Basics of regularization theory

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Abstract

We consider the dynamics of three point masses, where we assume that the mass of the third body is so small that it does not affect the motion of the primaries. In the framework of the *restricted* three–body problem, we investigate the collisional trajectories, which correspond to a singularity of the equations of motion. We investigate the regularizing techniques known as Levi–Civita, Kustaanheimo–Stiefel and Birkhoff transformations. The Levi–Civita regularization is adapted to the study of the *planar* restricted three–body problem, when considering a single collision with one of the primaries. The Kustaanheimo–Stiefel method concerns the same problem when the bodies are allowed to move in the 3–dimensional space. A simultaneous regularization with both primaries is achieved through the implementation of Birkhoff's transformation.

1 Introduction

The motion of the celestial bodies of the solar system is ruled by Newton's law, which states that the attraction between massive bodies is directly proportional to the product of the masses and inversely proportional to the square of their distance. Indeed, a collision between any two objects is marked by the fact that their distance becomes zero, which corresponds to a singularity of Newton's equations. The aim of regularization theory is to transform the singular differential equations into regular ones, thus providing an efficient mathematical tool to analyze motions leading to collisions. We shall be concerned with the *restricted three–body problem,* dealing with the motion of a small body in the gravitational field of two massive primaries. It is assumed that the primaries move on circular orbits around their common center of mass. In this framework, we start our discussion by introducing the basic technique to regularize the dynamics ([5], [7]), i.e. the so–called *Levi–Civita transformation,* which is convenient when dealing with the planar three–body problem (i.e., when neglecting the mutual inclinations). When the three bodies are allowed to move in the space, a different method must be adopted, i.e. the so–called Kustaanheimo–Stiefel regularization theory ([3]), often denoted as KS–transformation. Both Levi–Civita and KS methods are *local* transformations, in the sense that their application allows to regularize collisions with only one of the two primaries. A suitable extension of such techniques allows to obtain a simultaneous regularization with both primaries, thus obtaining a *global* transformation, known as Birkhoff's method ([5], [7]). All these techniques rely on a common procedure, which consists in performing a suitable change of variables, a rescaling of time and in using the preservation of energy (see also [1], [4], [5], [6], [7], [8]).

Before entering into the intriguing world of the regularizing transformations, we premise some *facts about concrete* collisional events that occurred in the solar system (see also [2]). When thinking to impacts of heavy objects with the Earth, one is immediately led back to 65 million years ago: it is widely accepted that the disappearance of dinosaurs was caused by the collision of a large body with the Earth. The astroblame is the so–called Chicxulub's crater of about 180 km of diameter, which was located in the depth of the ocean, close to the Yucatan peninsula. Beside this catastrophic event, many other impacts marked the life of the Earth. Just to give a few examples, other astroblames were found throughout our planet: from Arizona (the Meteor Crater of about 1200 m of diameter), to Australia (the Wolf Creek crater of about 850 m), to Arabia (the Waqar crater of about 100 m of diameter). Live images of an impact in the solar system were provided in 1994 by the spectacular collision of the Shoemaker–Levy 9 comet with Jupiter, which fragmented in several pieces before the impact with the giant planet.

This paper is organized as follows. In section 2 we provide the application of the Levi–Civita regularization theory to a simple example, namely the motion of two bodies on a straight line. In section 3 we recall the equations of motion governing the two–body problem, while in section 4 we describe the equations concerning the planar, circular, restricted 3–body problem. The Levi–Civita and Kustaanheimo–Stiefel regularization techniques are presented, respectively, in sections 5 and 6. Birkhoff's global transformation is outlined in section 7.

2 The idea of regularization theory

The standard technique concerning regularization theory is essentially based on three steps:
- a change of coordinates, known as the Levi–Civita transformation;
- the introduction of a *fictitious time*, in order to get rid of the fact that the velocity becomes infinite at the singularity;
- the use of the conservation of energy, in order to transform the singular differential equations into regular ones, through the introduction of the *extended phase space*.

