A relativized Dobrushin uniqueness condition and applications to Pirogov-Sinai models

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Abstract. The paper is divided in three parts: in part 1, we establish a Dobrushin like criterion for uniqueness of DLR measures, in a relatively abstract and general context. In part 2 we show that in "generalized" Pirogov-Sinai models the finite volume corrections to the pressure can be reduced to the analysis of restricted ensembles which naturally fit in the general scheme of Part 1. In Part 3 we apply the previous theory to systems with Kac potentials, namely the ferromagnetic Ising spins, the LMP models of particles in the continuum, [**14**], and their quantum version in the Ginibre loops representation. Our results on the latter are used in a companion paper, [**1**], to prove phase transitions.

Keywords: Pirogov-Sinai; Dobrushin uniqueness; Phase transitions

1. Introduction

Dobrushin's high temperature, uniqueness theorem,[**7**], has a central role in this paper. Dobrushin showed that if the Vaserstein distance between single spin Gibbs conditional probabilities with different boundary conditions satisfies a smallness requirement, then there is a unique Gibbs measure and its correlations decay at infinity. In Sections 2 and 3 we will recall and extend the Dobrushin's theorem.

Since the Vaserstein distance between single spin conditional probabilities is in many cases explicitly computable, the uniqueness criterion in the Dobrushin's theorem has concrete applicability and indeed it has been and it still is very much used.

As shown in [**4**], the regime of validity of the Dobrushin uniqueness condition is the semi-infinite interval $(T^{\rm mf}, \infty)$, $T^{\rm mf}$ the mean field critical temperature, even though there may still be uniqueness below $T^{\rm mf}$, namely if $T^{\rm mf} > T_c$, T_c the critical temperature. To study the temperature interval $(T_c, T^{\rm mf})$, Dobrushin and Shlosman, [**8**], have introduced weaker criteria, essentially based on coarse graining ideas:

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even though they made it clear that the conditions are still checkable with a finite number of operations, their verification, from an analytical point of view, has been problematic, with noticeable exceptions.

Below the critical temperature there is no longer uniqueness, yet uniqueness questions may become relevant when studying a restricted class of boundary conditions. Correspondingly one may look for Dobrushin like conditions to investigate decay of correlations in the extremal DLR states, or to verify that a given DLR measure is extremal. The problem in all such cases is that the Dobrushin single spin condition cannot be verified for all possible boundary conditions, as it would yield global uniqueness. One however expects that the boundary conditions for which things go bad have small probability (w.r.t. the Gibbs measures produced by the given class of boundary conditions). One thus need a relativized Dobrushin criterion, where the lack of validity of the original condition is compensated by a priori estimates on the probability of a bad conditioning. Even though the idea is very simple and convincing a theorem along these lines has escaped the efforts of many authors.

Van der Berg and Maes, [**6**], gave important contributions to this subject with their studies of systems with random temperature fluctuations. They considered models where typically the local temperature lies inside the Dobrushin regime, but, due to fluctuations, there are always sites where the uniqueness condition is not verified. The Dobrushin method then fails and van der Berg and Maes have introduced an alternative algorithm to construct couplings which gives agreement away from the boundaries, if a disagreement percolation estimate is verified. The ideas have been applied to several other systems and in particular to Ising systems at low temperatures with $+$ (or $-$) boundaries conditions. There are also very interesting applications to problems of random walks in random sceneries, by den Hollander and Steif, [**12**], where the coupling techniques of disagreement percolation are used to establish Bernoulli properties of the corresponding dynamical systems.

Closer to the content of this paper are the works by Buttà, Merola and Presutti, [**3**], and Baffioni, Merola and Presutti, [**2**], on the structure of the plus and minus DLR measures in Ising systems with Kac potentials, where disagreement percolation has been applied after a coarse graining transformation, to exploit the scaling properties of the potential. Here we consider an alternative approach, which applies to a large class of systems, in particular those obtained from general Pirogov Sinai models, once described in terms of "restricted ensembles". The purpose of reducing the configuration ensemble, is to exclude a priori a class of Gibbs measures and hence to reduce to a single one, the one we want to examine. Besides the problem of reconstructing the original measure in the full configuration ensemble, a true proof of uniqueness in the restricted ensemble is however not obvious, because of the appearence of an extra hard core interaction (intrinsic to the definition of restricted ensembles) which does not fit well with the Dobrushin uniqueness condition. In this paper we will give conditions for uniqueness in restricted ensembles and check their validity in a class of systems with Kac potentials.

The paper is divided into three parts. In part 1, Sections 2 and 3, we establish a criterion for uniqueness, in a relatively abstract and general context. In part 2 , Sections 4 and 5, we consider "generalized" Pirogov-Sinai models showing that the finite volume corrections to the pressure can be reduced to the analysis of restricted ensembles which naturally fit in the general scheme of Part 1. In Part 3 (consisting of the remaining sections) we further focus the analysis on systems with Kac potentials, by explicitly considering (see Section 6) the ferromagnetic Ising spins, the LMP models of particles in the continuum, [**14**], and their quantum version in the Ginibre loops representation. We then show that, once described in terms of restricted ensembles, all these systems satisfy the general criterion of Sections 2 and 3, thus getting the decay properties necessary for computing the finite volume corrections to the pressure. With such estimates, it is then possible, following the Pirogov Sinai scheme, to prove that, at any temperature below the mean field critical one, there is a phase transition, at least when the Kac scaling parameter γ is sufficiently small, see [**5**], [**14**], [**1**].

2. The Dobrushin uniqueness criterion

Content of Section 2: • An abstract definition of "restricted ensembles"; • statement without proof (postponed to Section 3) of Theorem 2.2, which is an extended version of the Dobrushin uniqueness theorem, with some measurability assumptions dropped; • proof of decay of correlations in systems where the assumptions of Theorem 2.2 are verified.

By "restricted ensembles" we will mean probability measures on product spaces $S^{\mathbb{Z}^d}$, S a Polish space, which satisfy the DLR equations wity respect to a family of prescribed conditional probabilities $\{p_\Lambda(\cdot|\underline{s})\}$, Λ running over the bounded sets in $\mathbb{Z}^d$, having the following properties. Denoting by $\underline{s}$ the elements of $S^{\mathbb{Z}^d}$ and by s_Λ their restrictions to sets $\Lambda \sqsubset \mathbb{Z}^d$, the probabilities $p_\Lambda(\cdot|\underline{s})$ are chosen as the Gibbs specifications with respect to a product measure $\prod_{x\in\mathbb{Z}^d} \nu(ds_x)$ and an energy given by a stable Hamiltonian H plus "a hard core interaction". We start describing the latter. Let $\mathcal{D} = \{C_i, i \in \mathbb{Z}^d\}$ be a partition of $\mathbb{Z}^d$ into equal cubes C_i of side length $\ell \in \mathbb{N}$, let $\mathcal{R}_i \sqsubset S^{\mathbb{Z}^d}$ be sets which depend on $\{s_x, x \in C_i\}$ and are one the translate of the other; then the hard core interaction restricts the configuration space to

$$\mathcal{X} = \sqcap_i \mathcal{R}_i \tag{2.1}$$

in the sense that all $p_\Lambda(\cdot|\underline{s})$ and hence all DLR measures must be supported by $\mathcal{X}$, which is thus the space of allowed configurations. We will suppose that the sets $\mathcal{R}_i$ are compact and have positive free measure:

$$\int_{\mathcal{R}_i} \prod_{x \in C_i} \nu(ds_x) > 0 \tag{2.2}$$

About the hamiltonian H, we suppose that it is invariant by translations which, along each coordinate direction, are integer multiples of ℓ', with ℓ' an integer multiple of ℓ.

The setup is just the one which arises when studying "generalized Pirogov Sinai models", as recalled in Section 4, where we will also see that a step in their analysis requires a proof of uniqueness and decay of correlations. Such results will be derived by using an extension of the Dobrushin's uniqueness theorem stated in Theorem 2.2 below and proved in the next section.

Let $d(s,s')$ be a distance on S, $\Delta \sqsubset \mathbb{Z}^d$, $d_\Delta(\underline{s},\underline{s}') = \sum_{x\in\Delta} d_x(\underline{s},\underline{s}')$, $d_x(\underline{s},\underline{s}') = d(s_x, s'_x)$. We will suppose that

$$\|d\| := \sup_{x\in\mathbb{Z}^d} \sup_{\underline{s},\underline{s}'\in\mathcal{X}} d_x(\underline{s},\underline{s}') < \infty \tag{2.3}$$

For any two probabilities μ and μ' on $S^{\mathbb{Z}^d}$, we define

$$R_\Delta(\mu,\mu') = \inf_Q \int_{S^{\mathbb{Z}^d}\times S^{\mathbb{Z}^d}} d_\Delta(\underline{s},\underline{s}')\; Q(d\underline{s}, d\underline{s}') \tag{2.4}$$

where the inf is over all joint representations (couplings) $Q(d\underline{s}, d\underline{s}')$ of μ and μ', namely $Q(d\underline{s}, d\underline{s}')$ is a probability on $S^{\mathbb{Z}^d} \times S^{\mathbb{Z}^d}$ whose marginal on the first [respectively the second] variable is μ [μ']. $R_\Delta(\mu,\mu')$ is the Vaserstein distance of μ and μ' relative to the distance d_Δ.

The uniqueness criterion, which is the main assumption in Theorem 2.2 below, is based on the existence of a function $r : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$ such that for all $i \in \mathbb{Z}^d$ and $\underline{s}, \underline{s}' \in S^{\mathbb{Z}^d}$

$$R_{C_i}\big(p_{C_i}(\cdot|\underline{s}), p_{C_i}(\cdot|\underline{s}')\big) \le \sum_j r(i,j) d(s_{C_j}, s'_{C_j}) \tag{2.5}$$

$$\sup_i \sum_{j\neq i} r(i,j) \le \delta, \quad 0 < \delta < 1 \tag{2.6}$$

Without any loss of generality, we have set $r(i,i)=0$. Let $I \sqsubset \mathbb{Z}^d$ and $r_I : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$,

$$r_I(i,j) = r(i,j)\mathbf{1}_{i\in I} \quad i,j \in \mathbb{Z}^d \tag{2.7}$$

We then define iteratively for all $n \geq 1$ the "convolutions"

$$\begin{aligned} r_I^1(i,j) &= r(i,j) \\ r_I^n(i,j) &= \sum_{j'\in\mathbb{Z}^d} r(i,j')\, r_I^{n-1}(j',j) \quad i,j \in \mathbb{Z}^d \end{aligned}$$

and have:

Proposition 2.1. *If* (2.6) *holds, then, for any bounded set* I *in* $\mathbb{Z}^d$ *and any two configurations* $\underline{s}$, $\underline{s}' \in \mathcal{X}$, *there is a unique solution of the equation*

$$u(i) = \begin{cases} \displaystyle\sum_{j\neq i} r(i,j)\, u(j) & \text{if } i \in I \\ d_{C_i}(s_{C_i}, s'_{C_i}) & \text{if } i \notin I \end{cases} \tag{2.8}$$

The solution is

$$u_{I,\underline{s},\underline{s}'}(i) = \sum_{j\in I^c} \sum_{n>0} r_I^n(i,j) d_{C_j}(s_{C_j}, s'_{C_j}) \qquad \text{for } i \in I \tag{2.9}$$

$u_{I,\underline{s},\underline{s}'}(i) \geq 0$,

$$\sup_{I,\underline{s},\underline{s}',i} u_{I,\underline{s},\underline{s}'}(i) = \ell^d \|d\|, \quad (\|d\| \text{ as in } (2.3)) \tag{2.10}$$

and, for any i,

$$\lim_{I\nearrow\mathbb{Z}^d} \sup_{\underline{s},\underline{s}'\in\mathcal{X}} u_{I,\underline{s},\underline{s}'}(i) = 0 \tag{2.11}$$

Proof. Since

$$\sum_{j\in\mathbb{Z}^d} r_I^n(i,j) \leq \delta^n, \quad \sum_{j\in\mathbb{Z}^d} \sum_{n>0} r_I^n(i,j) \leq \frac{\delta}{1-\delta} \tag{2.12}$$

Then the r.h.s. of (2.9) is well defined and the function $v(i)$ equal for $i \in I$ to the r.h.s. of (2.9) and equal to $d_{C_i}(s_{C_i}, s'_{C_i})$ elsewhere, solves (2.8): indeed, for any $i \in I$,

$$\begin{aligned} v(i) &= \sum_{j\notin I} r_I(i,j) d_{C_j}(s_{C_j}, s'_{C_j}) + \sum_{j\notin I} \sum_{n>1} r_I^n(i,j) d_{C_j}(s_{C_j}, s'_{C_j}) \\ &= \sum_{j\notin I} r_I(i,j) d_{C_j}(s_{C_j}, s'_{C_j}) + \sum_{j\notin I} \sum_{k\in I} r_I(i,k) \sum_{n>0} r_I^n(k,j) d_{C_j}(s_{C_j}, s'_{C_j}) \\ &= \sum_{j\notin I} r(i,j) d_{C_j}(s_{C_j}, s'_{C_j}) + \sum_{k\in I} r(i,k) v(k) = \sum_j r(i,j) v(j) \end{aligned}$$

To prove uniqueness, suppose that v and u are two solutions, then $w := u - v$ vanishes for $j \notin I$, so that

$$w(i) \quad = \quad \sum_{k\in I} r_I(i,k)w(k) = \sum_{k\in I} r_I^n(i,k)w(k)$$

Hence $|w(i)| \le \delta^n \sup_{k\in I} |w(k)|$ and by the arbitrariness of n, $w = 0$.

Existence, uniqueness and the formula (2.9) are thus proved. By (2.3), $d_{C_i}(s_{C_i}, s'_{C_i}) \le \|d\|$ and (2.10) follows from (2.8). To prove (2.11), let $\epsilon > 0$ and N_ϵ such that

$$\sup_{i\in\mathbb{Z}^d} \sum_{|j-i|\ge N_\epsilon} r(i,j) < \epsilon$$

Calling $r_{I,\epsilon}(i,j) := r_I(i,j)\mathbf{1}_{|i-j|<N_\epsilon}$, and $u = u_{I,\underline{s},\underline{s}'}$, we have

$$u(i) = \sum_{n\ge0}\sum_{k\in I} r^n_{I,\epsilon}(i,k) \sum_{h:|h-k|\ge N_\epsilon} r(h,k)u(k) + \sum_{n>0}\sum_{j\notin I} r^n_{I,\epsilon}(i,j)u(j)$$

so that,

$$u_{I,\underline{s},\underline{s}'}(i) \le \frac{1}{1-\delta}\Big(\epsilon + \delta^{\mathrm{dist}(i,I^c)/N_\epsilon}\Big)(\ell^d\|d\|)$$

$$\lim_{I\nearrow\mathbb{Z}^d} \sup_{\underline{s},\underline{s}'\in\mathcal{X}} u_{I,\underline{s},\underline{s}'}(i) \le \epsilon\,\frac{\ell^d\|d\|}{1-\delta}$$

because $\mathrm{dist}(i,I^c) \to \infty$ as $I \nearrow \mathbb{Z}^d$. By letting $\epsilon \to 0$, we then get (2.11). Proposition 2.1 is proved. □

Theorem 2.2. *Suppose that* (2.5)-(2.6) *are satisfied, let*

$$\Delta = \bigsqcup_{i\in J} C_i, \quad \Lambda = \bigsqcup_{i\in I} C_i, \quad J \sqsubset I \sqsubset \mathbb{Z}^d,\ |I| < \infty \tag{2.13}$$

then, for any $\underline{s}$, $\underline{s}' \in S^{\mathbb{Z}^d}$

$$R_\Delta\big(p_\Lambda(\cdot|\underline{s}), p_\Lambda(\cdot|\underline{s}')\big) \le \sum_{i\in J} u_{I,\underline{s},\underline{s}'}(i) \tag{2.14}$$

Theorem 2.2 is proved in the next section. Take notice that no measurability assumption is stated, namely we have not supposed that the inf in the definition of the Vaserstein distance is over joint representations $Q(ds_{C_i}, ds'_{C_i}|\underline{s},\underline{s}')$ which depend measurably on $\underline{s}$ and $\underline{s}'$. With such an extra assumption the classical proof of Dobrushin would apply, but, as we will see in the next section, the hypothesis can be dropped and the assumptions (2.5)-(2.6) as stated are easier to check in some of the applications we will consider.

Theorem 2.2 and Proposition 2.1 yield uniqueness:

Corollary 2.3. *If* (2.6) *holds then there is one and only one DLR measure (corresponding to the specification* $\{p_\Lambda(\cdot, s_{\Lambda^c})\}_{\Lambda\sqsubset\mathbb{Z}^d, \underline{s}\in S^{\mathbb{Z}^d}}$*).*

Proof. Let I, J, Λ and Δ be as in (2.13). Call f a Δ-Lipschitz function with constant c_f if

$$\sup_{\underline{s},\underline{s}'} \big|f(\underline{s}) - f(\underline{s}')\big| \le c_f \sum_{i\in J} d_{C_i}(s_{C_i}, s'_{C_i}) \tag{2.15}$$

By Theorem 2.2,

$$\big|p_\Lambda(f|\underline{s}) - p_\Lambda(f|\underline{s}')\big| \le c_f R_\Delta\big(p_\Lambda(\cdot|\underline{s}), p_\Lambda(\cdot|\underline{s}')\big) \le c_f \sum_{i\in J} u_{I,\underline{s},\underline{s}'}(i) \tag{2.16}$$

Thus, if μ and μ' are two DLR measures

$$|\mu(f) - \mu'(f)| \le \int_{\mathcal{X}\times\mathcal{X}} \mu(d\underline{s})\mu'(d\underline{s}')\big|p_\Lambda(f|\underline{s}) - p_\Lambda(f|\underline{s}')\big|$$

By (2.16) and (2.11), letting $I \nearrow \mathbb{Z}^d$, we conclude that $\mu(f) = \mu(f')$, hence, by the arbitrariness of f, $\mu = \mu'$. Uniqueness of DLR measures is thus proved.

To prove existence, we consider an increasing sequence $I_n \nearrow \mathbb{Z}^d$ and call $\Lambda_n = \bigsqcup_{i\in I_n} C_i$. Fix $\underline{s} \in \mathcal{X}$, then, for any Lipschitz f as above

$$\lim_{n\to\infty} p_{\Lambda_n}(f|\underline{s}) =: \mu(f) \tag{2.17}$$

because, as argued above, $\{p_{\Lambda_n}(f|\underline{s})\}$ is Cauchy. By the Kolmogorov's theorem for projective limits, the limits (2.17) are identified as expectations of a probability measure μ on $\mathcal{X}$. We will next show that $\mu(f) = \mu\big(p_\Lambda(f|\cdot)\big)$, f and Λ as above.

Let $I' \sqsupset I$, $\Lambda' := \bigsqcup_{i\in I'} C_i$ and observe that by (2.16) and (2.9)–(2.10)

$$\sup_{\underline{s},\underline{s}'\in\mathcal{X}: s_{\Lambda'}=s'_{\Lambda'}} \big|p_\Lambda(f|\underline{s}) - p_\Lambda(f|\underline{s}')\big| \le c_f \ell^d \|d\| \sum_{j\notin I'} \sum_{n>0} r^n_{I,\underline{s},\underline{s}'}(i,j)$$

which thus vanishes as $I' \nearrow \mathbb{Z}^d$. Then, for any $\epsilon > 0$ there is I' and a Λ'-Lipschitz function $g_{\Lambda'}(\underline{s})$, so that

$$\sup_{\underline{s}\in\mathcal{X}} \big|p_{\Lambda'}(f|\underline{s}) - g_{\Lambda'}(\underline{s})\big| \le \epsilon$$

For n large enough, $p_{\Lambda_n}(f|\underline{s}) = p_{\Lambda_n}\big(p_{\Lambda'}(f|\cdot)|\underline{s}\big)$, then, by (2.17),

$$|p_{\Lambda_n}(f|\underline{s}) - p_{\Lambda_n}\big(g_{\Lambda'}|\underline{s}\big)| < \epsilon, \quad |\mu(f) - \mu\big(p_{\Lambda'}(f|\cdot)\big)| < 2\epsilon$$

which, by the arbitrariness of ϵ proves that μ is DLR. Corollary 2.3 is proved. □

We will next state some properties of the solution u of (2.8) under additional assumptions on the coefficients $r(i,j)$. We assume there is a metric $\psi(i,j)$ on $\mathbb{Z}^d$

such that $\psi(i,j) \ge d_0 > 0$ for all $i \neq j \in \mathbb{Z}^d$, and

$$\sup_i \sum_{j \neq i} r(i,j) e^{\psi(i,j)} \le 1 \tag{2.18}$$

Notice that (2.18) implies (2.6), with $\delta \le e^{-d_0}$.

Theorem 2.4. *Suppose that* (2.18) *holds. Then, calling u the solution* (2.9) *of* (2.8)*, for any $i \in I$,*

$$u(i) \le \sum_{j \notin I} e^{-\psi(i,j)} d_{C_j}(s_{C_j}, s'_{C_j}) \tag{2.19}$$

Proof. By (2.9)

$$u(i) = \sum_{j \notin I} \sum_{n>0} r_I^n(i,j)\, d_{C_j}(s_{C_j}, s'_{C_j})$$

Calling

$$\pi(h,k) = r_I(h,k) e^{\psi(h,k)}$$

since $\psi(i,j)$ satisfies the triangular inequality,

$$\sum_{n>0} r_I^n(i,j) \le e^{-\psi(i,j)} \sum_{n>0} \pi^n(i,j)$$

By (2.18), $\sum_j \pi(i,j) \le 1$, hence $\sum_{n>0} \pi^n(i,j) \le 1$. Theorem 2.4 is proved. □

Remarks. If $r(i,j)$ decays exponentially, there exists $\lambda > 0$ such that $\psi(i,j) = \lambda |i-j|$ satisfies (2.18): indeed, at $\lambda = 0$, the l.h.s. of (2.18) is equal to $\delta < 1$ and the statement follows by a continuity argument. Other decay rates can be studied similarly, for a power decay we would take $\psi(i,j) = \lambda \ln(|i-j|+1)$.

As a corollary of Theorem 2.4, if f is Δ-Lipschitz with constant c_f, then

$$\left| p_\Lambda(f|\underline{s}) - p_\Lambda(f|\underline{s}') \right| \le c_f \sum_{i \in I,\, j \notin I} e^{-\psi(i,j)} d_{C_j}(s_{C_j}, s'_{C_j}) \tag{2.20}$$

and if $\mu(d\underline{s})$ denotes the unique DLR measure, then

$$\left| p_\Lambda(f|\underline{s}) - \mu(f) \right| \le c_f\, \ell^d \|d\| \sum_{i \in I,\, j \notin I} e^{-\psi(i,j)} \tag{2.21}$$

3. A uniqueness criterion

Content of Section 3: • Proof of Theorem 2.2 (given after Proposition 3.2); • Theorem 3.1 which gives sufficient conditions (Assumptions 1–4) for the validity of (2.5)-(2.6) and (Assumptions 1–5) for the validity of (2.18); • construction of successful couplings when Assumptions 1–5 are verified.

By regarding s_{C_i} as a single spin, the uniqueness criterion (2.6) becomes the usual Dobrushin uniqueness condition, except for the already remarked absence of measurability. The beauty of the Dobrushin condition is that it can be explicitly checked, a fact which relies on the single spin distributions being usually quite easy to handle. In our applications instead, s_{C_i} is a large collection of spins and the validity of (2.6) cannot be checked by a direct and explicit computation. The crucial step in this section is a proof that if the true single spin distribution satisfies the original Dobrushin condition for "boundary conditions" in a set of large probability, then (2.6) holds.

