INTERFACES OF GROUND STATES IN ISING MODELS WITH PERIODIC COEFFICIENTS

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Abstract. We study the interfaces of ground states of ferromagnetic Ising models with external fields. We show that, if the coefficients of the interaction and the magnetic field are periodic, the magnetic field has zero flux over a period and is small enough, then for every plane, we can find a ground state whose interface lies at a bounded distance of the plane. This bound on the width of the interface can be chosen independent of the plane.

We also study the average energy of the plane-like interfaces as a function of the direction. We show that there is a well defined thermodynamic limit and that it enjoys several convexity properties.

1. Introduction

The goal of this paper is to study the interfaces of Ising models in which the material has a periodic structure and is subject to a weak magnetic field, also periodic, with zero mean flux and not too strong.

Roughly speaking, we will show (see Theorem 2.5 for a precise statement) that such models possess ground states whose interface is plane-like (i.e., contained between two parallel planes). The orientation of these interfaces is arbitrary and furthermore the width of the strip containing the interface can be chosen to be independent of the orientation. We will also show that there is a well defined limit of the average energy of the interface.

Results of similar to those above have been proved for minimal surfaces in [CdlL01]. The results presented here are very similar to those above because the energy of an Ising model is closely related to the area of the interface. Indeed, the proofs follow roughly the same lines.

Nevertheless, because of the discrete nature of the model, some of the technical arguments are easier In particular, the density estimates needed in the present paper are trivial. Of course, other arguments related to calculus are not present. And indeed, as we will see, several of the results that are established for continuous models in [CdlL03a], [CdlL03b] are false even for the standard Ising model.

2. Notation and statement of results

2.1. **Notation on Ising models.** We refer to [Rue99],[Isr79], [Sim93] for more information on statistical mechanics models. Nevertheless, in this paper, we will consider only ground states – zero temperature – and we will include most of the notation that we use.

The Ising models we will consider will be defined on a lattice $\mathbb{Z}^d$ which, for convenience in some geometric arguments, we will consider as contained in $\mathbb{R}^d$. We will consider the lattice endowed with the usual ℓ^1 distance.

A configuration s will be a mapping $s : \mathbb{Z}^d \to \{+1, -1\}$. We will denote by $\mathcal{C}$ the space of configurations.

Given a configuration s, we will denote by ∂s the interface of the configuration. That is

$$\partial s = \{ i \in \mathbb{Z}^d \mid s_i = +1, \exists\, j \quad \text{s. t. } |i - j| = 1, s_j = -1 \} \tag{1}$$

The behavior of an Ising model is described by a (formal) functional on configurations.

$$H(s) = \sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j| \le R}} J_{ij}(s_i s_j - 1) + \sum_{i \in \mathbb{Z}^d} h_i s_i \tag{2}$$

In the classical Ising model, $J_{ij} = 1$ but in this paper, we want to consider more general models, in particular, we do not want to keep translation invariance by all vectors, even if we will assume some periodicity by some sublattice of vectors.

Given $\omega \in \mathbb{R}^d$, we denote $\Pi_\omega = \{x \in \mathbb{R}^d | \omega \cdot x = 0\}$. Clearly, $\Pi_\omega = \Pi_{\omega'}$ when ω is a multiple of ω'.

Remark 2.1. Some of the results that we will discuss go through for somewhat more general models in which the interaction may be three or more bodies or the lattice does not need to be an Euclidean lattice but rather in a richer geometric framework considered in [CdlL98].

We will not discuss such generalizations here, nevertheless, we point out that these generalizations could be necessary to make contact with continuum models

The number R is referred to as the range of the interaction. In the classical Ising models, the range is 1, which corresponds to only nearest neighbor interactions. In this paper, we will only consider finite range interactions, but the existence results go through for infinite range interactions by taking limits.

Given a set $\Gamma \subset \mathbb{Z}^d$ and a number R we denote Γ^R the set of points in Γ whose distance to Γ is smaller or equal than R. When R is the range of the interaction, Γ^R is the collection of sites that can interact with the sites in Γ.

Given a finite set $\Gamma \subset \mathbb{Z}^d$, we define

$$H_\Gamma(s) = \sum_{\substack{i \in \Gamma,\, j \in \mathbb{Z}^d \\ |i-j| \le R}} J_{ij}(s_i s_j - 1) + \sum_{i \in \Gamma} h_i s_i \ .$$

The most important definition for us is

Definition 2.1.1. *We say that a configuration s is a ground state when*

$$H_\Gamma(u) \ge H_\Gamma(s)$$

for all u that agree with s in $(\mathbb{Z}^d - \Gamma)^R$.

Note that Definition 2.1.1 only uses finite sums, so that the formal character of the sums (2), does not matter. We also note that the notion of ground state – quite customary in Physics – is also very similar to the notion of class A minimizer in [Mor24].

We recall that there is an equivalent description of the energy of Ising models, which makes the connection with geometric questions clearer, namely, the description of a state in terms of contours.

A configuration can be described by indicating the set

$$\mathcal{S}(s) = \{ j \in \mathbb{Z}^d \mid s_j = +1 \} \tag{3}$$

As it is customary in statistical mechanics, the boundaries of the set $\mathcal{S}$ can be described geometrically by placing a unit plaque perpendicularly across each bond joining $\mathcal{S}$ with its complement.

Notice that ∂s, the interface of the configuration s, is very similar to the boundary of the set $\mathcal{S}(s)$.

Remark 2.2. In the language of contours, the theory of ground states is very similar to the theory of minimal surfaces as formulated in the language of *sets of finite perimeter*.

As an illustrative example, when $J_{ij} = 1$

$$\sum_{\substack{|i-j|=1 \\ i,j \in \mathbb{Z}^d}} J_{ij}(s_i s_j - 1) \ ,$$

is twice the area of the contour describing $\mathcal{S}$, and ground states correspond to surfaces whose area cannot be decreased by making local modifications. Hence, ground states can be considered as discrete analogues of minimal surfaces. The terms h can be interpreted as some volume terms, so that the ground states are discrete analogues of prescribed curvature.

For the experts, we also mention that there is a theory of minimal surfaces based on studying surfaces as boundaries of sets (an account of this theory can be found in [Giu84] and it was the basic language of [CdlL01].) The analogue of the sets of finite perimeter in the geometric theory is the sets $\mathcal{S}(s)$ associated to the configurations.

As it turns out, the proofs of several of the results will be follow the strategy for the results in [CdlL01], which were formulated in this language. Of course the details of the proofs will have to be different since many methods from calculus are not available. Indeed, as we will see in Section 5, there are examples that show that the straightforward analogues of the results in [CdlL03a], [CdlL03b] in this discrete situation are false.

Remark 2.3. We recall that there are two physical interpretations that are reasonable for these models. One is that the s_i are the states of spin of an atom at site i. The other – usually called lattice gases – is that the s_i describe whether a site is occupied or not. In the first interpretation, the average energy of the ground state has the interpretation of a magnetic energy near a wall. In the second, it is a surface tension.

The physical interpretation in terms of lattice gases is remarkably close to being a discrete version of the theory of sets of finite perimeter.

2.2. **The assumptions of this paper.** We will consider systems of the form (2) such that they are

H1. Periodic of period N. That is:

$$J_{i+e,j+e} = J_{ij} \ \forall\ e \in N\mathbb{Z}^d$$
$$h_{i+e} = h_{i+e} \ \forall\ e \in N\mathbb{Z}^d$$

H2.
- Weakly ferromagnetic. That is:
$$J_{ij} \le 0.$$
- There is a $c < 0$ such that for each site i, there is one j such that
$$J_{ij} \le c.$$

H3. The magnetic field h has zero flux

$$\sum_{i \in F} h_i = 0$$

where F is a fundamental domain for $\mathbb{Z}^d/N\mathbb{Z}^d$. That is, $F = \{0, 1, \ldots, N-1\}^d \subset \mathbb{Z}^d$.

