PERIODIC ORBITS AND SCALING LAWS FOR A DRIVEN DAMPED QUARTIC OSCILLATOR

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ABSTRACT. In this paper we investigate the conditions under which periodic solutions of the nonlinear oscillator $\ddot{x} + x^3 = 0$ persist when the differential equation is perturbed so as to become $\ddot{x} + x^3 + \varepsilon x^3 \cos t + \gamma \dot{x} = 0$. We conjecture that for any periodic orbit, characterized by its frequency ω, there exists a threshold for the damping coefficient γ, above which the orbit disappears, and that this threshold is infinitesimal in the perturbation parameter, with integer order depending on the frequency ω. Some rigorous analytical results toward the proof of these conjectures are provided. Moreover the relative size and shape of the basins of attraction of the existing stable periodic orbits are investigated numerically, giving further support to the validity of the conjectures.

1. INTRODUCTION

We study the existence of periodic solutions of the ordinary differential equation

$$\ddot{x} + x^3 + \varepsilon f(t) x^3 + \gamma \dot{x} = 0, \qquad f(t) = \cos t, \tag{1.1}$$

where $\gamma > 0$ is the friction coefficient and ε is a real parameter characterizing the strength of the external driving force; both are assumed to be small.

As the driving function $f(t)$ has fixed period 2π this means that the period $2\pi/\omega$ of the motion has to be commensurable with 2π, that is with $\omega = p/q$, where p and q are relatively prime integers. We refer to such a case as a $p\!:\!q$ resonance. Problems of this type appear in Celestial Mechanics when studying the problem of resonance locking between the revolution and rotation periods of satellites via the mechanism of spin-orbit coupling [7, 8, 5].

It is clear that a given periodic orbit can possibly exist only if the friction coefficient γ is not too large: in fact, if γ is large enough it is easy to show that any trajectory is attracted to the origin [2].

Numerically one finds that for values of γ not too small only a few periodic orbits appear, and they are asymptotically stable. Moreover the union of (the closure of) the corresponding basins of attraction seems to fill the entire phase space.

By decreasing the value of γ new periodic orbits can arise. They are less relevant than the old ones, in the sense that their basins of attraction are very small compared with those of the orbits already appearing for larger values of γ. In practice it is very difficult to see the orbits arising for very small values of γ; as we shall see the corresponding basins of attraction are very thin and the convergence to the orbit is obviously very slow.

We therefore make and investigate the following conjectures.

(1) *A periodic orbit of given rational frequency ω, if it exists at all, exists only if γ is less than a suitable threshold $\gamma_0(\omega, \varepsilon)$.*

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(2) *The threshold* $\gamma_0(\omega,\varepsilon)$ *is infinitesimal in* ε.
(3) *The order of magnitude in* ε *of* $\gamma_0(\omega,\varepsilon)$ *is given by an integer exponent* $m(\omega)$*, that is* $\gamma_0(\omega,\varepsilon) = O(\varepsilon^{m(\omega)})$.

Therefore the thresholds $\gamma_0(\omega,\varepsilon)$ are characterized by a scaling law in terms of the perturbation parameter ε, with a scaling exponent which depends on the frequency ω.

In this paper we provide both rigorous and numerical results which support these conjectures. For fixed ε and γ we can write

$$\gamma = \varepsilon^m C_m \tag{1.2}$$

(or more generally $\varepsilon^m C_m$ + higher order terms, which are irrelevant anyway), and we find that periodic solutions with given rational frequency ω either are not possible at all or appear only if C_m is less than some threshold value depending on ω, say $C_m < C_{m,0}(\omega)$.

Inspired by the aforementioned conjectures and numerical results we study analytically the system (1.1) by writing γ as in (1.2), so that, for fixed m and γ given by (1.2), we can consider ε and C_m as independent parameters. We shall discuss in detail the cases $m = 1$ and $m = 2$; the other cases could be treated in the same way, and they add nothing new from a qualitative point of view.

For fixed C_m we prove that only a finite number of periodic orbits are present, as an upper bound on the values of p and q arises. Once a periodic orbit with frequency ω has appeared, it remains for all values of C_m less than the threshold value $C_{m,0}(\omega)$. In particular this will yield that by increasing the value of m in (1.2) the periodic orbits found for smaller m still survive, while new ones appear. We call the resonances appearing for $m = 1$ primary resonances, those appearing for $m = 2$ secondary resonances, and so on.

Note that in this way we more or less reverse the point of view explained initially: in fact, given m, we determine the periodic orbits which are possible using the parameterization (1.2) of the friction coefficient, and the lowest such m for a given frequency $\omega = p/q$ is therefore $m(\omega)$.

2. Perturbative theory of periodic solutions for primary resonances

We start by considering $\ddot{x} + x^3 + \varepsilon f(t)x^3 + \varepsilon C\dot{x} = 0$, with $f(t) = \cos t$, for small $\varepsilon \in \mathbb{R}$: this corresponds to $m = 1$ in (1.2). Here C is a real parameter; we shall be interested in the case $\varepsilon C > 0$, so that $\varepsilon C\dot{x}$ can be interpreted as a dissipative term.

For $\varepsilon = 0$ the system reduces to

$$\ddot{x} + x^3 = 0, \tag{2.1}$$

and the solution $x^{(0)}(t)$ can be written in terms of Jacobi elliptic functions. If we set $\alpha = (4E)^{1/4}$, where the energy

$$E = H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4, \tag{2.2}$$

is a constant of motion for (2.1), we can write the solution as $x^{(0)}(t) = \alpha\,\mathrm{cn}(\alpha(t-t_0))$, where $\mathrm{cn}\, t \equiv \mathrm{cn}(t, 1/\sqrt{2})$ is the cosine-amplitude function with elliptic modulus $k = 1/\sqrt{2}$, and $-\alpha t_0$ is the initial phase. In Appendix A we recall some basic properties of elliptic functions.

We can also write $f(t+t_0)$ instead of $f(t)$ in (1.1), so obtaining for $\gamma = \varepsilon C$

$$\ddot{x} + x^3 + \varepsilon f(t+t_0)x^3 + \varepsilon C\dot{x} = 0, \qquad f(t) = \cos t. \tag{2.3}$$

The advantage of introducing t_0 is that we can suitably choose t_0 in order to fix as zero the phase of the solution, which becomes

$$x^{(0)}(t) = \alpha \,\mathrm{cn}(\alpha t). \tag{2.4}$$

In other words, as in principle we have an infinite number of unperturbed solutions of energy E, all differing just by a phase, we prefer to fix this phase at zero by using implicitly an initial condition $x^{(0)}(0) = \alpha$, and move the freedom of choice in the initial condition to the phase of the driving force (the only one which is time-dependent). In this way no generality is lost in fixing the initial condition as in (2.4).

If we denote with $K(k)$ the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \tag{2.5}$$

which for $k = 1/\sqrt{2}$ gives $K(1/\sqrt{2}) = [\Gamma(1/4)]^2/(4\sqrt{\pi}) \approx 1.85407$, we have that the solution $x^{(0)}(t)$ is periodic with period $T_0 = 4K/\alpha$, with $K = K(1/\sqrt{2})$. Its derivative is given by

$$y^{(0)}(t) \equiv \dot{x}^{(0)}(t) = -\alpha^2 \,\mathrm{sn}(\alpha t)\, \mathrm{dn}(\alpha t), \tag{2.6}$$

with $\mathrm{sn}\, t \equiv \mathrm{sn}(t, 1/\sqrt{2})$ and $\mathrm{dn}\, t \equiv \mathrm{dn}(t, 1/\sqrt{2})$, and the constant α is determined by the initial conditions $(\bar{x}^{(0)}, \bar{y}^{(0)})$ as

$$\bar{x}^{(0)} \equiv x^{(0)}(0) = \alpha, \qquad \bar{y}^{(0)} \equiv y^{(0)}(0) = 0. \tag{2.7}$$

It is more convenient to work with action-angle variables. A straightforward calculation gives (see Appendix B)

$$\begin{cases} x = (3I)^{1/3} \,\mathrm{cn}\,\varphi, \\ y = -(3I)^{2/3} \,\mathrm{sn}\,\varphi \,\mathrm{dn}\,\varphi, \end{cases} \tag{2.8}$$

where $(\varphi, I) \in \mathbb{R}/4K\mathbb{Z} \times \mathbb{R}_+$ are conjugate variables. Then (2.2) becomes

$$E = \frac{1}{4}(3I)^{4/3} \equiv \mathcal{H}_0(I), \tag{2.9}$$

which yields $3I = (4E)^{3/4}$, and the equations of motion (2.1) can be written as (see Appendix B)

$$\begin{cases} \dot{\varphi} = (3I)^{1/3} + \varepsilon\,(3I)^{1/3} f(t+t_0)\, \mathrm{cn}^4\varphi - \varepsilon\, C\, \mathrm{cn}\,\varphi\, \mathrm{sn}\,\varphi\, \mathrm{dn}\,\varphi, \\ \dot{I} = \varepsilon\,(3I)^{4/3} f(t+t_0)\, \mathrm{cn}^3\varphi\, \mathrm{sn}\,\varphi\, \mathrm{dn}\,\varphi - \varepsilon\, C\,(3I)\, \mathrm{sn}^2\varphi\, \mathrm{dn}^2\varphi. \end{cases} \tag{2.10}$$

In terms of the variables (φ, I) the unperturbed solution ($\varepsilon = 0$) becomes $(\varphi^{(0)}(t), I^{(0)}(t)) = (\alpha t, I^{(0)}) = (\alpha t, \alpha^3/3)$, so that one has $\alpha = (3I^{(0)})^{1/3}$.

The Wronskian matrix $W(t)$ is a solution of the unperturbed linear system

$$\dot{W} = M(t)\, W, \qquad M(t) = \begin{pmatrix} 0 & (3I^{(0)})^{-2/3} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^{-2} \\ 0 & 0 \end{pmatrix}, \tag{2.11}$$

so that the matrix $W(t)$ can be taken as

$$W(t) = \begin{pmatrix} 1 & \alpha^{-2} t \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

and one can check that $W(0) = \mathbb{1}$ and $\det W(t) \equiv 1$.

We look for solutions $z(t) = (\varphi(t), I(t))$ of the form

$$z(t) = \sum_{n=0}^{\infty} \varepsilon^n z^{(n)}(t) = \sum_{n=0}^{\infty} \varepsilon^n \sum_{\boldsymbol{\nu} \in \mathbb{Z}^2} e^{i\boldsymbol{\omega}\cdot\boldsymbol{\nu} t} z^{(n)}_{\boldsymbol{\nu}}, \tag{2.13}$$

where $\boldsymbol{\omega} = (\omega, 1)$ and $z^{(n)}$ is given by

$$z^{(n)}(t) = \begin{pmatrix} \varphi^{(n)}(t) \\ I^{(n)}(t) \end{pmatrix} = W(t) \begin{pmatrix} \bar\varphi^{(n)} \\ \bar I^{(n)} \end{pmatrix} + W(t) \int_0^t \mathrm{d}\tau \, W^{-1}(\tau) \begin{pmatrix} F_1^{(n)}(\tau) \\ F_2^{(n)}(\tau) \end{pmatrix}, \tag{2.14}$$

with

$$\begin{aligned} F_1^{(n)}(t) &= \Big[(3I(t))^{1/3} f(t+t_0) \, \mathrm{cn}^4 \varphi(t) - C \, \mathrm{cn}\, \varphi(t) \, \mathrm{sn}\, \varphi(t) \, \mathrm{dn}\, \varphi(t) \Big]^{(n-1)}, \\ F_2^{(n)}(t) &= \Big[(3I(t))^{4/3} f(t+t_0) \, \mathrm{cn}^3 \varphi(t) \, \mathrm{sn}\, \varphi(t) \, \mathrm{dn}\, \varphi(t) - C \, (3I(t)) \, \mathrm{sn}^2 \varphi(t) \, \mathrm{dn}^2 \varphi(t) \Big]^{(n-1)}, \end{aligned} \tag{2.15}$$

where with $[\ldots]^{(n)}$ we denote the terms of order n in ε within $[\ldots]$. Of course if $\omega = p/q \in \mathbb{Q}$ we obtain a periodic solution.