Assumption 1. For any $x \in \mathbb{Z}^d$ there is a measurable set $G_x \sqsubset \mathcal{X}$, determined only by the variables $\{s_y, y \in C(x) \setminus x\}$, $C(x)$ the cube in $\mathcal{D}$ which contains x, such that

$$R_x\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}')\big) \le \sum_y b(x,y) d(s_y, s'_y), \quad \text{for all } \underline{s},\ \underline{s}' \text{ in } G_x \tag{3.1}$$

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in C(x)} b(x,y), \quad b(x,y) \ge 0,\, b(x,x) = 0 \tag{3.2}$$

Remarks. (3.1)-(3.2) reminds of the usual Dobrushin condition, from which it differs because of (1), non uniformity on the boundary conditions, as $\underline{s}$ and $\underline{s}'$ are constrained to G_x; (2), the sum over y is limited to $y \in C(x)$; (3), the Vaserstein distance is defined over joint representations which are not required to depend measurably on the conditioning spins. The next assumptions will cover (1) and (2).

As we will see, in a Pirogov-Sinai scheme, (3.1) is essentially a consequence of the definition of $\mathcal{X}$, which is chosen in such a way that only small deviations are allowed from a stable state, so that the Dobrushin condition is satisfied except when "close to the boundaries of the constraint". In such cases the conditional distribution becomes "rigid" and may not satisfy (3.1)-(3.2).

Assumption 2. There is $\epsilon > 0$ so that for any $x \in C_i$,

$$4\{\int_{G_x^c} p_{C(x)}(d\underline{t}|\underline{s})\}\{\int |t_x|^2 p_{C(x)}(d\underline{t}|\underline{s})\} \le \epsilon^2, \quad \text{for all } \underline{s} \in \mathcal{X} \tag{3.3}$$

Conditions on the smallness of ϵ will be stated in equations (3.8) and (3.9).

Assumption 3. Calling

$$\mathcal{B}_\xi(i) = \{j : \text{dist}(C_i, C_j) \le \xi\}, \quad \xi > 0 \tag{3.4}$$

for any $\underline{s}^{(1)}$ and $\underline{s}^{(2)}$ in $\mathcal{X}$, which agree on any C_j, $j \in \mathcal{B}_\xi(i) \setminus i$,

$$R_{C_i}\big(p_{C_i}(\cdot|\underline{s}^{(1)}), p_{C_i}(\cdot|\underline{s}^{(2)})\big) \le \sum_{j \notin \mathcal{B}_\xi(i)} r(i,j) d_{C_j}(\underline{s}^{(1)}, \underline{s}^{(2)}) \tag{3.5}$$

$$\sup_i \sum_{j \notin \mathcal{B}_\xi(i)} r(i,j) < 1 \tag{3.6}$$

Remarks. This is a condition on the tail of the interaction, which will follow in the applications from assuming a fast decay of the interaction and by choosing ξ large enough. The assumption is stated separately from the previous ones, because in our applications, the estimates for $b(x,y)$ do not have good decay properties for $|i-j| \to \infty$.

Theorem 3.1 below states that if the above three assumptions plus a fourth one (involving the smallness of the parameters and stated in the Theorem 3.1 itself) are satisfied, then the conditional probabilities $\{p_{C_i}(\cdot|\underline{s})\}$ satisfy the uniqueness condition (2.6). Let

$$b_{C_i}(x,y) = b(x,y)\mathbf{1}_{x \in C_i} \tag{3.7}$$

$b(x,y)$ as in (3.1), and let for $j \in B_\xi(i) \setminus i$,

$$r(i,j) = \sup_{y \in C_j} \sum_{x \in C_i} \sum_{n>0} b^n_{C_i}(x,y) + 2\epsilon\Big(|C_i| + \sum_{x,z \in C_i} \sum_{n>0} b^n_{C_i}(x,z)\Big) \tag{3.8}$$

where ϵ is as in (3.3). Thus $r(i,j)$, is defined for all i and j via (3.1), (3.5), (3.7), (3.8) and by setting $r(i,i) = 0$.

Theorem 3.1. *If,* **Assumption 4***,*

$$\sup_i \sum_j r(i,j) \le \delta < 1 \tag{3.9}$$

then (2.5) *and* (2.6) *are satisfied. If additionally,* **Assumption 5**, *there is a metric* $\psi(i,j)$ *on* $\mathbb{Z}^d$ *such that* $\psi(i,j) \geq d_0 > 0$ *for all* $i \neq j \in \mathbb{Z}^d$, *and*

$$\delta e^{\psi(|\xi|)} + \sup_i \sum_{j \notin B_\xi(i)} r(i,j) e^{\psi(|i-j|)} < 1 \tag{3.10}$$

δ *as in* (3.9), *then* (2.18) *is satisfied.*

Theorem 3.1 and Theorem 2.2 will be consequence of Proposition 3.2 that we state after introducing some notation and definitions.

Denote by $\mathcal{P}$ a partition of $\mathbb{Z}^d$ into cubes K_i, which, in a first case, is the partition $\mathcal{D}$ and, in a second one, $\mathbb{Z}^d$ itself (i.e. K_i are the sites of $\mathbb{Z}^d$). Let also I_Λ be a finite set in $\mathbb{Z}^d$ and

$$\Lambda = \bigsqcup_{i \in I_\Lambda} K_i \tag{3.11}$$

By an abuse of notation, given a non negative function v on I_Λ, we set

$$R_v(p_\Lambda(\cdot|\underline{s}), p_\Lambda(\cdot|\underline{s}')) = \inf_Q \int \sum_{i \in I_\Lambda} v_i d_{K_i}(s_{K_i}, s'_{K_i}) \, dQ \tag{3.12}$$

Thus if v is the characteristic function of a set $I_\Delta \sqsubset I_\Lambda$ where Δ is the union of K_i, $i \in I_\Delta$, then if $K_i = C_i$, R_v is what we have called R_Δ in Section 2. We also write $R\big(p_{K_i}(\cdot|\underline{s}), p_{K_i}(\cdot|\underline{s}')\big)$ for the Vaserstein distance relative to $d_{K_i}(s_{K_i}, s'_{K_i})$.

Proposition 3.2. *Suppose that:*

$$R\big(p_{K_i}(\cdot|\underline{s}), p_{K_i}(\cdot|\underline{s}')\big) \leq \alpha_i(\underline{s}, \underline{s}') \tag{3.13}$$

with $\alpha_i(\underline{s}, \underline{s}')$ *a measurable function of* $\underline{s}$ *and* $\underline{s}'$ *which depends only on* $s_{K_i^c}$ *and* $s'_{K_i^c}$. *Then for any non-negative function* v *on* I_Λ *and any boundary conditions* $\underline{s}^{(1)}, \underline{s}^{(2)}$

$$R_v\big(p_\Lambda(\cdot|\underline{s}^{(1)}), p_\Lambda(\cdot|\underline{s}^{(2)})\big) \leq \inf_Q \int \tau_{\alpha,\Lambda,v}(\underline{s}, \underline{s}') \, dQ(d\underline{s}, d\underline{s}') \tag{3.14}$$

where the inf is over all couplings Q *of* $p_\Lambda(\cdot|s^{(1)}_{\Lambda^c})$ *and* $p_\Lambda(\cdot|s^{(2)}_{\Lambda^c})$ *and*

$$\tau_{\alpha,\Lambda,v}(\underline{s}, \underline{s}') := \frac{|I_\Lambda| - 1}{|I_\Lambda|} \{v \cdot d\}(\underline{s}, \underline{s}') + \frac{1}{|I_\Lambda|} \{v \cdot \alpha\}(\underline{s}, \underline{s}') \tag{3.15}$$

$$\{v \cdot d\}(\underline{s}, \underline{s}') := \sum_{i \in I_\Lambda} v_i \, d_{K_i}(\underline{s}, \underline{s}'), \quad \{v \cdot \alpha\}(\underline{s}, \underline{s}') := \sum_{i \in I_\Lambda} v_i \, \alpha_i(\underline{s}, \underline{s}') \tag{3.16}$$

Proof. By the Kantorovich-Rubinstein duality, for any two probabilities μ_1 and μ_2 on S^Λ,

$$R_v(\mu_1,\mu_2) := \inf_Q \int d_v(\underline{s},\underline{s}')\, dQ = \sup_{f:\|f\|_{\mathrm{Lip},v}\le 1} \Big| \int f(\underline{s})\ \{\mu_1(d\underline{s}) - \mu_2(d\underline{s})\}\Big|$$

where the inf is over the joint representations Q of μ_1 and μ_2 and

$$\|f\|_{\mathrm{Lip},v} := \sup_{\substack{\underline{s},\underline{s}'\in S^\Lambda\\ d_v(\underline{s},\underline{s}')>0}} \frac{|f(\underline{s})-f(\underline{s}')|}{d_v(\underline{s},\underline{s}')}$$

is the Lipschitz-norm corresponding to d_v. Calling

$$A_f := \Big| \int f(\underline{t})\{p_\Lambda(d\underline{t}|\underline{s}^{(1)}) - p_\Lambda(d\underline{t}|\underline{s}^{(2)})\}\Big|, \quad \|f\|_{\mathrm{Lip},v}\le 1 \tag{3.17}$$

and since $p_\Lambda(d\underline{t}|\underline{s}^{(i)})$ are Gibbs measures, we get

$$A_f = \Big| \int f(\underline{t}) \frac{1}{|I_\Lambda|}\sum_{i\in I_\Lambda} \int p_{K_i}(d\underline{t}|\underline{s})\{p_\Lambda(d\underline{s}|\underline{s}^{(1)}) - p_\Lambda(d\underline{s}|\underline{s}^{(2)})\}\Big| \tag{3.18}$$

Let Q be a joint distribution of $p_\Lambda(\cdot|\underline{s}^{(1)})$ and $p_\Lambda(\cdot|\underline{s}^{(2)})$, then

$$A_f = \Big| \frac{1}{|I_\Lambda|}\sum_{i\in I_\Lambda} \iiint f(\underline{t})\{p_{K_i}(d\underline{t}|\underline{s}) - p_{K_i}(d\underline{t}|\underline{s}')\}\ Q(d\underline{s},d\underline{s}')\Big| \tag{3.19}$$

For each $K_i \sqsubset \Lambda$ and $\underline{s},\underline{s}'\in\mathcal{X}$, take a joint representation $P^{\underline{s},\underline{s}'}_{K_i}$ of $p_{K_i}(d\underline{t}|\underline{s})$ and $p_{K_i}(d\underline{t}|\underline{s}')$ on S^{K_i}, then

$$\begin{aligned}\Big| \int f(\underline{t})\{p_{K_i}(d\underline{t}|\underline{s}) - p_{K_i}(d\underline{t}|\underline{s}')\}\Big| &= \Big| \iint \big(f(\underline{t}) - f(\underline{t}')\big) P^{\underline{s},\underline{s}'}_{K_i}(d\underline{t},d\underline{t}')\Big| \\ &\le \|f\|_{\mathrm{Lip},v}\Big(\sum_{j\in I_\Lambda\setminus i} v_j\, d_{K_j}(\underline{s},\underline{s}') + v_i \int d_{K_i}(\underline{t},\underline{t}') P^{\underline{s},\underline{s}'}_{K_i}(d\underline{t},d\underline{t}')\Big)\end{aligned}$$

Recalling that $\|f\|_{\mathrm{Lip},v}\le 1$, taking the infimum over all $P^{\underline{s},\underline{s}'}_{K_i}(d\underline{t},d\underline{t}')$ and using (3.13), we get

$$\Big| \int f(\underline{t})\{p_{K_i}(d\underline{t}|\underline{s}) - p_{K_i}(d\underline{t}|\underline{s}')\}\Big| \le \sum_{j\in I_\Lambda\setminus i} v_j\, d_{K_j}(\underline{s},\underline{s}') + v_i\,\alpha_i(\underline{s},\underline{s}') \tag{3.20}$$

which, inserted in (3.19), yields

$$A_f \le \frac{|I_\Lambda|-1}{|I_\Lambda|}\int \sum_{j\in I_\Lambda} v_j\, d_{K_j}(\underline{s},\underline{s}')\ Q(d\underline{s},d\underline{s}') + \frac{1}{|I_\Lambda|}\int\sum_{i\in I_\Lambda} v_i\,\alpha_i(\underline{s},\underline{s}')\ Q(d\underline{s},d\underline{s}')$$

Same bound holds for the sup over f of A_f, so that taking the inf over all possible Q we get (3.14) with $\tau_{\alpha,\Lambda,v}$ given by (3.15). □

Proof of Theorem 2.2. We choose $\mathcal{P} = \mathcal{D}$, i.e. $K_i = C_i$. By (2.5), we can take α_i in (3.13) as

$$\alpha_i(\underline{s}, \underline{s}') = \sum_{j \neq i} r(i,j) d_{C_j}(\underline{s}, \underline{s}') \tag{3.21}$$

Let

$$\mathfrak{T} := \mathbf{1} + \frac{1}{|I_\Lambda|}(\mathfrak{R}_\Lambda - \mathbf{1}) \quad ; \quad [\mathfrak{R}_\Lambda]_{i,j} = r(j,i)\mathbf{1}_{j \in I_\Lambda} \ ; \ d_j \equiv d_{C_j}(\underline{s}^{(1)}, \underline{s}^{(2)})$$

By (2.6)

$$\|\mathfrak{T}\| := \sup_j \sum_i |\mathfrak{T}_{ij}| < 1, \quad \|\mathfrak{R}_\Lambda\| \le \delta < 1$$

and

$$(\mathbf{1} - \mathfrak{T})^{-1} \equiv |I_\Lambda|(1 - \mathfrak{R}_\Lambda)^{-1} = |I_\Lambda| \sum_{n=0}^{\infty} \mathfrak{R}_\Lambda{}^n$$

Then (3.14) becomes

$$R_v\big(p_\Lambda(\cdot|\underline{s}^{(1)}), p_\Lambda(\cdot|\underline{s}^{(2)})\big) \le R_{\mathfrak{T}v}\big(p_\Lambda(\cdot|\underline{s}^{(1)}), p_\Lambda(\cdot|\underline{s}^{(2)})\big) + \frac{1}{|I_\Lambda|} \sum_{j \in I_\Lambda^c} (\mathfrak{R}_\Lambda v)_j \, d_j$$

which iterated n times yields when $n \to \infty$, (since $[\mathfrak{T}, \mathfrak{R}_\Lambda] = 0$)

$$R_v\big(p_\Lambda(\cdot|\underline{s}^{(1)}), p_\Lambda(\cdot|\underline{s}^{(2)})\big) \le \frac{1}{|I_\Lambda|} \sum_{j \in I_\Lambda^c} \sum_{n=0}^{\infty} (\mathfrak{T}^n \mathfrak{R}_\Lambda v)_j d_j = \frac{1}{|I_\Lambda|} \sum_{j \in I_\Lambda^c} ((\mathbf{1} - \mathfrak{T})^{-1} \mathfrak{R}_\Lambda v)_j d_j$$

and, finally,

$$R_v\big(p_\Lambda(\cdot|\underline{s}^{(1)}), p_\Lambda(\cdot|\underline{s}^{(2)})\big) \le \sum_{j \in I_\Lambda^c} \sum_{n=1}^{\infty} (\mathfrak{R}_\Lambda{}^n v)_j \ d_j \tag{3.22}$$

Setting $v_i = \mathbf{1}_{i \in I_\Delta}$, (3.22) becomes (2.14), (recall the expression of $u(i)$ given by (2.9)). Theorem 2.2 is proved.

□

Proof of Theorem 3.1. We take here $\mathcal{P} = \mathbb{Z}^d$, i.e. $K_i = x$, $\Lambda \equiv \Delta = C_i$, C_i a cube of the partition $\mathcal{D}$. Then by (3.1) condition (3.13) holds for

$$\alpha_x(\underline{s}, \underline{s}') = \sum_{y \neq x} b(x,y) d(s_y, s'_y) + (\varphi_x(\underline{s}) + \varphi_x(\underline{s}'))\left(\mathbf{1}_{\underline{s} \in G_x^c} + \mathbf{1}_{\underline{s}' \in G_x^c}\right) \tag{3.23}$$

where

$$\varphi_x(\underline{s}) = \int |t_x| p_x(dt_x|\underline{s}) \tag{3.24}$$

Since for any coupling Q between $p_{C_i}(\cdot|\underline{s}^{(1)}), p_{C_i}(\cdot|\underline{s}^{(2)})$,

$$\int (\varphi_x(\underline{s}) + \varphi_x(\underline{s}'))\left(\mathbf{1}_{\underline{s}\in G_x^c} + \mathbf{1}_{\underline{s}'\in G_x^c}\right)\, dQ$$
$$\leq\ 2 \max_{\underline{s}'\in\{\underline{s}^{(1)},\underline{s}^{(2)}\}} p_{C_i}(\varphi_x(\underline{s})\mathbf{1}_{\underline{s}\in G_x^c}|\underline{s}') + 2Q(\varphi_x(\underline{s})\mathbf{1}_{\underline{s}'\in G_x^c})$$

After using Cauchy-Schwartz and Assumption 2, (3.14) becomes

$$R_v\Big(p_{C_i}(\cdot|\underline{s}^{(1)}), p_{C_i}(\cdot|\underline{s}^{(2)})\Big) \leq R_{\mathfrak{T}v}\Big(p_{C_i}(\cdot|\underline{s}^{(1)}), p_{C_i}(\cdot|\underline{s}^{(2)})\Big)$$
$$+\frac{1}{|C_i|}\sum_{y\in C_i^c}(\mathcal{R}_{C_i}v)_y d_y + \frac{2\epsilon}{|C_i|}\sum_{x\in C_i} v_y$$

Proceeding as in the proof of Theorem 2.2

$$R_v\Big(p_{C_i}(\cdot|\underline{s}^{(1)}), p_{C_i}(\cdot|\underline{s}^{(2)})\Big) \leq \sum_{n\geq 1}\sum_{y\in C_i^c}(\mathcal{R}^n_{C_i}v)_y d_y + 2\epsilon\sum_{n\geq 0}\sum_{x\in C_i}(\mathcal{R}^n_{C_i}v)_x$$

and when $v_x = \mathbf{1}_{x\in C_i}$ we get

$$R_v\Big(p_{C_i}(\cdot|\underline{s}^{(1)}), p_{C_i}(\cdot|\underline{s}^{(2)})\Big) \leq \sum_{y\in C_i^c}\sum_{x\in C_i}\sum_{n\geq 1} b^{(n)}_{C_i}(x,y)d_y$$
$$+2\epsilon\left(|C_i| + \sum_{x,z\in C_i}\sum_{n\geq 1} b^{(n)}_{C_i}(x,z)\right)$$

which coincides with (3.8). Theorem 3.1 then follows from the triangular inequality using Assumption 3.

Proof of (2.18). Since ψ is an increasing function,

$$\sum_{j\in B_\xi(i)} r(i,j)e^{\psi(|i-j|)} \leq \sum_{j\in B_\xi(i)} r(i,j)e^{\psi(\xi)} \leq \delta e^{\psi(\xi)}$$

the last inequality being a consequence of Assumption 4; (2.18) then follows from Assumption 5. Theorem 3.1 is proved.

□

By Theorem 3.1, if Assumptions 1–4 are satisfied, then (2.5)–(2.6) hold, and, by Corollary 2.3, we can conclude that there is only one DLR measure, which we denote by $p(d\underline{s})$. To describe its properties we will next construct couplings of $p(d\underline{s})$ and finite volume Gibbs measures and between finite volume Gibbs measures as well. By adding some extra assumptions, we will prove that these couplings are successful, i.e. give "large probability to the diagonal".

Theorem 3.3. *If Assumption 1–5 are verified, the latter with* $\psi(i,j)=\omega_0|i-j|$, $\omega_0>0$, *and if* $d_x(\cdot,\cdot)\ge 1$, *then there exists* $c>0$ *so that the following holds. Let* Λ_i, $i=1,2$, *and* Δ *be bounded,* $\mathcal{D}$*-measurable sets,* $\underline{s}^{(i)}\in\mathcal{X}$, *then there is a coupling* P *of* $p_\Lambda(d\underline{s}|s^{(i)}_{\Lambda_i^c})$, $i=1,2$, *and a coupling* Q *of* $p_\Lambda(d\underline{s}|s^{(1)}_{\Lambda_1^c})$ *and* $p(d\underline{s})$ *such that*

$$P(s_\Delta\ne s'_\Delta)\le c|\Delta|e^{-\omega_0\, dist(\Delta,\Lambda_1^c\sqcup\Lambda_2^c)},\quad Q(s_\Delta\ne s'_\Delta)\le c|\Delta|e^{-(\omega_0/2)\, dist(\Delta,\Lambda_1^c)} \tag{3.25}$$

Proof. For any $\epsilon>0$, there is a coupling P_ϵ such that:

$$\begin{aligned}\int \mathbf{1}_{s_\Delta\ne s'_\Delta}dP_\epsilon(d\underline{s},d\underline{s}') &\le \int d(s_\Delta,s'_\Delta)dP_\epsilon(d\underline{s},d\underline{s}')\\ &= R_\Delta\big(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p_{\Lambda_2}(\cdot|\underline{s}^{(2)})\big)+\epsilon \end{aligned}\tag{3.26}$$

the first inequality follows from the fact that $d(s_\Delta,s_\Delta)\ge 1$ for any $s_\Delta\ne s'_\Delta$. Supposing, without loss of generality, that $\Delta\sqsubset\Lambda$, $\Lambda=\Lambda_1\sqcap\Lambda_2$ and calling $I_\Delta=\{i:C_i\sqsubset\Delta\}$ and $I_{\Lambda^c}=\{i:C_i\sqsubset\Lambda^c\}$, by Theorem 2.4

$$R_\Delta\big(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p_{\Lambda_2}(\cdot|\underline{s}^{(2)})\big)\le\sum_{i\in I_\Delta}\sum_{j\in I_{\Lambda^c}}e^{-\omega_0|j-i|}d_{C_j}(s^{(1)}_{C_j},s^{(2)}_{C_j})\le c|\Delta|e^{-(\omega_0/2)\mathrm{dist}(\Delta,\Lambda^c)} \tag{3.27}$$

recalling the assumption that $d_{C_j}(\cdot,\cdot)$ is uniformly bounded; the first inequality in (3.25) then follows by the arbitrariness of ϵ in (3.26). To prove the second one, we use the Kantorovich-Rubinstein duality to write

$$R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p(\cdot))=\sup_{f:\|f\|_{\mathrm{Lip},\Delta}\le1}\Big|\int f(\underline{s})\ \{p_{\Lambda_1}(d\underline{s}|\underline{s}^{(1)})-p(d\underline{s})\}\Big| \tag{3.28}$$

$$R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p_{\Lambda_1}(\cdot|\underline{s}^{(2)}))=\sup_{f:\|f\|_{\mathrm{Lip},\Delta}\le1}\Big|\int f(\underline{s})\ \{p_{\Lambda_1}(d\underline{s}|\underline{s}^{(1)})-p_{\Lambda_1}(d\underline{s}|\underline{s}^{(2)})\}\Big| \tag{3.29}$$

(3.29) shows that $R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p_{\Lambda_1}(\cdot|\underline{s}^{(2)}))$ is a measurable function of $(\underline{s}^{(1)},\underline{s}^{(2)})$. We use the DLR property in (3.28) to write

$$\begin{aligned}&R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p(\cdot))\\ &\qquad\le\int\sup_{f:\|f\|_{\mathrm{Lip},\Delta}\le1}\Big|\int f(\underline{s})\ \{p_{\Lambda_1}(d\underline{s}|\underline{s}^{(3)})-p_{\Lambda_1}(d\underline{s}|\underline{s}^{(4)})\}\Big|\mathbf{1}_{s^{(3)}_{\Lambda_1^c}=s^{(1)}_{\Lambda_1^c}}p(d\underline{s}^{(4)})\end{aligned}$$

so that, by (3.29),

$$R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p(\cdot))\le\sup_{\underline{s}^{(3)},\underline{s}^{(4)}\in\mathcal{X}}R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(3)}),p_{\Lambda_1}(\cdot|\underline{s}^{(4)})) \tag{3.30}$$

and, by (3.27),

$$R_\Delta(p_{\Lambda_1}(\cdot|\underline{s}^{(1)}),p(\cdot))\le c|\Delta|e^{-(\omega_0/2)\mathrm{dist}(\Delta,\Lambda^c)}$$

The second inequality in (3.25) then follows from the analogue of (3.26) with $p_{\Lambda_1}(\cdot|\underline{s}^{(1)})$ and $p(\cdot)$. Theorem 3.3 is proved. □

4. Contour models and restricted ensembles

Content of Section 4: • Definition of "generalized Pirogov-Sinai models"; • Definition of abstract contour models; • Derivation of restricted ensembles from contour models, Theorem 4.1; • A formula for the finite volume corrections to the pressure; • The Pirogov Sinai scheme.