H4. $\sup |h_i|$ sufficiently small.

Remark 2.4. Sometimes, in statistical mechanics one uses a hypothesis significantly stronger that *H2*. namely

There is a $c < 0$ such that for each site i, for all the j such that $|i - j| = 1$,

$$J_{ij} \leq c$$

Which is clearly satisfied by the classical Ising model. This hypothesis does not lead to improvements in our results.

The first main result of this paper is the following

Theorem 2.5. *Given any Ising model satisfying* H1, H2, H3, H4 *above there is M such that for every hyperplane $\Pi_\omega \subset \mathbb{R}^d$ of normal vector ω, we can find a ground state s_ω whose interface ∂s_ω is contained in a strip of width M around the plane Π_ω.*

That is:

$$d(\partial s_\omega, \Pi_\omega) \leq M \tag{4}$$

As we will see later, there are other properties which we will prove about the interface of the ground states with appear in the conclusions of Theorem 2.5 notably that the interfaces satisfy a so-called "Birkhoff property" (see Proposition 3.1.2) which plays an important role in Aubry-Mather theory. It was introduced in [Mat82], [ALD83] and similar properties appear in [Mor24], [Hed32]. As it turns out, the Birkhoff property for some minimizers does not need even the full H2 and it suffices that the system is weakly ferromagnetic.

Note that Theorem 2.5 only claims that there exists ground states satisfying the conclusion. As we will see, even for the classical Ising model in $d = 2$, when the interface is not oriented along the coordinate axis, it is possible to obtain ground states which do not satisfy the conclusions of Theorem 2.5

We will refer to ground states satisfying (4) for some M as plane-like ground states. Note that in Theorem 2.5 we show that the M can be chosen uniformly for all orientations.

We will also prove another result giving the existence of an average interface energy for all the plane like minimizers.

Theorem 2.6. *In the assumptions of Theorem 2.5.*

Let Σ be a compact set of $\mathbb{R}^d$ with C^1 boundary. For $\lambda \in \mathbb{R}^+$, Denote by $\lambda\Sigma = \{x \in \mathbb{R}^d | (1/\lambda)x \in \Sigma\}$.

For $\omega \in \mathbb{R}^d$, $|\omega| = 1$,

Let s be a ground state whose interface lies at a bounded distance from the plane Π_ω.

Then, we have:

$$\lim_{\lambda\to\infty} H_{\lambda\Sigma}(s)/\lambda^{d-1} = |\Sigma \cap \Pi_\omega|_{d-1} A(\omega) \tag{5}$$

where $|\ \ |_{d-1}$ *denotes the* $d-1$ *surface area.*

Note that $A(\omega)$ *is independent of* Σ *and* s*. It is a property only of the model.*

Moreover, the function A*, when extended to* $\mathbb{R}^d$ *as a positively homogeneous function of degree* 1 *(i.e.* $A(\lambda\omega) = \lambda A(\omega)$ *for* $\lambda \in \mathbb{R}^+$*) is convex.*

The limit in (5)*, is reached very uniformly. If* Σ *is* C^1*. There exists a constant* Ω_Σ *depending on* Σ *but independent of* ω*,* s *such that*

$$|H_{\lambda\Sigma}(s)/\lambda^{d-1} - |\Sigma \cap \Pi_\omega|_{d-1} A(\omega)| \le \Omega_\Sigma \lambda^{-1/2} \tag{6}$$

The exponent $-1/2$ in the remainder in (6) is not optimal. Also it seems that one can relax the regularity requirements on the surface Σ. The only thing required is that one can approximate it well by cubes.

The physical meaning of $A(\omega)$ is the density of magnetic energy of the interface. In the lattice gas interpretation of the model, $A(\omega)$ is a surface tension. The homogeneity is natural if we think of ω as being a *"surface element"*. That is a vector oriented along the normal and with modulus the area.

We note that $A(\omega)$ is also related to the average action in Aubry-Mather theory or to the stable-norm in the calculus of variations. Note, however that the discrete nature of the problem makes it impossible to use many of the arguments customary in these theories. Indeed, some of the results obtain in the continuous cases are false for the discrete cases considered here.

3. Proof of Theorem 2.5

The strategy of proof will be very similar to that of [CdlL01]. We will establish the existence of some particular minimizers first for rational ω but we will establish enough uniform bounds for the interface of these special ground states so that we will be able to pass to the limit of irrational frequencies.

The first step will be to consider minimizers among configurations which are periodic and which satisfy some constraints. Among them, we will consider a particular one, which will enjoy special properties.

3.1. **Notation.**

First we will develop some notation which will allow us to work comfortably with translations, periodicities, fundamental domains, multiplying fundamental domains, etc.

3.1.1. *Translations.* We introduce the translation operators $\mathcal{T}_k$, $k \in \mathbb{Z}^d$ acting on configurations

$$(\mathcal{T}_k s)_{i+k} = s_i \ \forall \ i$$

An important property of the models as in (2) satisfying periodicity is that formally for all configurations s ,

$$H(\mathcal{T}_k s) = H(s) \ \forall \ k \in N\mathbb{Z}^d \ .$$

in the sense that all the terms that appear on one side appear on the other.

A precise form of the above is that, for every finite set Γ and for every configuration s we have

$$H_\Gamma(\mathcal{T}_k s) = H_{\Gamma+k}(s) \ \forall \ k \in N\mathbb{Z}^d \ . \tag{7}$$

The equation (7) can be established readily noting that it is just a change in the dummy variables in the sum.

For sets Γ, we introduce the notation

$$\mathcal{T}_k \Gamma = \Gamma + k$$

Note that this consistent with the application of $\mathcal{T}_k$ to the characteristic function of Γ. With this notation, (7) can be written as

$$H_{\mathcal{T}_k \Gamma}(\mathcal{T}_k s) = H_\Gamma(s)$$

3.1.2. *Symmetries.* From now on and until further notice, we will consider $\omega \in (LN)^{-1}\mathbb{Z}^d)$ where N is the period of the model and $L \in \mathbb{N}$. The frequencies of $N^{-1}\mathbb{Z}^d$ are the frequencies that correspond to planes in the lattice given by fundamental domains of the symmetries of the model. The L^{-1} factor is means that we will be considering subharmonics.

We will prove our results for frequencies of this type and obtain estimates which are rather uniform. This will allow to extend the results to $\omega \in \mathbb{R}^d$.

We denote by $\mathcal{R}_\omega$ the module

$$\mathcal{R}_\omega = \{k \in N\mathbb{Z}^d \mid \omega \cdot k = 0\}$$

where $\omega \cdot k$ denotes the usual inner product. We note that $\mathcal{R}_\omega$ is a $d-1$ dimensional module.

Given a module $\mathcal{R} \subset \mathbb{Z}^d$ we denote by $\mathcal{F}_\mathcal{R} = \mathbb{Z}^d/\mathcal{R}$ a fundamental domain of the translations in $\mathcal{R}$. If $\mathcal{R}$ is a $d-1$ dimensional module $\mathcal{F}_\mathcal{R}$ can be considered as a discrete version of $\mathbb{R}^{d-1}/\mathcal{R} = \mathbb{T}^{d-1} \times \mathbb{R}$.

In the case of $\mathcal{R} = \mathcal{R}_\omega$ we will denote simply $\mathcal{F}_\omega$ rather than $\mathcal{F}_{\mathcal{R}_\omega}$. In the case of $\mathcal{R} = L\mathbb{Z}^d$, $L \in \mathbb{N}$, we will denote $\mathcal{F}_{L\mathbb{Z}^d}$ as $\mathcal{F}_L$. Note that with this notation $\mathcal{F}_N$ is just a fundamental domain for the system under the translations assumed to exists in *H1*.