If $\varepsilon = 0$ one has $\omega = 2\pi/T_0 = 2\pi\alpha/4K$, so that one has a periodic solution with period commensurate with 2π if $2\pi\alpha/4K \in \mathbb{Q}$, which imposes a condition on the energy (2.9); the value of t_0 can be arbitrarily chosen.

If $\varepsilon \neq 0$ we study the conditions under which the unperturbed free solution is preserved, i.e. we look for a solution of the form (2.13), with

$$\omega = \frac{2\pi\alpha}{4K} = \frac{p}{q} \in \mathbb{Q}, \tag{2.16}$$

which reduces to the unperturbed case for $\varepsilon = 0$, only for a suitable choice of the initial time t_0. Note that if ω satisfies (2.16), i.e. if $\alpha = \alpha(p,q) \equiv 4Kp/2\pi q$, then the solution $z(t)$ is periodic with period $T = 2\pi q$.

In (2.14) we have

$$W(t) W^{-1}(\tau) \begin{pmatrix} F_1^{(n)}(\tau) \\ F_2^{(n)}(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \alpha^{-2}(t-\tau) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_1^{(n)}(\tau) \\ F_2^{(n)}(\tau) \end{pmatrix} = \begin{pmatrix} F_1^{(n)}(\tau) + \alpha^{-2}(t-\tau) F_2^{(n)}(\tau) \\ F_2^{(n)}(\tau) \end{pmatrix}, \tag{2.17}$$

so that, by taking into account that one has

$$\int_0^t \mathrm{d}\tau \, (t-\tau) \, F_2^{(n)}(\tau) = \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \, F_2^{(n)}(\tau'), \tag{2.18}$$

we can write the first component $\varphi^{(n)}(t)$ of $z^{(n)}(t)$ in (2.14) as

$$\varphi^{(n)}(t) = \bar\varphi^{(n)} + \alpha^{-2} t \, \bar I^{(n)} + \int_0^t \mathrm{d}\tau \, F_1^{(n)}(\tau) + \alpha^{-2} \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \, F_2^{(n)}(\tau'), \tag{2.19}$$

while the second component $I^{(n)}(t)$ is given by

$$I^{(n)}(t) = \bar I^{(n)} + \int_0^t \mathrm{d}\tau \, F_2^{(n)}(\tau). \tag{2.20}$$

Given a periodic function g, let us denote with $g_0 = \langle g \rangle$ the average of g and with $\tilde g$ the zero average function $g - \langle g \rangle$. Suppose that one has

$$\left\langle F_2^{(n)} \right\rangle = \frac{1}{T} \int_0^T \mathrm{d}t \, F_2^{(n)}(t) = 0, \tag{2.21}$$

so that we can write

$$\begin{aligned} \mathcal{F}_1^{(n)}(t) &= \int_0^t \mathrm{d}\tau \, F_1^{(n)}(\tau) = \left\langle F_1^{(n)} \right\rangle t + \int_0^t \mathrm{d}\tau \, \tilde F_1^{(n)}(\tau), \\ \mathcal{F}_2^{(n)}(t) &= \int_0^t \mathrm{d}\tau \, F_2^{(n)}(\tau) = \int_0^t \mathrm{d}\tau \, \tilde F_2^{(n)}(\tau). \end{aligned} \tag{2.22}$$

Then in (2.20) we can write

$$I^{(n)}(t) = \bar{I}^{(n)} + \mathcal{F}_2^{(n)}(t) = \bar{I}^{(n)} + \int_0^t \mathrm{d}\tau \tilde{F}_2^{(n)}(\tau), \tag{2.23}$$

while in (2.19) we have

$$\varphi^{(n)}(t) = \bar{\varphi}^{(n)} + \alpha^{-2} t\, \bar{I}^{(n)} + \left\langle F_1^{(n)} \right\rangle t + \int_0^t \mathrm{d}\tau\, \tilde{F}_1^{(n)}(\tau) + \alpha^{-2} \left\langle \mathcal{F}_2^{(n)} \right\rangle t + \alpha^{-2} \int_0^t \mathrm{d}\tau\, \tilde{\mathcal{F}}_2^{(n)}(\tau), \tag{2.24}$$

where all terms which are not linear in time are periodic.

Then, by choosing the initial conditions $\bar{I}^{(n)}$ such that one has

$$\bar{I}^{(n)} = -\left(\alpha^2 \left\langle F_1^{(n)} \right\rangle + \left\langle \mathcal{F}_2^{(n)} \right\rangle \right), \tag{2.25}$$

we obtain

$$\begin{cases} \varphi^{(n)}(t) = \bar{\varphi}^{(n)} + \displaystyle\int_0^t \mathrm{d}\tau\, \tilde{F}_1^{(n)}(\tau) + \alpha^{-2} \int_0^t \mathrm{d}\tau\, \tilde{\mathcal{F}}_2^{(n)}(\tau), \\ I^{(n)}(t) = \bar{I}^{(n)} + \displaystyle\int_0^t \mathrm{d}\tau \tilde{F}_2^{(n)}(\tau), \end{cases} \tag{2.26}$$

so that $z^{(n)}(t)$ is a periodic function with period T.

So we are left with the problem of proving (2.21). We shall see that this will require fixing also the initial phase t_0 and the initial conditions $\bar{\varphi}^{(n)}$ (which are corrections to t_0).

We shall prove first that the series expansion (2.13) is formally defined, that is that the coefficients $z^{(n)}(t)$ are well defined to all perturbation orders. Then we shall show that the coefficients admit a bound $|z^{(n)}(t)| < Z^n$ for some constant Z, so that, by taking ε and C small enough, the series (2.13) converges to a $2\pi/\omega$-periodic function, analytic in t, ε and C.

3. Existence of periodic solutions for primary resonances

The validity of assumption (2.21) is guaranteed by the following result.

Lemma. *Consider the formal series* (2.13)*. If $p/q = 1/2n$, $n \in \mathbb{N}$, and C is small enough, it is possible to fix the initial conditions $(\bar{\varphi}^{(n)}, \bar{I}^{(n)})$ in such a way that* (2.21) *holds for all $n \geq 1$. If $p/q \neq 1/2n$ for all n then* (2.21) *can be satisfied only for $C = 0$.*

Proof. For $n = 1$ one has

$$\begin{aligned} F_2^{(1)}(t) &= \left(3I^{(0)}\right)^{4/3} f(t+t_0)\, \mathrm{sn}\, \varphi^{(0)}(t)\, \mathrm{dn}\, \varphi^{(0)}(t)\, \mathrm{cn}^3\, \varphi(t) - C \left(3I^{(0)}(t)\right) \mathrm{sn}^2\, \varphi^{(0)}(t)\, \mathrm{dn}^2\, \varphi^{(0)}(t) \\ &= \alpha^4 f(t+t_0)\, \mathrm{sn}(\alpha t)\, \mathrm{dn}(\alpha t)\, \mathrm{cn}^3(\alpha t) - C\alpha^3\, \mathrm{sn}^2(\alpha t)\, \mathrm{dn}^2(\alpha t) \end{aligned} \tag{3.1}$$

Define

$$\Delta = \frac{1}{T_0} \int_0^{T_0} \mathrm{d}t\, \mathrm{sn}^2(\alpha t)\, \mathrm{dn}^2(\alpha t) = \frac{1}{4K} \int_0^{4K} \mathrm{d}t\, \mathrm{sn}^2\, t\, \mathrm{dn}^2\, t = \frac{1}{3}, \tag{3.2}$$

(see Appendix C) and

$$\begin{aligned} \Gamma_1(t_0; p, q) &= \frac{1}{T} \int_0^T \mathrm{d}t\, \mathrm{sn}(\alpha t)\, \mathrm{dn}(\alpha t)\, \mathrm{cn}^3(\alpha t)\, f((t+t_0) \\ &= \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \mathrm{sn}\, t\, \mathrm{dn}\, t\, \mathrm{cn}^3\, t\, f((t/\alpha) + t_0) \\ &= \cos t_0 \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \mathrm{sn}\, t\, \mathrm{dn}\, t\, \mathrm{cn}^3\, t\, \cos(t/\alpha) - \sin t_0 \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \mathrm{sn}\, t\, \mathrm{dn}\, t\, \mathrm{cn}^3\, t\, \sin(t/\alpha) \end{aligned} \tag{3.3}$$

$$= \quad -\sin t_0 \frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \; \mathrm{cn}^3\, t \; \sin(t/\alpha),$$

(as the integral which multiplies $\cos t_0$ is 0 because of parity) which we rewrite as

$$\Gamma_1(t_0;p,q) = -\sin t_0\, G_1(p,q), \tag{3.4}$$

where

$$G_1(p,q) \equiv \frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \; \mathrm{cn}^3\, t \; \sin(t/\alpha). \tag{3.5}$$

Then we obtain that, by choosing (if possible) t_0 such that

$$C = \mathcal{G}_1(t_0;p,q) \equiv -\alpha \sin t_0 \frac{G_1(p,q)}{\Delta} \equiv -\sin t_0 \frac{4K}{2\pi\Delta}\left(\frac{p}{q} G_1(p,q)\right), \tag{3.6}$$

one has $\left\langle F_2^{(1)} \right\rangle = 0$.

The function $G_1(p,q)$ is identically vanishing for odd q, while for even q one has $G_1(p,q) = 0$ for all $p \neq 1$ and $G_1(1,q) \neq 0$; see Appendix B for a proof of this assertion. Moreover the function $G_1(p,q)$ is decreasing in q, so that, for a fixed value of C, there will be $q_0 = q_0(C)$ such that (3.5) can be satisfied only for $q \le q_0$: in other words only a finite number of periodic orbits will exist. Again we refer to Appendix B for details.

A list of values of nontrivial values for $G_1(p,q)$ up to $q = 10$ is given in table 1.

Table 1: Values of $G_1(1,q)$ for $q = 2,4,6,8,10$. All the other values of $G_1(p,q)$, $q \le 10$, are vanishing. The corresponding threshold values $C_0(p/q)$ for C are computed according to (3.8).

q	$G_1(1,q)$	$\alpha(1/q)$	$C_0(1/q)$
2	0.100773	0.590170	0.178442
4	0.069555	0.295085	0.061574
6	0.015217	0.196723	0.008980
8	0.002078	0.147543	0.000920
10	0.000220	0.118034	0.000078

Note that the existence of a value t_0 satisfying (3.6) is possible only if

$$\min_{t_0\in[0,2\pi]} \mathcal{G}_1(t_0;p,q) \le C \le \max_{t_0\in[0,2\pi]} \mathcal{G}_1(t_0;p,q), \tag{3.7}$$

that is only if

$$|C| \le C_0(p/q) \equiv \frac{4K}{2\pi\Delta}\left(\frac{p}{q} G_1(p,q)\right) \approx 3.54102\left(\frac{p}{q} G_1(p,q)\right), \tag{3.8}$$

which represents a smallness condition on C. See table 1 for some *thresholds values* $C_0(p/q)$.

Once t_0 has been set according to (3.5) one has to fix $\bar{I}^{(1)}$ as prescribed by (2.25) for $n = 1$.

To obtain (2.22) for $n \ge 2$ one has to fix the initial conditions $\bar{\varphi}^{(n)}$ for $n \ge 1$: notice that in order to eliminate the terms diverging in time we had to fix only the initial conditions $\bar{I}^{(n)}$.

For all $n \geq 2$ one has

$$\begin{aligned} F_2^{(n)}(t) = & \left(3I^{(0)}\right)^{4/3} f(t+t_0) \frac{\partial}{\partial\varphi} \left(\text{sn } \varphi \text{ dn } \varphi \text{ cn}^3 \varphi\right)\Big|_{\varphi=\alpha t} \bar{\varphi}^{(n-1)} \\ & - C \left(3I^{(0)}\right) \frac{\partial}{\partial\varphi} \left(\text{sn}^2 \varphi \text{ dn}^2 \varphi\right)\Big|_{\varphi=\alpha t} \bar{\varphi}^{(n-1)} + R_2^{(n)}(t), \end{aligned} \tag{3.9}$$

where $R_2^{(n)}(t)$ is a suitable function which does not depend on $\bar{\varphi}^{(n-1)}$.