Purpose of this section is to focus on one of the main steps of the Pirogov Sinai scheme, namely the analysis of the finite volume corrections to the pressure in restricted ensembles. Using an interpolation formula this is reduced to studying decay of correlations for which the theory of Sections 2 and 3 can be used.

We start with a system in the full space $S^{\mathbb{Z}^d}$ with a stable, translational invariant, finite range hamiltonian h. Take notice that h is not the hamiltonian H of Section 2 and 3, which only later will enter into the game.

Phase indicators. With in mind a Pirogov-Sinai scenario, we introduce two functions, $\eta(\underline{s};x)$, $\Theta(\underline{s};x)$, $\underline{s}\in S^{\mathbb{Z}^d}$, $x\in\mathbb{Z}^d$, with values in $\{\pm 1, 0\}$. As functions of x with $\underline{s}$ fixed, η and Θ are respectively constant on the cubes of two partitions $\mathcal{D}$ and $\mathcal{D}'$ of $\mathbb{Z}^d$ with $\mathcal{D}$ finer than $\mathcal{D}'$. We will denote by $\{C_i, i\in\mathbb{Z}^d\}$ and by $\{C'_i, i\in\mathbb{Z}^d\}$ the cubes of the two partitions $\mathcal{D}$ and $\mathcal{D}'$, C_i having side ℓ, C'_i side $\ell'>\ell$, $\ell'/\ell\in\mathbb{N}$ and ℓ' larger than the interaction range.

$\Theta(\underline{s};x)$ has the meaning of a phase indicator and we will say that a site x is in the plus or in the minus phase if $\Theta(\underline{s};x)=\pm 1$, respectively. The values of $\Theta(\cdot;\cdot)$ are determined by those of $\eta(\cdot;\cdot)$ as follows. $\Theta(\underline{s};x)=1$ ($\Theta(\underline{s};x)=-1$) if and only if $\eta(\underline{s};y)=1$ $[=-1]$ for all y in the cube C'_i which contains x as well as in those cubes of $\mathcal{D}'$ contiguous to C'_i, whose union is denoted by $\delta^{\ell'}_{\rm out}[C'_i]$. $\Theta(\underline{s};x)=0$ if neither one of the above is verified. As a consequence of the definition the plus and minus regions, i.e. $\{x:\Theta(\underline{s};x)=1\}$ and $\{x:\Theta(\underline{s};x)=-1\}$ are not connected to each other, in fact each one of their connected components is surrounded by $\{x:\Theta(\underline{s};x)=0\}$ (the connection above is $*$ connection in $\mathbb{Z}^d$, i.e. x and y are connected if $|x_i-y_i|\le 1$, $i=1,..,d$).

Remarks. In most applications, $\eta(\underline{s};x)$ depends only on the values of $\underline{s}$ on the cube C_i of $\mathcal{D}$ which contains x and $\Theta(\underline{s};x)$ on the values of $\underline{s}$ on $C'_i\sqcup\delta^{\ell'}_{\rm out}[C'_i]$, C_i the cube of $\mathcal{D}'$ which contains x. However such measurability conditions may fail, as in the quantum version of the LMP model, see Section 6.

In a Pirogov-Sinai picture the plus and minus DLR measures (with hamiltonian h), which are obtained as thermodynamic limits of finite volume Gibbs measures with "plus and minus boundary conditions", are supported by configurations with mostly $\Theta = 1$ (resp. $\Theta = -1$), and only rare and small "islands" where $\Theta < 1$ (resp. $\Theta > -1$). Thus, in the plus case, for most cubes $C_i \in \mathcal{D}$, with large probability, $\underline{s}$ is in the set

$$\mathcal{R}_i^+ = \big\{ \underline{s} : \eta(\underline{s}; x) = 1 \text{ for any } x \in C_i \big\} \tag{4.1}$$

which will be later identified to the constraint introduced in Section 2. However with probability one there will always exist regions where $\Theta < 1$, which are present in the original model and absent in the restricted ensemble $\mathcal{X}^+ = \sqcap \mathcal{R}_i^+$. The main point here is to show that the presence of regions with $\Theta < 1$ can be taken into account by only changing the hamiltonian h into a new hamiltonian H, and then working in the restricted ensemble $\mathcal{X}^+$, see Theorem 4.1 below. In this way we recover the setup of Section 2.

Contours. A contour Γ of a configuration $\underline{s}$ is a pair $\Gamma = \{\mathrm{sp}(\Gamma), \eta_\Gamma\}$, where $\mathrm{sp}(\Gamma)$ is the closure of a maximal, connected component of $\{x : \Theta(\underline{s}; x) = 0\}$ and η_Γ is the restriction of $\eta(\underline{s}; x)$ to $\mathrm{sp}(\Gamma)$. An abstract contour is a pair which becomes a contour in some configuration $\underline{s}$.

Given a contour Γ, we decompose $\mathrm{sp}(\Gamma)^c = \mathrm{ext}(\Gamma) \sqcup \mathrm{int}(\Gamma)$, where $\mathrm{ext}(\Gamma)$ is the unbounded maximal connected component of $\mathrm{sp}(\Gamma)^c$ and call $c(\Gamma) = \mathrm{sp}(\Gamma) \sqcup \mathrm{int}(\Gamma)$. We call $A = \delta_{\mathrm{out}}^{\ell'}[\mathrm{sp}(\Gamma)]$ the union of all cubes of $\mathcal{D}'$ not in $\mathrm{sp}(\Gamma)$, but contiguous to $\mathrm{sp}(\Gamma)$. By definition of contour, $\Theta(s; x) \neq 0$ on A and constant on its connected components, we then denote by $A^\pm$ the subsets of A where $\Theta(s; x) = \pm 1$. The union of the connected components of $\mathrm{int}(\Gamma)$ which intersect A^+ are called $\mathrm{int}^+(\Gamma)$, the others $\mathrm{int}^-(\Gamma)$. We call Γ a plus contour if $A^+ \sqcap \mathrm{ext}(\Gamma) \neq \emptyset$ and minus otherwise and denote

$$\{\Gamma\}^\pm = \text{ the collection of all } \pm \text{ contours with } \mathrm{sp}(\Gamma) \text{ bounded} \tag{4.2}$$

$$\mathcal{B}^\pm = \big\{ \underline{\Gamma} = (\Gamma_1, .., \Gamma_n) : n \in \mathbb{N}_+,\ \Gamma_i \in \{\Gamma\}^\pm,\ \mathrm{sp}(\Gamma_i) \sqcap \mathrm{sp}(\Gamma_j) \neq \emptyset,\ i \neq j \big\} \tag{4.3}$$

$$\mathcal{X}^\pm = \{ \underline{s} : \eta(\underline{s}; x) \equiv \pm 1,\ x \in \mathbb{Z}^d \} \tag{4.4}$$

$\{\Gamma\}_\Lambda^\pm$ and $\mathcal{B}_\Lambda^\pm$ are defined similarly, but with the condition that all contours should have spatial support in Λ.

Contour models. Contour models are systems defined on the product space $\mathcal{X}^+ \times \mathcal{B}^+$, (to have lighter notation we hereafter restrict to the plus case). Contours are given a strictly positive statistical weight $w^+(\Gamma, \underline{s})$, $\Gamma \in \{\Gamma\}^+$, which depends on the configuration $\underline{s} \in \mathcal{X}^+$ only through the restriction of $\underline{s}$ to $\mathrm{sp}(\Gamma)$. We shorthand

$$w^+(\underline{\Gamma}, \underline{s}) = \prod_{\Gamma \in \underline{\Gamma}} w^+(\Gamma, \underline{s}), \quad \underline{\Gamma} \in \mathcal{B}^+ \tag{4.5}$$

We then introduce the "dilute", finite volume Gibbs measures on $\mathcal{X}^+ \times \mathcal{B}^+_\Lambda$, Λ a bounded $\mathcal{D}'$ measurable region, by setting

$$\mu^+_{\text{abs},\Lambda}(d\underline{s}', \underline{\Gamma}|\underline{s}) = \frac{\mathbf{1}_{\underline{s}'\in\mathcal{X}^+}}{Z^+_{\text{abs}}(\Lambda|\underline{s})} w^+(\underline{\Gamma}, \underline{s}') e^{-h_\Lambda(s'_\Lambda|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda)\delta(s'_{\Lambda^c} - s_{\Lambda^c}) ds'_{\Lambda^c} \tag{4.6}$$

where "abs" stands for abstract; $\underline{s} \in \mathcal{X}^+$ and

$$Z^+_{\text{abs}}(\Lambda|\underline{s}) = \sum_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} \int_{\mathcal{X}^+_\Lambda} w^+(\underline{\Gamma}, \underline{s}) e^{-h_\Lambda(s'_\Lambda|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda) \tag{4.7}$$

having used the shorthand notation $\nu_\Lambda(ds_\Lambda) = \prod_{x\in\Lambda} \nu(ds_x)$, for the free measure on S^Λ. By $h_\Lambda(s'_\Lambda|s_{\Lambda^c})$ we denote the conditional energy, formally defined as $h(s'_\Lambda s_{\Lambda^c}) - h(s_{\Lambda^c})$.

Remarks. Notice that the elements of a pair $(\underline{s}, \underline{\Gamma})$ in a contour model are totally unrelated, as the configuration $\underline{s}$, being in $\mathcal{X}^+$, has no contour, so that $\underline{\Gamma}$ are not at all contours in the configuration $\underline{s}$.

In the classical examples of Pirogov-Sinai models, like in Ising models, the restricted ensemble $\mathcal{X}^+$ consists of a single configuration, the plus ground state, and the (plus) contour model is the space of compatible contours $\mathcal{B}^+$. The original theory has then be generalized to cases where "colors" can be added so that the restricted ensemble is no longer a singleton; with Kac potentials the restricted ensembles are the outcome of a coarse graining procedure and they are spaces with a full and rich structure, we will see examples in Section 6.

The above definitions are not coming from nowhere as it may look, we will in fact see that, with a proper choice of the weights, we can relate the original Gibbs measures and partition functions to those of the contour models.

Peierls bounds. The weights $\{w^+(\Gamma, \underline{s})\}$ satisfy the Peierls bounds with positive constant C_P if, for all Γ and $\underline{s}$,

$$w^+(\Gamma, \underline{s}) \le e^{-C_P N_\Gamma} \tag{4.8}$$

where N_Γ is the number of cubes from $\mathcal{D}'$ contained in $\text{sp}(\Gamma)$.

Theorem 4.1. *If the weights $w^+(\Gamma, \underline{s})$ satisfy the Peierls bounds with a constant C_P large enough (in particular $C_P > 2b$, b as in* (4.11) *below) then there is a hamiltonian $H^{+,o}_\Lambda$ so that for any bounded, $\mathcal{D}'$ measurable region Λ and any $\underline{s} \in \mathcal{X}^+$,*

$$Z^+_{abs}(\Lambda|\underline{s}) = Z^{+,o}_{\text{abs},\Lambda}(\underline{s}) := \int_{\mathcal{X}^+_\Lambda} e^{-H^{+,o}_\Lambda(s'_\Lambda|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda) \tag{4.9}$$

$$H_\Lambda^{+,o}(s'_\Lambda|s_{\Lambda^c}) = h_\Lambda(s'_\Lambda|s_{\Lambda^c}) + K_\Lambda^+(s'_\Lambda), \quad K_\Lambda^+(s'_\Lambda) = \sum_{\Delta\sqsubset\Lambda} U_\Delta^+(s'_\Delta) \tag{4.10}$$

where the potentials $U_\Delta^+(s_\Delta)$*, defined in* (4.14) *and* (4.15) *below, vanish unless* Δ *is a bounded, connected,* $\mathcal{D}'$*-measurable subset of* Λ*. Moreover,*

$$\|U_\Delta^+(\cdot)\|_\infty = \sup_{s_\Delta} |U_\Delta^+(s_\Delta)| \le e^{-(C_P-b)N_\Delta} \tag{4.11}$$

where N_Δ *is the number of* $\mathcal{D}'$*-cubes in* Δ*;* $b>0$ *a dimension dependent constant, such that* $\sum_{\Delta\ni x} e^{-bN_\Delta} \le 1$*.*

Finally, denoting by $p^{+,o}_{\mathrm{abs},\Lambda}(\cdot|\underline{s})$ *the marginal on* $\mathcal{X}^+$ *of* $\mu^+_{\mathrm{abs},\Lambda}(\cdot|\underline{s})$*,*

$$p^{+,o}_{\mathrm{abs},\Lambda}(d\underline{s}'|\underline{s}) = \frac{1}{Z^+_{\mathrm{abs},\Lambda}(\underline{s})} e^{-H_\Lambda^{+,o}(s'_\Lambda|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda)\delta(s'_{\Lambda^c} - s_{\Lambda^c}) ds'_{\Lambda^c} \tag{4.12}$$

Proof. (4.9) follows from (4.7) and the first equality in (4.10), by setting

$$e^{-K_\Lambda^{+,o}(s_\Lambda)} = \sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda^+} w^+(\underline{\Gamma},\underline{s}) \tag{4.13}$$

To prove the remaining statements we will use cluster expansion to express the energy $K_\Lambda^{+,o}(s_\Lambda)$ in terms of a sum of weights of polymers, which will then identify the many-body potentials $U_\Delta^+(s_\Delta)$.

Polymers are finite, connected subsets of $\{\Gamma\}^+$, two elements in $\{\Gamma\}^+$ being called connected if their spatial supports have non empty intersection. Denoting by $\mathcal{P}^+$ the collection of all polymers and by $\mathcal{P}^+_\Lambda$ those made by contours in $\{\Gamma\}^+_\Lambda$, if C_P is large enough, it follows from Kotecki and Preis, [**16**], that there are numbers $\varpi(\underline{\Gamma},\underline{s})$, such that

$$\log \sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda^+} w^+(\underline{\Gamma},\underline{s}) = \sum_{\underline{\Gamma}\in\mathcal{P}_\Lambda^+} \varpi^+(\underline{\Gamma},\underline{s}) \tag{4.14}$$

Calling $\mathrm{sp}(\underline{\Gamma}) = \bigsqcup_{\Gamma\in\underline{\Gamma}} \mathrm{sp}(\Gamma)$, we then set

$$-U_\Delta^+(s_\Delta) = \sum_{\underline{\Gamma}\in\mathcal{P}_\Lambda^+,\, \mathrm{sp}(\underline{\Gamma})=\Delta} \varpi^+(\underline{\Gamma},s_\Delta) \tag{4.15}$$

and (4.9) follows from (4.7) and (4.13), while (4.11) follows from (4.15) and [**16**].

The proof of (4.12) is similar and omitted. Theorem 4.1 is proved. □

Finite volume corrections to the pressure. Theorem 4.1 reduces the analysis of $Z^+_{\mathrm{abs}}(\Lambda|\underline{s})$ to the study of a partition function in the restricted ensemble, for which the theory of Sections 2 and 3 can be used. As it will be discussed at the end of this section, the main step in the Pirogov Sinai scheme is the computation of the surface corrections to the pressure in contour models, which, by Theorem 4.1, requires to study $\log Z^{+,o}_{\mathrm{abs},\Lambda}(\underline{s})$ separating volume and surface effects (for large regions Λ). With this in mind and following Dobrushin and Shlosman, [**9**], see also [**14**], [**17**], we use an interpolation procedure to write

$$\log Z^{+,o}_{\mathrm{abs},\Lambda}(\underline{s}) = \log Z^{+}_{\mathrm{abs},\Lambda;0}(\underline{s}) - \int_0^1 \int_{\mathcal{X}^+} \big(H^{+,o}_{\Lambda}(s'_\Lambda|s_{\Lambda^c}) - H_{0;\Lambda}(s'_\Lambda|s_{\Lambda^c})\big)\, p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s})\, du \tag{4.16}$$

where H_0 is a "reference hamiltonian" and $Z^{+}_{\mathrm{abs},\Lambda;0}(\underline{s})$ the partition function on $\mathcal{X}^+_\Lambda$ with hamiltonian H_0 and b.c. given by $\underline{s}$; $p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s})$ is given by (4.12) with $H^{+,o}_{\Lambda}(s'_\Lambda|s_{\Lambda^c})$ replaced by

$$H^{+,o}_{\Lambda;u}(s'_\Lambda|s_{\Lambda^c}) = uH^{+,o}_{\Lambda}(s'_\Lambda|s_{\Lambda^c}) + (1-u)H^{+}_{0;\Lambda}(s'_\Lambda|s_{\Lambda^c}) \tag{4.17}$$

In agreement with the examples of Section 6, we choose H_0^+ as a free hamiltonian:

$$H^{+}_{0,\Lambda}(s_\Lambda) = \sum_{x\in\Lambda} v_{x;H_0^+}(s_x) \tag{4.18}$$

$v_{x;H_0^+}$ the translate by x of $v_{0;H_0^+}$. Writing the energy difference in (4.16) as a sum of potential terms,

$$H^{+,o}_{\Lambda}(s'_\Lambda|s_{\Lambda^c}) - H^{+}_{0;\Lambda}(s'_\Lambda|s_{\Lambda^c}) = \sum_{x\in\mathbb{Z}^d} D_{x,\Lambda}(s'_\Lambda, s_{\Lambda^c}) \tag{4.19}$$

we then need a thermodynamic limit result:

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} D_{x,\Lambda}(\underline{s}) = D_x(\underline{s}); \qquad \lim_{\Lambda\nearrow\mathbb{Z}^d} p^{+,o}_{\mathrm{abs},\Lambda;u}(\cdot|\underline{s}) = p^{+}_{\mathrm{abs};u}(\cdot) \tag{4.20}$$

with the assumption of invariance by translations by multiples of ℓ': namely any D_x is the translate by $x-y$ of D_y, if $y = x$ modulo $\mathcal{D}'$ and $p^{+}_{\mathrm{abs};u}$ is a measure invariant under translations by multiples of ℓ'. Then

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\log Z^{+,o}_{\mathrm{abs},\Lambda}(\underline{s})}{|\Lambda|} = P^{+}_{\mathrm{abs};0} - \int_0^1 \int \{\frac{1}{|C'_0|}\sum_{x\in C'_0} D_x(\underline{s})\} p^{+}_{\mathrm{abs};u}(d\underline{s})\, du := P^{+}_{\mathrm{abs}} \tag{4.21}$$

where C'_0 and C_0 are the cubes of $\mathcal{D}'$ and $\mathcal{D}$ which contain the origin and

$$P^{+}_{\mathrm{abs};0} = \frac{1}{|C_0|}\log \int_{\eta(s_{C_0};x)=1, x\in C_0} \prod_{x\in C_0} \{e^{-v_{x;H_0^+}(s_x)}\nu_x(ds_x)\} \tag{4.22}$$

Supposing Λ a bounded, $\mathcal{D}'$ measurable region, we then have

$$\log Z^{+,o}_{\mathrm{abs},\Lambda}(\underline{s}) = P^{+}_{\mathrm{abs}}|\Lambda| - \sum_{x\in\mathbb{Z}^d}\int_0^1 \Big(\int_{\mathcal{X}^+} [D_{x,\Lambda}(s'_\Lambda, s_{\Lambda^c}) - \mathbf{1}_{x\in\Lambda} D_x(s'_\Lambda, s_{\Lambda^c})] p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s}) + \int_{\mathcal{X}^+} \mathbf{1}_{x\in\Lambda} D_x(s'_\Lambda, s_{\Lambda^c})[p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s}) - p^{+}_{\mathrm{abs};u}(d\underline{s}'|\underline{s})]\Big)\, du \tag{4.23}$$

$P^+_{\rm abs}|\Lambda|$ is the volume term, correctly given by the pressure times the volume; the remaining terms are the finite volume corrections to the pressure, they grow like the surface if the convergence in (4.20) is exponential, as we will see in Section 5. Here is where the theory of Sections 2 and 3 enters into play.

Restricted ensembles. Summarizing the discussion of the previous subsection, we need to study restricted ensembles on $\mathcal{X}^+$ (and $\mathcal{X}^-$ as well !) with hamiltonian

$$H^+_u = u(h + K^+) + (1-u)H^+_0, \quad u \in [0,1], \;\; K^+ \text{ as in (4.15)} \tag{4.24}$$

We denote by $p^+_{{\rm abs},\Lambda;u}(\cdot|s_{\Lambda^c})$ the finite volume Gibbs measures with hamiltonian H^+_u, noticing that $p^+_{{\rm abs},\Lambda;u}(\cdot|\underline{s}) \neq p^{+,o}_{{\rm abs},\Lambda;u}(\cdot|\underline{s})$.

The measure $p^{+,o}_{{\rm abs},\Lambda}(d\underline{s}'|\underline{s})$ (i.e. putting $u=0$) is the marginal of $\mu^+_{{\rm abs},\Lambda}(d\underline{s}', \underline{\Gamma}|\underline{s})$, obtained by integrating out the variables $\underline{\Gamma}$. The latter is recovered by the formula

$$\mu^+_{{\rm abs},\Lambda}(d\underline{s}', \underline{\Gamma}|s_{\Lambda^c}) = \pi^+_\Lambda(\underline{\Gamma}; s'_\Lambda) p^{+,o}_{{\rm abs},\Lambda}(d\underline{s}'|s_{\Lambda^c}) \tag{4.25}$$

with $\pi^+_\Lambda(\underline{\Gamma}; \underline{s}')$ the conditional probability of $\underline{\Gamma}$ given $\underline{s}'$, given by

$$\pi^+_\Lambda(\underline{\Gamma}; \underline{s}') = \frac{\mathbf{1}_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda}}{\Xi^+(\Lambda; s'_\Lambda)} w(\underline{\Gamma}; s'_\Lambda), \quad \Xi^+(\Lambda; s'_\Lambda) = \sum_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} w(\underline{\Gamma}; s'_\Lambda) \tag{4.26}$$

Thus $\pi^+_\Lambda(\underline{\Gamma}; \underline{s}')$ depends on $\underline{s}'$ only through s'_Λ and makes the contours $\underline{\Gamma}$ pairwise independent except for the exclusion rule, their distribution however depends on "the environment" $\underline{s}'$.

