)}\right)$$

where $h(e)$ is bounded and does not depend on the sign $\pm$. This gives, for $q_2(\theta_1) < 1$,

$$f(\psi(\sqrt{e}\theta_1),e) \underset{e\to 0^+}{\sim} -\frac{1}{|\log e|}.$$

Integrating over θ_1 and e, we obtain

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} -\frac{2\mathrm{Vol}(\mathbb{S}^{d_1-1})}{d_1(d_1+2)(2\pi)^{d_1}\sqrt{\mathrm{Det}\,(Q_1-RQ_2^{-1}R^*)}} \frac{E^{1+d_1/2}}{|\log E|}.$$

- If $1+tI(0,0)=0$ and $d_2 \geq 5$: we now compute $\partial_{\theta_1}^2 Q(0,0)$. Therefore, we continue the computation done above to obtain

$$\begin{aligned}\partial_{\theta_1}^2 G(0,0) &= -\partial_{\theta_1}\left(\int_{\mathbb{T}^{d_2}} \frac{\partial_{\theta_1}(h(\theta_1,\theta_2)-h_2(\theta_1))}{(h(\theta_1,\theta_2)-h_2(\theta_1))^2}d\theta_1\right)_{|\theta_1=0} \\ &= -\left(\int_{\mathbb{T}^{d_2}} \frac{\partial_{\theta_1}^2(h(\theta_1,\theta_2)-h_2(\theta_1))}{(h(\theta_1,\theta_2)-h_2(\theta_1))^2}d\theta_1\right)_{|\theta_1=0} + 2\left(\int_{\mathbb{T}^{d_2}} \frac{[\partial_{\theta_1}(h(\theta_1,\theta_2)-h_2(\theta_1))]^2}{(h(\theta_1,\theta_2)-h_2(\theta_1))^3}d\theta_1\right)_{|\theta_1=0} \qquad (5.24) \\ &= -\int_{\mathbb{T}^{d_2}} \frac{\partial_{\theta_1}^2 h(0,\theta_2)}{(h(0,\theta_2))^2}d\theta_1 + 2\int_{\mathbb{T}^{d_2}} \frac{[\partial_{\theta_1} h(0,\theta_2)]^2}{(h(0,\theta_2))^3}d\theta_1 + \left(\int_{\mathbb{T}^{d_2}} \frac{1}{(h(0,\theta_2))^2}d\theta_1\right) Q_2\end{aligned}$$

where Q_2 is defined in (5.8). On the other hand, one has

$$\partial_z G(0,0) = -\partial_z I(0,z)_{|z=0} = -J \quad \text{where} \quad J := \int_{\mathbb{T}^{d_2}} \frac{1}{(h(0,\theta_2))^2} d\theta_1.$$

Plugging this and (5.24) into (5.21), we obtain

$$1 + tI(\psi(\sqrt{e}\theta_1), e \pm i\varepsilon) = -teJ + o(e) + tS(e\cdot(q_2(\theta_1)-1) \mp i\varepsilon)(H(0,0)+o(1)).$$

where $o(e)$ does not depend of $\pm$. This gives, for $q_2(\theta_1) < 1$,

$$f(\psi(\sqrt{e}\theta_1), e) \underset{e\to 0^+}{\sim} -\frac{s(e\cdot(q_2(\theta_1)-1))}{J}.$$

Integrating over θ_1 and e, we obtain

$$\int_0^E dN_s^t(e) \underset{E\to 0^+}{\sim} \frac{c(d_1,d_2)\mathrm{Vol}(\mathbb{S}^{d_2-1})\mathrm{Vol}(\mathbb{S}^{d_1-1})}{d(2\pi)^d\sqrt{\mathrm{Det}\, Q}} \cdot \frac{-1}{J} \cdot s(E)E^{d_1/2}$$

where $c(d_1,d_2)$ is defined in (5.20).