Therefore one has $\left\langle F_2^{(n)} \right\rangle = 0$ if and only if one has

$$\begin{aligned} \left\langle R_2^{(n)} \right\rangle = \alpha^3 & \left(\frac{\alpha}{T} \int_0^T \mathrm{d}t \, f(t+t_0) \frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left(\text{sn}\,(\alpha t) \text{ dn}(\alpha t) \text{ cn}^3(\alpha t)\right) \right. \\ & \left. - \frac{C}{T} \int_0^T \mathrm{d}t \, \frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left(\text{sn}^2(\alpha t) \text{ dn}^2(\alpha t)\right) \right) \bar{\varphi}^{(n-1)}. \end{aligned} \tag{3.10}$$

An easy computation shows that one has

$$\left\langle R_2^{(n)} \right\rangle = \mathcal{M}_1(t_0; p, q) \, \bar{\varphi}^{(n-1)}, \qquad \mathcal{M}_1(t_0; p, q) = \alpha^3 \cos t_0 \, G_1(p, q), \tag{3.11}$$

so that $\mathcal{M}_1(t_0; p, q)$ is nonvanishing for t_0 such that (3.5) is satisfied; for a proof of (3.11) see Appendix C.

Therefore we can fix the initial conditions $\bar{\varphi}^{(n)}$ in such a way that one has $\langle F_2^{(n)} \rangle$ to all orders. This completes the proof of the lemma. □

The analysis performed so far shows that the coefficients $z^{(n)}(t)$ of the series expansions in (2.13) are well defined to all orders. To complete the proof of the theorem one has still to show that the series expansions converge. This is assured by the following result.

Theorem 1. *Fix $\omega = p/q = 1/2n$, $n \in \mathbb{N}$. Then there exists $C_0 = C_0(1/q)$, decreasing to zero in q, and, for all $|C| < C_0$, a value $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ the system (2.3) admits a $2\pi/\omega$-periodic solution $x_1(t, \varepsilon, C)$ analytic in (t, ε, C). The analyticity domain in (ε, C) contains the region*

$$\left\{ (\varepsilon, C) \in \mathbb{R}^2 \, : \, \left(\frac{\varepsilon}{a}\right)^2 + \left(\frac{C}{C_0}\right)^2 < 1 \text{ and } |\varepsilon| < b \right\}, \tag{3.12}$$

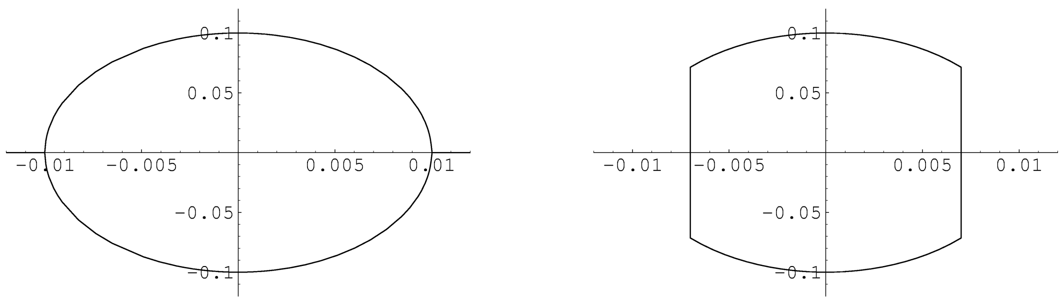
for two suitable positive constant a and b. Furthermore there are no periodic solutions with frequency $\omega \neq 1/2n$.

Proof. As we are looking for periodic solutions there are no small divisors, and one easily shows that the periodic functions $z^{(n)}(t)$ are analytic in t and admit a bound like Z^k for some positive constant Z; for instance one could use the tree expansion as in [6] and [5]. Then it is sufficient to take ε small enough, say less than some value ε_0, and analyticity in ε for $|\varepsilon| < \varepsilon_0$ follows. The constant ε_0 depends on C; to make explicit such a dependence first of all we can express t_0 in terms of C, as $\sin t_0 = -C/C_0$, with $C_0 = \alpha \, G_1(p, q)/\Delta$ (see (3.6)), and hence $\cos t_0 = \sqrt{1 - (C/C_0)^2}$. The study of the perturbation series (again we refer to [5] for details) shows that, for $\omega = 1/2n$ and $|C| < C_0(1/2n)$, as far as the dependence on C is concerned, to all perturbation orders n the functions $z^{(n)}(t)$ are just polynomials of order n in C and $\bar{\varphi}^{(n')}$, $n' < n$, and the initial conditions $\bar{\varphi}^{(n)}$ can be written as $(1/\cos t_0)$ times a quantity which again is a polynomial in C and $\bar{\varphi}^{(n')}$, $n' < n$. At the end we find that $z^{(n)}(t)$ can be written as a polynomial of degree n in C and $\sqrt{1 - (C/C_0)^2}$, so that for $|C| < C_0$ analyticity in C follows.

The condition of smallness on ε is of the form $\varepsilon_0 < \min\{b, a|\cos t_0|\}$, for suitable positive constants a and b (again see [5] for details), so that we can write

$$\varepsilon_0 < \min\left\{b, a\sqrt{1-(C/C_0)^2}\right\}, \tag{3.13}$$

and we find that for $\varepsilon < \min\{b, a\}$ we can choose any value of C such that $|C| < C_0\sqrt{1-(\varepsilon/a)^2}$; see figure 1. Then the relation (3.12) between ε and C follows.



(a) Case $a \leq b$.

(b) Case $a > b$.

Figure 1: Projection on the real plane of the (estimated) analyticity domain in (ε, C) of the periodic solution $x_1(t, \varepsilon, C)$.

Finally, from the proof of the lemma (see comments after (3.6)), we obtain that only periodic solutions with frequency $\omega = p/q$ with $p = 1$ and q even are possible, and the larger the value of q is the smaller is $C_0(1/q)$. □

If one performs explicitly the calculations of the constants a and b appearing in the estimates, one finds $a > b$, so that figure 1b gives a shape of the estimated analyticity domain which is more realistic. The existence of the formal series (that is the fact that the coefficients are well defined to all orders), which follows from the Lemma, requires a threshold value which is independent of ε, whereas the analysis of the convergence of that series gives a dependence on ε. An approximately square form of the analyticity domain would imply that with good approximation one can take the same threshold value for all ε in the domain.

As the condition (3.6) shows for $|C| < C_0(1/2n)$ there are two periodic solutions of (2.13) with frequency $\omega = 1/2n$: one of them is asymptotically stable, hence attracting, while the other one is unstable, hence repelling, so that, when performing numerical analysis, only the first one can actually be detected: this means that numerically, by letting the system evolve in the future for a long time, at most only one periodic orbit with a frequency ω can be found.

4. Secondary resonances

Consider now the equation

$$\ddot{x} + x^3 + \varepsilon f(t+t_0)x^3 + \varepsilon^2 D\dot{x} = 0, \qquad f(t) = \cos t, \tag{4.1}$$

and look for periodic solutions analytic in ε.

We can repeat the analysis performed in the previous section: the only difference is that now the contributions due to the dissipative terms start arising from the second order on.

To first order we find

$$\left\langle F_2^{(1)} \right\rangle = -\sin t_0\, \alpha^4\, G_1(p,q), \tag{4.2}$$

as immediately follows from (3.1) by taking into account definitions (3.4) to (3.5) and the fact that there is no term proportional to C.

Existence of the formal solution requires $\langle F_2^{(1)} \rangle = 0$. Therefore if $G_1(p,q) = 0$, that is if $p/q \neq 1/2n$, we have that (4.2) is identically satisfied and t_0 is arbitrary, while if $G_1(p,q) \neq 0$, that is if $p/q = 1/2n$, we have to fix $\sin t_0 = 0$.

The latter solutions are dealt with through the following result.

Theorem 2. *Fix $\omega = p/q = 1/2n$. There are $2\pi/\omega$-periodic solutions $x_2(t,\varepsilon,D)$ of* (4.1)*, analytic in (t,ε,D), and they coincide with the solutions $x_1(t,\varepsilon,\varepsilon D)$ of* (2.3) *given by Theorem 1.*

Proof. Fix $\omega = p/q = 1/2n$. By reasoning as for the proof of Theorem 1, we find that there exist $2\pi/\omega$-periodic solutions of (4.1) for ε small enough and D not too large. The equation (4.2) requires $\sin t_0 = 0$. A second order computation gives

$$\left\langle F_2^{(2)} \right\rangle = \alpha^3 \cos t_0\, G_1(p,q)\, \bar{\varphi}^{(1)} + \alpha^4\, \Gamma_2(t_0;p,q) - \frac{1}{3}\alpha^3 D, \tag{4.3}$$

where the first term is that obtained in the previous case, the last one is due to the dissipative term, and the second one takes into account all the contributions which depend neither on $\bar{\varphi}^{(1)}$ nor on D.

It is easy to see that, just by parity properties, one has $\Gamma_2(t_0;p,q) = 0$ for $\sin t_0 = 0$, so that (4.3) requires

$$\cos t_0\, G_1(p,q)\, \bar{\varphi}^{(1)} - \frac{D}{3} = 0, \qquad \cos t_0 = \pm 1, \tag{4.4}$$

which imposes a constraint on the two parameters D and $\bar{\varphi}^{(1)}$.

From higher order computations (as in [6] and [5]) one finds also that, in order to have convergence of the perturbation series, one has to require

$$\varepsilon \bar{\varphi}^{(1)} c_1 < 1, \tag{4.5}$$

for some positive constant c_1. In the end one obtains the result that there exists $\varepsilon_1 > 0$ such that for $|\varepsilon| < \varepsilon_1$ and $D < D_0(1/2n) = O(1/\varepsilon)$ the equation (4.1) admits a solution $x_2(t,\varepsilon,D)$ analytic in t, ε and D.

For ε small enough choose D so small that, by fixing $C = \varepsilon D$, one has $|C| < C_0$ and (3.12) is satisfied. Then the function $x_1(t,\varepsilon,\varepsilon D)$ is analytic in ε, as composition of two analytic functions; note also that for $C = \varepsilon D$ the equation (2.3) reduces to (4.1). The analysis of Section 3 shows that there are two periodic solutions analytic in ε (as there are two possible values of t_0 such that (3.6) is satisfied). On the other hand there are also two periodic solutions of (4.1), corresponding to the two values $t_0 = 0$ and $t_0 = \pi$, so that the solutions have to be pairwise equal to each other. By the uniqueness of analytic continuations, such solutions have to be equal as long as they are defined. □

For the new periodic solutions (the ones corresponding to frequencies $\omega = p/q \neq 1/2n$) the following result applies.

Theorem 3. *Fix $\omega = p/q = 1/(2n+1)$, $n \in \mathbb{Z}_+$. There exists $D_0 = D_0(1/q)$, decreasing to zero in q, and, for all $|D| < D_0$, a value $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ the system (4.1) admits a $2\pi/\omega$-periodic solution analytic in (t, ε, D). There are no periodic solutions with frequency $\omega = p/q$, for $p \neq 1$.*

Proof. Fix $\omega = p/q \neq 1/2n$. As explained in the remark after (4.2) in this case $G_1(p,q) = 0$, hence t_0 is left arbitrary,

To second order a tedious computation (see Appendix C) gives

$$\Gamma_2(t_0; p, q) = \sin 2t_0 \, G_2(p,q), \tag{4.6}$$

where

$$\begin{aligned} G_2(p,q) = -\frac{1}{8Kp} \int_0^{4Kp} \mathrm{d}t \Big\{ & \sin(t/\alpha) \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{cn}^3\, t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \right) \int_0^t \mathrm{d}t' \, \cos(t'/\alpha) \; \mathrm{cn}^4\, t' \\ & + \cos(t/\alpha) \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{cn}^3\, t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \right) \int_0^t \mathrm{d}t' \, \sin(t'/\alpha) \; \mathrm{cn}^4\, t' \\ & + \sin(t/\alpha) \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{cn}^3\, t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \right) \int_0^t \mathrm{d}t' \left[\int_0^{t'} \mathrm{d}t'' \, \cos(t''/\alpha) \; \mathrm{cn}^3\, t'' \; \mathrm{sn}\, t'' \; \mathrm{dn}\, t'' \right. \\ & \qquad \left. - \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t \int_0^t \mathrm{d}t' \, \cos(t'/\alpha) \; \mathrm{cn}^3\, t' \; \mathrm{sn}\, t' \; \mathrm{dn}\, t' \right] \\ & + \cos(t/\alpha) \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{cn}^3\, t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \right) \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \, \sin(t''/\alpha) \; \mathrm{cn}^3\, t'' \; \mathrm{sn}\, t'' \; \mathrm{dn}\, t'' \\ & + 4 \Big(\sin(t/\alpha) \; \mathrm{cn}^3\, t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \int_0^t \mathrm{d}t' \, \cos(t'/\alpha) \; \mathrm{cn}^3\, t' \; \mathrm{sn}\, t' \; \mathrm{dn}\, t' \\ & + \cos(t/\alpha) \; \mathrm{cn}^3\, t \; \mathrm{sn}\, t \; \mathrm{dn}\, t \int_0^t \mathrm{d}t' \, \sin(t'/\alpha) \; \mathrm{cn}^3\, t' \; \mathrm{sn}\, t' \; \mathrm{dn}\, t' \Big) \Big\}, \end{aligned} \tag{4.7}$$

where the sum of the last two terms gives a vanishing contribution, as it is the average of a total derivative, so that (4.6) can be satisfied only if

$$|D| \leq D_0(p/q) \equiv \frac{4K}{2\pi\Delta} \left(\frac{p}{q} G_2(p,q) \right) \approx 3.54102 \left(\frac{p}{q} G_2(p,q) \right), \tag{4.8}$$

which defines the threshold values $D_0(p/q)$.