Remarks. Take notice that the partition function $Z^{+,o}_{{\rm abs},\Lambda;u}(\underline{s})$ is not the partition function with hamiltonian H^+_u, because $H^+_\Lambda(s_\Lambda|s_{\Lambda^c}) \neq H^{+,o}_\Lambda(s_\Lambda|s_{\Lambda^c})$, as in $H^+_\Lambda(s_\Lambda|s_{\Lambda^c})$ contribute terms $U^+_\Delta(s_\Delta)$ having $\Delta \sqcap \Lambda \neq \emptyset$ without $\Delta \sqsubset \Lambda$ which are not present in $H^{+,o}_\Lambda(s_\Lambda|s_{\Lambda^c})$.

In contrast to the original one, the new Hamiltonian H^+ has infinite range, but this is due to K^+, which being "small" and with exponential decay is easy to handle. Other hamiltonians enter into play when studying the finite volume corrections to the pressure, as we will see next when outlining the Pirogov-Sinai scheme.

There are mainly two reasons to study these abstract contour models and restricted ensembles, but first let us explain the terminology. Abstract underlines the arbitrariness of the weights, which are not related to the structure of the original hamiltonian. The whole point however is that there is a special choice of weights (which are then called the "true weights") for which the abstract partition function becomes equal to the original one (in the space $S^{\mathbb{Z}^d}$ with hamiltonian h and suitable boundary conditions). Thus the first reason to study abstract versions of a system is because one of these version is the "true one". But, in order to apply Theorem 4.1

to such a case, we need the a priori knowledge that the true weights fulfill the Peierls bounds with large enough C_P.

The beauty of the Pirogov-Sinai theory is that this crucial step can be established by studying only contour models. The theory shows that if the computations done using fictitious weights with cutoffs, which then automatically satisfy the Peierls bounds, are "consistent" producing bounds which are below the cutoff values, then the fictitious weights are equal to the true weights and everything works. All that is discussed in some more details in the sequel.

Dilute measures and partition functions. Let Λ be a bounded $\mathcal{D}'$ measurable region; $\delta_{\rm in}^{\ell'}[\Lambda] = \delta_{\rm out}^{\ell'}[\Lambda^c]$ the union of all cubes of $\mathcal{D}'$ which are in Λ and contiguous to Λ^c; $\underline{s} \in S^{\mathbb{Z}^d}$, $\eta(\underline{s}, x) = 1$, $x \in \Lambda^c$. We then introduce the "plus, dilute Gibbs measure"

$$\mu_\Lambda^+(d\underline{s}'|s_{\Lambda^c}) = \frac{1}{Z^+(\Lambda|s_{\Lambda^c})} \mathbf{1}_{\eta(\underline{s}',x)=1, x\in\delta_{\rm in}^{\ell'}[\Lambda]} e^{-h_\Lambda(\underline{s}'|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda)\delta(s_{\Lambda^c} - s'_{\Lambda^c}) ds'_{\Lambda^c} \quad (4.27)$$

$$Z^+(\Lambda|\underline{s}) = \int_{\eta(s'_\Lambda s_{\Lambda^c},x)=1, x\in\delta_{\rm in}^{\ell'}[\Lambda]} e^{-h_\Lambda(s'|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda) \quad (4.28)$$

calling $Z^+(\Lambda|\underline{s})$ the "plus dilute partition function": the "plus" referring to the plus boundary conditions while "dilute" refers to the constraint on $\delta_{\rm in}^{\ell'}[\Lambda]$. The "minus, dilute Gibbs measure" $\mu_\Lambda^-(d\underline{s}'|s_{\Lambda^c})$ and the "minus dilute partition function" $Z^-(\Lambda|\underline{s})$ are defined analogously.

The true weight of a contour. Given a contour $\Gamma \in \{\Gamma\}^+$, call $c(\Gamma) = \mathrm{sp}(\Gamma) \sqcup \mathrm{int}^-(\Gamma)$ and $K = \{C_i \in \mathcal{D} : C_i \sqsubset \mathrm{sp}(\Gamma), \mathrm{dist}(C_i, A^+) \le \ell'/2\}$, ℓ' the side length of the cubes of $\mathcal{D}'$. We suppose that $\mathcal{D}$ and $\mathcal{D}'$ have been chosen in such a way that K is $\mathcal{D}$ measurable. Then "the true weight" of the plus contour Γ is

$$w^+(\Gamma, \underline{s}) = \frac{\mu_{c(\Gamma)\setminus K}\Big(\{\underline{s}' : \eta(\underline{s}'; x) = \eta_\Gamma(x), x \in \mathrm{sp}(\Gamma); \Theta(\underline{s}'; x) = -1, x \in A^-\}\Big|\underline{s}\Big)}{\mu_{c(\Gamma)\setminus K}\Big(\{\underline{s}' : \eta(\underline{s}'; x) = 1, x \in \mathrm{sp}(\Gamma) \sqcup A^-\}\Big|\underline{s}\Big)} \quad (4.29)$$

where $\mu_{c(\Gamma)\setminus K}(d\underline{s}'|\underline{s})$ is the Gibbs measure on $S^{\mathbb{Z}^d}$ with hamiltonian h (in the region $c(\Gamma) \setminus K$ and with b.c. given by $\underline{s}$). Notice that $w^+(\Gamma, \underline{s})$ depends only on s_K and in the sequel we may write $w^+(\Gamma, s_B)$ if $B \sqsupset K$.

Lemma 4.2. *The plus dilute partition function $Z^+(\Lambda|\underline{s})$ can be written as*

$$Z^+(\Lambda|\underline{s}) = \sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda^+} \int_{\eta(s';x)=1, x\in\Lambda} w^+(\underline{\Gamma}, s'_\Lambda) e^{-h_\Lambda(s'_\Lambda|s_{\Lambda^c})} \nu_\Lambda(ds'_\Lambda) \quad (4.30)$$

Proof. Calling $\Delta_{\underline{\Gamma}} = \text{int}(\underline{\Gamma}) \setminus A(\underline{\Gamma})$ we have

$$Z^+(\Lambda|\underline{s}) = \sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda^{+,\text{ext}}} \int_{\eta(\underline{s}',x)=1, x\in\Lambda\setminus\Delta_{\underline{\Gamma}}} w^+(\underline{\Gamma},\underline{s}')e^{-h_{\Lambda\setminus\Delta_{\underline{\Gamma}}}(\underline{s}'|s_{\Lambda^c})} Z^+(\Delta_{\underline{\Gamma}}|\underline{s}')\nu_{\Lambda\setminus\Delta_{\underline{\Gamma}}}(d\underline{s}') \tag{4.31}$$

where $\mathcal{B}_\Lambda^{+,\text{ext}}$ denotes the set of all collections of external contours: namely, all collections $\underline{\Gamma} \in \mathcal{B}_\Lambda^+$ such that $\text{sp}(\Gamma_j) \sqsubset \text{ext}(\Gamma_i)$ and $\text{sp}(\Gamma_i) \sqsubset \text{ext}(\Gamma_j)$ for any $\Gamma_i \neq \Gamma_j \in \underline{\Gamma}$. By iterations of (4.31) we then derive (4.30). Lemma 4.2 is proved.

□

The above reduction to a partition function of a contour model is however only apparent, as we do not know if the true weights satisfy the Peierls bounds with a suitably large constant C_P, which is the fundamental property of the contour models. Since the purpose of the Pirogov Sinai scheme was to prove phase transitions and phase transitions are consequence of such Peierls bounds, it seems we are back to the starting point. The way to implement the Pirogov-Sinai scheme, as proposed by Zahradnik, is to introduce an artificial contour model obtained by replacing in (4.30) the "true weights" by fictitious ones, which satisfy the Peierls condition:

$$w^{\pm,\text{cf}}(\Gamma,\underline{s}) := \min\{w^\pm(\Gamma,\underline{s}), e^{-C_P N_\Gamma/2}\} \tag{4.32}$$

If C_P is large enough, we can then apply Theorem 4.1, reduce to a restricted ensemble and then apply the theory of Sections 2 and 3. However, these are not the true weights, but only cutoff ones.

Let us go back to (4.29) which can be written as the ratio of a numerator N and a denominator D, where

$$N = \int e^{-h_{A^-}(s_{A^-})} Z^-(\text{int}^-(\Gamma) \setminus A^- |s_{A^-}) Z(\text{sp}(\Gamma) \setminus K\,; \eta = \eta_\Gamma | s_{A^-}, s_K)\nu_{A^-}(ds_{A^-}) \tag{4.33}$$

$$D = \int e^{-h_{A^-}(s_{A^-})} Z^+(\text{int}^-(\Gamma) \setminus A^- |s_{A^-}) Z(\text{sp}(\Gamma) \setminus K\,; \eta = 1 | s_{A^-}, s_K)\nu_{A^-}(ds_{A^-}) \tag{4.34}$$

where the last partition functions in (4.33) and (4.34) are defined with the constraint that $\eta = \eta_\Gamma$ and, respectively, $\eta = 1$ on $\text{sp}(\Gamma) \setminus K$.

The model for which the approach works, are such that the ratio of the partition functions on $\text{sp}(\Gamma) \setminus K$ is bounded by $e^{-C_P N_\Gamma}$ with C_P large enough. If we knew that the ratio of the other two partition functions is $< e^{(C_P/2)N_\Gamma}$, then overall $w^\pm(\Gamma,\underline{s}) < e^{-(C_P/2)N_\Gamma}$, i.e. smaller than the cutoff value. Now, the beautiful point of the theory is that it is sufficient to prove that it is $< e^{(C_P/2)N_\Gamma}$ the ratio of the partition functions defined by (4.30) with the weights replaced by the cutoff weights (4.32). If such a ratio is $< e^{(C_P/2)N_\Gamma}$, then automatically the cutoff weights are equal to the true ones and these satisfy the Peierls bounds and we are thus in business. Thus the computations which are needed involve the finite volume corrections to the pressure in a restricted ensemble as outlined earlier.

5. Couplings and decay of correlations

Content of Section 5: • Construction of "successful couplings" for restricted ensembles; • and for contour models.

We consider in this section the model of Section 4 with hamiltonians H^+_u, $u \in [0,1]$, and suppose that the assumptions in Theorem 4.1 are satisfied uniformly in the interpolating parameter u. We also suppose that the hamiltonians H^+_u satisfy Assumptions 1–5 with $\psi(i,j) = \omega_0|i-j|$, $\omega_0 > 0$, and with $d_x(\cdot,\cdot) \geq 1$ for all x (again uniformly in u). Thus the assumptions in Theorem 3.3 are all verified, and its statements can be applied to the finite volume Gibbs measures $p^+_{\text{abs},\Lambda;u}(d\underline{s}'|s_{\Lambda^c})$. In particular there is a unique thermodynamic limit, denoted by $p^+_{\text{abs};u}(d\underline{s})$, which is the candidate for the limit measure in (4.20)-(4.21). To prove it, we need to extend the analysis to the measures $p^{+,o}_{\text{abs},\Lambda;u}(d\underline{s}'|s_{\Lambda^c})$. We state a preliminary lemma:

Lemma 5.1. *Under the above assumptions, there are c' and ω_1' positive so that the following holds. Let Λ, Δ and Δ' be bounded, $\mathcal{D}$-measurable sets, $\Delta \sqsubset \Delta' \sqsubset \Lambda$, $\underline{s}^{(1)}$ in $\mathcal{X}^+$; call $p^{+,o}_{\text{abs},\Delta';\Lambda,\underline{s}^{(1)};u}(\cdot|\underline{s}^{(2)})$ the conditional probability of $p^{+,o}_{\text{abs},\Lambda;u}(\cdot|\underline{s}^{(1)})$ given that the configuration outside Δ' agrees with $\underline{s}^{(2)}$; then, for any $u \in [0,1]$,*

$$R_\Delta\big(p^+_{\text{abs},\Delta';u}(\cdot|\underline{s}^{(2)}); p^{+,o}_{\text{abs},\Delta';\Lambda,\underline{s}^{(1)};u}(\cdot|\underline{s}^{(2)})\big) \leq c'|\Delta||\Delta'|e^{-\omega_1'\text{dist}(\Delta',\Lambda^c)} \tag{5.1}$$

Proof. By Proposition A.1,

$$\begin{aligned} &R_\Delta\big(p^+_{\text{abs},\Delta';u}(\cdot|\underline{s}^{(2)}); p^{+,o}_{\text{abs},\Delta';\Lambda,\underline{s}^{(1)};u}(\cdot|\underline{s}^{(2)})\big) \\ &\qquad \leq 2c_1 \sup_{s_{\Delta'}:\eta(s_{\Delta'};\cdot)=1} \big|H^+_{\Delta';u}(s_{\Delta'}|s^{(2)}_{\Delta'^c}) - H^{+,0}_{\Delta';\Lambda;u}(s_{\Delta'}|s^{(2)}_{\Delta'^c})\big| \end{aligned} \tag{5.2}$$

where $c_1 = \sup_{s,s'\in S} d(s,s')$ and

$$H^{+,0}_{\Delta';\Lambda;u}(s_{\Delta'}|s^{(2)}_{\Delta'^c}) = u\{h(s_{\Delta'}|s^{(2)}_{\Delta'^c}) + \sum_{\Delta''\sqsubset\Lambda,\Delta''\sqcap\Delta'\neq\emptyset} U^+_{\Delta''}(s_{\Delta''})\} + (1-u)H^+_0(s_{\Delta'}) \tag{5.3}$$

Then, by (4.11),

$$\big|H^+_{\Delta';u}(s_{\Delta'}|s^{(2)}_{\Delta'^c}) - H^{+,0}_{\Delta';\Lambda;u}(s_{\Delta'}|s^{(2)}_{\Delta'^c})\big| \le \sum_{\Delta''\sqcap\Lambda^c\neq\emptyset,\Delta''\sqcap\Delta'\neq\emptyset} \|U^+_{\Delta''}\|_\infty$$

$$\le \{\sup_{\Delta''\sqcap\Lambda^c\neq\emptyset,\Delta''\sqcap\Delta'\neq\emptyset} e^{-(C_p-2b)N_{\Delta''}}\}\{\sum_{\Delta''\sqcap\Delta\neq\emptyset} e^{-bN_{\Delta''}}\} \tag{5.4}$$

The second bracket is bounded proportionally to $|\Delta'|$ while the first bracket decays exponentially with an exponent proportional to the distance of Δ' from Λ^c. Lemma 5.1 is proved.

□

Theorem 5.2. *Under the same assumptions of Lemma 5.1, there are c and ω_1 positive so that for any bounded, $\mathcal{D}$-measurable sets Λ and Δ and any $\underline{s}^{(1)} \in \mathcal{X}$, there is a coupling Q of $p^{+,o}_{\text{abs},\Lambda}(\cdot|\underline{s}^{(1)})$ and $p^+_{\text{abs}}(\cdot)$ such that*

$$Q(s_\Delta \neq s'_\Delta) \le c|\Delta|e^{-\omega_1\, dist(\Delta,\Lambda_1^c)} \tag{5.5}$$

Proof. As in (3.26), for any $\epsilon > 0$ there is a coupling Q_ϵ such that

$$Q_\epsilon(s_\Delta \neq s'_\Delta) \le R_\Delta\big(p^{+,o}_{\text{abs},\Lambda;u}(\cdot|\underline{s}^{(1)}), p^+_{\text{abs};u}(\cdot)\big) + \epsilon \tag{5.6}$$

For Λ large enough, let Δ' be $\mathcal{D}$-measurable set such that

$$\text{dist}(\Delta,\Delta'^c) \ge \frac{\text{dist}(\Delta,\Lambda^c)}{3}, \quad \text{dist}(\Delta',\Lambda^c) \ge \frac{\text{dist}(\Delta,\Lambda^c)}{3}$$

Then, by the analogue of (3.30),

$$R_\Delta\big(p^{+,o}_{\text{abs},\Lambda;u}(\cdot|\underline{s}^{(1)}), p^+_{\text{abs};u}(\cdot)\big) \le \sup_{\underline{s}^{(2)},\underline{s}^{(3)}} R_\Delta\big(p^{+,o}_{\text{abs},\Delta';\Lambda,\underline{s}^{(1)};u}(\cdot|\underline{s}^{(2)}), p^+_{\text{abs},\Delta';u}(\cdot|\underline{s}^{(3)})$$

$$\le \sup_{\underline{s}^{(2)},\underline{s}^{(3)}} R_\Delta\big(p^+_{\text{abs},\Delta';u}(\cdot|\underline{s}^{(2)}), p^+_{\text{abs},\Delta';u}(\cdot|\underline{s}^{(3)})\big)$$

$$+\sup_{\underline{s}^{(2)}} R_\Delta\big(p^+_{\text{abs},\Delta';u}(\cdot|\underline{s}^{(2)};u), (p^{+,o}_{\text{abs},\Delta';\Lambda,\underline{s}^{(1)};u}(\cdot|\underline{s}^{(2)})\big)$$

Due to Theorem 3.3 the first term on the r.h.s. is bounded by $c|\Delta|e^{-(\omega_0/2)\text{dist}(\Delta,(\Delta')^c)}$ while the second one is bounded by the r.h.s. of (5.1). Theorem 5.2 is proved.

□

Corollary 5.3. *Under the same assumptions of Theorem 5.2 and with c and ω_1 as in Theorem 5.2, for any bounded, $\mathcal{D}$-measurable sets Λ_i, $i = 1, 2$, and Δ and any $\underline{s}^{(i)} \in \mathcal{X}$, there is a coupling P of $p^{+,o}_{\text{abs},\Lambda;u}(\cdot|\underline{s}^{(i)})$ such that*

$$P(s_\Delta \neq s'_\Delta) \le 2c|\Delta| \max\{e^{-\omega_1\, dist(\Delta,\Lambda_1^c)}, e^{-\omega_1\, dist(\Delta,\Lambda_2^c)}\} \tag{5.7}$$

Proof. By the triangular inequality,

$$R_\Delta\big(p^{+,o}_{\mathrm{abs},\Lambda_1;u}(\cdot|\underline{s}^{(1)}), p^{+,o}_{\mathrm{abs},\Lambda_2;u}(\cdot|\underline{s}^{(2)}\big) \le R_\Delta\big(p^{+,o}_{\mathrm{abs},\Lambda_1;u}(\cdot|\underline{s}^{(1)}), p^{+}_{\mathrm{abs};u}(\cdot)\big)$$
$$+R_\Delta\big(p^{+,o}_{\mathrm{abs},\Lambda_2;u}(\cdot|\underline{s}^{(2)}), p^{+}_{\mathrm{abs};u}(\cdot)\big)$$

The terms on the r.h.s. have been bounded in the course of the proof of Theorem 5.2. Then, the same argument used in the proofs of Theorems 3.3 and 5.2 leads to (5.7). Corollary 5.3 is proved. □

As explained in the last part of Section 4, a crucial point in the Pirogov Sinai scheme is to prove that the second term on the r.h.s. of (4.23) goes like a surface. In our setup, this is a consequence of Theorem 5.2, as we are going to see. Using the notation

$$h_\Lambda(s_\Lambda) = \sum_{\Delta\sqsubset\Lambda} U_{\Delta;h}(s_\Delta), \quad K^+_\Lambda(s_\Lambda) = \sum_{\Delta\sqsubset\Lambda} U_{\Delta;K^+}(s_\Delta) \tag{5.8}$$

(with $U_{\Delta;K^+}$ the term previously denoted by U^+_Δ), we write $D_{x,\Lambda}$ in (4.19) as

$$D_{x,\Lambda} = \sum_{\Delta\ni x:\Delta\sqcap\Lambda\neq\emptyset} \frac{1}{|\Delta|}\{U_{\Delta;h} + U_{\Delta;K^+}\mathbf{1}_{\Delta\sqsubset\Lambda}\} - v_{x;H_0^+}\mathbf{1}_{x\in\Lambda} \tag{5.9}$$

Its thermodynamic limit is then

$$D_x = \sum_{\Delta\ni x}\frac{1}{|\Delta|}\{U_{\Delta;h} + U_{\Delta;K^+}\} - v_{x;H_0^+} \tag{5.10}$$

and, recalling (4.23),

$$\log Z^{+,o}_{\mathrm{abs},\Lambda}(\underline{s}) - P^+_{\mathrm{abs}}|\Lambda| = \Sigma^{(1;+)}_{\Lambda,\underline{s}} + \Sigma^{(2;+)}_{\Lambda,\underline{s}} + \Sigma^{(3;+)}_{\Lambda,\underline{s}} \tag{5.11}$$

where

$$\Sigma^{(1;+)}_{\Lambda,\underline{s}} = -\sum_{x\notin\Lambda}\sum_{\Delta\ni x:\Delta\sqcap\Lambda\neq\emptyset}\int_0^1\int_{\mathcal{X}^+}\frac{U_{\Delta;h}}{|\Delta|}\, p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s})du \tag{5.12}$$

$$\Sigma^{(2;+)}_{\Lambda,\underline{s}} = \sum_{x\in\Lambda}\sum_{\Delta\ni x:\Delta\sqcap\Lambda^c\neq\emptyset}\int_0^1\int_{\mathcal{X}^+}\frac{U_{\Delta;K^+}}{|\Delta|}\, p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s})du \tag{5.13}$$

$$\Sigma^{(3;+)}_{\Lambda,\underline{s}} = -\sum_{x\in\Lambda}\int_0^1\int_{\mathcal{X}^+} D_x(s'_\Lambda, s_{\Lambda^c})[p^{+,o}_{\mathrm{abs},\Lambda;u}(d\underline{s}'|\underline{s}) - p^+_{\mathrm{abs};u}(d\underline{s}'|\underline{s})]\,du \tag{5.14}$$

$\Sigma^{(1;+)}_{\Lambda,\underline{s}}$ clearly grows like the surface of Λ because the interaction has finite range; $\Sigma^{(2;+)}_{\Lambda,\underline{s}}$ is also a surface term because of the exponential decay of U^+_Δ, see (4.11); finally $\Sigma^{(3;+)}_{\Lambda,\underline{s}}$ by Theorem 5.2.

Theorem 5.2 and Corollary 5.3 prove the existence of couplings in the context of the restricted ensembles. As we will see in Appendix B they can be used to prove the existence of successful couplings also for contour models.

6. The three basic models

Content of Section 6: • Definition of the ferromagnetic Ising model with Kac potential, the LMP particle model and its Euclidean quantum version; • Characterization of their restricted ensembles; • Statement of main results.

Systems with Kac potentials are the systems for which the above theory has been devised. As an example, we will consider in the rest of the paper three specific models where in fact the previous theory can be successfully applied. They are the Ising model with ferromagnetic Kac interactions, the LMP particles model and the Euclidean representation of the quantum LMP model (QLMP for brevity). The scaling parameter of the Kac interaction will be denoted by $\gamma > 0$, the range of the interaction scaling as γ^{-1}.