If $\mathcal{R} = L\mathcal{R}_\omega$, $L \in \mathbb{N}$ we will denote $\mathcal{F}_{L\mathcal{R}_\omega} = \mathcal{F}_{L,\omega}$. The sets

$$\mathcal{F}_\omega^A = \{i \in \mathcal{F}_\omega \mid 0 \le \omega \cdot i \le A|\omega|\}$$
$$\mathcal{F}_{L,\omega}^A = \{i \in \mathcal{F}_{L,\omega} \mid 0 \le \omega \cdot i \le A|\omega|\}$$

are finite sets. We note that $\mathcal{F}_\omega^A$, $\mathcal{F}_{L,\omega}^A$ are invariant under translations in $\mathcal{R}_\omega$ and $L\mathcal{R}_\omega$, respectively.

Again, we note that $\mathcal{F}_{L,\omega}^A$ is a covering — in the directions perpendicular to ω of $\mathcal{F}_L^A$.

3.1.3. *Symmetric configurations.* Given a $\mathbb{Z}$-module $\mathcal{R}$ we denote by $\mathcal{P}_\mathcal{R}$ the set of configurations which are invariant under translations in $\mathcal{R}$

$$\mathcal{P}_\mathcal{R} = \left\{ s \in \mathcal{C} \;\middle|\; s_{i+k} = s_i \;\forall\, i \in \mathbb{Z}^d,\, k \in \mathcal{R} \right\}$$

In the case of $\mathcal{R} = \mathcal{R}_\omega$ we will denote $\mathcal{P}_{\mathcal{R}_\omega} = \mathcal{P}_\omega$. Similarly $\mathcal{P}_{L,\omega} = \mathcal{P}_{L\mathcal{R}_\omega}$.

We will also consider

$$\mathcal{P}_{L,\omega}^A = \left\{ s \in \mathcal{P}_{L,\omega} \,\middle|\, s_i = -1 \text{ when } \omega \cdot i > A|\omega|, \quad s_i = +1 \text{ when } \omega \cdot i < 0 \right\}$$

These classes of configurations consist of configurations which are periodic in the directions parallel to the plane and satisfy boundary conditions on the top and the bottom of the slab of width A parallel to the plane Π.

When $L = 1$ we will simply write $\mathcal{P}_\omega^A$.

Note that a configuration in $\mathcal{P}_{L,\omega}^A$ is determined when we prescribe it in the finite set $\mathcal{F}_{L,\omega}^A$. (We can determine for all the other points either by using the periodicity in the translations or by the boundary conditions.)

Note that the classes $\mathcal{P}_\omega$ above involve not only periodicity but also some boundary conditions. (We have taken the convention that ω is oriented in the sense in which the conditions go from positive to negative. Of course, since we are considering ω an arbitrary vector, taking the opposite convention just amounts to changing ω into $-\omega$.

When the magnetic field is not present, it is easy to see that changing s into $-s$ does not change the energy, hence, all the results will be the same when we change ω into $-\omega$ nevertheless, when $h \not\equiv 0$, in general, the results could change when ω changes into $-\omega$.

We will eventually take A to ∞ but, as it is well known in statistical mechanics some information about the boundary remains.

3.1.4. *Operations on configurations.* We introduce the notation

$$(s \wedge t)_i = \min(s_i, t_i) \qquad i \in \mathbb{Z}^d$$
$$(s \vee t)_i = \max(s_i, t_i) \qquad i \in \mathbb{Z}^d \ .$$

Given any configurations s, t we can write:

$$\begin{aligned} s &= s \wedge t + \alpha \\ t &= s \wedge t + \beta \\ s \vee t &= s \wedge t + \alpha + \beta \end{aligned} \tag{8}$$

with $\alpha, \beta \geq 0$.

Note that $s + t = s \vee t + s \wedge t$. We also note that if $s, t \in \mathcal{P}^A_{L,\omega}$, then $s \wedge t$, $s \vee t \in \mathcal{P}^A_{L,\omega}$.

In comparing with [CdlL01] it is useful to observe that if we use the description of configurations by sets as in (3), we have

$$\begin{aligned} \mathcal{S}(s \vee t) &= \mathcal{S}(s) \cap \mathcal{S}(t) \\ \mathcal{S}(s \wedge t) &= \mathcal{S}(s) \cup \mathcal{S}(t) \end{aligned} \tag{9}$$

3.2. **Minimizers and infimal minimizers.** Now, we turn our attention to the problem of producing minimizers in spaces of periodic configurations. The goal of this section is to produce a minimizer that enjoys some remarkable properties.

We call attention to the fact that the results of this section work under the assumptions of weak ferromagnetism and do not require the fact that the interaction is non-degenerate.

Since configurations on $\mathcal{P}^A_{L,\omega}$ are determined by the values on a finite set on $\mathcal{P}^A_{L,\omega}$ it is natural to consider the functional $H_{\mathcal{F}^A_{L,\omega}}(S)$.

Since $\mathcal{F}^A_{L,\omega}$ is finite it is clear that $H_{\mathcal{F}^A_{L,\omega}}$ reaches its minimum. It can well happen that there are several configurations which achieve the minimum.

Note that the minimizers, minimizes the functional $\mathcal{F}^A_{L,\omega}$ among configurations in $\mathcal{P}^A_{L,\omega}$. but at this state of the argument, there is not reason why they should be minimizers with respect to more general perturbations that have less periodicity or that violate the other constraints. Hence, the minimizers could fail to be ground states. This is, of course, a manifestation of symmetry breaking.

Hence, we will select a particular minimizer (infimal minimizer) that enjoys special properties. In particular, we will show that this infimal minimizer does not experience symmetry breaking and that enjoys a property analogous to the property called Birkhoff property in dynamical systems.

A good deal of the argument later will be precisely showing that there is no symmetry breaking for the infimal minimizer. This will have as a consequence that all the minimizers remain as minimizers under multiplication of the period. This is not completely obvious because, as we will see there are more minimizers when we increase the period. We hope that the examples in Section 5 will clarify this situation.

This will require that we take advantage of properties of the functional. We start by observing that the functional H defining the models has a quadratic part, a linear part, and a constant. Namely:

$$\begin{aligned} Q_\Gamma(s) &= \sum_{\substack{i\in\Gamma \\ j\in\mathbb{Z}^d \\ |i-j|\le R}} J_{ij} s_i s_j \\ L_\Gamma(s) &= \sum_{i\in\Gamma} h_i s_i \\ C_\Gamma &= \sum_{i\in\Gamma} 1 \end{aligned}$$

We will also introduce the notation

$$Q_\Gamma(s,t) = \sum_{\substack{i\in\Gamma \\ j\in\mathbb{Z}^d \\ |i-j|\le R}} s_i t_j$$

so that $Q_\Gamma(s) = Q_\gamma(s,s)$.

With the notations above, we have the following identity

$$H_\Gamma(s\wedge t) + H(s\vee t) = H_\Gamma(s) + H_\Gamma(t) + Q_\Gamma(\alpha,\beta) \tag{10}$$

where α, β are given in (8).

Under the hypothesis of ferromagnetism, for all $\alpha, \beta \ge 0$ we have:

$$Q_\Gamma(\alpha,\beta) \le 0 \tag{11}$$

because $\alpha_i\beta_i \ge 0$, $J_{ij} \le 0$.