By reasoning as in Appendix B for $G_1(p,q)$, one can show that one can have $G_2(p,q) \neq 0$ only if $p/q = 1/n$; see Appendix C. As we are excluding $p/q = 1/2n$, we obtain the result that p has to be 1, and q has to be odd. See table 2 for the quantities $G_2(1,q)$ and the corresponding threshold values $D_0(1/q)$, for the first odd values of q.

So far we have seen that the first order gives no condition, while the second order fixes the value of the initial phase t_0. Then one can show that the higher order contributions fix the values of the corrections $\bar{\varphi}^{(n)}$: contrary to what happens in the case discussed in Section 3 now the condition on $\bar{\varphi}^{(n)}$, $n \geq 1$, is found at step $n+2$ instead than at step $n+1$. We omit the proof, which is rather cumbersome. □

An important difference with respect to the primary resonances discussed in Section 3 is that, as implied by the condition (4.7) for t_0, for $|D| < D_0(p/q)$ we now have four periodic orbits with frequency $\omega = p/q$:

Table 2: Values of $G_2(1,q)$ for $q = 1,3,5,7,9$. All the other values of $G_2(p,q)$, with odd $q \le 10$, are vanishing. The corresponding threshold values $D_0(p,q)$ for D are computed according to (4.7).

q	$G_2(1,q)$	$\alpha(1,q)$	$D_0(1/q)$
1	0.041322	1.180341	0.146322
3	0.055069	0.393447	0.065001
5	0.009161	0.236068	0.006488
7	0.000351	0.168620	0.000177
9	0.000006	0.131149	0.000002

two of them will be asymptotically stable, while the other two will be unstable, so that, numerically, by considering only evolution in forward time, only two periodic orbits can be detected. This is in agreement with the numerical results presented in Section 5.

We can also consider models (1.1) with γ given by (1.2), with other values of m. The general scenario is that, by increasing the value of m, one still finds all the periodic orbits found for the previous values of m, and then new periodic orbits appear, with a threshold of order 1 in ε. For fixed m and C_m, only a finite number of periodic orbits exist, as the following result shows.

Theorem 4. *Consider the system* (1.1)*, with* γ *given by* (1.2)*. For fixed* C_m *only a finite number of periodic orbits exist, and the corresponding frequencies* $\omega = p/q$ *are such that* $|p| \le m$ *and* $1 \le q \le q_m(C_m)$*, where* $q_m(C_m)$ *goes to infinity when* C_m *goes to zero.*

Proof. We give only a sketch of the proof, which can be performed by induction. For $m = 1$ the statement follows from theorem 1.

Then assume that the statement is true for all $m' < m$: one has to check that only a finite number of new periodic orbits appear. By using the perturbation expansion envisaged in the previous sections one looks for periodic orbits with frequencies ω which were not possible for any previous value m'. Therefore, up to order m, no condition has arisen for such new periodic orbits. To order m one obtains a condition of the form

$$\alpha^4\,\Gamma_m(t_0;p,q) - \frac{1}{3}\alpha^3 C_m = 0, \tag{4.9}$$

where $\Gamma_m(t_0;p,q)$ is the average (in t) of a function which depends on t_0 as a trigonometric polynomial of order at most m:

$$\Gamma_m(t_0;p,q) = \sum_{r_1=-m}^{m}\sum_{r_2=-m}^{m}\sum_{n\in\mathbb{Z}} P_{r_1,r_2,n} e^{ir_1t_0}\frac{1}{4Kp}\int_0^{4kp} \mathrm{d}t\, e^{i\pi q r_2 t/2Kp} e^{i\pi nt/2K}, \tag{4.10}$$

so that one can prove (see Appendix C) that (4.10) can be different from zero only if $q/n = n/r_2$, which yields $|p| \le r_2 \le m$. For fixed p only the component with $r_2 = p$, hence with $n = q$, can contribute to the average in (4.10). On the other hand the coefficients $P_{r_1,r_2,n}$ tend to zero exponentially for $n \to \infty$, by the analyticity of the elliptic functions, so that, for a fixed value of C_m, there is a value $q = q_m(C_m)$ such that the corresponding $\alpha\gamma(t_0;p,q)$ is less than $C_m/3$, so that (4.9) can not be satisfied. □

What emerges numerically is that the size of the basins of attraction of the attracting periodic orbits increases when γ is decreased: we can interpret such a phenomenon by saying that if we let γ decrease, when it crosses some value $\gamma_0(p/q, \varepsilon)$ an attracting periodic orbit with frequency p/q appears, and its basin of attraction enlarges as γ continues to decrease.

The periodic orbits obtained for $m = 1$ in (1.2), which were the first to appear, have the largest basins of attraction, the orbits appeared at $m = 2$ have smaller basins of attraction, and so on, until the orbits which appear for the largest values of n will be the less relevant ones, that is the ones with the smallest basins of attraction.

Such a scenario is well accounted by the numerical results given in the next Section.

5. Numerical results

In this Section we give some numerical results for the model (1.1), with $\varepsilon = 0.1$. For numerical purposes it is more convenient to have the same initial phase for all solutions: so, if there is some periodic solution $x(t)$ with frequency ω, then t_0 will be defined by the condition that, by denoting with $x^{(0)}(t)$ the corresponding unperturbed solution, one has $x^{(0)}(0) = \alpha\,\mathrm{cn}(-\alpha t_0)$.

If one tries to obtain explicit bounds for the constants a and b appearing in the statement of theorem 1, by proceeding as outlined in the proof without looking for optimal estimates, the value of ε_0 as given by (3.13) turns out to be very small. However one expects that such bounds can be highly improved through a more careful analysis, so that we assume that the analytical results found in the previous sections still apply to the chosen value of ε. Taking ε really small would make any numerical analysis difficult, as the system would become a very small perturbation of the free system (for $\varepsilon = 0$) and the corresponding value of the damping coefficient γ would also be small, so that decay of the transient, and hence attraction to the periodic orbit, would become very slow and hence difficult to detect without increasing the numerical precision and the running times of the programs.

First of all note that if the value of γ is large enough all trajectories tend to the origin, which is a global attractor according to the analysis performed in [2]. By decreasing γ new attractors appear.

For instance if we fix $\gamma = 0.005$, hence $C = 0.05$ in (2.3) and $D = 0.5$ in (4.1), according to table 1 we can have only periodic orbits with frequency 1/2 and 1/4, while no periodic orbit among the ones described in Section 4 is possible, as the corresponding value of D is above the threshold value (cf. table 2. By taking a grid of 1024×1024 initial conditions in the square $\mathcal{Q} = [-1, 1] \times [-1, 1]$ around the origin, we find indeed that all trajectories are captured either by the origin or by one of the two periodic orbits represented in figure 2.

The parts contained in the square $\mathcal{Q}$ of the basins of attraction of the origin and the periodic orbits with frequencies $\omega = 1/2$ and $\omega = 1/4$ are represented in figures 6 to 8 in Appendix D.

Of course there are faster and more sophisticated methods one could use to study the basins of attraction, such as the *straddle orbit method* or its variants [10, 4, 11, 1]. However we are mostly interested in the relative sizes of the basins, so the method we use, which consists in just following the evolution of the initial data point by point, even if very simple and slow, is better suited for our purposes.

If we fix $\gamma = 0.001$, hence $C = 0.01$ in (2.3) and $D = 0.1$ in (4.1), the analysis in Section 3 predicts that only the periodic orbits with frequency 1/2 and 1/4 appear, according to table 1, while by using the

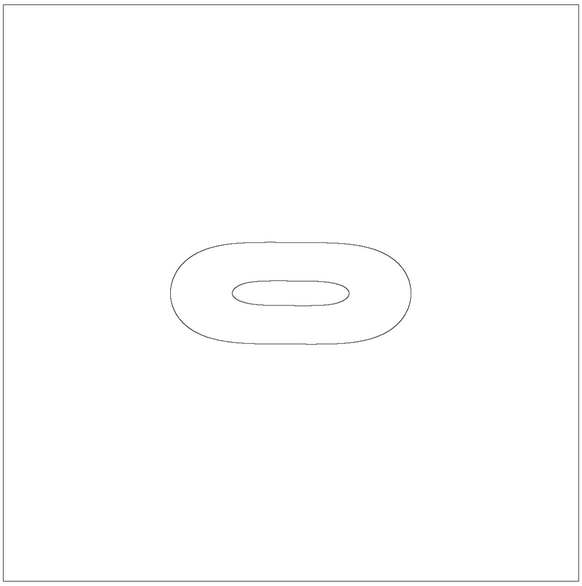


Figure 2: Attracting periodic orbits in the square $[-1.4, 1.4] \times [-1.4, 1.4]$. The frequencies of the two periodic orbits are $1/4$ for the inner one and $1/2$ for the external one. Here $\varepsilon = 0.1$ and $\gamma = 0.005$.

threshold values in table 2 we see that only the periodic orbits with frequency 1 has to be added to the previous one. We expect that other models (1.1) with γ given by (1.2) for $m \geq 3$ do not imply other periodic orbits than the considered ones, for the fixed value of γ, so that, in the end, we see that the only attractors which are possible for $\varepsilon = 0.1$ and $\gamma = 0.001$ are, besides the origin, the periodic orbits with frequencies $1/2$, $1/4$ and 1. This is in agreement with the numerical results. Indeed if we take as before a grid of 1024×1024 initial conditions in the square $\mathcal{Q} = [-1, 1] \times [-1, 1]$ around the origin we find that all trajectories are captured either by the origin or by one of the periodic orbits represented in figure 3, which have exactly the frequencies predicted by the theory. As anticipated there are two attracting periodic orbits with frequency $\omega = 1$, while, for $\omega = 1/2$ and $\omega = 1/4$, there is only one attracting orbit.

The parts contained in the square $\mathcal{Q}$ of the basins of attraction of the origin and of the four periodic orbits are represented in figures 9 to 13 in Appendix D.

A natural criterion to measure the relative relevance of the basins of attraction is provided by the respective areas (as given by the number of points of the grid of initial data evolving toward the corresponding attractor). For instance, for $\gamma = 0.001$, one finds that the basin of attraction of the origin is still the largest one, as it covers 44.39% of the square $\mathcal{Q}$. On the other hand the size of the basin of attraction of the periodic orbit with frequency $1/2$ is comparable, as it covers 40.94% of $\mathcal{Q}$. The relative measures of the basins of attraction of the periodic orbit with frequency $1/2$ and of the two periodic orbits with frequency 1 are, respectively, given by 13.32%, 0.67% and 0.67% of the overall area of the square $\mathcal{Q}$.

An analogous numerical analysis performed for higher and lower values of γ shows that, when γ is decreasing, the basin of attraction of the origin, which at the beginning (that is for γ such that $C = \gamma/\varepsilon$ is above the critical threshold $C_0(1/2)$) filled the entire phase space (see also [2]), begins to decrease, to the advantage of the basins of attraction of the newly appearing attractors.