Single-spin state space. (Denoted in Section 2 by S). In the Ising model $S = \{-1, 1\}$. In the classical LMP model, which is a model of point particles in $\mathbb{R}^d$, we introduce spins in the following way. We start from a partition $\mathcal{D}^{(\ell_\gamma)}$ of $\mathbb{R}^d$ into cubes $C_x^{(\ell_\gamma)}$, $x \in \mathbb{Z}^d$, $C_x^{(\ell_\gamma)}$ being the cube of the partition $\mathcal{D}^{(\ell_\gamma)}$ which contains the point $\ell_\gamma x$. We then associate to any particle configuration a spin configuration $\underline{s}$, where the spin s_x is the restriction of the particle configuration to the cube $C_x^{(\ell_\gamma)}$. Thus S is the space of configurations of point particles in a cube of side ℓ_γ, which will be denoted by $(n; q_1, .., q_n)$, $n \in \mathbb{N}$, $(q_1, .., q_n)$ a sequence of points in the cube. $\ell_\gamma > 0$ is a free parameter which is chosen suitably small, see (6.10) below, so that typically in each cube there will be at most one particle and the analysis (when checking Assumption 1 of Section 3) becomes similar to the Ising case.

Similarly, in QLMP the elements of S are $(n, q_1, .., q_n, \omega_1, .., \omega_n)$ where $(n, q_1, .., q_n)$ are as in the classical case, while $(\omega_1, .., \omega_n)$ are loops, namely each $\omega_i(\cdot)$ is a continuous function on $[0, \beta]$ with values in $\mathbb{R}^d$ such that $\omega_i(0) = \omega_i(\beta) = 0$. We then set $q_i(t) = q_i + \omega_i(t)$ and call it a loop with origin q_i.

Free measure. This is a measure (denoted by ν) on S. In the Ising model $\nu(\pm 1) = 1$. In LMP, instead, the probability density of ν at $(n; q_1, .., q_n)$ is

$$\frac{e^{-\ell_\gamma^d}}{n!} \mathbf{1}_{q_1,..,q_n \in C^{(\ell_\gamma)}} dq_1 \dots dq_n$$

while in QLMP (we are considering systems which obey the Maxwell-Boltzmann statistics), it is

$$\frac{e^{-\ell_\gamma^d}}{n!} \mathbf{1}_{q_1,..,q_n \in C^{(\ell_\gamma)}} dq_1 \dots dq_n W(d\omega_1) \dots W(d\omega_n)$$

where W is the law of a Brownian bridge starting from 0 and coming back, at time β in 0.

Metric. In Ising $d(s, s') := |s - s'|$. In LMP, take two configurations $s = (n; q_1, \dots, q_n)$ and $s' = (n'; q'_1, \dots, q'_{n'})$, supposing without loss of generality that $n \le n'$, then

$$d(s, s') := n' - n + \sum_{i=1}^{n} \mathbf{1}_{q_i \neq q'_{\pi(i)}}$$

$\{\pi\}$ the collection of all subsets of cardinality n in $\{1, .., n'\}$.

In QLMP, calling $s = (n; q_1, \dots, q_n, \omega_1, \dots, \omega_n)$, $s' = (n'; q'_1, \dots, q'_{n'}, \omega_1, \dots, \omega_n)$ and supposing again $n \le n'$,

$$d(s, s') := n' - n + \sum_{i=1}^{n} \mathbf{1}_{(q_i,\omega_i) \neq (q'_{\pi(i)},\omega'_{\pi(i)})}$$

Hamiltonian. The Hamiltonian $h = h_\gamma$, which depends on the scaling parameter γ, is translational invariant and with finite range, it incorporates the inverse temperature $\beta > 0$ as a factor. In Ising it is given by a spin-spin interaction of the form

$$-\beta J_\gamma(x, y)\sigma(x)\sigma(y) \tag{6.1}$$

where

$$J_\gamma(x, y) = \gamma^d J(\gamma x, \gamma y), \qquad \gamma > 0 \tag{6.2}$$

with $J(r, r')$ a symmetric probability kernel, translational invariant and with range 1 (i.e. $J(0, r) = 0$ for $|r| \ge 1$) which is differentiable with bounded derivative (a weaker assumption as in Appendix A of [**1**] could be used instead). The range of h is γ^{-1}.

In the LMP model the energy of a configuration s_Λ which corresponds to n particles in Λ at positions $q=(q_1,..,q_n)$ is

$$h(s_\Lambda)=\int_{\mathbb{R}^d}\beta\, e_\lambda\left(J_\gamma * q(r)\right)dr,\qquad J_\gamma * q(r)=\sum_{i=1}^n J_\gamma(r,q_i) \tag{6.3}$$

where

$$e_\lambda(x):=-\lambda x-\frac{x^2}{2!}+\frac{x^4}{4!} \tag{6.4}$$

with $\lambda\in\mathbb{R}$ having the meaning of a chemical potential. By writing explicitly the r.h.s. of (6.3) we can check that h is given by up to four body potentials and that the range of the interaction is $2\gamma^{-1}$. Since e_λ is bounded below, the corresponding h is strongly stable, i.e. $h_\Lambda(s'_\Lambda|s_{\Lambda^c})=h(s'_\Lambda s_{\Lambda^c})-h(s_{\Lambda^c})\ge -B|s_\Lambda|$ for $\underline{s},\underline{s}'\in S^{\mathbb{Z}}$ see e.g. Appendix A of [**1**].

In QLMP,

$$h(s_\Lambda)=\int_0^\beta\int_{\mathbb{R}^d} e_\lambda\left(J_\gamma * q_t(r)\right)dr\,dt,\qquad q_t=(q_i+\omega_i(t),\ i=1,..,n) \tag{6.5}$$

where s_Λ represents n loops $q_i(t)=q_i+\omega_i(t)$, $i=1,..,n$, all $q_i\in\Lambda$, By the same argument as in the classical case, h is again strongly stable.

Mean field limit. In a Pirogov-Sinai scheme, the plus and minus Gibbs states when $\gamma>0$ is sufficiently small, are regarded as perturbations of the mean field ground states, defined as the functions constantly equal to a minimizer of the mean field free energy density. In the Ising case, the latter is

$$f_\beta(m)=-\frac{1}{2}m^2-\frac{1}{\beta}I(m),\qquad I(m)=-\frac{1-m}{2}\log\frac{1-m}{2}-\frac{1+m}{2}\log\frac{1+m}{2} \tag{6.6}$$

with $m\in[-1,1]$. The first term corresponds to the energy (6.1) the second one is proportional to the entropy, which is the entropy of a Bernoulli measure on $\{-1,1\}^{\mathbb{Z}^d}$ with average m.

If $\beta>1$, $f_\beta(m)$ has a double well shape with minimum achieved at $\pm m_\beta$, where $m_\beta>0$ solves the mean field equation

$$m_\beta=\tanh\{\beta m_\beta\} \tag{6.7}$$

We will therefore restrict in the Ising case to $\beta>1$. We also notice that $f_\beta(m)$ is quadratic at the minimizers $\pm m_\beta$ and $|d\tanh\{\beta m\}/dm|$ at $m=\pm m_\beta$ is <1, a contraction property which is behind the validity of Assumptions 1–2 of Section 3.

In both LMP and QLMP the mean field free energy is

$$f_{\beta,\lambda}(\rho)=e_\lambda(\rho)-\frac{1}{\beta}I(\rho),\qquad I(\rho)=-\rho\big(\log\rho-1\big) \tag{6.8}$$

$\rho \geq 0$ having the meaning of a particles density. For $\beta > (3/2)^{3/2}$, there is λ_β so that $f_\beta(\rho) \equiv f_{\beta,\lambda_\beta}(\rho)$ is a double well function with equal minima at $\rho_\beta^\pm$, solutions of

$$\rho = \exp\big\{ -\beta e'_{\lambda_\beta}(\rho)\big\}$$

The analysis in [**14**] (for the classical case) and in [**1**] (for the quantum case) are restricted to the case $\Big|\frac{d}{d\rho}\exp\big\{ -\beta e'_{\lambda_\beta}(\rho)\big\}\Big| < 1$, which holds for $\beta \in ((3/2)^{3/2}, \beta_0)$, β_0 a value larger than $(3/2)^{3/2}$, and since we rely here on that analysis we restrict as well to $\beta \in ((3/2)^{3/2}, \beta_0)$.

Basic partitions. The partitions $\mathcal{D}$ and $\mathcal{D}'$ of Section 4 are defined in terms of two scale lengths,

$$\gamma^{-(1\pm\alpha)}, \qquad \alpha > 0 \tag{6.9}$$

$\alpha < 1$. Both α and γ will be very small, so that $1 \ll \gamma^{-(1-\alpha)} \ll \gamma^{-(1+\alpha)}$.

In Ising the sides of the cubes of the partitions $\mathcal{D}$ and $\mathcal{D}'$ are chosen equal to $\ell_{\pm,\gamma}$, with $\ell_{\pm,\gamma} \in [\gamma^{-(1\pm\alpha)}, 2\gamma^{-(1\pm\alpha)}]$: $\ell_{+,\gamma}$ is an integer multiple of $\ell_{-,\gamma}$, to enforce the fact that $\mathcal{D}$ should be finer than $\mathcal{D}'$, as required in Section 4.

In LMP and QLMP we consider three partitions of $\mathbb{R}^d$, $\mathcal{D}^{(\ell_\gamma)}$, $\mathcal{D}^{(\ell_{-,\gamma})}$ and $\mathcal{D}^{(\ell_{+,\gamma})}$ each one finer than the successive one. We choose

$$\ell_\gamma = \gamma^{3d} \tag{6.10}$$

$\ell_{\pm,\gamma} \in [\gamma^{-(1\pm\alpha)}, 2\gamma^{-(1\pm\alpha)}]$ and such that $\ell_{+,\gamma}/\ell_{-,\gamma}$ and $\ell_{-,\gamma}/\ell_\gamma$ are integers. Then the map of $\mathbb{R}^d$ onto $\mathbb{Z}^d$, which associates to a point $r \in \mathbb{R}^d$ the site $x \in \mathbb{Z}^d$ such that $C_x^{(\ell_\gamma)} \ni r$, transforms the partitions $\mathcal{D}^{(\ell_\gamma)}$, $\mathcal{D}^{(\ell_{-,\gamma})}$ and $\mathcal{D}^{(\ell_{+,\gamma})}$ into $\mathbb{Z}^d$, $\mathcal{D}$ and $\mathcal{D}'$.

Note that the length of the sides of the cubes of $\mathcal{D}$ and $\mathcal{D}'$ (which are measured in lattice units) are different than those of $\mathcal{D}^{(\ell_{-,\gamma})}$ and $\mathcal{D}^{(\ell_{+,\gamma})}$, from which they differ by a factor ℓ_γ.

Phase indicators. In the Ising case

$$\eta(\underline{s};x) = \pm 1 \quad \Leftrightarrow \quad \frac{1}{|C_x^{(\ell_{-,\gamma})}|}\Big|\sum_{y\in C_x^{(\ell_{-,\gamma})}} \big(s_y \mp m_\beta\big)\Big| \leq \gamma^a \tag{6.11}$$

and $= 0$ otherwise; in (6.11), $C_x^{(\ell_{-,\gamma})}$ is the cube of $\mathcal{D} \equiv \mathcal{D}^{(\ell_{-,\gamma})}$ which contains x, $a > 0$ is a parameter such that $a \ll \alpha$; m_β is defined in (6.7).

In LMP,

$$\eta(\underline{s};x) = \pm 1 \quad \Leftrightarrow \quad \Big|\frac{1}{|C_x^{(\ell_{-,\gamma})}|}\,|q \sqcap C_x^{(\ell_{-,\gamma})}| \mp \rho_\beta^\pm\Big| \leq \gamma^a \tag{6.12}$$

and $= 0$ otherwise; in (6.12) q is the particle configuration described by $\underline{s}$ and $|q \sqcap C|$ the number of particles in the cube C.

In QLMP we first define a variable $\eta'(\underline{s};x)$ by the r.h.s. of (6.12) with q the configuration made of all particles in $\underline{s}$ whose loops are "short", where a loop $\omega(t)$ is short if $|\omega(t)-\omega(0)|<\gamma^{-1/2}$ for $0\le t\le\beta$. Then $\eta(\underline{s};x)=\pm1$ if $\eta'(\underline{s};x)=\pm1$ and moreover all trajectories $q_i(t)=q_i+\omega_i(t)$ described by $\underline{s}$ which pass through $C_x^{(\ell_-,\gamma)}$ are short. In all other cases $\eta(\underline{s};x)=0$.

In each one of the three models the variable $\Theta(\underline{s};x)$ is defined in terms of the corresponding variable $\eta(\underline{s};x)$ according to what said in Section 4.

Reference hamiltonian. There is a reference hamiltonian in the plus and one in the minus restricted ensembles, as in Section 4 we restrict to the plus case and denote it by H_0^+. In Ising

$$H^+_{0,\Lambda}(s_\Lambda)=-\sum_{x\in\Lambda}m_\beta s_x \tag{6.13}$$

in LMP and QLMP, denoting by n the number of particles in the configuration s_Λ,

$$H^+_{0,\Lambda}(s_\Lambda)=e'_{\lambda_\beta}(\rho^+_\beta)n \tag{6.14}$$

In Ising, the mean field free energy for the system $uh+(1-u)H_0^+$ is, recalling (6.6),

$$-\frac{u}{2}m^2-(1-u)m_\beta m-\frac{1}{\beta}I(m) \tag{6.15}$$

for which m_β is again a minimizer (and the only one if $u<1$). Analogously, in LMP and QLMP, we have from (6.8),

$$ue_{\lambda_\beta}(\rho)+(1-u)e'_{\lambda_\beta}(\rho^+_\beta)\rho-\frac{1}{\beta}I(\rho) \tag{6.16}$$

which has again ρ^+_β as a minimizer.

Peierls bounds. We suppose that the weights of the contours satisfy the bound (4.8) with C_P as follows. In Ising there is a positive constant c and for all γ small enough, $C_P=c\gamma^{-(1-\alpha)d+2a}$. The same expression (but with a different value of c) holds in LMP for all λ which differ from λ_β by $\le\gamma^{a'}$, $0<a'<a$. In QLMP, $C_P=c\gamma^{-1}$ with λ varying in the same interval as in the classical case.

Remarks. Note that in the three models C_P increases to infinity as $\gamma\searrow0$. The Peierls bounds should be regarded in LMP and QLMP above as assumptions, but as explained in Section 4, the assumption must be consistent within the Pirogov Sinai scheme. This happens for a proper choice of the positive constant c. We will not prove it here, but establish, for the above weights, the validity of Assumptions 1–5, which is the main ingredient in the proof of self consistency of the Peierls bounds.

In the Ising case the Pirogov Sinai theory is not needed, due to the spin flip symmetry of the system. However the introduction of restricted ensembles and the estimates we prove here are useful when studying the surface corrections to the pressure for plus and minus dilute partition functions.

Choice of parameters. In all three models, the parameter ξ of Section 3 is chosen equal to

$$\xi = 3\gamma^{-1} \tag{6.17}$$

The choice is such that the hamiltonian h does not give contribution to the interaction at distances larger than ξ (3 above is not optimal).

Let ω_0 be the parameter introduced in Theorem 3.3. In Ising,

$$\omega_0 = c_0 \gamma \ell_{-,\gamma} \tag{6.18}$$

where $c_0 > 0$ is a suitably small constant, independent of γ. The choice (6.18) has the following motivation: In Section 3, ω_0 enters in the definition of the metric $\psi(i,j) = \omega_0 |i-j|$. i and j are labels for the cubes C_i and C_j of $\mathcal{D}$, whose sides in Ising scale as $\ell_{-,\gamma}$. Thus $\psi(i,j) = c_0\gamma(\ell_{-,\gamma}|i-j|)$ is proportional by $c_0\gamma$ to the distance between C_i and C_j; the factor γ says that this distance is measured in terms of interaction lengths (as γ is the inverse of the interaction range).

In LMP and QLMP,

$$\omega_0 = c_0 \gamma \frac{\ell_{-,\gamma}}{\ell_\gamma} \tag{6.19}$$

with c_0 a suitably small constant. The choice (6.19) has the same motivation as (6.18), the extra factor ℓ_γ comes in because the lattice unit in LMP and QLMP corresponds to a distance ℓ_γ in $\mathbb{R}^d$.

Restricted ensembles. We consider in the sequel the plus restricted ensembles in the above three models, with λ in LMP and QLMP ranging in $[\lambda_\beta - \gamma^{a'}, \lambda_\beta + \gamma^a]$ and $0 < a' < a$.

Theorem 6.1. *There is* c_0 *in* (6.18) *(for Ising) and* (6.19) *(for LMP and QLMP) so that Assumption 1–5 (the latter with* $\psi(i,j) = \omega_0|i-j|$*) are satisfied in the restricted ensembles of the three models uniformly on the interpolation parameter* $u \in [0,1]$.

We will prove Theorem 6.1 in the remaining sections, considering only the case $u = 1$, as the other cases can be handled in a similar way. We conclude this section stating the results about phase transitions for the three models, which are already known in Ising and LMP, while they are proved in [**1**] with the help of the above Theorem 6.1.

Theorem 6.2. *In the Ising case, for all γ small enough, the plus and minus dilute Gibbs measures converge in the thermodynamic limit to mutually distinct measures. The same occurs in LMP and QLMP, provided the chemical potential λ is suitably chosen ($\lambda = \lambda_{\beta,\gamma}$, $|\lambda_\beta - \lambda_{\beta,\gamma}| \le c\gamma^{1/2}$).*

7. Validity of Assumption 1

In this section we will check the validity Assumption 1 of Section 2 in the three models of Section 6, deriving explicit bounds for the coefficients.

Ising model. For $x \in \mathbb{Z}^d$, define

$$G_x = \left\{ \underline{s} \in \mathcal{X} : \underline{s}^{(x)} \in \mathcal{X} \right\} \tag{7.1}$$

where $\underline{s}^{(x)}_z = s_z$, for $z \neq x$ and $\underline{s}^{(x)}_x = -s_x$.

Recall that $\mathcal{X} = \{\underline{s} : \eta(\underline{s}; x) = 1,\, x \in \mathbb{Z}^d\}$ with $\eta(\underline{s}; x)$ as in (6.11). Thus, if for example $\underline{s} \in \mathcal{X}$ is such that

$$\sum_{y \in C_x^{(\ell_-,\gamma)} \setminus x} s_y > (m_\beta + \gamma^a)|C_x^{(\ell_-,\gamma)}| - 1 \tag{7.2}$$

with $C_x^{(\ell_-,\gamma)}$ the cube of $\mathcal{D}$ which contains x, then, necessarily, $s_x = -1$ and $\underline{s} \notin G_x$. On the other hand, if we replace $\underline{s}$ by a new configuration $\underline{s}'$ obtained by changing a plus into a minus at some $y \neq x$ in $C_x^{(\ell_-,\gamma)}$, then s_x is free in this new configuration, i.e. $\underline{s}' \in G_x$. Therefore, the Vaserstein distance between the conditional probabilities of s_x given $\underline{s}$ and $\underline{s}'$ outside x is not "small". The first assumption will come by an almost explicit evaluation of the Vaserstein distance. This is possible because of the simple nature of the space S, which consists of only two points. Closeness to mean field, as guaranteed by the configurations being in $\mathcal{X}$, and the stability properties of the mean field ground states, will then yield (3.2).

Proposition 7.1. *For all* $\gamma > 0$ *small enough,* (3.1) *holds with*

$$b(x,y) = r[J_\gamma(x,y) + e^{-(C_P - 2b)N_\gamma(x,y)}] \tag{7.3}$$

where $r < 1$ *and* b, C_P *as in* (4.11) *with* $C_P = c\gamma^{-(1-\alpha)d+2a}$, $c > 0$, *while*

$$N_\gamma(x,y) \ge 1 + \textit{integer part of } \frac{|x-y|}{\ell_{+,\gamma}} \tag{7.4}$$

Thus

$$\lim_{\gamma\to 0} \sup_{x\in\mathbb{Z}^d} \sum_{y\in C_x^{(\ell_-,\gamma)}\setminus x} b(x,y) = 0 \tag{7.5}$$

and (3.2) *holds for all* γ *small enough.*

Proof. Let $\underline{s}$ and $\underline{s}'$ be in G_x, then the conditional probabilities $p_x(\cdot|\underline{s})$ and $p_x(\cdot|\underline{s}')$ are just the usual Gibbs measures, as the constraint of being in $\mathcal{X}$ does not affect the values of s_x. Thus

$$p_x(s_x = \sigma|\underline{s}) = \frac{e^{\beta k(\underline{s})\sigma}}{e^{\beta k(\underline{s})} + e^{-\beta k(\underline{s})}}, \qquad k(\underline{s}) = \sum_{z\neq x} J_\gamma(x,z)s_z + h_x^+(\underline{s})$$

where

$$h_x^+(\underline{s}) = \frac{1}{2}\sum_{\Delta\ni x} [U_\Delta(1, s_{\Delta\setminus x}) - U_\Delta(-1, s_{\Delta\setminus x})] \tag{7.6}$$

$(\sigma, s_{\Delta\setminus x})$ being the configuration in $\{-1,1\}^\Delta$ equal to σ at x and which agrees with $\underline{s}$ on the other sites, U_Δ being as in (4.11). Indeed, the interaction energy of σ with all the other spins due to h^+ is $h_x^+(\sigma|\underline{s}) = \sum_{\Delta\ni x} U_\Delta(\sigma, s_{\Delta\setminus x})$. Since

$$h_x^+(\sigma|\underline{s}) = \frac{1+\sigma}{2} h_x^+(1|\underline{s}) + \frac{1-\sigma}{2} h_x^+(-1|\underline{s}) = \sigma h_x^+(\underline{s}) + \text{ const}$$

where the last term is independent of σ and does not contribute to the conditional Gibbs measure, hence (7.6).

We will next show that for $\underline{s}$ and $\underline{s}'$ both in G_x,

$$R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}')\big) = |\tanh\{\beta k(\underline{s})\} - \tanh\{\beta k(\underline{s}')\}| \tag{7.7}$$

The Vaserstein distance is attained by the joint representation which has the maximal mass on the diagonal, namely $\min\{p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}')\}$ and is triangular, (for the first statement we have used that d is also the variational distance and for the second statement that s_x takes only two values). On the other hand the integral of $d(s_x, s'_x) = |s_x - s'_x|$ over a triangular joint representation is the same as the absolute value of the integral of $(s_x - s'_x)$ (without modulus), hence (7.7).