Therefore we have:

Proposition 3.0.1. *If s, t are minimizers of $H_{\mathcal{F}^A_{L,\omega}}$ in $\mathcal{P}^A_{L,\omega}$, then so are $s\vee t$, $s\wedge t$. In particular, there is an infimal minimizer defined by:*

$$s^A_{L,\omega} = \min_{s\in\text{Minimizers}} s \tag{12}$$

Proof. Note that $s\vee t$, $s\wedge t$ are configurations with the same periodicity as s, t. hence, by s, t being minimizers, we have

$$\begin{aligned} H_\Gamma(s\wedge t) &\ge H_\Gamma(s) = H_\Gamma(t) \\ H_\Gamma(s\vee t) &\ge H_\Gamma(s) = H_\Gamma(t) \end{aligned}$$

On the other hand, using (10) and (11), we have:

$$H_\Gamma(s \wedge t) + H_\Gamma(s \vee t) \le H_\Gamma(s) + H_\Gamma(t)$$

Therefore,

$$H_\Gamma(s \wedge t) = H_\Gamma(s \vee t) = H_\Gamma(s) + H_\Gamma(t)$$

and $s \wedge t$, $s \vee t$ are minimizers. □

Clearly, once we prescribe $\mathcal{R}$, A, the infimal minimizer is unique since it is given by the formula (12).

This has the important consequence that there is no symmetry breaking (Proposition 3.0.2) which in turn will lead to the fact that S^A_ω is a minimizer against configurations that respect the boundary conditions (Proposition 3.1.1).

The physical interpretation of the infimal minimizer is that it would be the minimizer if we introduced a very small magnetic field (or an small pressure in the lattice gas interpretation) but maintained the lower constraint.

3.2.1. *Absence symmetry breaking.* In the following proposition, we show that for any $K \in \mathbb{N}$, if we consider perturbations with K-times the period, the infimal minimizer is also a minimizer among those. Indeed, it is the infimal minimizer for functions with K period.

Proposition 3.0.2. *Let $K, M \in \mathbb{N}$. Denote $L = K \cdot M$. Let $A \in \mathbb{R}^+$. Then*

$$s^A_{L,\omega} = s^A_{M,\omega} \tag{13}$$

Proof. We define

$$\widetilde{s} = \bigwedge_{k \in M\mathcal{R}_\omega / L\mathcal{R}_\omega} \mathcal{T}_k s^A_{L,\omega}$$

since $0 \in M\mathcal{R}_\omega / L\mathcal{R}_\omega$ we have $\widetilde{s} \le s^A_{L,\omega}$.

It is important to note that $\widetilde{s} \in \mathcal{P}^A_{M,\omega}$.

Since $\mathcal{T}_k$, $s^A_{L,\omega}$ are minimizers in $\mathcal{P}^A_{L,\omega}$, we obtain, applying Proposition 3.1.1, that $\widetilde{s}$ is a minimizers in $\mathcal{P}^A_{L,\omega}$.

From the definition of infimal minimizer we obtain

$$\widetilde{s} \ge s^A_{L,\omega}$$

which with the observation after the definition implies $\widetilde{s} = s^A_{L,\omega}$.

Using that $s^A_{L,\omega}$ and $s^A_{M,\omega}$ are minimizers of their respective functionals we obtain

$$H_{\mathcal{F}^A_{L,\omega}}(s^A_{L,\omega}) \le H_{\mathcal{F}^A_{L,\omega}}(s^A_{M,\omega})$$

$$H_{\mathcal{F}^A_{M,\omega}}(s^A_{M,\omega}) \le H_{\mathcal{F}^A_{M,\omega}}(s^A_{L,\omega})$$

On the other hand, for configurations $s \in \mathcal{P}^A_{M,\omega}$ we have

$$H_{\mathcal{F}^A_{L,\omega}}(s) = \#\left(\frac{M\mathcal{R}_\omega}{L\mathcal{R}_\omega}\right) H_{\mathcal{F}^A_{L,\omega}}(s) \tag{14}$$

Using (16) and (14) we obtain

$$H_{\mathcal{F}^A_{L,\omega}}(s^A_{L,\omega}) = H_{\mathcal{F}^A_{L,\omega}}(s^A_{M,\omega})$$
$$H_{\mathcal{F}^A_{M,\omega}}(s^A_{L,\omega}) = H_{\mathcal{F}^A_{M,\omega}}(s^A_{M,\omega})$$

Hence we obtain that $s^A_{L,\omega}$ is a minimizer in $\mathcal{P}^A_{M,\omega}$ and $s^A_{M,\omega}$ is a minimizer in $\mathcal{P}^A_{L,\omega}$.

Therefore, using the definition of infimal minimizer, we obtain

$$s^A_{L,\omega} \geq s^A_{M,\omega}$$
$$s^A_{M,\omega} \geq s^A_{L,\omega}$$

and therefore, the claim of Proposition 3.0.2. □

As a corollary of Proposition 3.0.2, we obtain:

Corollary 3.0.1. *All minimizers in $\mathcal{P}^A_{\mathcal{R}_\omega}$ are minimizers in $\mathcal{P}^A_{K\mathcal{R}_\omega}$*

The proof is simply observing that the energy of a minimizer with a certain period is the same as that of the infimal minimizer.

Hence, if we consider a minimizer u with unit period its energy in the unit period will be the same as that of the infimal minimizer of unit period. Since the infimal minimizer of period K is just K^d copies of the infimal minimizer, we obtain that the energy of the minimizers with period K is K^d times the energy of a minimizer of period 1, which is the same as the energy of considering u in period K. Hence, u is also a minimizer in period K. □

Remark 3.1. The phenomenon that minimizers under perturbations of one period are not minimizers under perturbations of a longer period – hence the energy of the minimizer decreases with the period – happens in many variational problems. It appears already in [Hed32].

This phenonmenon often prevents to take the limit of minimizers when we change the period to an irrational period.

Note that the argument above implies that if there is a way of selecting a unique minimizer, the Hedlund phenomena does not happen.

The Corollary 3.0.1 is somewhat surprising since we will see in Section 5 that, even in the classical Ising model, there are more minimizers in $\mathcal{P}^A_{K\mathcal{R}_\omega}$ than in $\mathcal{P}^A_{\mathcal{R}_\omega}$.

Notice that, since K is arbitrary, it immediately follows from Proposition 3.0.2. given any perturbation of $s^A_{L,\omega}$ of bounded support, we can find a K large enough so that it can be considered as a perturbation in a fundamental domain of the KN perturbation. Hence, we have established:

Proposition 3.1.1. *$s^A_{L,\omega}$ is a class-A minimizer among the configurations in $\mathcal{P}^A_\omega$.*

That is, s^A_ω is a minimizer for all the functions that satisfy the boundary conditions, irrespective of periodicity. Given the fact that $S^A_{L,\omega}$ is independent of L we will just use the notation S^A_ω from now on.

3.2.2. *The Birkhoff property.* The following property of the infimal minimizer is quite analogous to a property that is commonly called "Birkhoff property" in dynamical systems.

In the following Proposition 3.1.2 we prove it for the infimal minimizer.

Proposition 3.1.2. *Let s^A_ω be the infimal minimizer as before (in particular, recall that $\omega \in \frac{1}{N}\mathbb{Z}^d$)*

Let $k \in N\mathbb{Z}^d$ then,

$$\begin{aligned} \mathcal{T}_k s^A_\omega &\le s^A_\omega \qquad k\cdot\omega \le 0 \\ \mathcal{T}_k s^A_\omega &\ge s^A_\omega \qquad k\cdot\omega \ge 0 \end{aligned} \tag{15}$$

Proof. Because of (7) $\mathcal{T}_k s^A_\omega$ is a minimizer for $H_{\mathcal{T}_k\mathcal{F}^A_\omega}$. We note that $\mathcal{T}_k s^A_\omega \in \mathcal{P}^A_\omega$.

We will prove the inequality (15) for $\omega\cdot k \le 0$. The other case will be identical.