6. Appendix

Pick $E < -d$. We now prove that, for h taken as in Remark 1.1, the function $\tilde{h}$ defined in (1.7) is not constant. For the purpose of this argument, we write $\theta_1 = (\theta^1, \cdots, \theta^{d_1})$. To check that $\tilde{h}$ is not constant, by (1.2) and (1.7), it suffices to check that the function $\theta_1 \mapsto I(\theta_1, E)$ is not constant, hence, that the function $\theta^1 \mapsto J(\theta^1)$ defined by

$$J(\theta^1) = \frac{1}{(2\pi)^{d_1-1}} \int_{[0,2\pi]^{d_1-1}} I(\theta^1,\theta^2,\cdots,\theta^{d_1},E)d\theta^2\cdots d\theta^{d_1} = \frac{1}{(2\pi)^{d-1}} \int_{[0,2\pi]^{d-1}} \frac{1}{h(\theta^1,\theta')-E} d\theta' \tag{6.1}$$

is not constant. We used the notation $\theta = (\theta_1,\theta_2) = (\theta^1,\theta')$.
Recall from Remark 1.1 that $h(\theta) = h_0(G'\cdot\theta)$ where $G' \in \mathrm{GSL}(\mathbb{Z})$ and h_0 is defined in (1.1). So, the n-th Fourier coefficient of J is given by

$$\begin{aligned}\hat{J}_n &= \frac{1}{(2\pi)^d} \int_{[0,2\pi]} \int_{[0,2\pi]^{d-1}} \frac{e^{in\theta^1}}{h(\theta^1,\theta')-E} d\theta' d\theta^1 = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{e^{in\theta^1}}{h_0(G'\cdot\theta)-E} d\theta \\ &= \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{e^{in(G'^{-1}\cdot\theta)^1}}{h_0(\theta)-E} d\theta = \frac{e^{in(G'^{-1}\cdot\theta_\pi)^1}}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{e^{in(G'^{-1}\cdot\theta)^1}}{-h_0(\theta)-E} d\theta\end{aligned}$$

where $(G'^{-1}\cdot\theta)^1$ denotes the first coordinate of the vector $G'^{-1}\cdot\theta$, and θ_π, the vector $(\pi,\ldots,\pi)$ in $\mathbb{R}^d$. So to prove that $\hat{J}_n$ does not vanish for any n which implies that J is not constant, it suffices to prove that the Fourier coefficients of $(h_0(\theta)-E)^{-1}$ do not vanish. This is a consequence of the Neuman expansion

$$\frac{1}{-h_0(\theta)-E} = \frac{-1}{E} \sum_{k\geq 0} \left(\frac{h_0(\theta)}{-E}\right)^k.$$

Indeed, the n-th Fourier coefficient in each of the terms of order k larger than n in this series is positive : it is easily seen as $-E > 0$ and the multiplication operator $(h_0)^n$ is unitarily equivalent through Fourier transformation to $(-\frac{1}{2}\Delta)^n$; so the Fourier coefficients of $(h_0)^n$ are the entries of the zeroth row of the matrix $(-\frac{1}{2}\Delta)^n$ and, the n first super- and sub-diagonals of this convolution matrix are positive.

References

[1] M. Sh. Birman and D. R. Yafaev. The spectral shift function. The papers of M. G. Kreĭn and their further development. *Algebra i Analiz*, 4(5):1–44, 1992.
[2] A. Chahrour. Densité intégrée d'états surfaciques et fonction généralisée de déplacement spectral pour un opérateur de Schrödinger surfacique ergodique. *Helv. Phys. Acta*, 72(2):93–122, 1999.
[3] A. Chahrour and J. Sahbani. On the spectral and scattering theory of the Schrödinger operator with surface potential. *Rev. Math. Phys.*, 12(4):561–573, 2000.
[4] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators*. Springer Verlag, Berlin, 1987.
[5] A. Dembo and O. Zeitouni. *Large deviation techniques and applications*. Jones and Bartlett Publishers, Boston, 1992.