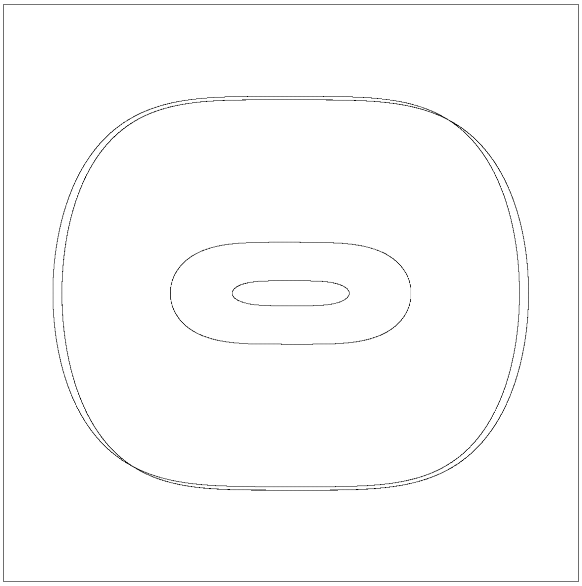


Figure 3: Attracting periodic orbits in the square $[-1.4, 1.4] \times [-1.4, 1.4]$. The frequencies of the four periodic orbits are $1/4$ for the inner one $1/2$ for the middle one and 1 for the two external ones, denoted as I and II. Here $\varepsilon = 0.1$ and $\gamma = 0.001$.

For $\gamma = 0.0005$ there are seven attracting periodic orbits: the ones with frequency 1 and $1/3$ appear in pairs, while there is only one periodic orbit for each frequency of the form $\omega = 1/q$, with $q = 2, 4, 6$; see figure 4). The two periodic orbits with frequencies $1/3$, which appear as indistinguishable in figure 4, can be detected as different if the figure is suitably enlarged.

The parts contained in the square $\mathcal{Q}$ of the basins of attraction of the origin and of the seven periodic orbits are represented in figure 14 to 21. in Appendix D.

The relative sizes of the (parts contained in $\mathcal{Q}$) of the basins of attraction of the origin and of the attracting periodic orbits for some values of γ are given in table 3.

Table 3: Relative sizes of the parts of the basins of attraction contained inside the square $\mathcal{Q} = [-1, 1] \times [-1, 1]$ for some values of γ; the value of ε is kept fixed to $\varepsilon = 0.1$. The periodic orbits are labelled by the corresponding frequencies, and 0 denotes the origin. Vanishing percentages mean that there is no corresponding attracting orbit.

γ	0	1/2	1/4	1/6	1-I	1-II	1/3-I	1/3-II
0.0200	100.00%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0150	91.08%	08.92%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0100	79.12%	20.88%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0050	64.83%	31.84%	03.34%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0010	44.39%	40.94%	13.32%	00.00%	00.67%	00.67%	00.00%	00.00%
0.0005	34.03%	41.83%	14.56%	06.44%	01.29%	01.29%	00.29%	00.29%

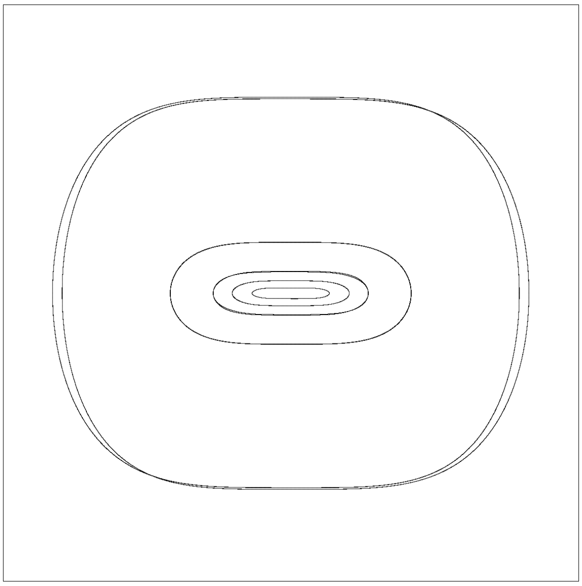


Figure 4: Attracting periodic orbits in the square $[-1.4, 1.4] \times [-1.4, 1.4]$. Starting from the outer to the inner ones, the frequencies of the orbits are 1 (I and II), 1/2, 1/3 (I and II), 1/4,1/6. Here $\varepsilon = 0.1$ and $\gamma = 0.0005$.

It emerges from table 3 that as γ decreases new attracting orbits appear and, once either $C = \gamma/\varepsilon$ or $D = \gamma/\varepsilon^2$ (or whatever else) have became smaller than the corresponding critical threshold, their basins of attraction get larger at expenses of the basin of attraction of the origin. For example the relative measure of the basin of attraction of the periodic orbit with frequency 1/2 increases: for instance for $\gamma = 0.001$ (that is for $C = 0.01$ and $D = 0.1$) it becomes almost equal to the relative measure of the basin of attraction of the origin, and for $\gamma = 0.0005$ it becomes positively larger. Analogous considerations hold for the other attracting orbits.

In general to see orbits with frequency p/q, where either q or p or both of them are large, one needs a small value for the friction coefficient γ: in the limit in which such a constant is negligible (that is if we put $\gamma = 0$) then all periodic orbits appear, as a byproduct of the analysis of the previous sections.

Note the appearence of the moiré-like patterns in some of the basins. We find the phenomenon interesting, though we do not have any explanations for it.

Numerically we find that that each initial condition in our grid belongs to the basin of attraction of one of the coexisting periodic orbits. This suggests that the union of the closure of all basins of attraction fills the entire phase space (of course approximation errors in the numerical integration of the ordinary differential equation makes it impossible to study the forward evolution of the boundaries of the basins of attraction).

6. Open problems

The analysis performed in Sections 3 and 4 deals with periodic orbits which are obtained for (1.1) in a very particular way. We expressed γ as an analytic function of ε (in fact as a power) and looked for periodic solutions analytic in ε. Numerically all periodic orbits which have been detected are of this kind: it could be interesting to find a mathematical justification for this phenomenon.

As a byproduct we found that all attractors are, in the investigated cases, periodic orbits. This is unlikely to be an accident. In general, introducing a dissipation term into Hamiltonian equations can produce other kind of attractors; see for instance [3]. In our case the system, in the absence of friction, is a quasi-integrable system (that is a perturbation of an integrable system), whereas in [3], if we look at the model considered there as a perturbation of an integrable system, strange attractors appear when the values of the perturbation parameters are large enough, beyond the perturbation regime. Perhaps it is natural to expect that only periodic orbits appear when adding a dissipative term to a quasi-integrable system and confining ourselves to small values of the parameters: also such an issue would deserve further study. Note also that in our case, the unperturbed system has a very simple structure, as there is only a stable equilibrium point, while in the model studied in [3] there is also a saddle-point, with the corresponding stable and unstable manifolds, which can be responsible for strange attractors appearing for large values of the parameters.

It would be also interesting to see what happen in our case when increasing the value of ε, in particular when we are definitely beyond the range of validity of perturbation theory. Of course an analytic study in such a case is too difficult, but numerically the problem can be easily approached.

In this paper we limited ourselves to perturbations of the quartic oscillator. Of course one could consider more general systems, for instance any analytic anharmonic potential. We think that analogous results can be expected in such cases, even if analytically the unperturbed solutions would no longer have the nice properties of the Jacobi elliptic functions we have used to perform the calculations of perturbation theory.

Finally a more detailed study of the basins of attraction, for instance with the techniques quoted in Section 5, could be a further topic of investigation. In particular it would be interesting to elucidate the moiré-like patterns and the fractal-like boundaries of the basins.

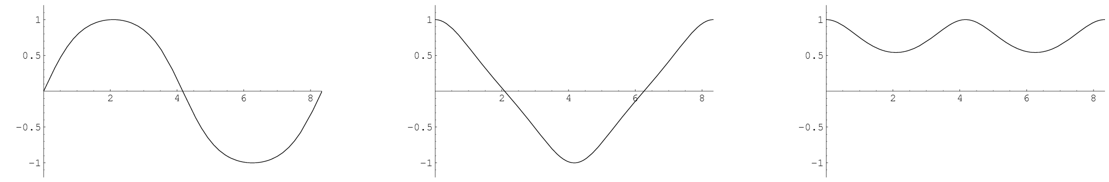
Acknowledgments. All calculations have been done on Compaq Alpha computers using Fortran 77 and Mathematica. We thank the Department of Physics of the University of Rome "La Sapienza" for providing us with some of the computing resources.

Appendix A. Basic properties of the elliptic functions

Let us denote by $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ the Jacobi elliptic functions sine-amplitude, cosine-amplitude and delta-amplitude, respectively; see for instance [9, 12]. Here $k \in (0,1)$ is the elliptic modulus, and $k' = \sqrt{1-k^2}$ is the complementary modulus. See figure 5.

The Jacobi elliptic functions are doubly periodic functions with periods $4K(k)$ and $4K'(k)i$, where

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad K'(k) = K(k'), \tag{A.1}$$



(a) Sine-amplitude $\mathrm{sn}(u,k)$.

(b) Cosine-amplitude $\mathrm{cn}(u,k)$.

(c) Delta-amplitude $\mathrm{dn}(u,k)$.

Figure 5: Jacobi elliptic functions with elliptic modulus $k = 1/\sqrt{2}$.

are the complete elliptic integral of the first kind. More precisely one has

$$\begin{aligned} \mathrm{sn}(u + 2mK + 2niK', k) &= (-1)^m \,\mathrm{sn}(u,k), \\ \mathrm{cn}(u + 2mK + 2niK', k) &= (-1)^{m+n} \,\mathrm{cn}(u,k), \\ \mathrm{dn}(u + 2mK + 2niK', k) &= (-1)^n \,\mathrm{dn}(u,k), \end{aligned} \tag{A.2}$$

where $K = K(k)$ and $K' = K'(k)$, so that, for real values of the arguments, one has

$$\begin{aligned} \mathrm{sn}(u + 2mK, k) &= (-1)^m \,\mathrm{sn}(u,k), \\ \mathrm{cn}(u + 2mK, k) &= (-1)^m \,\mathrm{cn}(u,k), \\ \mathrm{dn}(u + 2mK, k) &= \mathrm{dn}(u,k), \end{aligned} \tag{A.3}$$

which means that $\mathrm{cn}(u,k)$ and $\mathrm{sn}(u,k)$ are periodic functions with period $4K$, while $\mathrm{dn}(u,k)$ is periodic with period $2K$.

One has

$$\mathrm{cn}(-u,k) = \mathrm{cn}(u,k), \qquad \mathrm{sn}(-u,k) = -\,\mathrm{sn}(u,k), \qquad \mathrm{dn}(-u,k) = \mathrm{dn}(u,k), \tag{A.4}$$

and

$$\begin{aligned} \frac{\partial}{\partial u}\mathrm{cn}(u,k) &= -\,\mathrm{sn}(u,k)\;\mathrm{dn}(u,k), \\ \frac{\partial}{\partial u}\mathrm{sn}(u,k) &= \mathrm{cn}(u,k)\;\mathrm{dn}(u,k), \\ \frac{\partial}{\partial u}\mathrm{dn}(u,k) &= -k^2\,\mathrm{sn}(u,k)\;\mathrm{cn}(u,k). \end{aligned} \tag{A.5}$$

Moreover the following identities hold

$$\begin{aligned} &\mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = 1, \\ &\mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1, \\ &\mathrm{dn}^2(u,k) - k^2\,\mathrm{cn}^2(u,k) = 1 - k^2. \end{aligned} \tag{A.6}$$

Finally one has

$$\begin{aligned} \mathrm{sn}(u+v,k) &= \frac{\mathrm{sn}(u,k)\ \mathrm{cn}(v,k)\ \mathrm{dn}(v,k) + \mathrm{cn}(u,k)\ \mathrm{dn}(u,k)\ \mathrm{sn}(v,k)}{1-k^2\,\mathrm{sn}^2(u,k)\ \mathrm{sn}^2(v,k)}, \\ \mathrm{cn}(u+v,k) &= \frac{\mathrm{cn}(u,k)\ \mathrm{cn}(v,k) - \mathrm{sn}(u,k)\ \mathrm{dn}(u,k)\ \mathrm{sn}(v,k)\ \mathrm{dn}(v,k)}{1-k^2\,\mathrm{sn}^2(u,k)\ \mathrm{sn}^2(v,k)}, \\ \mathrm{dn}(u+v,k) &= \frac{\mathrm{dn}(u,k)\ \mathrm{dn}(v,k) - k^2\,\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)\ \mathrm{sn}(v,k)\ \mathrm{cn}(v,k)}{1-k^2\,\mathrm{sn}^2(u,k)\ \mathrm{sn}^2(v,k)}. \end{aligned} \tag{A.7}$$

If $k = 1/\sqrt{2}$ is fixed, we can write, for simplicity, $\mathrm{cn}(u) = \mathrm{cn}(u,1/\sqrt{2})$, $\mathrm{sn}(u) = \mathrm{sn}(u,1/\sqrt{2})$ and $\mathrm{dn}(u) = \mathrm{dn}(u,1/\sqrt{2})$.