Calling $u = \min\{|k(\underline{s})|, |k(\underline{s}'|\}$, we have

$$|\tanh\{\beta k(\underline{s})\} - \tanh\{\beta k(\underline{s}')\}| \le \frac{\beta}{\cosh^2(\beta u)}\big(\sum_{y\neq x} J_\gamma(x,y)|s_y - s'_y| + |h^+_x(\underline{s}) - h^+_x(\underline{s}')|\big) \tag{7.8}$$

To get the Lipschitz norm of $h^+_x(\underline{s})$, we write, recalling (7.6),

$$|h^+_x(\underline{s}) - h^+_x(\underline{s}')| \le \frac{1}{2}\sum_{\Delta\ni x}\Big(|U_\Delta(1, s_{\Delta\setminus x}) - U_\Delta(1, s'_{\Delta\setminus x})| + |U_\Delta(-1, s_{\Delta\setminus x}) - U_\Delta(-1, s'_{\Delta\setminus x})|\Big)$$

and

$$|U_\Delta(\pm 1, s_{\Delta\setminus x}) - U_\Delta(\pm 1, s'_{\Delta\setminus x})| \le \|U_\Delta(\cdot)\|_\infty \sum_{y\in\Delta\setminus x}|s_y - s'_y|$$

Hence

$$|h^+_x(\underline{s}) - h^+_x(\underline{s}')| \le \sum_{y\neq x}\{\sum_{\Delta\ni x,y}\|U_\Delta(\cdot)\|_\infty\}|s_y - s'_y|$$

From (7.8) we then get (3.1) with

$$b(x,y) \le \frac{\beta}{\cosh^2(\beta u)}\big(J_\gamma(x,y) + \sum_{\Delta\ni x,y}\|U_\Delta(\cdot)\|_\infty\big) \tag{7.9}$$

To bound the fraction on the r.h.s. we recall that $u = \min\{|k(\underline{s})|, |k(\underline{s}'|\}$ and that

$$k(\underline{s}) = \sum_{z\neq x} J_\gamma(x,z)s_z + h^+_x(\underline{s})$$

so that

$$|k(\underline{s}) - m_\beta| \le |\sum_{z\neq x} J_\gamma(x,z)(s_z - m_\beta)| + \sum_{\Delta\ni x}\|U_\Delta(\cdot)\|_\infty$$

By the assumptions on J_γ (see Section 6) and (7.2), the first term is bounded by $c(\gamma\ell_{-,\gamma} + \gamma^a + \ell^d_{-,\gamma})$, c a suitable constant. The first term comes from the variations of $J_\gamma(x,z)$ when z varies in a cube of $\mathcal{D}^{(\ell_{-,\gamma})}$, the second one by (6.11). Using (4.11)

$$\sum_{\Delta\ni x}\|U_\Delta(\cdot)\|_\infty \le \sum_{\Delta\ni x} e^{-(C_p - b)N_\Delta} \le e^{-(C_p - 2b)} \tag{7.10}$$

for $b > 0$ so large that $\sum_{\Delta\ni x} e^{-bN_\Delta} \le 1$. Recall the condition for b in Theorem 4.1; $C_P = c\gamma^{-(1-\alpha)d+2a}$. Thus

$$\lim_{\gamma\to 0}\sup_{\underline{s}\in G_x}|k(\underline{s}) - m_\beta| = 0$$

so that, given any $\epsilon > 0$ there is $\gamma_\epsilon > 0$ and for $\gamma \le \gamma_\epsilon$ we get from (7.9)

$$b(x,y) \le [\frac{\beta}{\cosh^2(\beta m_\beta)} + \epsilon]\big(J_\gamma(x,y) + \sum_{\Delta\ni x,y}\|U_\Delta(\cdot)\|_\infty\big) \tag{7.11}$$

The square bracket is < 1 for $\epsilon > 0$ small enough, because

$$\frac{\beta}{\cosh^2(\beta m_\beta)} < 1$$

which is the mean field condition of stability of the solution m_β (at $\beta > 1$). It thus only remains to bound the last term in (7.11). We write

$$\sum_{\Delta \ni x,y} \|U_\Delta(\cdot)\|_\infty \le \sum_{\Delta \ni x,y} e^{-(C_p - b)N_\Delta} \le e^{-(C_p-2b)N_\gamma(x,y)} \sum_{\Delta \ni x} e^{-bN_\Delta} \tag{7.12}$$

Hence (7.3) and (7.5). The Proposition is proved. □

Classical LMP model. In LMP, the single spin has much more structure than the simple, ± 1 valued Ising spin, being a whole configuration of particles in a cube. However by choosing small enough the side ℓ_γ of the cubes of the partition $\mathcal{D}^{(\ell_\gamma)}$, we will see that with large probability there will be at most one particle in the cube, and the analysis will then become similar to the previous one for the Ising case and, in the end, based on stability properties of the mean field solution.

We will use the following notation. The spin s_x, $x \in \mathbb{Z}^d$, represents the collection of particles in the cube $C_{\gamma,x}$ of $\mathcal{D}^{(\ell_\gamma)}$ of $\mathbb{R}^d$, the choice of the latter such that $\ell_\gamma x \in C_{\gamma,x}$. We will also denote by $n(s_x)$ the number of particles in the configuration s_x. Given $x \in \mathbb{Z}^d$, let $C \sqsubset \mathbb{Z}^d$ be the cube of $\mathcal{D}$ (seen as made up from cubes of $\mathcal{D}^{(\ell_\gamma)}$, such that $\sqcup_{x\in C} C_{\gamma,x}$ is a cube of side length $\ell_{-,\gamma}$) which contains x and $|C|$ its cardinality, we then define

$$G_x = \Big\{ \underline{s} \in \mathcal{X} : -\gamma^a \le |C|^{-1} \sum_{y \in C\setminus x} n(s_y) - \rho^+_\beta \le \gamma^a - \frac{1}{|C|} \Big\} \tag{7.13}$$

so that G_x is the set of configurations $\underline{s} \in \mathcal{X}$ which remain in $\mathcal{X}$ if we replace s_x by s'_x with $n(s'_x) = 0, 1$.

Lemma 7.2. *Let* $d(\underline{s}, \underline{s}') = \sum_z d(s_z, s'_z)$ *and*

$$\hat r(x,y) := \sup_{\underline{s},\underline{s}' \in G_x, d(\underline{s},\underline{s}') = d(s_y, s'_y) = 1} R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}')\big) \tag{7.14}$$

then

$$\hat r(x,y) \le b(x,y) \qquad \Longleftrightarrow \qquad (3.1) \textit{ holds} \tag{7.15}$$

Proof. If (3.1) holds then for $\underline{s}$ and $\underline{s}'$ as in (7.14),

$$R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}')\big) \le b(x,y)$$

hence $\hat r(x,y) \le b(x,y)$. To prove the reverse thesis, by the triangular inequality, it is enough to show that there is a sequence $\underline{s}^{(i)}$ of elements in G_x so that

$$\sum_{y \ne x} d(s^{(i)}_y, s^{(i+1)}_y) = 1 \tag{7.16}$$

and such that, for any $y \neq x$,

$$\sum_i d(s_y^{(i)}, s_y^{(i+1)}) = d(s_y, s'_y) \tag{7.17}$$

The existence of $\underline{s}^{(i)}$ can be proved for one cube $C_{y,\gamma}$ after another and we just explain for one of them.

Let then s_y, s'_y respectively, be described by the configurations $q \sqcup q'$ and $q \sqcup q''$ in $C_{y,\gamma}$. Thus, the configuration q is in common to both, while the other particles are in different positions, respectively q' and q''. We are going to change s_y with at each step adding or removing a particle and in such a way that at the end we get s'_y: the number of steps must be equal to $n(q') + n(q'') - n(q') \wedge n(q'') = d(s_y, s'_y)$ ($n(q)$ the number of particles in q). If $n(q') > n(q'')$, we change successively the configuration q' by removing one particle at a time, till the number of particles left is equal to $n(q'')$. Since initially both configurations were in G_x, the new configuration at any of the above steps is still in G_x, by (7.13). If instead $n(q') < n(q'')$ we add particles, putting them at the positions of particles in q'', till we reach parity. Thus it remains to define the modifications when the number of particles are equal. Then simultaneously we add a particle to the first configuration $q \sqcup q'$ in a position of those of the second which are not matched, i.e. q'', and we subtract a particle among the mismatched of the first configuration, i.e. q'. This leads again to an equal number of particles in the two configurations and one less to match. Thus the distance between this two resulting configurations is 1 and they both are in G_x in agreement with (7.17). By iterating the procedure we then prove (7.16)-(7.17).

$$\begin{aligned} R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}')\big) &\le \sum_y \sum_i R\big(p_x(\cdot|s_y^{(i)}), p_x(\cdot|s_y^{(i+1)})\big) \le \sum_i \hat{r}(x, y_i) \\ &\le \sum_{y \neq x} \sum_i b(x,y) d(s_y^{(i)}, s_y^{(i+1)}) = \sum_{y \neq x} b(x,y) d(s_y, s'_y) \end{aligned}$$

where y_i denotes the indey y of the cube $C_{\gamma,x}$ at which $s_y^{(1)}$ and $s_y^{(2)}$ disagree. (3.1) is proved and hence Lemma 7.2. □

We might have used the analogue of Lemma 7.2 also in the proof of Proposition 7.1. We state in fact without proofs the following result:

Lemma 7.3. *Let $\underline{s}$ and $\underline{s}'$ be two configurations of the Ising model which are both in $\mathcal{X}$ and which differ in a finite set Y. Then $d(\underline{s}, \underline{s}') = 2|Y|$ and there are $|Y| + 1$ configurations $\underline{s}^{(i)}$, $i = 0, .., |Y|$, all in $\mathcal{X}$ with $\underline{s}^{(0)} = \underline{s}$ and $\underline{s}^{(|Y|)} = \underline{s}'$ such that, for any $i = 1, .., |Y|$, $\underline{s}^{(i+1)}$ differs from $\underline{s}^{(i)}$ by a spin flip in a site in Y.*

We will next reduce to $n(s_x) \le 1$. Let $\underline{s} \in G_x$, $\bar{q}$ the collection of all the particles of s_y, $y \neq x$, and

$$V_{\underline{s},x}(r) = \int \Big(e_{\lambda_{\beta,\gamma}}\big((J_\gamma * \bar{q})(r') + J_\gamma(r',r)\big) - e_{\lambda_{\beta,\gamma}}\big(J_\gamma * \bar{q}(r')\big)\Big) dr', \quad r \in C_{\gamma,x} \quad (7.18)$$

Let then $m_{x;\le 1}(ds_x|\underline{s})$ be the Gibbs probability on S, supported by $n(s_x) \le 1$, and defined by

$$m_{x;\le 1}(ds_x|\underline{s}) = \frac{1}{Z(x,\underline{s})}\Big(\mathbf{1}_{n(s_x)=0} + e^{-\beta V_{\underline{s},x}(\ell_\gamma x)}\mathbf{1}_{n(s_x)=1} dr\Big) \quad (7.19)$$

with $r \in C_{\gamma,x}$ the position of the particle of s_x when $n(s_x) = 1$, and

$$Z(x,\underline{s}) = 1 + e^{-\beta V_{\underline{s},x}(\ell_\gamma x)}\ell_\gamma^d \quad (7.20)$$

Note that instead of the real interaction of a particle moving in $C_{\gamma,x}$ we just use the potential at the center of the cube $C_{\gamma,x}$.

Lemma 7.4. *For all $\gamma > 0$ small enough, $x \in \mathbb{Z}^d$ and $\underline{s} \in G_x$,*

$$R\big(p_x(\cdot|\underline{s}), m_{x;\le 1}(\cdot|\underline{s})\big) \le c\gamma\ell_\gamma^{d+1} \quad (7.21)$$

with $c > 0$ independent of γ and $\underline{s}$.

Proof. By the triangular inequality,

$$R\big(p_x(\cdot|\underline{s}), m_{x;\le 1}(\cdot|\underline{s})\big) \le R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}; \le 1)\big) + R\big(p_x(\cdot|\underline{s}; \le 1), m_{x;\le 1}(\cdot|\underline{s})\big) \quad (7.22)$$

where $p_x(\cdot|\underline{s}; \le 1)$ is $p_x(\cdot|\underline{s})$ conditioned on $n(s_x) \le 1$.

For the first term on the r.h.s. we use (A.3) bounding $d(s_x, s'_x) \le n(s_x) + n(s'_x)$,

$$R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}; \le 1)\big) \le 2\int_{n(s_x)\ge 2} n(s_x)\, p_x(ds_x|\underline{s})$$

Denoting by $\bar{q}$ the collection of all the particles of $s_y, y \neq x$, and by q those of s_x, the conditional energy in $p_x(ds_x|\underline{s})$ is

$$H^+(s_x|\underline{s}) = K^+(s_x|\underline{s}) + \int \Big(e_{\lambda_{\beta,\gamma}}(J_\gamma * \bar{q}(r) + J_\gamma * q(r)) - e_{\lambda_{\beta,\gamma}}(J_\gamma * \bar{q}(r))\Big) dr$$

As in (7.10) using (4.11),

$$|K^+(s_x|\underline{s})| \le \sum_{\Delta \ni x} e^{-(C_p - b)N_\Delta} \le e^{-(C_p - 2b)} \quad (7.23)$$

Then, by the strong stability of the hamiltonian h, (see Section 6) there is $B > 0$ so that

$$H^+(s_x|\underline{s}) \ge -Bn(s_x)$$

Since

$$\int_{n(s_x)\ge 2} n(s_x)\, p_x(ds_x|\underline{s}) \le \sum_{n\ge 2} \frac{(\ell_\gamma^d e^{\beta B})^n}{n!} \le c\ell_\gamma^{2d}$$

we obtain the following bound for the first term on the r.h.s. of (7.22):

$$R\big(p_x(\cdot|\underline{s}), p_x(\cdot|\underline{s}; \le 1)\big) \le 2c\ell_\gamma^{2d} \tag{7.24}$$

We will next bound the second term on the r.h.s. of (7.22). The two measures $p_x(\cdot|\underline{s}; \le 1)$ and $m_{x;\le 1}(\cdot|\underline{s})$ are two measures supported on $n(s_x) \le 1$.

$$p_x(\cdot|\underline{s} \le 1) = \frac{1}{\tilde{Z}(x,\underline{s})}\Big[e^{-K^+(s_x|\underline{s})}\mathbf{1}_{n(s_x)=0} + e^{-K^+(s_x|\underline{s})+V_{\underline{s},x}(r)}\mathbf{1}_{n(s_x)=1}\Big] \tag{7.25}$$

where r denotes the position of the particle in the configuration s_x when $n(s_x) = 1$ and $V_{\underline{s},x}(r)$ is defined in (7.18). The conditional energy of $m_{x;\le 1}(\cdot|\underline{s})$ is instead 0 when $n(s_x) = 0$ and otherwise equal to $V_{\underline{s},x}(\ell_\gamma x)$, independently of r.

We apply Proposition A.1 with h the energy of $m_{x;\le 1}(\cdot|\underline{s})$ and v the difference between the energy of $p_x(\cdot|\underline{s}; \le 1)$ and $m_{x;\le 1}(\cdot|\underline{s})$. With the notation of Proposition A.1, we then have as before that, for a suitable constant c,

$$\mu_t\big(n(s_x) = 1\big) \le c\ell_\gamma^d, \quad 0 \le t \le 1$$

and by (A.2) and (7.23),

$$R\big(p_x(\cdot|\underline{s}; \le 1), m_{x;\le 1}(\cdot|\underline{s})\big) \le 2c\ell_\gamma^d\big(e^{-(C_p-2b)} + \sup_{r\in C_{x,\gamma}} |V_{\underline{s},x}(r) - V_{\underline{s},x}(\ell_\gamma x)|\big)$$

By the assumptions on J_γ (see Section 6), $|V_{\underline{s},x}(r) - V_{\underline{s},x}(\ell_\gamma x)| \le c\gamma\ell_\gamma$ hence (7.21), recalling $C_p = c\gamma^{-(1-\alpha)d+2a}$. Lemma 7.4 is proved. □

By the triangular inequality, Lemma 7.4 reduces the estimate of (3.1) to that of measures of the kind $m_{x,\le 1}(ds_x|\underline{s})$, for which we proceed as in the Ising case.

Proposition 7.5. *For all $\gamma > 0$ small enough,* (3.1) *holds with*

$$b(x,y) \le \delta\ell_\gamma^d J_\gamma^2(\ell_\gamma x, \ell_\gamma y) + c\ell_\gamma^d\gamma^{2d}, \quad \delta < 1 \tag{7.26}$$

where $J_\gamma^2 = J_\gamma * J_\gamma$ and $c > 0$ independent of γ.

Proof. By Lemma 7.2 we only need to prove (7.26), with $b(x,y)$ as in (7.14). By Lemma 7.4 and the triangular inequality, we then have

$$b(x,y) \le \sup_{\underline{s},\underline{s}'\in G_x: d(\underline{s},\underline{s}')=d(s_y,s'_y)=1} R\big(m_{x;\le 1}(\cdot|\underline{s}), m_{x;\le 1}(\cdot|\underline{s}')\big) + 2c\gamma\ell_\gamma^{d+1} \tag{7.27}$$

Conditioned on $n(s_x)$, $m_{x;\le 1}(\cdot|\underline{s})$ and $m_{x;\le 1}(\cdot|\underline{s}')$ become equal, then, by Proposition A.3, their Vaserstein distance is the same as that of their marginals on $n(s_x)$

$$\frac{1}{Z(x,\underline{s})}\Big(\mathbf{1}_{n(s_x)=0}+e^{-\beta V_{\underline{s},x}(\ell_\gamma x)}\ell_\gamma^d \mathbf{1}_{n(s_x)=1}\Big) \tag{7.28}$$

Since $n(s_x)=0,1$, the same argument used for Ising tells us that the Vaserstein distance is equal to the absolute value of the difference of the expectations of $n(s_x)$. Let us then suppose that $\bar q$ are the particles in all $s_z, z\ne x$, of the configuration $\underline{s}$ and that $\underline{s}'$ is obtained by a adding a particle at r^*, with $r^*\in C_{y,\gamma}$. We then have, by a Taylor expansion to third order

$$V_{\underline{s},x}(\ell_\gamma x)=\int e'_{\lambda_{\beta,\gamma}}\big(J_\gamma * \bar q(r)\big)J_\gamma(\ell_\gamma x,r)+\frac{1}{2}e''_{\lambda_{\beta,\gamma}}\big(J_\gamma*\bar q(r)\big)J_\gamma(\ell_\gamma x,r)^2+L(r)\,dr$$

$$\begin{aligned}V_{\underline{s}',x}(\ell_\gamma x)=&\int e'_{\lambda_{\beta,\gamma}}\big(J_\gamma * \bar q(r)\big)J_\gamma(\ell_\gamma x,r)\\ &+\frac{1}{2}e''_{\lambda_{\beta,\gamma}}\big(J_\gamma*\bar q(r)\big)\{J_\gamma(\ell_\gamma x,r)^2+2J_\gamma(\ell_\gamma x,r)J_\gamma(r^*,r)\}+\tilde L(r)\,dr\end{aligned}$$

with L and L' third order remainders:

$$|L(r)|+|\tilde L(r)|\le c\big(J_\gamma(r^*,r)+J_\gamma(\ell_\gamma x,r)\big)^3$$

Then, calling $K_\gamma=\beta\int e'_{\lambda_{\gamma,\beta}}\big(J_\gamma*\bar q(r)\big)J_\gamma(\ell_\gamma x,r)dr$,

$$\begin{aligned}R\big(m_{x;\le 1}(\cdot|\underline{s}),m_{x;\le 1}(\cdot|\underline{s}')\big)&=\ell_\gamma^d\Big|\frac{e^{-\beta V_{\underline{s},x}(\ell_\gamma x)}}{Z(x,\underline{s})}-\frac{e^{-\beta V_{\underline{s}',x}(\ell_\gamma x)}}{Z(x,\underline{s}')}\Big|\\ &\le\frac{\ell_\gamma^d e^{-K_\gamma}}{(1+\ell_\gamma^d e^{-K_\gamma})}\Big(\Big|\int e''_{\lambda_{\beta,\gamma}}\big(J_\gamma*\bar q(r)\big)J_\gamma(\ell_\gamma x,r)J_\gamma(r^*,r)dr\Big|+c\gamma^{2d}\Big)\end{aligned}$$

As $\gamma\to 0$ and recalling that $|J_\gamma*\bar q-\rho_\beta^+|\le c'\gamma^a$, cf. Lemma D.1 in [**1**], and that for $\beta<\beta_0$, $-1<\frac{d}{d\rho}e^{-\beta e'_{\lambda_\beta}(\rho)}\Big|_{\rho=\rho_\beta^+}<1$,

$$\lim_{\gamma\to 0}e^{-K_\gamma}|e''_{\lambda_{\beta,\gamma}}\big(J_\gamma*\bar q(r)\big)|=e^{-\beta e'_{\lambda_\beta}(\rho_\beta^+)}|e''_{\lambda_\beta}(\rho_\beta^+)|<1$$

we finally get, recalling that r^* differs from $\ell_\gamma y$ at most by ℓ_γ,

$$R\big(m_{x;\le 1}(\cdot|\underline{s}),m_{x;\le 1}(\cdot|\underline{s}')\big)\le\delta\ell_\gamma^d J_\gamma^2(\ell_\gamma x,\ell_\gamma y)+c\ell_\gamma^d(\ell_\gamma\gamma^{d+1}+\gamma^{2d}),\quad \delta<1$$

where $J_\gamma^2=J_\gamma*J_\gamma$. Proposition 7.5 is proved. □

Quantum LMP model. The same argument used in the classical case applies to the quantum case as well, showing that we can reduce to a measure with at most one particle in $C_{x,\gamma}$. However, we can only localize, with such an accuracy, the initial position r of the particle, while the conditional energy depends on the loop originating at r and it is given by $\int_0^\beta V_{\underline{s},x}(r+\omega(t))\,dt$, see (7.18) for notation. With large probability, the trajectory $r+\omega(t)$ starting at $r \in C_{x,\gamma}$ will spend most of its time outside the small cube $C_{x,\gamma}$ and, despite having localized the initial position, the particle is effectively delocalized. This is a purely quantum effect, not present in the classical case and which requires a new analysis.

We define G_x as in (7.13), G_x being therefore independent of the loops in s_x and, exactly as in the classical case, we get to (7.29) below:

Proposition 7.6. *In the quantum LMP model,* (3.1) *holds using* C_P *defined as before Theorem 6.1 and*

$$b(x,y) = \sup_{\underline{s},\underline{s}' \in G_x : d(\underline{s},\underline{s}')=d(s_y,s'_y)=1} R\big(p_x(\cdot|\underline{s};\le 1), p_x(\cdot|\underline{s}';\le 1)\big) + 2c\ell_\gamma^{2d} \tag{7.29}$$

Moreover

$$b(x,y) \le \quad \delta \ell_\gamma^d J^2(\ell_\gamma x, \ell_\gamma y) + c\ell_\gamma^d \gamma^{d+1/2}, \qquad \delta < 1 \tag{7.30}$$

Proof. As already mentioned (7.29) is proved just as (7.24) in the classical case. In Lemma 7.4 besides that, we also approximated the interaction with a constant one, yielding in this model to the too large error $\ell_\gamma^d \gamma^{1/2}$. However, the problem can be reduced to the classical LMP model in the following manner.

Let $\underline{s}$ and $\underline{s}'$ be both in G_x, call $(\bar q, \bar\omega)$ and $(\bar q', \bar\omega')$ the initial positions and loops of the particles in $\{s_z, z \ne x\}$ and $\{{s'}_z, z \ne x\}$. By (7.29), we need to consider the case where $(\bar q', \bar\omega')$ is obtained from $(\bar q, \bar\omega)$ by adding a particle (r^*, ω^*), with $r^* \in C_{y,\gamma}$.