Note that $i\cdot\omega \le 0$ implies $(i+k)\cdot\omega = 0$. Hence, $(\mathcal{T}_k s^A_\omega)_i = (s^A_\omega)_{i+k} = +1$. Therefore,

$$s^A_\omega \wedge \mathcal{T}_k s^A_\omega \in \mathcal{P}^A_\omega \ .$$

Similarly, we obtain that

$$s^A_\omega \vee \mathcal{T}_k s^A_\omega \subset \mathcal{T}_k \mathcal{P}^A_\omega$$

We have therefore

$$\begin{aligned} H_{\mathcal{F}^A_\omega}(s^A_\omega \wedge \mathcal{T}_k s^A_\omega) &\ge H_{\mathcal{F}^A_\omega}(s^A_\omega) \\ H_{\mathcal{T}_k\mathcal{F}^A_\omega}(s^A_\omega \wedge \mathcal{T}_k s^A_\omega) &\ge H_{\mathcal{T}_k\mathcal{F}^A_\omega}(\mathcal{T}_k s^A_\omega) \end{aligned} \tag{16}$$

We note that

$$\begin{aligned} \mathcal{F}^A_\omega &\subset \mathcal{F}^{A-k\cdot\omega/|\omega|}_\omega \\ \mathcal{T}_k\mathcal{F}^A_\omega &\subset \mathcal{F}^{A-k\cdot\omega/|\omega|}_\omega \end{aligned}$$

Moreover, denoting $\Gamma \equiv \mathcal{F}_\omega^{A-k\cdot\omega/|\omega|}$, the periodicity and the zero flux condition imply that

$$
\begin{aligned}
H_\Gamma(s) &= H_{\mathcal{F}_\omega^A}(s)\ \forall\ s \in \mathcal{P}_\omega^A \\
H_\Gamma(s) &= H_{\mathcal{T}_k\mathcal{F}_\omega^A}(s)\ \forall\ s \in \mathcal{T}_k\mathcal{P}_\omega^A
\end{aligned} \tag{17}
$$

The reason for this equality is that in $\Gamma - \mathcal{F}_\omega^A$, because of the boundary conditions, the quadratic interaction term does not give any contribution. The contribution of the magnetic field term is zero because of the zero flux condition. Hence (16) becomes:

$$
\begin{aligned}
H_\Gamma(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) &\geq H_\Gamma(s_\omega^A) \\
H_\Gamma(s_\omega^A \vee \mathcal{T}_k s_\omega^A) &\geq H_\Gamma(\mathcal{T}_k s_\omega^A)
\end{aligned}
$$

Using (11), we obtain:

$$
\begin{aligned}
H_\Gamma(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) &\geq H_\Gamma(s_\omega^A) \\
H_\Gamma(s_\omega^A \vee \mathcal{T}_k s_\omega^A) &\geq H_\Gamma(s_\omega^A)
\end{aligned}
$$

Using again (17) we obtain:

$$
H_{\mathcal{F}_\omega^A}(s_\omega^A \wedge \mathcal{T}_k s_\omega^A) \geq H_{\mathcal{F}_\omega^A}(s_\omega^A)
$$

Therefore $s_\omega^A \wedge \mathcal{T}_k s_\omega^A$ is a minimizer. Since s_ω^A is the infimal minimizer, we obtain

$$
s_\omega^A \wedge \mathcal{T}_k s_\omega^A \geq s_\omega^A
$$

Therefore

$$
\mathcal{T}_k s_\omega^A \geq s_\omega^A
$$

which is the desired conclusion.

The case $\omega \cdot k \leq 0$ is proved exactly in the same way. □

Remark 3.2. We note that Propositions 11 and 3.1.2 have a natural geometric interpretation in terms of perimeters of contours. For example, the conclusion Proposition 11 reads:

$$
\operatorname{Per}(\mathcal{S}_1 \cup \mathcal{S}_2) + \operatorname{Per}(\mathcal{S}_1 \cap \mathcal{S}_2) \leq \operatorname{Per}(\mathcal{S}_1) + \operatorname{Per}(\mathcal{S}_2)
$$

Such interpretations appear naturally in the geometric measure theory problems considered in [CdlL01].

3.3. **Bounding the oscillation of the infimal minimizer.** To finish the proof of Theorem 2.5, we will just need to show that, if we take A large enough – but independent of the orientation –, the infimal minimizer will be an unconstrained minimizer and will not touch the boundaries.

The basic idea is that a minimizer cannot oscillate too much in an small scale since this will force it to have a very large energy in this scale and one can easily produce configurations with smaller energy. Using the Birkhoff property, we will use this information to control the large scale limit.

More precisely, our goal is to show

Lemma 3.2.1. *There exists an M large enough (independent of ω) such that for any $A \geq M$, we have:*

$$s^A_\omega = s^M_\omega$$

Lemma 3.2.1 shows that the infimal minimizer s^M_ω is completely unconstrained.

Indeed, if there was a periodic configuration u such that $H(u) \leq H(s)$ and $\partial u \subset \{i | -A|\omega| \leq i \cdot \omega \leq A|\omega|\}$, we see that for some $k \in \mathbb{Z}^d$ we have $\partial \mathcal{T}_k u \subset \{i| - A|\omega| \leq i \cdot \omega \leq A|\omega|\}$.

Hence,

$$H(u) \leq H(s^{2A}_\omega) = H(s^M_\omega).$$

In other words, the energy of the configuration s^M_ω cannot be lowered by compact perturbations.

Once we have that s^M_ω is a ground state and that its interface is contained in a strip of width independent of ω, we see that, given $\omega^* = \lim_{n\to\infty} \omega_n$ with $\omega_n \in \mathbb{Q}^d$ we can – by passing to a subsequence – obtain $s_{\omega^*} = \lim s_{\omega_n}$. This s_{ω^*} will be a ground estate and therefore, we have established Theorem 2.5 as soon as we prove Lemma 3.2.1.

The rest of this section is devoted to proving Lemma 3.2.1.

We introduce the notation for $\ell \in \mathbb{N}$, $x \in \mathbb{Z}^d$

$$\mathcal{C}^\ell_x \equiv \{0, \dots, \ell - 1\}^d + x\ .$$

That is $\mathcal{C}^\ell_x$ is a cube of side ℓ with the lower vertex at x.

A corollary of Proposition 3.1.2 is:

Proposition 3.2.1. *If $(s^A_\omega)_i = -1$ for all $i \in \mathcal{C}^N_x$, then,*

$$(s^A_\omega)_i = -1 \quad \forall\ i\ |\ |\omega \cdot i \geq x \cdot \omega + N\sqrt{d} \cdot |\omega|$$

Proof. By the Birkhoff property

$$(s^A_\omega)_i = -1\ \forall\ i \in \bigcup_{\substack{k \in N\mathbb{Z}^d \\ k\cdot\omega \geq 0}} \mathcal{C}^N_{x+k}$$