Appendix B. Action-angle variables for the quartic potential

To show that the coordinates (φ, I) given by (2.8) are canonical it is sufficient to show that one has

$$\{x,y\} \equiv \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial y}{\partial \varphi} = 1, \tag{B.1}$$

and this is an easy computation.

To see that the coordinates (φ, I) can be interpreted as action-angle variables, just note that, by defining

$$A \equiv \frac{1}{4K}\oint y\,\mathrm{d}x = \frac{1}{2\sqrt{2}K}\,(4E)^{3/4}\int_{-1}^{1}\mathrm{d}x\,\sqrt{1-x^4} = \frac{1}{2\sqrt{2}K}\,(4E)^{3/4}\,\frac{2\sqrt{2}K}{3}, \tag{B.2}$$

one obtains $I = A$ (we defer the computation of the integral to Appendix C).

Actually (φ, I) are not strictly speaking action-angle variables as φ is not an angle (it is defined modulo $4K$ instead than 2π); formulae are slightly simpler with our choice of φ.

In the new variables the Hamiltonian for $\varepsilon = 0$ is given by (2.9). If we neglect the dissipative term, then the equation of motion can be derived by the Hamiltonian

$$\mathcal{H}(\varphi, I) = \mathcal{H}_0(I) + \frac{1}{4}\varepsilon\,(3I)^{4/3}\,f(t+t_0)\ \mathrm{cn}^4\,\varphi, \tag{B.3}$$

which yields (2.10) for $C = 0$.

To take into account the dissipative term can just write

$$\begin{cases} \dot\varphi = \dfrac{\partial \varphi}{\partial x}\dot x + \dfrac{\partial \varphi}{\partial y}\dot y, \\ \\ \dot I = \dfrac{\partial I}{\partial x}\dot x + \dfrac{\partial I}{\partial y}\dot y, \end{cases} \tag{B.4}$$

where the partial derivatives can be computed in terms of the entries of the Jacobian matrix of the inverse transformation:

$$\begin{pmatrix} \dfrac{\partial \varphi}{\partial x} & \dfrac{\partial \varphi}{\partial y} \\ \\ \dfrac{\partial I}{\partial x} & \dfrac{\partial I}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial x}{\partial I} \\ \\ \dfrac{\partial y}{\partial \varphi} & \dfrac{\partial y}{\partial I} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\partial y}{\partial I} & -\dfrac{\partial x}{\partial I} \\ \\ -\dfrac{\partial y}{\partial \varphi} & \dfrac{\partial x}{\partial \varphi} \end{pmatrix}, \tag{B.5}$$

as the determinant of a symplectic matrix is 1, so that (2.10) is immediately obtained.

Appendix C. Some useful integrals

Given Δ as defined in (3.2), using obvious notational shorthands, one has

$$\begin{aligned}\Delta &= \langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle = -\langle(\mathrm{cn})'\,(\mathrm{sn}\,\mathrm{dn})\rangle = \langle \mathrm{cn}\,(\mathrm{sn}\,\mathrm{dn})'\rangle \\ &= \left\langle \mathrm{cn}\left(\mathrm{cn}\,\mathrm{dn}^2 - \frac{1}{2}\,\mathrm{cn}\,\mathrm{sn}^2\right)\right\rangle = \left\langle \mathrm{cn}^2\,\mathrm{dn}^2 - \frac{1}{2}\,\mathrm{cn}^2\,\mathrm{sn}^2\right\rangle \\ &= \left\langle \mathrm{cn}^2\left(1 - \frac{1}{2}\,\mathrm{sn}^2\right) - \frac{1}{2}\,\mathrm{cn}^2\,\mathrm{sn}^2\right\rangle \\ &= \langle \mathrm{cn}^2 - \mathrm{cn}^2\,\mathrm{sn}^2\rangle = \langle \mathrm{cn}^2 - \left(2\,\mathrm{dn}^2 - 1\right)\mathrm{sn}^2\rangle = \langle \mathrm{cn}^2 + \mathrm{sn}^2\rangle - 2\,\langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle \\ &= 1 - 2\,\langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle = 1 - 2\Delta,\end{aligned} \tag{C.1}$$

where the prime denotes the derivative, and (A.6) have been repeatedly used; hence $\Delta = 1/3$.

One has

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}x\,\sqrt{1-x^4} &= \int_0^{2K} \mathrm{d}t\;\mathrm{sn}\,t\;\mathrm{dn}\,t\,\sqrt{1-\mathrm{cn}^4\,t} \\ &= \int_0^{2K} \mathrm{d}t\;\mathrm{sn}\,t\;\mathrm{dn}\,t\,\sqrt{(1+\mathrm{cn}^2\,t)(1-\mathrm{cn}^2\,t)} \\ &= \int_0^{2K} \mathrm{d}t\;\mathrm{sn}\,t\;\mathrm{dn}\,t\,\sqrt{2\;\mathrm{sn}^2\,t\;\mathrm{dn}^2\,t} = 2\sqrt{2}K\left(\frac{1}{4K}\int_0^{4K}\mathrm{d}t\;\mathrm{sn}^2\;\mathrm{dn}^2\,t\right) \\ &= 2\sqrt{2}K\Delta = \frac{2\sqrt{2}K}{3},\end{aligned} \tag{C.2}$$

which implies the last identity in (B.2).

The Fourier series of the Jacobi elliptic functions considered in Appendix A are given by

$$\begin{aligned}\mathrm{sn}(u,k) &= \frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^{n-1/2}}{1-\mathfrak{q}^{2n-1}}\sin\frac{(2n-1)\pi u}{2K(k)}, \\ \mathrm{cn}(u,k) &= \frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^{n-1/2}}{1+\mathfrak{q}^{2n-1}}\cos\frac{(2n-1)\pi u}{2K(k)}, \\ \mathrm{dn}(u,k) &= \frac{\pi}{2K(k)} + \frac{2\pi}{K(k)}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^{n}}{1-\mathfrak{q}^{2n}}\cos\frac{2n\pi u}{2K(k)},\end{aligned} \tag{C.3}$$

where $\mathfrak{q} = \exp(-\pi K'(k)/K(k))$, so that $\mathfrak{q} = e^{-\pi}$ for $k = 1/\sqrt{2}$, while we can write

$$\begin{aligned}G_1(p,q) &= \frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\;\mathrm{sn}\,t\;\mathrm{dn}\,t\;\mathrm{cn}^3\,t\;\sin(t/\alpha) \\ &= \frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\left(-\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{cn}^4\,t\right)\sin(t/\alpha) \\ &= \frac{1}{4\alpha}\left(\frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\;\mathrm{cn}^4\,t\;\cos(t/\alpha)\right),\end{aligned} \tag{C.4}$$

with

$$\cos(t/\alpha) = \cos\frac{\pi t}{2K(k)}\frac{q}{p}, \tag{C.5}$$

so that one can have $G_1(p,q)$ only if, for suitable integers n_j one has

$$p\left(\pm(2n_1-1)\pm(2n_2-1)\pm(2n_3-1)\pm(2n_4-1)\right)\pm q = 0, \tag{C.6}$$

which requires for q to be of the form

$$q = 2np, \qquad n \in \mathbb{Z}. \tag{C.7}$$

Therefore first of all q has to be even. Moreover, for fixed q, one must have $p = q/2n$ for some $n \in \mathbb{Z}$. If we impose that (q, p) are relatively prime then the identity $p/q = 1/2n$ imposes $q = 2n$ and $p = 1$. Finally (C.4) also implies that for $p/q = 1/2n$ one has $G_1(1/2n) > 0$, as $G_1(p, q)$ is equal to the qth Fourier label of the function $\mathrm{cn}^4(t)$, which is strictly positive by the second of (C.3).

Note moreover that if we choose q to be large enough in (C.7) then also n has to be large, so that large Fourier labels of the elliptic functions have to be involved in order that the integral $G_1(p, q)$ be nonvanishing. This implies that the corresponding value of $G_1(p, q)$ has to be small enough (use the fact that in the expansions (C.3) one has $0 < \mathfrak{q} < 1$).

Now let us show how the condition (3.10) implies (3.11). One can write (3.10) as

$$\left\langle R_2^{(n)} \right\rangle = \alpha^3 \left(\alpha\, U(p, q) - C\, V(p, q)\right) \bar{\varphi}^{(n-1)}, \tag{C.8}$$

with

$$\begin{aligned} U(p, q) &\equiv \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \cos(t/\alpha)\, \frac{1}{\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{sn}\, t\ \mathrm{dn}\, t\ \mathrm{cn}^3\, t \right) \right) \cos t_0 \\ &\qquad - \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \sin(t/\alpha)\, \frac{1}{\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{sn}\, t\ \mathrm{dn}\, t\ \mathrm{cn}^3\, t \right) \right) \sin t_0 \\ &= \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \cos(t/\alpha) \frac{1}{\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{sn}\, t\ \mathrm{dn}\, t\ \mathrm{cn}^3\, t \right) \right) \cos t_0, \end{aligned} \tag{C.9}$$

and

$$V(p, q) \equiv \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \frac{1}{\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{sn}^2\, t\ \mathrm{dn}^2\, t \right) \right) = 0, \tag{C.10}$$

by parity properties. In (C.9), by integrating twice by parts, we obtain

$$\begin{aligned} U(p, q) &= -\frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t \left(\cos(t/\alpha) \frac{1}{\alpha} \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\frac{1}{4}\, \mathrm{cn}^4\, t \right) \right) \right) \cos t_0 \\ &= -\frac{1}{4\alpha} \left(\frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \sin(t/\alpha) \frac{1}{\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{cn}^4\, t \right) \right) \cos t_0 \\ &= \frac{1}{4\alpha^2} \left(\frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \cos(t/\alpha)\ \mathrm{cn}^4\, t \right) \cos t_0 = \frac{1}{\alpha}\, G_1(p, q)\, \cos t_0, \end{aligned} \tag{C.11}$$

which implies (3.11).

Now let us consider the case $\gamma = \varepsilon^2 D$ and $p/q \neq 1/2n$: first of all we want to prove (4.7). By shortening $O(t) = \mathrm{cn}^3(\alpha t)\ \mathrm{sn}\,(\alpha t)\ \mathrm{dn}\,(\alpha t)$, $E(t) = \dot{O}(t)$, $c(t) = \cos t$ and $s(t) = \sin t$, and by denoting with $I[F](t)$ the integral of F between 0 and t, we can write

$$\begin{aligned} \left\langle F_2^{(2)} \right\rangle = \alpha^4 \Big\{ &\cos t_0 \left\langle cE \right\rangle \bar{\varphi}^{(1)} + \cos t_0\, \sin t_0 \left(\left\langle sEI[cE^{(1)}] \right\rangle + \left\langle cEI[sE^{(1)}] \right\rangle \right) \\ &+ \cos t_0\, \sin t_0 \left(\left\langle sEI[I[cO^{(1)}] - \langle I[cO^{(1)}] \rangle] \right\rangle + \left\langle cEI[I[sO^{(1)}]] \right\rangle \right) \Big\} \\ &+ 4\,\alpha \left\{ -\sin t_0 \left\langle sO \right\rangle \bar{I}^{(1)} + \alpha^2 \cos t_0\, \sin t_0 \left(\left\langle sOI[cO^{(1)}] \right\rangle + \left\langle cOI[sO^{(1)}] \right\rangle \right) \right\} - \frac{1}{3} \alpha^3 D, \end{aligned} \tag{C.12}$$

for suitable functions $E^{(1)}$ (even) and $O^{(1)}$ (odd); an explicit computation gives

$$E^{(1)}(t) = -\alpha\ \mathrm{cn}^4(\alpha t), \qquad O^{(1)}(t) = -\alpha^2\, \mathrm{cn}^3(\alpha t)\ \mathrm{sn}\,(\alpha t)\ \mathrm{dn}\,(\alpha t) = -\alpha^2 O(t). \tag{C.13}$$

This simply follows from the parity properties of the unperturbed solution (2.4), from the remark that if F is an even function then $I[F]$ is odd and from the the Fourier expansions (C.3) of the Jacobi elliptic functions (which imply that the averages $\langle cE^{(1)}\rangle$ and $\langle I[sO]\rangle$ are vanishing for $p/q \neq 1/2n$).