Call $\bar q(t) = \bar q + \bar\omega(t)$, $q^*(t) = r^* + \omega^*(t)$ and $q(t) = r + \omega(t)$ the loop of a particle starting at $r \in C_{x,\gamma}$. Shorthand $a_\gamma(r,t) := (J_\gamma * \bar q(t))(r)$. We then define

$$U_{x,\bar q}(x,\omega) = \int_0^\beta \int_{\mathbb{R}^d} e'_{\lambda_{\beta,\gamma}}(a_\gamma(r,t)) J_\gamma(r, \ell_\gamma x + \omega(t)) + \frac{e''_{\lambda_{\beta,\gamma}}(a_\gamma(r,t))}{2} J_\gamma(r, \ell_\gamma x)^2 dr dt \tag{7.31}$$

$$U_{x,\bar q'}(x,\omega) = U_{x,\bar q}(x,\omega) + \int_0^\beta \int_{\mathbb{R}^d} e''_{\lambda_{\beta,\gamma}}(a_\gamma(r,t)) J_\gamma(r, q^*(t)) J_\gamma(r, \ell_\gamma x) dr dt \tag{7.32}$$

The r.h.s. of (7.31) and (7.32) collect terms of the Taylor expansion of

$$\int_0^\beta V_{x,\bar q(t)}(x+\omega(t))dt$$

and its analogue with $\underline{s}'$. Indeed we have,

$$\int_0^\beta \Big| V_{x,\bar q(t)}(r+\omega(t)) - U_{x,\bar q}(x,\omega) \Big| dt \le c(\gamma^{d+1/2} + \ell_\gamma \gamma) \tag{7.33}$$

There are three errors to be considered. The first error comes from the third order remainder, the second comes from having replaced in the second order, $J_\gamma(r,r'+\omega(t))^2$ $r\in C_{\gamma,x}$, by the term $J_\gamma(r,\ell_\gamma x)^2$, present in (7.31), and the third error comes from replacing r by $\ell_\gamma x$ in the first term of (7.31). The first error, due to the third order terms, is proportional to γ^{2d}. The second error is bounded proportionally to $\gamma^d\gamma\gamma^{-1/2}$: γ^d and γ come from the scaling: $J_\gamma(0,r)=\gamma^d J(0,\gamma r)$ while $\gamma^{-1/2}$ bounds the excursion $|\omega(t)|, 0\le t\le\beta$. The third error is bounded proportionally to $\ell_\gamma\gamma$, here ℓ_γ estimates the uncertainty of $r\in C_{\gamma,x}$ and γ comes from the scaling. Altogether (7.33) holds. An analogous computation shows that also

$$\int_0^\beta \Big|V_{x,\bar q(t)+q^*(t)}(r+\omega(t))-U_{x,\bar q(t)+q^*(t)}(x,\omega)\Big|dt\le c(\gamma^{d+1/2}+\ell_\gamma\gamma) \tag{7.34}$$

Analogously to (7.19), we then define $m_{x;\le1}(ds_x|\underline{s})$ as the Gibbs probability on S, supported by $n(s_x)\le1$, and defined by

$$m_{x;\le1}(ds_x|\underline{s})=\frac{1}{Z(x,\underline{s})}\Big(\mathbf{1}_{n(s_x)=0}+e^{-U_{x,\underline{s}}(x,\omega)}\mathbf{1}_{|\omega(\cdot)|\le\gamma^{-1/2}}W(d\omega)\mathbf{1}_{n(s_x)=1}dr\Big) \tag{7.35}$$

with $r\in C_{\gamma,x}$ the position of the particle of s_x when $n(s_x)=1$, and

$$Z(x,\underline{s})=1+\ell_\gamma^d\int_{|\omega(\cdot)|\le\gamma^{-1/2}}e^{-U_{x,\underline{s}}(x,\omega)}W(d\omega) \tag{7.36}$$

Similarly, we define $m_{x;\le1}(ds_x|\underline{s}')$ and $Z(x,\underline{s}')$ using $U_{x,\underline{s}'}(x,\omega)$ instead of $U_{x,\underline{s}}(x,\omega)$.

Using Proposition A.1 we then get from (7.29)

$$r(x,y)=\sup_{\underline{s},\underline{s}'\in G_x:d(\underline{s},\underline{s}')=d(s_y,s'_y)=1} R\big(m_{x;\le1}(\cdot|\underline{s}),m_{x;\le1}(\cdot|\underline{s}')\big)+2c(\ell_\gamma^{2d}+\ell_\gamma^d(\gamma^{d+1/2}+\ell_\gamma\gamma)) \tag{7.37}$$

We regard $m_{x;\le1}(\cdot|\underline{s})$ and $m_{x;\le1}(\cdot|\underline{s}')$ as probabilities on the space $X=\{0,(r,\omega)\}$ with 0 the state with no particles, (r,ω) the one particles states, with $r\in C_{x,\gamma}$ and ω a loop such that $|\omega(t)|\le\gamma^{-1/2}$. The important features of the two measures that we have to compare is uniformity in r and that the conditional distributions of ω, given that $n=1$ and the initial position r are identical. Indeed we can rewrite (7.31)-(7.32) as

$$U_{x,\underline{s}}(x,\omega)=u(\omega)+b,\qquad U_{x,\underline{s}'}(x,\omega)=u(\omega)+b' \tag{7.38}$$

where

$$u(\omega)=\int_0^\beta\int_{\mathbb{R}^d}e'_{\lambda_{\beta,\gamma}}(a_\gamma(r,t))J_\gamma(r,\ell_\gamma x+\omega(t))drdt \tag{7.39}$$

$$b=\int_0^\beta\int_{\mathbb{R}^d}\frac{e''_{\lambda_{\beta,\gamma}}(a_\gamma(r,t))}{2}J_\gamma(r,\ell_\gamma x)^2drdt \tag{7.40}$$

$$b'=b+\int_0^\beta\int_{\mathbb{R}^d}e''_{\lambda_{\beta,\gamma}}(a_\gamma(r,t))J_\gamma(r,q^*(t))J_\gamma(r,\ell_\gamma x)drdt \tag{7.41}$$

Thus, calling

$$C = \int_{|\omega(\cdot)|\le\gamma^{-1/2}} e^{-u(\omega)} W(d\omega) \tag{7.42}$$

the marginal distributions of $m_{x;\le 1}(\cdot|\underline{s})$ and $m_{x;\le 1}(\cdot|\underline{s}')$ on $X^{\rm class} = \{0\}\oplus\{r \in C_{x,(\ell_\gamma)}\}$ are for $n = 1$

$$m^{\rm class}_{x;\le 1}(dr|\underline{s}) = Z^{-1} C e^{-b} \mathbf{1}_{r\in C_{x,\gamma}} dr \tag{7.43}$$

and for $n = 0$ we define $m^{\rm class}_{x;\le 1}(0|\underline{s}) = Z^{-1}$ with $Z = 1 + Ce^{-b}\ell_\gamma^d$.

$m^{\rm class}_{x;\le 1}(\cdot|\underline{s}')$ is defined analogously with b replaced by b'. Finally the conditional probabilities of $m^{\rm class}_{x;\le 1}(\cdot|\underline{s})$ and $m^{\rm class}_{x;\le 1}(\cdot|\underline{s}')$ given the state with a particle in r are the same and equal to

$$C^{-1} e^{-u(\omega)} \mathbf{1}_{|\omega(\cdot)|\le\gamma^{-1/2}} W(d\omega) \tag{7.44}$$

The distances in X and $X_{\rm class}$ defined as in Section 6 satisfy the condition (A.4). Then, by Proposition A.3,

$$R\big(p_x^+(\cdot|\underline{s};\le 1), p_x^+(\cdot|\underline{s}';\le 1)\big) = R_1\big(m^{\rm class}_{x;\le 1}(\cdot|\underline{s}), m^{\rm class}_{x;\le 1}(\cdot|\underline{s}')\big) \tag{7.45}$$

where R_1 is the Vaserstein distance in $X_{\rm class}$. The computation of R_1 is exactly as in the classical case, and in this way we derive (7.30). Proposition 7.6 is proved. □

8. Validity of Assumption 2

In this section we will prove the validity of Assumption 2 of Section 3 in the three models of Section 6. The proofs are very similar to each other, the starting point, in all of them, being a bound on the probability of the event "not being in G_x" in terms of a variational problem involving a non local free energy functional. The link with a variational problem, goes back to the original works of Kac, Uhlenbeck and Hemmer, [**11**], and, in particular, to the analysis of Lebowitz and Penrose, [**15**]. The variational problem itself is then studied by exploiting the stability properties of the mean field ground states.

Ising model. We will prove here that there is a constant $c > 0$ so that for all γ small enough and for all $x \in \mathbb{Z}^d$ (calling below $C(x)$ the cube of $\mathcal{D}^{(\ell_{-,\gamma})}$ which contains x),

$$\sup_{\underline{s}\in\mathcal{X}} p_{C(x)}(G_x^c|\underline{s}) \le e^{-c\gamma^{2a}\ell^d_{-,\gamma}} \tag{8.1}$$

Since $\gamma^{2a}\ell^d_{-,\gamma} = \gamma^{-d+2a+2\alpha d}$ and recalling that $0 < a \ll \alpha \ll 1$. Using that $|t_x| \le 2$, (8.1) proves Assumption 2 of Section 3 for γ small enough with $\epsilon = 4e^{-c\gamma^{2a}\ell^d_{-,\gamma}/2}$.

Recalling the definition (7.1) of G_x, for any $\zeta > 0$ there is $\gamma(\zeta) > 0$ and for all $\gamma < \gamma(\zeta)$,

$$G_x^c \sqsubset \Big\{ \underline{s} \in \mathcal{X} : \Big| \frac{1}{|C|} \sum_{y \in C} (s_y - m_\beta) \Big| > (1-\zeta)\gamma^a \Big\} \tag{8.2}$$

where $C = C(x)$ is the cube of $\mathcal{D}^{(\ell_-,\gamma)}$ which contains x.

The non local free energy functional we referred to in the beginning of this section is the following. Let $D = \gamma C$, C as above, (thus D is a cube of side $\ell_- = \gamma^\alpha, \alpha > 0$). For $m_D \in L^\infty(D, [-1,1])$ and $m_{D^c} \in L^\infty(D^c, [-1,1])$, we then set

$$\begin{aligned} F_D(m_D|m_{D^c}) = &-\frac{1}{2}\int_D \int_D J(r,r') m_D(r) m_D(r') dr dr' - \frac{1}{\beta}\int_D S(m_D(r)) \\ &- \int_D \int_{D^c} J(r,r') m_D(r) m_{D^c}(r') dr dr' \end{aligned} \tag{8.3}$$

$$S(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2} \tag{8.4}$$

Associating to each $x = (x_1, .., x_d) \in \mathbb{Z}^d$ the cube $\{r = (r_1, .., r_d) \in \mathbb{R}^d : x_i \le r_i < x_i + 1\}$ the partition $\mathcal{D}^{(\gamma^{-1})}$ becomes a partition of $\mathbb{R}^d$, which, by an abuse of notation, we denote by the same symbol. Let

$$\mathcal{X} := \Big\{ m \in L^\infty(\mathbb{R}^d, [-1,1]) : \Big| \fint_{D(r)} m(r')dr' - m_\beta \Big| \le \gamma^a, r \in \mathbb{R}^d \Big\} \tag{8.5}$$

where $D(r)$ is the cube of $\gamma\mathcal{D}^{(\ell_-,\gamma)}$ which contains r and

$$\fint_A m(r)dr := \frac{1}{|A|}\int_A m(r)dr$$

We then call $\mathcal{X}_\Lambda$ the above expression (8.5) when the constrain is imposed on Λ, with $\Lambda \sqsubset \mathbb{R}^d$, a $\gamma\mathcal{D}^{(\ell_-,\gamma)}$-measurable set.

Proposition 8.1. *There is $c > 0$ so that, for all γ small enough, all x and calling $C(x)$ the cube of $\mathcal{D}^{(\ell_-,\gamma)}$ which contains x, and $D = \gamma C(x)$*

$$\begin{aligned} \sup_{\underline{s} \in \mathcal{X}} \log p_{C(x)}(G_x^c|\underline{s}) \le -\gamma^{-d} \inf_{m_{D^c} \in \mathcal{X}_{D^c}} \Bigg(&\inf_{\substack{m_D \in \mathcal{X}_D, \\ \left|\fint_D m(r)dr - m_\beta\right| > (1-\zeta)\gamma^a}} F_D(m_D|m_{D^c}) \\ &- \inf_{m_D \in \mathcal{X}_D} F_D(m_D|m_{D^c}) \Bigg) + c\gamma^{1/2}\ell_{-,\gamma}^d \end{aligned} \tag{8.6}$$

Proof. The proof is just as in Lebowitz and Penrose, [**15**]. We partition the space into cubes of side proportional to $\gamma^{-1/2}$ choosing such a partition, called $\mathcal{D}^{(\gamma^{-1/2})}$, finer than $\mathcal{D}^{(\ell_-,\gamma)}$. We then approximate the spin-spin interaction to make it constant on any of the cubes of $\mathcal{D}^{(\gamma^{-1/2})}$. The expression we get for the logarithmic partition function is then the free energy functional (8.3) while the error is proportional to the volume of the region involved times the error due to the piecewise constant approximation of the interaction, and it is given by the last term in (8.6). In this way Proposition 8.1 is proved.

The usefulness of (8.6) rests on the fact that the first term on the r.h.s. of (8.6) is so large to kill the error, i.e. the last term in (8.6). We approximate $F_D(m_D|m_{D^c})$ by $F^0_D(m_D|m_{D^c})$ where

$$F^0_D(m_D|m_{D^c}) = \int_D -h(r;m_{D^c})m_D(r) - \frac{1}{\beta}S(m_D(r))\,dr$$
$$h(r;m_{D^c}) = \int_{D^c} J(r,r')m_{D^c}(r')dr' \tag{8.7}$$

getting, for a suitable constant $c' > 0$,

$$\Big|F_D(m_D|m_{D^c}) - F^0_D(m_D|m_{D^c})\Big| \le c'|D|^2 = c'\gamma^{2\alpha d} \tag{8.8}$$

Indeed $F^0_D(m_D|m_{D^c})$ is $F_D(m_D|m_{D^c})$ without the self interaction energy, i.e. the first term on the r.h.s. of (8.3) which is bounded proportionally to $|D|^2$.

Being a convex functional on $L^\infty(D,[-1,1])$, $F^0_D(m_D|m_{D^c})$ has a unique minimizer given by

$$\bar m_D(r) = \tanh\{\beta h(r;m_{D^c})\} \tag{8.9}$$

We can write the difference $F^0_D(m_D|m_{D^c}) - F^0_D(\bar m_D|m_{D^c}) = \int_D \psi(m_D(r),\bar m_D(r))\,dr$, where, shorthanding m and $\bar m$ for $m_D(r)$ and $\bar m_D(r)$,

$$\psi(m,\bar m) = \frac{1-\bar m}{2\beta}\Big(\{\frac{1-m}{1-\bar m}\log\frac{1-m}{1-\bar m}\} - \frac{1-m}{1-\bar m} + 1\Big)$$
$$+\frac{1+\bar m}{2\beta}\Big(\{\frac{1+m}{1+\bar m}\log\frac{1+m}{1+\bar m}\} - \frac{1+m}{1+\bar m} + 1\Big)$$

Since

$$x\log x - x + 1 \ge \big(\sqrt{x}-1\big)^2$$

$$F^0_D(m_D|m_{D^c}) - F^0_D(\bar m_D|m_{D^c}) \ge \frac{1}{2\beta}\int_D \Big(\sqrt{1-m_D(r)} - \sqrt{1-\bar m_D(r)}\Big)^2$$
$$+\Big(\sqrt{1+m_D(r)} - \sqrt{1+\bar m_D(r)}\Big)^2 dr \tag{8.10}$$

Writing $|\sqrt{a}-\sqrt{b}| \ge |a-b|/(\sqrt{a}+\sqrt{b}) \ge |a-b|/2\sqrt{2}$, for a and b in $(0,2]$, we get

$$F^0_D(m_D|m_{D^c}) - F^0_D(\bar m_D|m_{D^c}) \ge \frac{1}{8\beta}\int_D \Big(m_D(r) - \bar m_D(r)\Big)^2 dr \tag{8.11}$$

We postpone the proof that there is $\epsilon > 0$ so that, for all $\gamma > 0$ small enough,

$$|\bar{m}_D(r) - m_\beta| \le (1-\epsilon)\gamma^a \tag{8.12}$$

We then get from (8.11), by Cauchy-Schwartz,

$$F^0_D(m_D|m_{D^c}) - F^0_D(\bar{m}_D|m_{D^c}) \ge \frac{1}{8\beta}|D|(\epsilon-\zeta)^2\gamma^{2a} \tag{8.13}$$

so that, going back to (8.6) and recalling that $\gamma^{-d}|D| = \ell^d_{-,\gamma}$,

$$p_{C(x)}(G_x|\underline{s}) \le \exp\Big\{ -\Big(\frac{(\epsilon-\zeta)^2}{8\beta}\gamma^{2a} - 2c'\gamma^{\alpha d} - c\gamma^{1/2}\Big)\ell^d_{-,\gamma}\Big\} \tag{8.14}$$

Proof of (8.12)

Calling $D(r)$ the cube of $\mathcal{D}^{(\ell_-)}$ which contains r, and

$$J^{(\ell_-)}(r,r') := \fint_{D(r')} J(r,r'')dr''$$

for γ, and hence $\ell_- = \gamma\ell_{-,\gamma} = \gamma^\alpha$, small enough,

$$\Big|J(r,r') - J^{(\ell_-)}(r,r')\Big| \le c\ell_- \mathbf{1}_{|r-r'|\le 2}, \qquad c := d\|\nabla J\|_\infty < \infty \tag{8.15}$$

d the space dimensions. Then, for any $m \in L^\infty(\mathbb{R}^d, [-1,1])$,

$$\Big|J * m - J^{(\ell_-)} * m\Big| \le 2^d c\ell_-$$

Letting $m(r) = m_D(r)\mathbf{1}_{r\in D} + m_{D^c}(r)\mathbf{1}_{r\in D^c}$

$$|J^{(\ell_-)} * m_{D^c} - J^{(\ell_-)} * u| \le c'\ell^d_-, \qquad u(r) = \fint_{D(r)} m(r')dr'$$

then, recalling from (8.7) that $h(r) \equiv h(r; m_{D^c}) = J * m_{D^c}(r)$.

$$\Big|h - J^{(\ell_-)} * u\Big| \le 2^d c\ell_- + c'\ell^d_-$$

On the other hand, since $m \in \mathcal{X}$, $|u(r) - m_\beta| \le \gamma^a$,

$$\Big|h(r) - m_\beta\Big| \le \gamma^a + 2^d c\ell_- + c'\ell^d_-, \qquad r \in D \tag{8.16}$$

Since

$$\frac{d}{ds}\tanh\{\beta s\}\Big|_{s=m_\beta} < 1$$

there are $b \in (0,1)$ and $\zeta^* > 0$ so that $d\tanh\{\beta s\}/ds < b$ for $|s - m_\beta| \le \zeta^*$, hence

$$\Big|\tanh\{\beta s\} - m_\beta\Big| \le b\Big|s - m_\beta\Big|, \qquad \text{for } |s-m_\beta| \le \zeta^* \tag{8.17}$$

For γ small enough, by (8.16)

$$\Big|\tanh\{\beta h(r)\} - m_\beta\Big| \le b(\gamma^a + 2^d c\ell_- + c'\ell^d_-), \qquad r \in D$$

On the other hand, since $\ell_- = \gamma^\alpha$, $\alpha > a$,

$$b(\gamma^a + 2^d c\ell_- + c'\ell_-^d) \le \gamma^a\Big(1 - [(1-b) - b2^d c\gamma^{\alpha-a} - bc'\gamma^{\alpha d - a}]\Big) \le \gamma^a(1-\epsilon)$$

with $\epsilon < (1-a)$ and provided γ is small enough. (8.12) is proved. □

Classical LMP model. We will prove that (8.1) holds for this case as well. Then Assumption 2 of Section 3 holds for small γ for

$$\epsilon = 2\ell_{-,\gamma}^d(\rho_\beta^+ + \gamma^a)e^{-c\gamma^{2a}\ell_{-,\gamma}^d/2} \tag{8.18}$$

using that in this case $|t_x| \le d(t_x, \emptyset) \le (\rho_\beta^+ + \gamma^a)|C^{(\ell_{-,\gamma})}|$. $\emptyset$ denotes the state with no particles. The analogue of (8.2) is

$$G_x^c \sqsubset \Big\{ \underline{s} \in \mathcal{X} : \Big| \frac{1}{|C|}\sum_{y\in C}(n(s_y) - \rho_\beta^+)\Big| > (1-\zeta)\gamma^a \Big\} \tag{8.19}$$

The non local free energy functional is here

$$F_D(\rho_D|\rho_{D^c}) = \int_{\mathbb{R}^d} e_{\lambda_{\beta,\gamma}}(J*\rho(r)) - e_{\lambda_{\beta,\gamma}}(J*\rho_{D^c}(r))\,dr - \frac{1}{\beta}\int_D S(\rho_D(r))dr \tag{8.20}$$

where $\rho_D \in L^\infty(D,\mathbb{R}_+)$, $\rho_{D^c} \in L^\infty(D^c,\mathbb{R}_+)$, $\rho(r) = \rho_D(r)\mathbf{1}_{r\in D} + \rho_{D^c}(r)\mathbf{1}_{r\in D^c}$ and

$$S(\rho) = -\rho\big(\log\rho - 1\big) \tag{8.21}$$

With such a functional, the analogue of Proposition 8.1 holds, see [**14**], and with the constraint that $\rho_D(r) \le C$. for a suitable constant C.

By expanding

$$\Big|\int e_{\lambda_{\beta,\gamma}}(J*\rho(r)) - e_{\lambda_{\beta,\gamma}}(J*\rho_{D^c}(r))\,dr - \int e'_{\lambda_{\beta,\gamma}}(J*\rho_{D^c}(r))J*\rho_D(r)\Big| \le c|D|^2$$

we get the analogue of (8.8), namely

$$\Big|F_D(\rho_D|\rho_{D^c}) - F_D^0(\rho_D|\rho_{D^c})\Big| \le c\gamma^{2\alpha d} \tag{8.22}$$

with

$$F_D^0(\rho_D|\rho_{D^c}) = \int_D -\lambda_{\text{eff}}(r)\rho_D(r) - \frac{1}{\beta}S(\rho_D(r))\,dr$$

$$\lambda_{\text{eff}}(r) = -\int J(r,r')e'_{\lambda_{\beta,\gamma}}(J*\rho_{D^c}(r'))dr' \tag{8.23}$$

$F_D^0(\rho_D|\rho_{D^c})$ has a unique minimizer given by

$$\bar\rho_D(r) = e^{\beta\lambda_{\text{eff}}(r)} \tag{8.24}$$

Using again the relative entropy, we get, similarly to (8.10),

$$F_D^0(\rho_D|\rho_{D^c}) - F_D^0(\bar\rho_D|\rho_{D^c}) \ge \frac{1}{2\beta}\int_D\Big(\sqrt{\rho_D(r)} - \sqrt{\bar\rho_D(r)}\Big)^2 \tag{8.25}$$

Recalling that $\rho_D(r) \le C$ we then get to the analogue of (8.11) and then to (8.13), with (8.12) proved in [**1**]. Thus Assumption 2 is proved to hold, for γ small enough.

Quantum LMP model. By exploiting the definition of restricted ensemble, where all loops are short, i.e. $|\omega(t)| \le \gamma^{-1/2}$, the delocalization of particles due to the loops, can be absorbed in the error due to the transition to the continuum, and indeed the analogue of Proposition 8.1 holds as well for the quantum case. With that the quantum problem is reduced to the classical one and the remaining of the proof is unchanged.