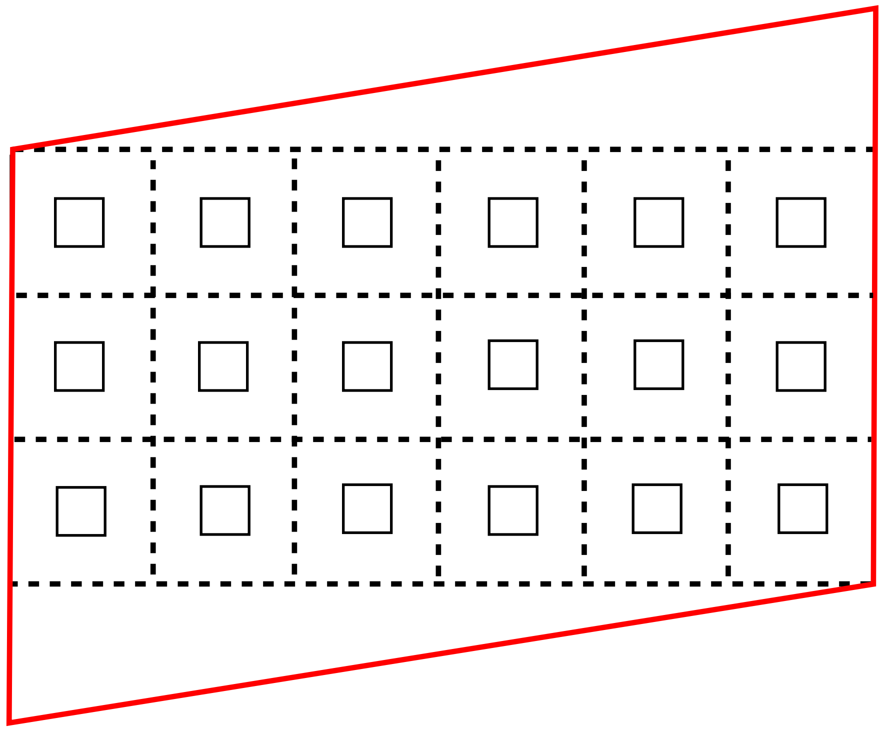

FIGURE 1. Illustration of the fundamental domain $\mathcal{F}^A_\omega$, the cubes $\mathcal{C}^N_x$ and the cubes $\mathcal{C}^N_y$ used in the proof of Proposition 3.2.2.

The set above is a collection of cubes of size N on a semilattice of size N. Hence, it contains a semiplane. □

In view of Proposition 3.2.1 to show that the interfaces of the infimal ground state s^A_ω is contained in a strip of uniform width M it suffices to show:

Proposition 3.2.2. *Assume that $A \geq M$ (where M can be chosen independent of ω) then there exists an $x \in \mathbb{Z}^d$ such that*

$$0 \leq \omega \cdot x \leq A - \sqrt{d}N$$
$$(S^A_\omega)_i = -1 \qquad i \in \mathcal{C}^N_x \ .$$

Proof. This will be a covering argument very similar to that used in [CdlL01] but somewhat simpler since the density estimates used in [CdlL01] are not needed in this case.

We will show that, we can bound the energy of a configuration from below by the number of cubes it touches multiplied by a constant. We also note that the energy of a configuration is bounded by the energy of a plane, which can be bounded from below by the area of the base of the strip times a constant. Moreover, the number of cubes in a strip is proportional to the area of the base of the strip multiplied by the height of the strip. The upshot of the discussion is that if the width of the strip is large enough (independent of the orientation), then there has to be a unit cube that is not touched by the interface. In the following we give a more formal proof.

Given a fundamental domain $\mathcal{F}_\omega^A$ we consider a collection of disjoint cubes centered in points x

$$\mathcal{C}_x^{3N}\ , \quad x \in 3N\mathbb{Z}^d$$
$$\mathcal{C}_x^{3N} \subset \mathcal{F}_\omega^A$$

such that $x \in 3N\mathbb{Z}^d$, $\mathcal{C}_x^{3N} \subset \mathcal{F}_\omega^A$. In each of the cubes $\mathcal{C}_x^{3N}$, we consider the cubes $\mathcal{C}_x^N$ with the same center x than $\mathcal{C}_x^{3N}$ but well inside $\mathcal{C}_x^{3N}$. We note that the cubes $\mathcal{C}_x^{3N}$ do not overlap and cover the fundamental domain rather completely except for a sliver near the edges.

We make several observations. The first two are purely geometric about the covering as indicated. The next two are involve the Hamiltonians and the properties of the ground states. Note that item *iii)* below uses the full strength of the assumption *H2*.

We can find a constant B (geometrically the area of the base of $\mathcal{F}_\omega^A$) such that

i) Denote by $\mathcal{B}$ the set

$$\mathcal{B} \equiv \mathcal{F}_\omega^A - \bigcup_{x\in\Sigma} \mathcal{C}_x^{3N}$$

(the set that is not covered by the cubes).

We have

$$\#\ \mathcal{B} < B\alpha$$

where $\#\mathcal{B}$ denotes the number of sites in $\mathcal{B}$.

This means that we can cover the whole fundamental domain by the cubes $\mathcal{C}_x^{3N}$ except for a thin sliver near the boundary.

The usual formula for the volume shows that it suffices to take $\alpha = \sqrt{d}\,3N$.

ii) Given M, we have that

$$\#\left\{x \mid d(\mathcal{C}_x^{3N}, \Pi) \leq M\right\} \geq BM\frac{1}{(3N)^d} - \frac{B\alpha}{(3N)^d}$$

Once we have item *i)* this result follows simply by noticing that each center has associated a cube of volume $(3N)^d$. So that the number of centers has to be bigger or equal than the total volume covered divided by the volume of each cube.

iii) Given any configuration s we have

$$H_{\mathcal{C}_x^{3N}}(s) \geq 0 \qquad H_{\mathcal{C}_{y(x)}^N}(s) \geq 0$$

If the cubes do not involve any interfaces, the result is obvious because we assumed that the flux of the magnetic field is zero, so that the final result is zero.

If there is an interface, by assumption *H2* there is one interaction term which is negative and bounded away from zero and the other interaction terms are positive. The other contribution to the energy is the the magnetic field over the incomplete box. By assumption *H4*, which says that the magnetic field is small enough, these terms cannot overcome the negative term which was bounded away from zero.

iv) In this term we make more precise the results before when there is an interface in the small cube.

Assume that

$$\partial s \cap \mathcal{C}_{y(x)}^N \neq \emptyset$$

Then

$$H_{\mathcal{C}_x^{3M}}(s) \geq \gamma$$

Observe that it suffices to take

$$\gamma = \inf_{|i-j|=1} J_{ij} - 3^d \Sigma |h_i|$$

v) Finally, we obtain a bound of the energy associated to the set $\mathcal{B}$ introduced in point i) which is not covered by the cubes.

$$H_{\mathcal{B}}(s) \geq -B\alpha \sup_i |h_i|$$

This is obvious because of the point i) and the interaction can be bounded from below by the magnetic field terms, which can be bounded from below as indicated.

Note that, under the assumption that h is small we have that the constant γ is strictly positive.

Note that the previous remarks give us a lower bound of the number of cubes (See item *ii*)). Note that this number grows with M. We also obtained a lower bound of the energy of the cubes $\mathcal{C}_x^{3N}$ for which $\mathcal{C}_x^{3N}$. Since the energy of the minimizer is bounded from above by a number independent of M, – by comparing e.g. with a plane –, we obtain that when M is large enough there is a cube that does not intersect the interface. We proceed to give some more details on the argument which will allow us to check that the width required is indeed independent on the orientation of the plane.

Proposition 3.2.3. *Denote by* $\mathcal{N}(s)^M$ *the number of cubes* $\mathcal{C}^N_{y(x)}$ *which intersect the interface of* s *and such that* $d(y(x),\pi) \le M$*. Then, we have for* $A \ge M$

$$H_{\mathcal{F}^A_\omega}(s) \ge \mathcal{N}^M(s)\gamma - B\alpha$$

The proof of Proposition 3.2.3 is obvious if we realize that

$$H_{\mathcal{F}^A_\omega}(s) = H_{\mathcal{B}}(s) + \sum_{x\in\Sigma} H_{\mathcal{C}^{3M}_x}(s)$$

We note that all the $H^{3N}_x(s) \ge 0$. Hence, we obtain a lower bound of the sum if we restrict it only to the cubes such that a $\mathcal{C}^N_{y(x)}$ intersects the interface and $d(\mathcal{C}^{3N}_s, \pi) \le M$. Moreover, a lower bound of the term $H_{\mathcal{B}}(s)$ is contained in the point v). □

We also observe that the test configuration s^* defined by

$$s^*_i = \begin{cases} +1 & \omega\cdot i \le |\omega| \\ -1 & \text{otherwise} \end{cases}$$

satisfies

$$H_{\mathcal{F}^A_\omega}(s^*) \le B\delta$$

where $\delta \le \sup|J_{ij}| + \sup h_i$. Therefore:

$$H_{\mathcal{F}^A_\omega}(s^A_\omega) \le B\delta \tag{18}$$

Comparing (18) with 3.2.3 we obtain

$$\mathcal{N}^M(s^A_\omega) \le \frac{B\delta}{\gamma} + \frac{B\alpha}{\gamma}$$

Since the number of cubes at a distance M is bounded from below in the point ii), we obtain that if

$$M \ge (3M)^d\left[(1-\alpha)^{-1}\left[\left(\frac{\delta+\alpha}{\gamma}\right)+1\right]\right] \tag{19}$$

there is one cube at a distance less than M such that it does not intersect the interfaced of s^A_ω. □

We emphasize that the condition (19) is independent of B and, hence, independent of ω.