In (C.12) the averages $\langle cE\rangle$ and $\langle sO\rangle$ are vanishing because of (C.6) and (C.7); note also that the first one is simply $\alpha^{-1}G_1(p,q)$ (see (C.9) and (C.11), so that (4.6) and (4.7) are proved, with $\gamma_1(p,q)$ defined according to (4.8).

Imposing $\langle F_2^{(2)}\rangle = 0$ gives

$$\begin{aligned}\frac{1}{3}\alpha^3\, D = \frac{1}{2}\alpha^3 \,\sin 2t_0 \Big\{ &\alpha \left(\left\langle sEI[cE^{(1)}] \right\rangle + \left\langle cEI[sE^{(1)}] \right\rangle \right) \\ &+\alpha \left(\left\langle sEI[I[cO^{(1)}]] \right\rangle + \left\langle cEI[I[sO^{(1)}]] \right\rangle \right) + 4 \left(\left\langle sOI[cO^{(1)}] \right\rangle + \left\langle cOI[sO^{(1)}] \right\rangle \right) \Big\},\end{aligned} \tag{C.14}$$

By writing

$$c(t) = \frac{1}{2}\sum_{\sigma=\pm 1} e^{i\sigma t}, \qquad s(t) = \frac{1}{2i}\sum_{\mu=\pm 1} \mu e^{i\mu t}, \qquad F(t) = \sum_{n\in\mathbb{Z}} e^{in\omega t} F_n, \tag{C.15}$$

for $F = E, O, E^{(1)}, O^{(1)}$, we can rewrite (C.13) as

$$\begin{aligned}&\frac{1}{3}\alpha^3\, D = \alpha^3\, \sin 2t_0\, G_2(p,q),\\ &G_2(p,q) = \frac{1}{4i}\sum_{\omega(n+n')+\mu+\sigma=0} \Big\{ \frac{\alpha}{i(\mu+\omega n')}\left(E_n E^{(1)}_{n'}\right)(\mu+\sigma) \\ &\qquad + \frac{\alpha}{i^2(\mu+\omega n')^2}\left(E_n O^{(1)}_{n'}\right)(\mu+\sigma) + \frac{4}{i(\mu+\omega n')}\left(E_n O^{(1)}_{n'}\right)(\mu+\sigma) \Big\},\end{aligned} \tag{C.16}$$

so that we immediately see that the contributions with $\mu+\sigma = 0$ disappear. As $\mu+\sigma \in \{-2,0,2\}$ then we have to retain in (C.16) only the contributions with n and n' such that $n+n' = \pm 2q/p$. As n and n' have to be even, as it is easy to check from the expressions (C.13) by relying on the Fourier expansions (C.3), we are left with $2n \pm 2q/p = 0$, which requires $p/q = 1/n$. As we are excluding values of p,q such that $p/q = 1/2n$ we have finally shown that the quantity $G_2(p,q)$ in (4.7) can be different from zero only for $p/q = 1/n$, with n odd.

Finally we want to prove that for $\gamma = \varepsilon^m C_m$ to order m the equation $\langle F_2^{(m)}\rangle = 0$ takes the form (4.9). First note that one has a function $f(t+t_0)$ for each perturbation order, so that $\langle F_2^{(m)}\rangle$ is a polynomial of order m in t_0: for the same reason it has to be a polynomial in t/α. Because of the presence of the Jacobi elliptic functions any further dependence on t has to be analytic and $4K$-periodic: hence the expansion (4.10) follows. In order to have $\gamma_m(t_0;p,q) \neq 0$ one has to require (by the same reasoning used to obtain (C.6) for $m = 1$) $qr_2/p = n \in \mathbb{Z}$: as p and q are relatively prime integers and $|r_2| \leq m$, then one must have $|p| \leq m$. The exponential decay in n of the coefficients $P_{r_1,r_2,n}$ can be proved as in the previous case $m = 1$, by using the analyticity of the elliptic functions (which in turn implies the exponential decay of the Fourier coefficients appearing in the expansions (C.3)).

Appendix D. Basins of attraction

In this section we show the images of the basins of attraction for the periodic orbits shown in fig. 2, 3, and 4. Note the singular moiré-like patterns particularly evident in fig. 9, 10, 14 and 15.

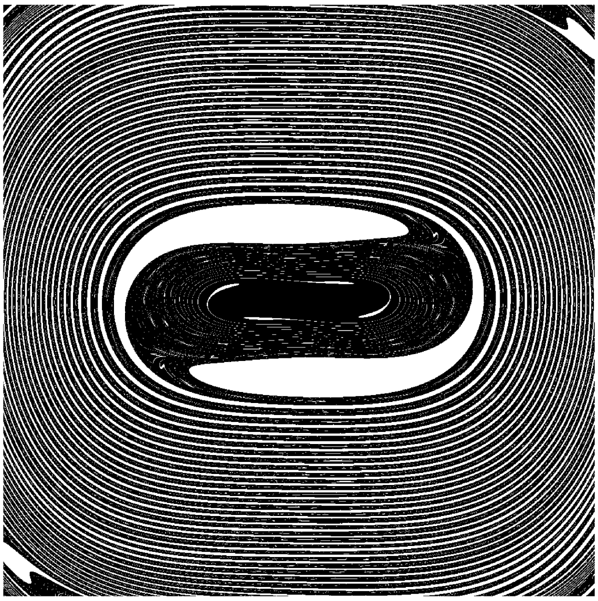


Figure 6: Basin of attraction of the origin contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.005$ in (1.1).

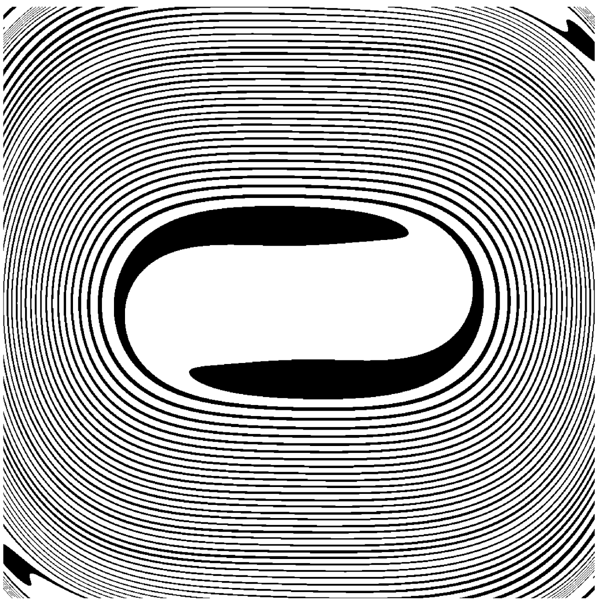


Figure 7: Basin of attraction of the periodic orbit with frequency $\omega = 1/2$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.005$ in (1.1).

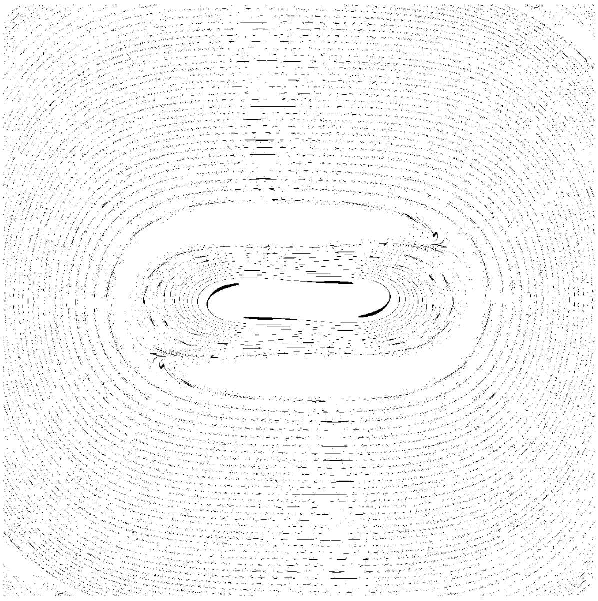


Figure 8: Basin of attraction of the periodic orbit with frequency $\omega = 1/4$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.005$ in (1.1).

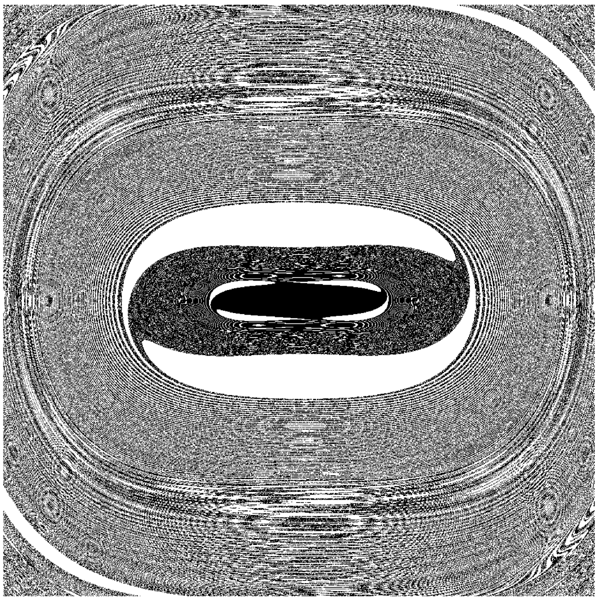


Figure 9: Basin of attraction of the origin contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.001$ in (1.1).

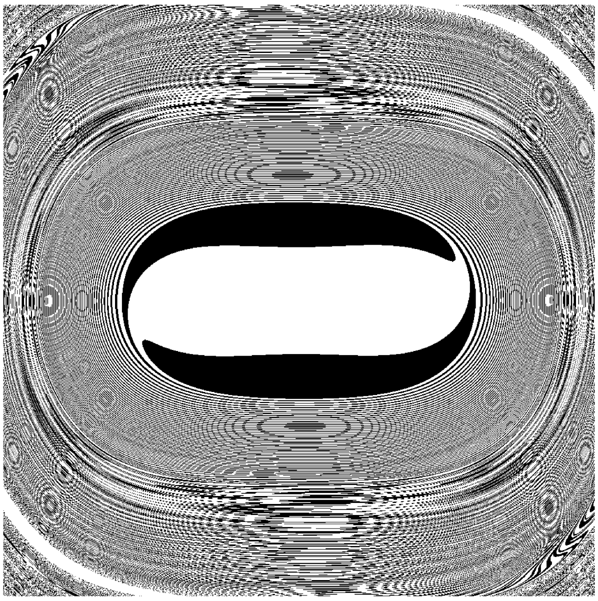


Figure 10: Basin of attraction of the periodic orbit with frequency $\omega = 1/2$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.001$ in (1.1).

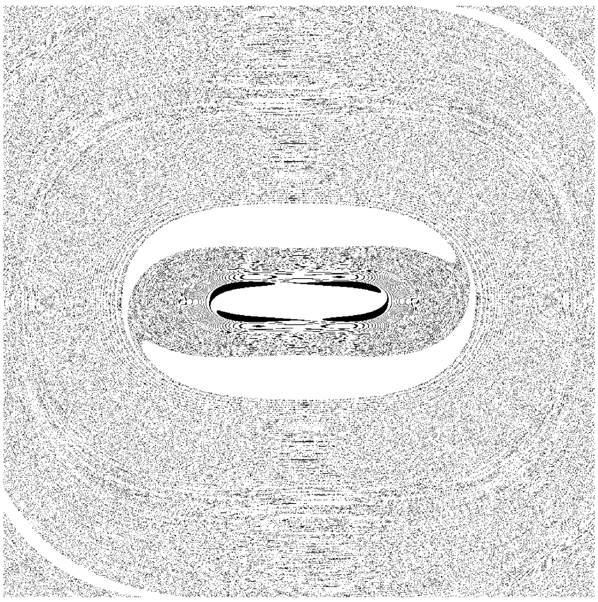


Figure 11: Basin of attraction of the periodic orbit with frequency $\omega = 1/4$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.001$ in (1.1).