We begin the presentation of regularization theory using a very simple example: consider two bodies, $\overline{P}_1$ and $\overline{P}_2$ (with masses, respectively, m_1 and m_2), which interact through Newton's law. We assume that the two bodies move on a straight line; as a consequence, we select a reference frame with the origin located in $\overline{P}_2$ and with the abscissa coinciding with the line of motion. Let us introduce the quantity $K = G(m_1 + m_2)$, where G denotes the gravitational constant. The motion of $\overline{P}_1$ with respect to $\overline{P}_2$ is ruled by the equation

$$\ddot{x} + \frac{K}{x^2} \ = \ 0 \ ;$$

the corresponding energy integral is provided by

$$h = \frac{K}{x} - \frac{1}{2}\dot{x}^2 .$$

We remark that the velocity $\dot{x} = \pm\sqrt{2(\frac{K}{x} - h)}$ becomes infinite at the collision, i.e. whenever $x = 0$.

We start by performing a change of coordinates, the Levi–Civita transformation, which can be written (in the present case) as

$$x = u^2 .$$

The equation of motion in the new coordinate becomes

$$\ddot{u} + \frac{1}{u}\dot{u}^2 + \frac{K}{2u^5} = 0 ,$$

while the energy integral takes the form

$$h = \frac{K}{u^2} - 2u^2\dot{u}^2 .$$

The new velocity is given by

$$\dot{u} = \pm\sqrt{\frac{K}{2u^4} - \frac{h}{2u^2}} ;$$

we notice that $\dot{u}$ becomes infinite at collision (i.e., at $u = 0$). Since the equation is still singular, we proceed to apply a change of time by introducing a *fictitious time:* to control the increase of speed at collision, we multiply the velocity by a suitable scaling factor which is zero at the singularity. Therefore, we introduce a fictitious time s defined by

$$dt = x\ ds = u^2\ ds \qquad \text{or} \qquad \frac{dt}{ds} = x = u^2 .$$

Denoting by $u' \equiv \frac{du}{ds}$, one has

$$\begin{aligned} \dot{u} &= \frac{du}{dt} = \frac{du}{ds}\frac{ds}{dt} = \frac{1}{u^2}u' , \\ \ddot{u} &= \frac{1}{u^2}\frac{d}{ds}(\frac{1}{u^2}u') = \frac{1}{u^4}u'' - \frac{2}{u^5}u'^2 . \end{aligned}$$

The equation of motion and the energy integral become

$$u'' - \frac{1}{u}\ u'^2 + \frac{K}{2u} = 0 , \qquad h = \frac{K}{u^2} - \frac{2u'^2}{u^2} .$$

Finally, we make use of the preservation of energy as follows. Let us rewrite the equation of motion as

$$u'' + \frac{1}{2}\Big(\frac{K}{u^2} - \frac{2u'^2}{u^2}\Big)\ u = 0 ;$$

using the expression of the energy integral, we get the differential equation

$$u'' + \frac{h}{2}\ u = 0 ,$$

which corresponds to the equation of the harmonic oscillator with frequency $\omega = \sqrt{\frac{h}{2}}$. We conclude by remarking that the above regularizing procedure allowed to obtain a regular differential equation, whose solution is a periodic function of the fictitious time s.

3 The two–body problem

We consider two massive bodies, say $\overline{P}_1$ and $\overline{P}_2$, which attract each other through Newton's law. Since the relative motion takes place on a plane (i.e., the motion is no more constrained on a straight line as in section 2), we select a reference frame coinciding with the plane of motion and we let (q_1, q_2) be the relative cartesian coordinates. Let us show that, after a suitable normalization of the units of measure, the Hamiltonian function describing the two–body problem is given by

$$H = H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{(q_1^2 + q_2^2)^{\frac{1}{2}}} \ , \tag{1}$$

where we defined the momenta as $p_j \equiv \dot{q}_j$, $j = 1, 2$. We remark that the equations of motion associated to (1) are given by