Applying Proposition 3.2.1 with Proposition 3.2.2 we obtain that $(s^A_\omega)_i = -1$ whenever $\omega\cdot i \ge M|\omega|$ independently of A. This establishes Lemma 3.2.1 and, by the arguments at the beginning of this section, it proves Theorem 2.5.

4. Proof of Theorem 2.6

4.1. **Existence of the limits.** We will first prove the existence of the limit of the average energy when we consider sequences of cubes.

Once we prove the result with enough uniformity with respect to the direction and with respect to the ground state, as well as with very explicit error estimates, the existence of the limits claimed in Theorem 2.6 will follow easily by approximating the domain $\lambda\Sigma$ by cubes.

The first result that we will prove that the average energy of a large cube is largely independent of which cube and which plane-like minimizer we are considering.

This will be the basis of much of the uniformity that we need later. Note that we establish that for cubes of size L, up to errors which are much smaller than the area of the boundary, the energy associated to the cube is determined by the area of the intersection.

Proposition 4.0.4. *There exists a constant Ω independent of the cubes, the strips and the ground states (it may depend on the model and the constant M) with the following property:*

Let s, s' be class-A minimizers, contained in strips Γ, Γ' of width M around parallel planes Π, Π' respectively.

Assume without loss of generality that $\Gamma + k = \Gamma'$ for some $k \in N\mathbb{Z}^d$.

Let Q, Q' be cubes of side L – L sufficiently large – Assume that

$$|\#(\Gamma \cap Q) - \#(\Gamma' \cap Q')| \le (\Omega/2)L^{d-2} \tag{20}$$

Then,

$$|H_Q(s) - H_{Q'}(s')| \le \Omega L^{d-2} \tag{21}$$

Note that in Theorem 2.5 we have shown that the constant M can be taken to be independent of the orientation for the infimal minimizer. Hence, if we apply Proposition 4.0.4 to the configurations produced in Theorem 2.5, we get that Ω depends only on the model. Of course, the way that we formulated it, applies to other ground states provided that they are plane-like.

The assumption that the strips are congruent under translations can always be arranged by making them slightly bigger. (so that the interfaces will always be contained) anyway. The amount is not bigger than $N\sqrt{d}$. Hence, for large L this is rather irrelevant.

Proof. The proof is very simple in the case that the cubes and the intersections are congruent by translations which are multiple of N, the period of the interaction. We can produce an configuration s''

that agrees with s outside of Q and whose intersection with Q is a translation by multiples of N of the intersection of s' with Q.

Since s is a ground state, we conclude that $H_{Q^R}(s'') \geq H_{Q^R}(s)$. But $|H_{Q^R}(s'') - H_{Q'^R}(s')| \leq \Omega L^{d-2}$ because the terms in the energy differ only in the boundary terms. Since the interface is contained in a strip of width M, the number of affected terms can be bounded by $CMRL^{d-1}$ where C is a constant that depends only on the dimension and the geometry and R is the range of the interaction.

By exchanging the role of s, s', we obtain the desired result.

When the cubes are not congruent by translations multiples of N, we note that we can discard some points in the cubes, which are at a distance not more than N from the boundary so that we obtain cubes $\tilde{Q}$, $\tilde{Q}'$ that are congruent under translations by N.

Clearly, we have $|H_Q(s) - H_{\tilde{Q}}(s)| \leq \Omega L^{d-2}$.

□

In view of Proposition 21, from now on, we will speak about the energy of a plane-like ground state in a cube of length L and we will not bother specifying which cube or which ground state. As Proposition 4.0.4 shows this is defined up to an additive term of size $\leq \Omega L^{d-2}$, which will not affect any of the subsequent arguments.

The following result gives us some crude bounds of a form similar to that of the desired limit. Later we will refine them.

Proposition 4.0.5. *Under the assumptions of Theorem 2.6.*

Let s be a plane-like minimizer. Let Q be a cube of length L. L.

For some suitable constants Ω_1,Ω_2, Ω_2 depending only on the model and on M, we have:

$$\Omega_1 |\Pi_\omega \cap Q|_{d-1} - \Omega_3 L^{d-2} \leq H_Q(s) \leq \Omega_2 |\Pi_\omega \cap Q|_{d-1} + \Omega_3 L^{d-2} \tag{22}$$

Proof. The upper bound is very similar comparing with that of a state with an interface along the plane.

The lower bound follows from noting that the interface is the boundary of a set, so that we can bound the number of points in the interface by the area of the intersection.

The arguments are very similar to the remarks that lead to a proof of Proposition 3.2.3. We refer there for more details.

The energy of interaction of a site in the boundary is bounded from below by a constant. Hence, the energy of the interaction is bounded from below by a constant times the number of points in the interface. Hence, by a constant times the area of the intersection of the plane with the cube.

By the assumption of zero magnetic flux, the absolute value of the energy due to the magnetic field can be bounded by the strength of the magnetic field times the number of N-cubes that contain the some point in the interface.

□

The following definition will be useful since it selects a particular class of intersections.

Definition 4.0.1. *Given a cube and a strip, we say that the intersection with the cube is clean if*

- *Whenever the intersection with one face of the cube is non-empty, the intersection with the parallel face of the cube is not empty.*
- *The intersection does not include any intersection of more than two faces.*

Note that for all the clean intersections between cubes of the same length and parallel planes have the same area.

Now, we study the limit of the cubes growing larger.

Proposition 4.0.6. *Let s be a plane-like minimizer. Let Q_L, Q_{2L} be cubes of size L, $2L$ respectively.*

Assume that the plane-like minimizer s intersects cleanly Q_L and that the minimizer s' intersects cleanly Q_{2L}.

Then

$$|2^{d-1} H_{Q_L}(s) - H_{Q_{2L}}(s')| \leq \Omega L^{d-2} \tag{23}$$

Proof. Given the uniformity properties proved in Proposition 4.0.4, it suffices to observe that the intersection in the cube Q_{2L} can be covered by 2^{d-1} disjoint cubes with with a clean intersection.

In effect, suppose without loss of generality that the plane of intersection is a graph over of a linear function of the first $d-1$ variables to the d one and that the angle with the horizontal is smaller than 1. (It suffices to reorder the components so that the d component is the largest one).

Take a dyadic decomposition of the base of Q_{2L}. For each of these $d-1$ cubes $\tilde{Q}$ of size L, we can find an interval I of size L so that the cube $\tilde{Q} \times I$ has a clean intersection with the plane.

□

We define

$$\begin{aligned} A^+(L) &= \sup L^{-d+1} H_{Q_L}(s) \\ A^-(L) &= \inf L^{-d+1} H_{Q_L}(s) \end{aligned} \tag{24}$$

where the sup, inf are taken over all the cubes of size L and all the plane like s that have a clean intersection with them.