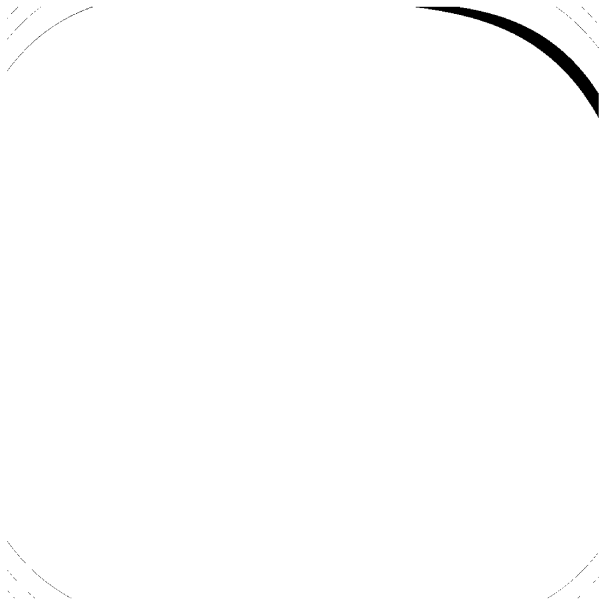


Figure 12: Basin of attraction of the first periodic orbit with frequency $\omega = 1$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.001$ in (1.1).

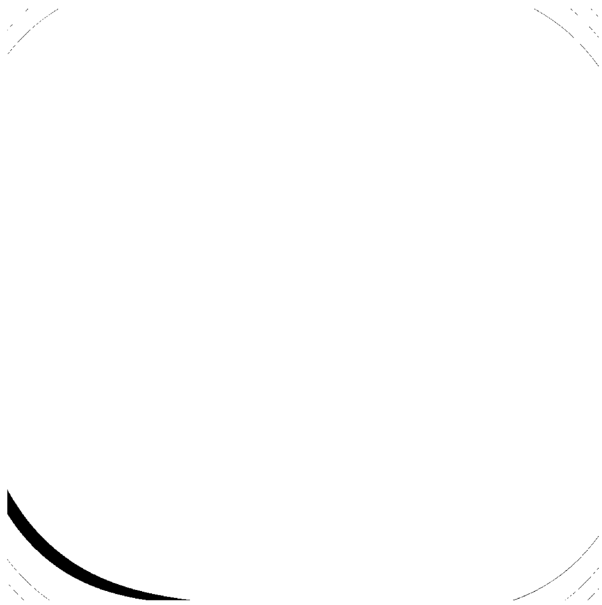


Figure 13: Basin of attraction of the second periodic orbit with frequency $\omega = 1$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.001$ in (1.1).

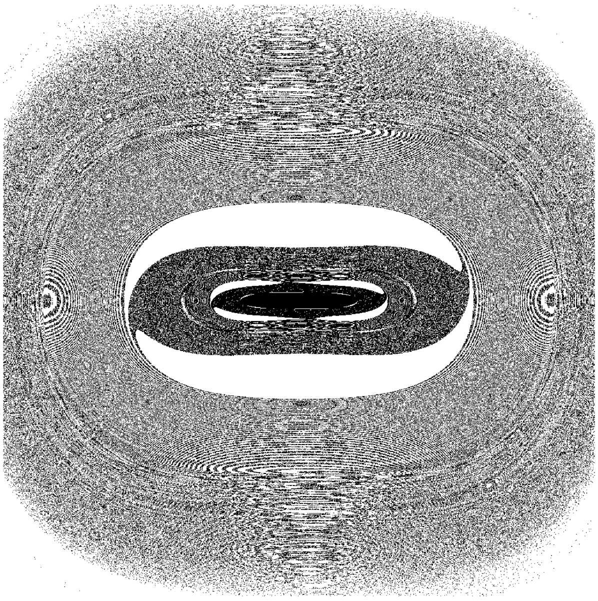


Figure 14: Basin of attraction of the origin contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

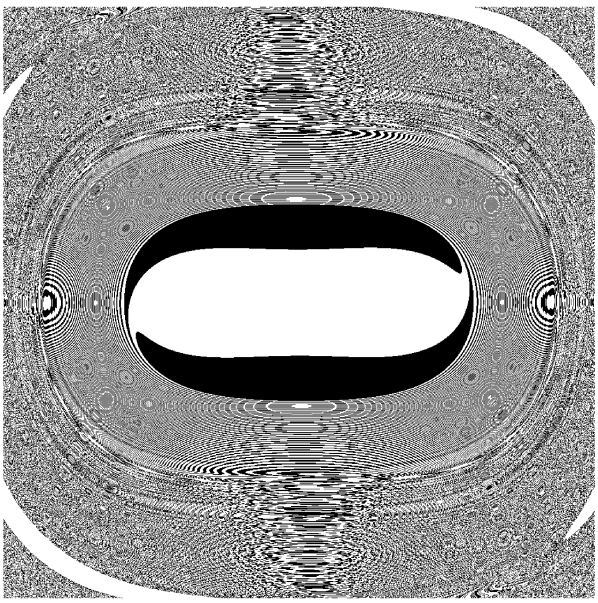


Figure 15: Basin of attraction of the periodic orbit with frequency $\omega = 1/2$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

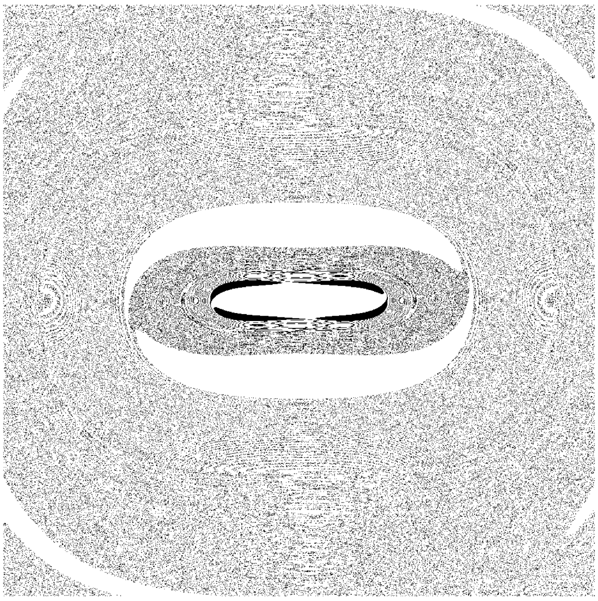


Figure 16: Basin of attraction of the periodic orbit with frequency $\omega = 1/4$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

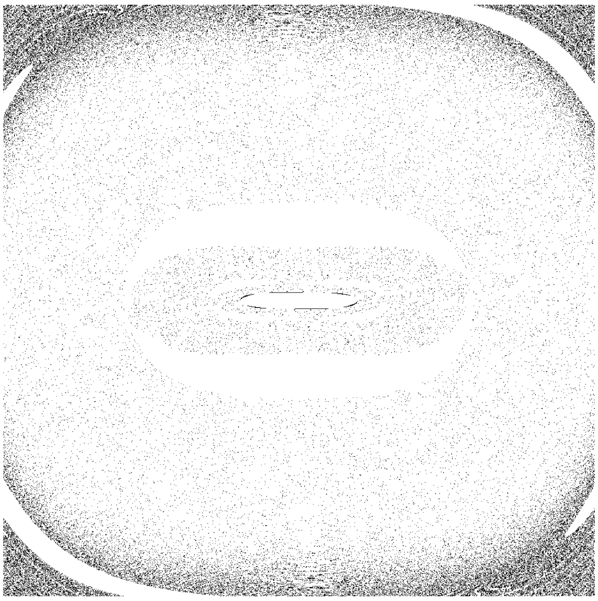


Figure 17: Basin of attraction of the periodic orbit with frequency $\omega = 1/6$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

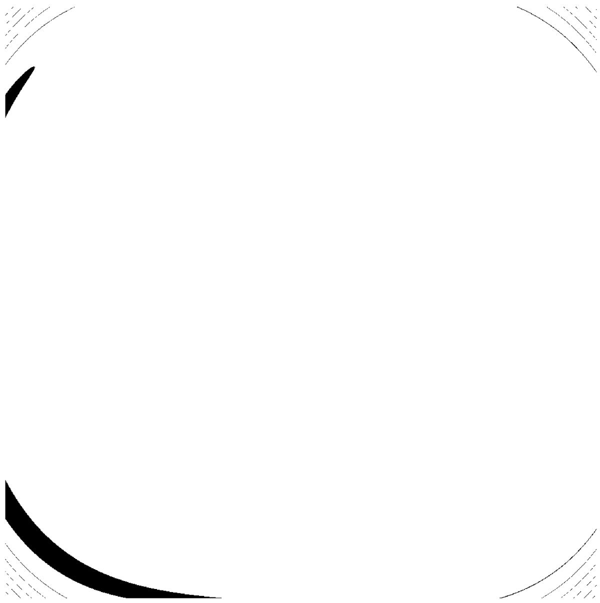


Figure 18: Basin of attraction of the first periodic orbit with frequency $\omega = 1$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

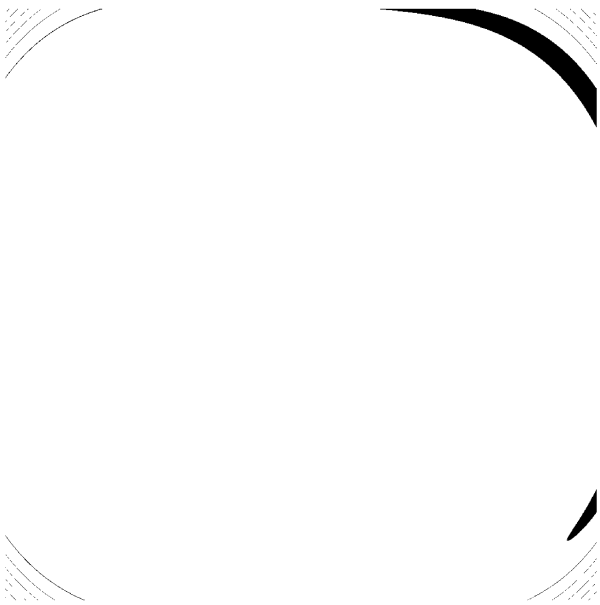


Figure 19: Basin of attraction of the second periodic orbit with frequency $\omega = 1$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

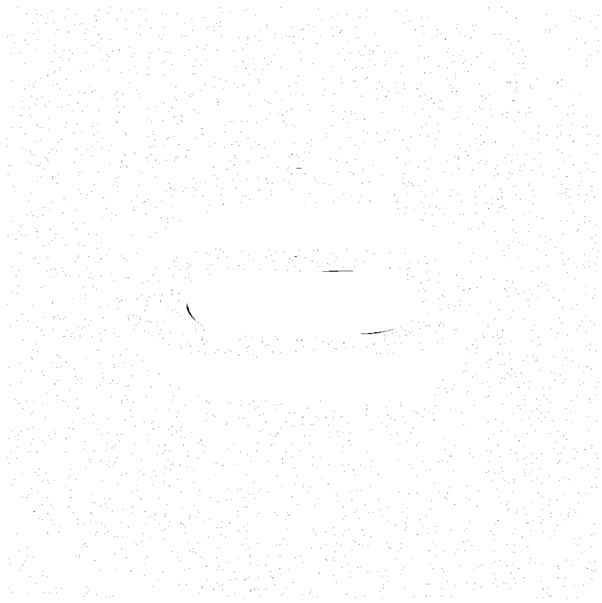


Figure 20: Basin of attraction of the first periodic orbit with frequency $\omega = 1/3$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

Figure 21: Basin of attraction of the second periodic orbit with frequency $\omega = 1/3$ contained in the square $\mathcal{Q} = \{(x, \dot{x}) \in [-1, 1] \times [-1, 1]\}$ for $\varepsilon = 0.1$ and $\gamma = 0.0005$ in (1.1).

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