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1 \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1} = -\frac{q_1}{(q_1^2 + q_2^2)^{\frac{3}{2}}} \qquad \frac{q_2}{+ q_2^2)^{\frac{3}{2}}} \ . \tag{2}$$

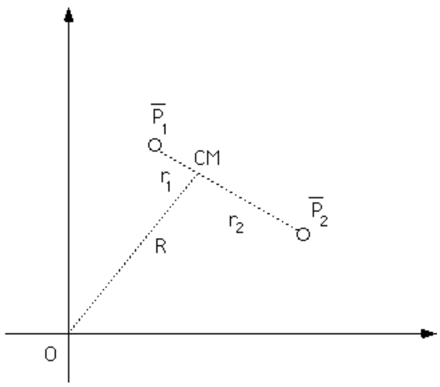

Figure 1: The positions of $\overline{P}_1$ and $\overline{P}_2$ in an inertial reference frame.

In order to derive the Hamiltonian (1), let us denote by R the distance between the center of mass between $\overline{P}_1$ and $\overline{P}_2$ and the origin of the reference frame; let r_1 and r_2 be the distances of the two bodies from the center of mass and define $r \equiv r_2 - r_1$ (see Figure 2). The kinetic energy is the sum of the contributions due to the motion of the center of mass and to the motions of $\overline{P}_1$ and $\overline{P}_2$ relative to the center of mass, i.e.

$$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 \ .$$

Since $r_1 = -\frac{m_2}{m_1+m_2}r$, $r_2 = \frac{m_1}{m_1+m_2}r$, then the Lagrangian function is given by

$$L = \frac{m_1 + m_2}{2}\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{r}^2 + V(r) \ ,$$

where $V(r)$ denotes the Newtonian potential. The first term does not contribute to the equations of motion; the remaining terms can be expressed in cartesian coordinates (q_1, q_2) as

$$L = \frac{1}{2}\mu(\dot{q}_1^2 + \dot{q}_2^2) + \frac{K}{(q_1^2 + q_2^2)^{\frac{1}{2}}} \ ,$$

where $\mu \equiv \frac{m_1 m_2}{m_1+m_2}$ is the *reduced mass* and K is a suitable constant. Normalizing the units of measure so that $\mu = 1$ and $K = 1$, one obtains the Hamiltonian (1).

Remark: Every elliptic solution of the classical Newtonian equation is *unstable* ([5]). In order to make this statement more precise, let us provide the following

Definition of Lyapunov stability: Consider a reference solution with given initial data at some time t_0; define a second solution, which is obtained by slightly varying the initial data. The reference solution is called *stable,* if for any $t \geq t_0$ the distance between the two solutions can be made smaller than ε by an appropriate choice of the variations of the initial conditions.

In the two–body approximation, Hamilton's equations (2) can be written as

$$\ddot{q} + \omega^2 q = 0 \ ,$$

where $q = (q_1, q_2)$, $r = \sqrt{q_1^2 + q_2^2}$ and $\omega = \omega(r) = \frac{1}{r^{3/2}}$. Consider a circular reference solution and a varied solution, which is also circular. The period of revolution varies with r as $\frac{2\pi}{\omega} = 2\pi \cdot r^{3/2}$; therefore, there exists a time when the two particles are opposite to each other with respect to the center of mass, which is incompatible with Lyapunov stability.

4 The planar, circular, restricted 3–body problem

Let S be a body with an infinitesimal mass, subject to the gravitational attraction of $\overline{P}_1$, $\overline{P}_2$ (whose masses are, respectively, μ_1 and μ_2). We assume that the primaries are not affected by S, i.e. we consider the so–called *restricted* three–body problem. Moreover, we assume that the motion takes place on the same plane (i.e., we neglect the relative inclinations) and that the trajectories of $\overline{P}_1$ and $\overline{P}_2$ are circular with origin coinciding with their common center of mass.

Let (x_1, x_2) be the coordinates of S in an inertial frame centered at the barycenter of $\overline{P}_1$ and $\overline{P}_2$ and let us normalize the units of measure so that

We denote by and $\overline{P}_2$ (see Figure 3).