9. Validity of Assumptions 3–5

We start with Assumption 3 and, recalling the definition (3.4) of $B_\xi(i)$, we choose $\xi > 0$ so that the interaction between a cube C_i of $\mathcal{D}$ and a cube C_j not in $B_\xi(i)$ is only due to the additional part of the hamiltonian H^+, namely the one arising from K^+. Thus $\xi = \gamma^{-1}$ in the Ising model, $2\gamma^{-1}$ in the classical LMP model and $< 3\gamma^{-1}$ in the quantum LMP model (the extra length because of the loops). To unify the cases we will thus choose $\xi = 3\gamma^{-1}$.

With this choice of ξ, we have to determine $r(i,j)$ from (3.5) and, at this point, it is convenient to examine the three models separately.

Ising model. By the triangular inequality and Lemma 7.3, we have (denoting by $d(\underline{s},\underline{s}')$ the quantity $d_\Lambda(\underline{s},\underline{s}'), \Lambda = \mathbb{Z}^d$)

$$r(i,j) = \frac{1}{2} \sup_{\underline{s},\underline{s}':d(\underline{s},\underline{s}')=d_{C_j}(\underline{s},\underline{s}')=2} R_{C_i}\big(p_{C_i}(\cdot|\underline{s}), p_{C_i}(\cdot|\underline{s}')\big) \tag{9.1}$$

and need to check that (3.6) holds.

With $\underline{s}$ and $\underline{s}'$ as in (9.1) (namely with $\underline{s}'$ obtained from $\underline{s}$ by flipping a spin in C_j), recalling Theorem 4.1,

$$|H^+_{C_i}(s_{C_i}|s'_{C_i^c}) - H^+_{C_i}(s_{C_i}|s_{C_i^c})| \le \sum_{\Delta \sqsupset C_i, C_j} 2\|U_\Delta(\cdot)\|_\infty \tag{9.2}$$

where the sets Δ are $\mathcal{D}^{(\ell_+,\gamma)}$-measurable, so that the cubes C_i and C_j which belong to the partition of $\mathcal{D}^{(\ell_-,\gamma)}$ are each one contained in a cube of $\mathcal{D}^{(\ell_+,\gamma)}$. Recalling (7.10) we get

$$\text{l.h.s. of (9.2)} \le 2 \sup_{x\in C_i, y\in C_j} e^{-(C_P-2b)N_\gamma(x,y)} \tag{9.3}$$

(9.3) is proved by the argument used in (7.12), to which we refer for details. By Proposition A.1 and since $d_{C_i}(\underline{s},\underline{s}') \le 2|C_i| = 2\gamma^{-(1-\alpha)d}$, we get from (9.1)-(9.3)

$$r(i,j) \le 2\gamma^{-(1-\alpha)d} 2 \sup_{x\in C_i, y\in C_j} e^{-(C_P-2b)N_\gamma(x,y)}, \quad C_P = c\gamma^{-(1-\alpha)d+2a} \tag{9.4}$$

hence

$$\lim_{\gamma\to 0} \sup_i \sum_{j\notin B_\xi(i)} r(i,j) = 0 \tag{9.5}$$

so that (3.6) holds if γ is small enough.

LMP models. Analogously to (9.1) we have

$$r(i,j) = \sup_{\underline{s},\underline{s}':d(\underline{s},\underline{s}')=d_{C_j}(\underline{s},\underline{s}')=1} R_{C_i}\big(p_{C_i}(\cdot|\underline{s}), p_{C_i}(\cdot|\underline{s}')\big) \tag{9.6}$$

namely $\underline{s}$ and $\underline{s}'$ differ by only one particle somewhere in C_j (or, in the quantum case, by a loop starting in C_j). The argument hereafter proceeds exactly as in the Ising model, with $C_P = c\gamma^{-1}$ in the quantum case, see before Theorem 6.1, and (9.4) holds in the two LMP models as well.

The validity of Assumption 4 is just a consequence of the estimates obtained so far. We need to prove that uniformly in $i \in \mathbb{Z}^d$,

$$\sum_{j\ne i} r(i,j) < 1 \tag{9.7}$$

By (9.4) (established in all the three models), it is enough to prove that there is $u < 1$ and $\gamma' > 0$, so that for all $\gamma < \gamma'$

$$\sum_{j\in B_\xi(i)\setminus i} r(i,j) \le u \tag{9.8}$$

To verify (9.8) we recall the values of $b(x,y)$ and of ϵ in the three models.

In Ising this is given by (7.3) and given just before Theorem 6.1:

$$b(x,y) = r[J_\gamma(x,y) + e^{-(C_P-2b)N_\gamma(x,y)}], \qquad C_P = c\gamma^{-(1-\alpha)d+2a},\ r<1 \tag{9.9}$$

$$\epsilon = 4e^{-c\gamma^{2a}\ell^d_{-,\gamma}/2} \tag{9.10}$$

Using (3.8) with (9.9) and (9.10) one obtains that

$$\sum_{j\in B_\xi(i)} |C_j| \sup_{y\in C_i} b(x,y) \le r\int dy\, J_\gamma(x,y) + r3^d\gamma\ell_{-,\gamma} + c\ell^d_{-,\gamma} e^{-C_P-2b} \tag{9.11}$$

The first error derives from replacing $\sup_{y\in C_j} J_\gamma(x,y)$ by $J_\gamma(x,y)$ and the second is the part due to the second summand in the expression for b in (9.9) recalling that

$\xi = 3\gamma^{-1}$. Hence one can consider $b(x, y)$ as a Markov chain. Using similar arguments as in Theorem 2.4 one sees that this implies

$$\sum_{j \in B_\xi(i)\setminus i} \sup_{y \in C_j} \sum_{x \in C_i} \sum_{n>0} b_{C_i}^{(n)}(x, y) < 1 \tag{9.12}$$

(9.8) follows from (9.10) as the bracket in (3.8) grows only polynomially in γ^{-1}.

In the classical LMP model, $b(x, y)$ is given in (7.26):

$$b(x, y) \leq \delta \ell_\gamma^d J_\gamma^2(\ell_\gamma x, \ell_\gamma y) + 2c\ell_\gamma^d \gamma^{2d}, \quad \delta < 1 \tag{9.13}$$

with ϵ given in (8.18)

$$\epsilon = 2e^{-c\gamma^{2a}\ell_{-,\gamma}^d/2}(\rho_\beta^+ + \gamma^a)|C_{\ell_{-,\gamma}}| \tag{9.14}$$

Using (9.13) one sees that the analog of (9.11) holds. In this case the second term in (9.13) gives rise to the error $\xi^d 2c\gamma^{2d} = 2c3^d\gamma^d$, because $B_\xi(i)$ contains $(\xi/\ell_\gamma)^d$ points. The factor ℓ_γ^d in (9.13) is needed to reconstruct the integral in (9.11). Arguing as before (9.8) follows.

Appendix A. Estimates of Vaserstein distance

In this appendix we will prove some elementary results about the Vaserstein distance between measures on a metric space Ω. We write $d(\omega, \omega')$ for the distance in Ω and $|\omega| = d(\omega, \omega_0)$, $\omega_0 \in \Omega$.

The reader should not be confused by the use of the symbol ω which refers in the text to loops, while here ω is an element of an abstract space Ω.

Proposition A.1. *Let $\nu(d\omega)$ be a probability on Ω and h, v be such that for all $t \in [0, 1]$, $e^{-[h+tv]} \in L^1(\Omega, \nu)$. Call*

$$\mu_t = Z_t^{-1} e^{-[h+tv]}\nu, \quad Z_t = \int e^{-[h+tv]} d\nu \tag{A.1}$$

then

$$R(\mu_1, \mu_0) \leq \sup_{0 \leq t \leq 1} \Big(\mu_t(|v||\omega|) + \mu_t(|v|)\mu_t(|\omega|)\Big) \tag{A.2}$$

Proof. Calling D_t the density of μ_t w.r.t. ν and

$$\alpha(\omega) = \min\{D_1(\omega), D_0(\omega)\}, \qquad c = 1 - \int \alpha(\omega)\nu(d\omega)$$

the probability on $\Omega \times \Omega$, given by

$$P(d\omega, d\omega') = \alpha(\omega)\delta(\omega - \omega')\nu(d\omega) + c^{-1}[D_1(\omega) - \alpha(\omega)][D_0(\omega') - \alpha(\omega')]\nu(d\omega)\nu(d\omega')$$

is a joint representation of μ_1 and μ_0. Thus,

$$\begin{aligned} R(\mu_1, \mu_0) &\leq \int d(\omega, \omega')P(d\omega, d\omega') \leq \int (|D_1(\omega) - \alpha(\omega)| + |D_0(\omega) - \alpha(\omega)|)|\omega|\nu(d\omega) \\ &= \int |D_1(\omega) - D_0(\omega)||\omega|\nu(d\omega) \end{aligned}$$

having bounded $d(\omega, \omega') \leq |\omega| + |\omega'|$ and integrated out the missing variable.

(A.2) is then obtained by writing $D_1(\omega) - D_0(\omega)$ as an integral of the t-derivative of D_t. Proposition A.1 is proved. □

Proposition A.2. *Let μ be a probability on Ω, $A \sqsubset \Omega$, $\mu(A) \in (0, 1)$ and μ_A the conditional probability given A. Then*

$$R(\mu_A, \mu) \leq \int_{\omega \in A^c, \omega' \in A} \mu(d\omega)\mu_A(d\omega')d(\omega, \omega') \tag{A.3}$$

Proof. The same argument used in the proof of Proposition A.1 shows that

$$P(d\omega, d\omega') = \delta(\omega - \omega')\mathbf{1}_{\omega \in A}\mu(d\omega) + (1 - \mu(A))^{-1}\mathbf{1}_{\omega \in A^c, \omega' \in A}\mu(d\omega)[\mu_A(d\omega') - \mu(d\omega')]$$

is a joint triangular representation of μ and μ_A. Hence (A.3) and Proposition A.2 are proved. □

Suppose $\Omega = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2(\omega_1)\}$ and that

$$d\big((\omega_1, \omega_2), (\omega_1', \omega_2')\big) \geq d_1(\omega_1, \omega_1') \quad \text{with equality if } \omega_1 \neq \omega_1' \tag{A.4}$$

where $d_1(\omega_1, \omega_1')$ is the distance on the space Ω_1. We denote by $R(\cdot, \cdot)$ and $R_1(\cdot, \cdot)$ the Vaserstein distances in Ω and Ω_1, the latter relative to d_1. Let also μ and μ' be probabilities on Ω, μ_1 and μ_1' their marginals on Ω_1, and $\mu(d\omega_2|\omega_1)$, $\mu'(d\omega_2|\omega_1)$ their conditional probabilities given ω_1.

Proposition A.3. *Let μ and μ' be as above and $\mu(d\omega_2|\omega_1) = \mu'(d\omega_2|\omega_1)$ on Ω_1, then*

$$R(\mu, \mu') = R_1(\mu_1, \mu_1') \tag{A.5}$$

Proof. Let $P_1(d\omega_1, d\omega_1')$ be a joint representation of μ_1 and μ_1', then

$$P(d\omega, d\omega') = P_1(d\omega_1, d\omega_1')\Big(\mathbf{1}_{\omega_1=\omega_1'}\delta(\omega_2-\omega_2')\mu(d\omega_2|\omega_1) + \mathbf{1}_{\omega_1\neq\omega_1'}\mu(d\omega_2|\omega_1)\mu(d\omega_2'|\omega_1')\Big) \tag{A.6}$$

is a joint representation of μ and μ', so that, by (A.4),

$$R(\mu,\mu') \le R_1(\mu_1,\mu_1') \tag{A.7}$$

To prove the reverse inequality, let $P(d\omega, d\omega')$ be a joint representation of μ and μ', then by (A.4)

$$\int d\big((\omega_1,\omega_2),(\omega_1',\omega_2')\big)P(d\omega,d\omega') \ge \int d_1\big(\omega_1,\omega_1'\big)P(d\omega,d\omega') = \int d_1\big(\omega_1,\omega_1'\big)Q(d\omega_1,d\omega_1')$$

where, denoting by $f(\omega_1,\omega_1')$ any bounded measurable function on $\Omega_1\times\Omega_1$ and by

$$g((\omega_1,\omega_2),(\omega_1',\omega_2')) = f(\omega_1,\omega_1')$$

then

$$\int f(\omega_1,\omega_1')Q(d\omega_1,d\omega_1') = \int g(\omega,\omega')P(d\omega,d\omega')$$

Thus Q is a joint representation of μ_1 and μ_1' and $R \ge R_1$. Together with (A.6) this proves (A.5). The proposition is proved. □

Appendix B. Couplings of Gibbs measures in contour models

Theorems 3.3 and 5.2 prove the existence of successful couplings (i.e. with a "large mass" on the diagonal) for Gibbs measures in restricted ensembles. The construction of successful couplings for the corresponding contour models requires a different strategy. Recall from Section 4, see (4.25), that the finite volume Gibbs measures in a contour model are the probabilities on $\mathcal{X}^+\times\mathcal{B}^+_\Lambda$, defined by

$$\mu^+_{\mathrm{abs},\Lambda}(d\underline{s}',\underline{\Gamma}|\underline{s}) = \pi^+_\Lambda(\underline{\Gamma};\underline{s}')p^{+,o}_{\mathrm{abs},\Lambda}(d\underline{s}'|\underline{s}) \tag{B.1}$$

with $\pi^+_\Lambda(\underline{\Gamma}; s'_\Lambda)$ as in (4.26). We thus start from two measures $\mu^+_{\mathrm{abs},\Lambda_i}(d\underline{s},\underline{\Gamma}|\underline{s}^{(i)})$, $i=1,2$, with Λ_i bounded, $\mathcal{D}'$ measurable sets and $\underline{s}^{(i)}\in\mathcal{X}^+$. By Corollary 5.3, there is a coupling P of $p^{+,o}_{\mathrm{abs},\Lambda_i}(\cdot|\underline{s}^{(i)})$ such that

$$P(s_{\Delta'}\neq s'_{\Delta'}) \le 2c|\Delta'|\max\{e^{-\omega_1\mathrm{dist}(\Delta',\Lambda_1^c)}, e^{-\omega_1\mathrm{dist}(\Delta',\Lambda_2^c)}\} \tag{B.2}$$

with Δ' a bounded, $\mathcal{D}'$ measurable set.

We are interested in the case where Λ_i are very large and, consequently, Δ' large. Except for a set of small probability, we can then reduce to pairs $\underline{s}^{(i)}$, $i=3,4$, which agree on Δ'. The next theorem starts from such a setup and uses the following notation. Let $\underline{\Gamma} \in \mathcal{B}^+$ and Δ a bounded, $\mathcal{D}'$ measurable set, then $\underline{\Gamma}_\Delta$ denotes the following element of $\mathcal{B}^+$:

$$\underline{\Gamma}_\Delta = \big\{\Gamma \in \underline{\Gamma} : \mathrm{sp}(\Gamma) \sqcap \Delta \neq \emptyset\big\} \tag{B.3}$$

We also write

$$\mathrm{sp}(\underline{\Gamma}_\Delta) = \bigsqcup_{\Gamma \in \Gamma_\Delta} \mathrm{sp}(\Gamma) \tag{B.4}$$

Theorem B.1. *There are positive constants c_2 and ω_2 so that if Λ_i, Δ' and $\underline{s}^{(i)}$ are as above, then there is a coupling $Q_{\underline{s},\underline{s}'}$ of $\pi^+_{\Lambda_1}(\underline{\Gamma}|\underline{s}^{(3)})$ and $\pi^+_{\Lambda_2}(\underline{\Gamma}'|\underline{s}^{(4)})$ such that*

$$Q_{\underline{s},\underline{s}'}\Big(\{\underline{\Gamma}_\Delta \neq \underline{\Gamma}'_\Delta\} \sqcap \{\mathrm{sp}(\underline{\Gamma}_\Delta) \sqsubset \Delta'\}\Big) \leq c_2|\Delta|e^{-\omega_2\, dist(\Delta, \Delta'^c)} \tag{B.5}$$

(the inclusion above being strict). Moreover, for any $(\underline{\Gamma}, \underline{\Gamma}')$, $Q_{\underline{s},\underline{s}'}$ depends measurably on $(\underline{s}, \underline{s}') \in \mathcal{X}^+ \times \mathcal{X}^+$.

Proof. We introduce spin configurations $\underline{\xi} = (\xi_i)_{i\in\mathbb{Z}^d} \in \{0,1\}^{\mathbb{Z}^d}$ as follows. Given a configuration $\underline{\Gamma} \in \mathcal{B}^+$, we define ξ_i as equal to 0, if C_i does not belong to $\mathrm{sp}(\underline{\Gamma})$, otherwise $\xi_i = 1$; $\underline{\xi}'$ is the configuration associated to $\underline{\Gamma}'$ and the distributions of $\underline{\xi}$ and $\underline{\xi}'$ are those inherited from $\pi^+_{\Lambda_1}(\underline{\Gamma}; \underline{s}^{(3)})$ and $\pi^+_{\Lambda_2}(\underline{\Gamma}; \underline{s}^{(4)})$, respectively.

While the interaction among the spins ξ_i has infinite range, the distribution of $\{\xi_i, i \in A\}$, A a bounded set, conditioned on $\{\xi^*_i, i \in A^c\}$, has the following independence property. If all $\xi^*_i = 0$ with $i \in A^c$ and contiguous to A, (we then call A "good") then the conditional distribution of $\underline{\Gamma}_{r(A)}$ is supported by $\mathcal{B}^+_{r(A)}$ and independent of the other values of ξ^*_i. Thus the conditional distribution of $(\xi_i)_{i\in A}$ is that inherited by $\pi^+_{r(A)}(\underline{\Gamma}|\underline{s}^{(3)})$ and depends only on $s^{(3)}_{r(A)}$. To prove the theorem, it is then enough to construct a coupling for which there is a set A simultaneously good for both $\underline{\xi}$ and $\underline{\xi}'$ and with $r(A) \sqsubset \Delta'$.

We will construct the coupling using the disagreement percolation algorithm of van der Berg and Maes, [**6**]. We call A unsuccessful for the pair $(\underline{\xi}, \underline{\xi}')$ unless A is good for both of them, in which case A is successful. We start from $A = \Delta'$ and we consider a sequence of sets till success is reached. If Δ is successful for $(\underline{\xi}, \underline{\xi}')$, we are finished. Otherwise, we define $A^{(j+1)}$ from an unsuccessful $A^{(j)}$ by removing all sites $i \in A^{(j)}$ which are contiguous to a site k outside $A^{(j)}$ where either $\xi_k = 1$ or $\xi'_k = 1$, or both. We call $A(\underline{\xi}, \underline{\xi}')$ the first successful set, noticing that it has the property of a stopping time, namely that $A(\underline{\xi}, \underline{\xi}') = A$ does not depend on the values ξ_i, ξ'_i with $i \in A$.

Let then $Q_{\underline{s},\underline{s}'}$ on $\mathcal{B}^+_\Lambda \times \mathcal{B}_{\Lambda'^+}$ be

$$\begin{aligned} Q_{\underline{s},\underline{s}'}(\underline{\Gamma}, \underline{\Gamma}') \;=\;& \mathbf{1}_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} \mathbf{1}_{\underline{\Gamma}'\in\mathcal{B}^+_{\Lambda'}} \mathbf{1}_{\underline{\Gamma}=\underline{\Gamma}' \mathrm{on} A_s(\Gamma,\Gamma')} w^+(\underline{\Gamma}_{A_s(\underline{\Gamma},\underline{\Gamma}')}; s_\Delta) \\ & \cdot\, \Xi^+(A_s(\underline{\Gamma}, \underline{\Gamma}'); s_\Delta) \pi^+_\Lambda(\underline{\Gamma}_{\Lambda\setminus A_s(\underline{\Gamma},\underline{\Gamma}')}|s_\Lambda) \pi^+_\Lambda(\underline{\Gamma}'_{\Lambda'\setminus A_s(\underline{\Gamma},\underline{\Gamma}'))}|s'_{\Lambda'}) \end{aligned} \tag{B.6}$$

where $A_s(\underline{\Gamma}, \underline{\Gamma}')$ is $r(A_s(\underline{\xi}, \underline{\xi}'))$ for ξ (ξ' respectively) corresponding to $\underline{\Gamma}$ ($\underline{\Gamma}'$ respectively) and $\underline{\Gamma}_A = \{\Gamma_i \in \underline{\Gamma} | \mathrm{sp}(\Gamma_i) \sqsubset A\}$.

By an argument similar to the one used by van der Berg and Maes, see also [**2**], one can show that the above is indeed a coupling of $\pi^+_\Lambda(\underline{\Gamma}|s_{\Lambda^c})$ and $\pi^+_\Lambda(\underline{\Gamma}'|s'_{\Lambda^c})$ and (B.5) follows from the assumptions on the weights (which fulfill the Peierls bounds with large enough constant C_P). Theorem B.1 is proved

□

As a immediate corollary of the Theorem 3.3 and Corollary 5.3, ?? we have the following result for the skew measures $\mu^+_{\mathrm{abs},\Lambda}(\underline{\Gamma}, ds_\Lambda|s_{\Lambda^c}) = \pi^\pm_\Lambda(\underline{\Gamma}; s_\Lambda) p^{\pm;\Lambda}_{\mathrm{abs}}(ds_\Lambda|s_{\Lambda^c})$ on the space $\mathcal{X}^\pm_\Lambda \times \mathcal{B}^\pm_\Lambda$, cf. (4.25). We call Δ a set of agreement for $(\underline{\Gamma}^{(1)}, s^{(1)})$, $(\underline{\Gamma}^{(2)}, s^{(2)})$ if $s^{(1)}_\Delta = s^{(2)}_\Delta$ and $\underline{\Gamma}^{(1)} = \underline{\Gamma}^{(2)}$ on Δ.

Corollary B.2. *There are positive constants c_2, ω_2 so that the following holds. Let $\Delta, \Lambda, \Lambda'$ be $\mathcal{D}'$ bounded measurable sets, with Δ contained in $\Lambda \sqcap \Lambda'$, for any two measures $\mu^+_{\mathrm{abs},\Lambda}(\underline{\Gamma}^{(1)}, ds^{(1)}_\Lambda|s_{\Lambda^c})$, $\mu^+_{\mathrm{abs},\Lambda'}(\underline{\Gamma}^{(2)}, ds^{(2)}_\Lambda|s'_{\Lambda^c})$ there exists a coupling $\mathcal{Q}$ so that:*

$$\mathcal{Q}\Big(\big((\underline{\Gamma}^{(1)}, s^{(1)}), (\underline{\Gamma}^{(2)}, s^{(2)})\big)\Big|\text{there is } \tilde{\Delta} \sqsupset \Delta, \text{ such that } \tilde{\Delta} \text{ is a set of agrement}\Big)$$
$$\le 1 - c_2|\Delta| e^{-\omega_2 \mathrm{dist}(\Delta, \Lambda^c)} \tag{B.7}$$

$\{\mu^+_{\mathrm{abs},\Lambda}(\cdot|s_{\Lambda^c})\}_{\Lambda,\underline{s}}$ converges to a unique limit probability measure μ^+_{abs}. This measure is ℓ'-translation invariant and given any bounded $\mathcal{D}'$-measurable sets Λ, Δ with $\Delta \sqsubset \Lambda$ and any $\underline{s} \in \mathcal{X}^+$, there is a coupling $\mathcal{Q}$ of $\mu^+_{\mathrm{abs},\Lambda}(d\underline{s}^{(1)}|s_{\Lambda^c})$ and $\mu^+_{\mathrm{abs}}(d\underline{s}^{(2)})$ such that (B.7) *holds. The analogous result holds for the minus case.*

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