[6] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*. Number 268 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1999.

[7] R. Durrett. *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition, 1996.

[8] H. Englisch, W. Kirsch, M. Schröder, and B. Simon. Density of surface states in discrete models. *Phys. Rev. Lett.*, 61(11):1261–1262, 1988.

[9] H. Englisch, W. Kirsch, M. Schröder, and B. Simon. Random Hamiltonians ergodic in all but one direction. *Comm. Math. Phys.*, 128(3):613–625, 1990.

[10] B. Helffer and J. Sjöstrand. On diamagnetism and the De Haas-Van Alphen effect. *Annales de l'Institut Henri Poincaré, série Physique Théorique*, 52:303–375, 1990.

[11] L. Hörmander. *The Analysis of Linear Partial Differential Operators*. Springer Verlag, Heidelberg, 1983.

[12] L. Hörmander. *The analysis of linear partial differential equations. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften*. Springer Verlag, 1990.

[13] V. Jakšić and Y. Last. Corrugated surfaces and a.c. spectrum. *Rev. Math. Phys.*, 12(11):1465–1503, 2000.

[14] W. Kirsch. Random Schrödinger operators. In A. Jensen H. Holden, editor, *Schrödinger Operators*, number 345 in Lecture Notes in Physics, Berlin, 1989. Springer Verlag. Proceedings, Sonderborg, Denmark 1988.

[15] F. Klopp. Resonances for perturbations of a semi-classical periodic Schrödinger operator. *Arkiv för Matematik*, 32:323–371, 1994.

[16] F. Klopp. Band edge behaviour for the integrated density of states of random Jacobi matrices in dimension 1. *Journal of Statistical Physics*, 90(3-4):927–947, 1998.

[17] F. Klopp. Internal Lifshits tails for random perturbations of periodic Schrödinger operators. *Duke Math. J.*, 98(2):335–396, 1999.

[18] F. Klopp. Weak disorder localization and Lifshitz tails. *Comm. Math. Phys.*, 232:125–155, 2002.

[19] F. Klopp and T. Wolff. Lifshitz tails for 2-dimensional random Schrödinger operators. *J. Anal. Math.*, 88:63–147, 2002. Dedicated to the memory of Tom Wolff.

[20] V. Kostrykin and R. Schrader. The density of states and the spectral shift density of random Schrödinger operators. *Rev. Math. Phys.*, 12(6):807–847, 2000.

[21] V. Kostrykin and R. Schrader. Regularity of the surface density of states. *J. Funct. Anal.*, 187(1):227–246, 2001.

[22] J. N. Mather. On Nirenberg's proof of Malgrange's preparation theorem. In *Proceedings of Liverpool Singularities-Symposium I*, number 192 in Lecture Notes in Mathematics, Berlin, 1971. Springer Verlag.

[23] L. Pastur and A. Figotin. *Spectra of random and almost-periodic operators*, volume 297 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[24] B. Simon. Spectral analysis of rank one perturbations and applications. In *Mathematical quantum theory. II. Schrödinger operators (Vancouver, BC, 1993)*, pages 109–149. Amer. Math. Soc., Providence, RI, 1995.

[25] D. R. Yafaev. *Mathematical Scattering Theory*, volume 105 of *Transaction of Mathematical Monographs*. American Mathematical Society, Providence, R.I, 1992.

(Werner Kirsch) Fakultät für Mathematik and SFB-TR 12, Ruhr Universität Bochum, D-44780 Bochum, Deutschland

E-mail address: werner@mathphys.ruhr-uni-bochum.de

(Frédéric Klopp) LAGA, U.M.R. 7539 C.N.R.S, Institut Galilée, Université de Paris-Nord, 99 venue J.-B. Clément, F-93430 Villetaneuse, France

E-mail address: klopp@math.univ-paris13.fr