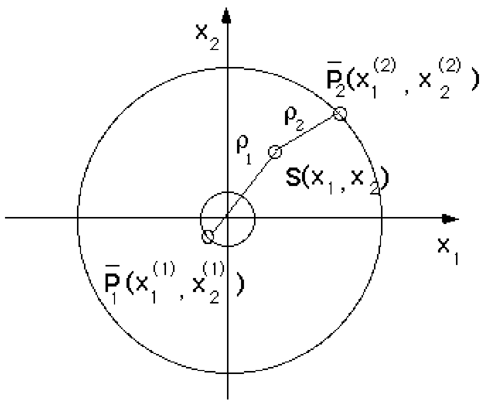

Figure 2: The coordinates of $\overline{P}_1$, $\overline{P}_2$ and S in a fixed reference frame.

The motion of S is described by the Lagrangian function

$$L = L(\dot{x}_1, \dot{x}_2, x_1, x_2, t) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + V(x_1, x_2, t) \ ,$$

where $V(x_1, x_2, t) \equiv \frac{\mu_1}{\rho_1} + \frac{\mu_2}{\rho_2}$ and

$$\begin{aligned} \rho_1 &\equiv \sqrt{(x_1 - x_1^{(1)})^2 + (x_2 - x_2^{(1)})^2} \ , \\ \rho_2 &\equiv \sqrt{(x_1 - x_1^{(2)})^2 + (x_2 - x_2^{(2)})^2} \ . \end{aligned}$$

We remark that $x_1^{(1)}$, $x_2^{(1)}$, $x_1^{(2)}$, $x_2^{(2)}$ are explicit functions of the time. Denoting by y_i $(i = 1, 2)$ the kinetic moments conjugated to x_i, the Hamiltonian function is given by

$$H(y_1, y_2, x_1, x_2, t) = \frac{1}{2}(y_1^2 + y_2^2) - V(x_1, x_2, t) \ .$$

Remark: In the spatial case, let $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)})$, $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$ be the coordinates of $\overline{P}_1$ and $\overline{P}_2$; denoting by y_i $(i = 1, 2, 3)$ the kinetic moments conjugated to x_i, the Hamiltonian function reads as

$$H(y_1, y_2, y_3, x_1, x_2, x_3, t) = \frac{1}{2}(y_1^2 + y_2^2 + y_3^3) - V(x_1, x_2, x_3, t) \ ,$$

where

$$
\begin{aligned}
V(x_1,x_2,x_3,t) &= \frac{\mu_1}{\sqrt{(x_1-x_1^{(1)})^2+(x_2-x_2^{(1)})^2+(x_3-x_3^{(1)})^2}} \\
&+ \frac{\mu_2}{\sqrt{(x_1-x_1^{(2)})^2+(x_2-x_2^{(2)})^2+(x_3-x_3^{(2)})^2}} \ .
\end{aligned}
$$

Let us consider a rotating or *synodic* reference frame centered at the barycenter of $\overline{P}_1$ and $\overline{P}_2$; assume th … e angular velocity of $\overline{P}_1$ and $\overline{P}_2$ is unity. Then … , $\overline{P}_1(\mu_2, 0)$ and $\overline{P}_2(-\mu_1, 0)$. Let (q_1, q_2) be the … e 4).

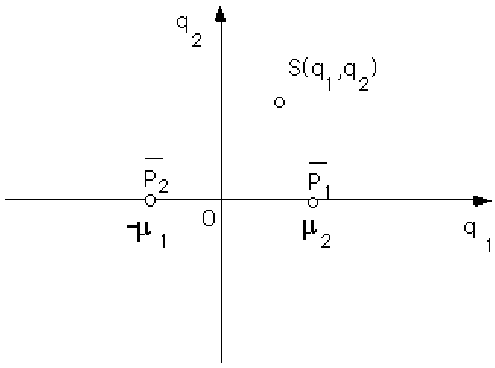

Figure 3: The coordinates of $\overline{P}_1$, $\overline{P}_2$ and S in a synodic reference frame.