Proposition 4.0.5 tells us that the functions $A^{\pm}$ are well defined and that we have

$$A^{+}(L) - A^{-}(L) \leq \Omega L^{-1}$$

Using Proposition 4.0.6 we have

$$|A^{\pm}(2L) - A^{\pm}(L)| \leq \Omega L^{-1}$$

From this, it clearly follows that $\lim_{L\to\infty} A^{\pm}(L)$ exists and that it is equal for both functions.

Moreover, the convergence is rather uniform.

If we approximate the set $\lambda\Sigma$ by cubes of size $\lambda^{1/2}$ we see that we can cover the intersection of $\lambda\Sigma \cap \Pi_\omega$ except for a set whose measure can be bounded by $\lambda^{d-2}\lambda^{1/2}$.

We have a number of cubes, each of which has an average energy $A(\omega)$ up to an error $\lambda^{-1/2}$.

Hence, the desired result follows.

It seems that, if one used coverings more efficient than the covering by uniform cubes, one could get better estimates for the remainder, but we will not pursue this here. □

4.2. **Convexity properties of the averaged energy.** To prove the convexity of the averaged energy, the argument used in [CdlL01] works without modification. For the convenience of the reader we repeat here the most salient steps. The argument is illustrated in Figure 4.2 which is reproduced from [CdlL01].

Given the uniformity properties established in the previous subsection, we can compute approximations of $A(\omega)$ just by taking a very large set and computing the energy of the intersection of this set with any of the plane-like ground states whose interface lies in a neighborhood of the plane Π_ω

By the homogeneity, it is enough to show that

$$A(\omega_1) + A(\omega_2) \geq A(\omega_1 + \omega_2)$$

There is only anything to prove in the case that ω_1 is not parallel to ω_2.

By the uniformity of the limits, it is enough to take very large sets. We just take very large cylinders sets whose transversal section is indicated in the figure. We see that taking the joining of the sets corresponding to ω_1 and ω_2 as comparisons with the infimal minimizer corresponding to ω_3 and noting that for all of them, the error from

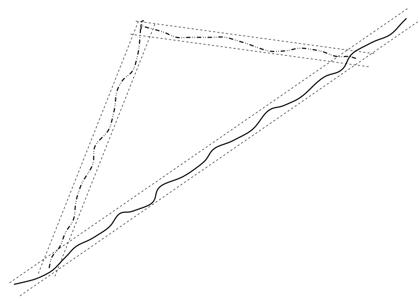

FIGURE 2. Illustration of the argument to show that A is convex

the average is uniformly small if the size is big enough, we obtain the desired result.

Since A is sublinear, it follows that it is Lipschitz. As we will see in Section 5 for the Ising model, it is not C^1.

5. Some examples

5.1. **The classical Ising model.** This corresponds to taking $J_{ij} = 1$ when $|i - j| = 1$ and 0 otherwise. In particular, this satisfies the very strong non-degeneracy assumption alluded to in Remark 2.4.

It is easy to see that the minimization problem in a periodic class admits minimizers that are not Birkhoff. For dimension $d = 2$, some of them are depicted in Figure 3.

Non-Birkhoff minimizers can be constructed by fixing two points in the interface as required by the periodicity. The interface consists of a path that joins these two points and consists of a horizontal segment and a vertical segment. (The fact that these are minimizers is obvious because if we consider the interface as a path, the length is just the taxicab distance.)

It is clear that if we multiply by K the periodicity allowed in the configurations, a similar construction will give an interface that recedes from the plane by an amount K times larger. Hence, in the classical Ising model, there is symmetry breaking for the ground states.

Note that in any dimension, including $d = 2$, given a box of size K, for periodic conditions which are not along the direction of the axis, it is possible to find ground states that are at a distance greater that $c(\omega)K$ from the boundary imposed by the boundary conditions.

Notice also that it is possible to chose a sequence of these minimizers so that their oscillations diverge, hence, it is impossible to make them converge to a limit even after translating them.

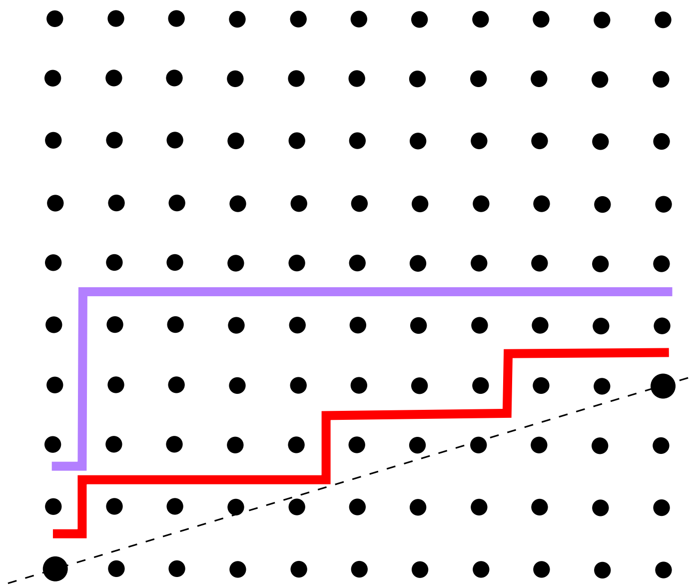

FIGURE 3. Non-Birkhoff minimizers and the infimal minimizer for two dimensional classical Ising models.

In contrast, we see that the infimal ground state can be obtained by removing squares with two sides in the interface from the minimizer above (this is a modification that does not change the energy of the interface) as much as possible compatible with the constraint that the interface should lie above the line $\{\omega \cdot x = 0\}$. The interface of these infimal ground states indeed, does not recede more that a fixed constant for the plane and, if we double the period, the minimizer is the same.

Note, however that for some special periodicities – when the plane Π_ω is a coordinate plane, all the minimizers consist only of straight lines. These minimizers are Birkhoff and do not exhibit symmetry breaking. Hence, for the classical Ising model, the symmetry breaking and the Birkhoff property for all periodic minimizers happen or not depending on the orientation of the boundary conditions.

The considerations here should serve as a counterpoint with the analogies with the theory of minimal surfaces mentioned in Remark 2.2. In [CdlL03b], is is shown all the periodic minimal surfaces are Birkhoff and in [CdlL03a] it is shown that all periodic minimizers in spin systems and in Dirichlet problems are Birkhoff and that there is no symmetry breaking.

This raises the question of whether there are discrete spin models for which the property that there is no symmetry breaking in ground states and that all ground states are Birkhoff is true. The results of the above papers suggest that this should be true for models which resemble more the continuous models. This suggests that absence of symmetry breaking for ground states could be true for models with a longer range interaction (or with several body interactions).

We also note that since the minimizers for a given period are just segments in the horizontal and vertical directions, the average energy can be readily computed and it is

$$A_{\text{Ising}}(\omega) = |\omega_1| + |\omega_2|$$

This function is, clearly Lipschitz but it is not C^1.

Remark 5.1. A classical problem in statistical mechanics is the study of the interfaces in Ising models for low temperature and the surface tension as a function of the temperature. A collection of classical papers in this area is [Sin91].

In comparing the results of the papers in [Sin91] with the results here, one has to note that that all the studies in [Sin91] are carried out for the case, in our notation, that the ω is oriented around one of the coordinate axis. Indeed, many of the papers in [Sin91] use as a starting assumption that the number of ground states satisfying the boundary condition is uniformly bounded as the size goes to infinity. This is clearly not the case for interfaces with other periodicities.

5.2. **Layered material.** Another example for which it is much easier to create complicated ground states is a layered material in which the layers do not interact.

That is $J_{ij} = -1$ if $|i - j| = 1$ and $e_d \cdot (i - j) = 0$ where e_d is the unit vector along the d coordinate. Otherwise, $J_{i,j} = 0$.

Clearly, a ground state can be obtained by choosing any ground state in each of the layers. Hence, it is possible to chose ground states which are not Birkhoff and which do not converge.

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