In order to derive the synodic Hamiltonian, we consider the generating function

$$
\begin{aligned}
W(y_1,y_2,q_1,q_2,t) &= y_1q_1\cos t - y_1q_2\sin t \\
&+ y_2q_1\sin t + y_2q_2\cos t \ ,
\end{aligned}
$$

which provides the characteristic equations

$$
\begin{aligned}
x_1 &= \frac{\partial W}{\partial y_1} = q_1\cos t - q_2\sin t \\
p_1 &= \frac{\partial W}{\partial q_1} = y_1\cos t + y_2\sin t \\
x_2 &= \frac{\partial W}{\partial y_2} = q_1\sin t + q_2\cos t \\
p_2 &= \frac{\partial W}{\partial q_2} = -y_1\sin t + y_2\cos t \ .
\end{aligned}
$$

Inverting the above equations, one obtains

$$\begin{aligned} q_1 &= x_1\cos t + x_2\sin t \qquad & y_1 = p_1\cos t - p_2\sin t \\ q_2 &= -x_1\sin t + x_2\cos t \qquad & y_2 = p_1\sin t + p_2\cos t\ . \end{aligned}$$

It is trivial to check that the synodic Hamiltonian takes the form

$$\begin{aligned} \tilde{H}(p_1,p_2,q_1,q_2,t) &= H - \frac{\partial W}{\partial t} \\ &= \frac{1}{2}(p_1^2+p_2^2) + q_2p_1 - q_1p_2 \\ &- V(q_1\cos t - q_2\sin t, q_1\sin t + q_2\cos t, t)\ . \end{aligned}$$

Recalling that in the fixed frame the bodies $\overline{P}_1$ and $\overline{P}_2$ describe circles of radius μ_2 and μ_1, we can write their coordinates as

$$\begin{aligned} x_1^{(1)} &= \mu_2\cos t \qquad & x_1^{(2)} = -\mu_1\cos t \\ x_2^{(1)} &= \mu_2\sin t \qquad & x_2^{(2)} = -\mu_1\sin t\ . \end{aligned}$$

The expression of the perturbing function in the rotating coordinates becomes

$$V(q_1,q_2) = \frac{\mu_1}{\sqrt{(q_1-\mu_2)^2+q_2^2}} + \frac{\mu_2}{\sqrt{(q_1+\mu_1)^2+q_2^2}}\ .$$

Therefore we are led to the following synodic Hamiltonian:

$$\tilde{H}(p_1,p_2,q_1,q_2) = \frac{1}{2}(p_1^2+p_2^2) + q_2p_1 - q_1p_2 - V(q_1,q_2)\ , \tag{3}$$

whose associated Hamilton's equations are

$$\begin{aligned} \dot{q}_1 &= p_1 + q_2 \qquad & \dot{p}_1 = p_2 + V_{q_1} \\ \dot{q}_2 &= p_2 - q_1 \qquad & \dot{p}_2 = -p_1 + V_{q_2}\ . \end{aligned}$$

Denoting by

$$\overline{\Omega} \equiv \frac{1}{2}(q_1^2+q_2^2) + V\ , \tag{4}$$

we can write the previous equations in the form

$$\begin{aligned} \ddot{q}_1 - 2\dot{q}_2 &= \dot{p}_1 + \dot{q}_2 - 2\dot{q}_2 = \dot{p}_1 - \dot{q}_2 = q_1 + V_{q_1} \equiv \overline{\Omega}_{q_1} \\ \ddot{q}_2 + 2\dot{q}_1 &= \dot{p}_2 - \dot{q}_1 + 2\dot{q}_1 = \dot{p}_2 + \dot{q}_1 = q_2 + V_{q_2} \equiv \overline{\Omega}_{q_2}\ . \end{aligned} \tag{5}$$

For notational convenience, we define $\Omega \equiv \overline{\Omega} + \frac{1}{2}\mu_1\mu_2$. The expression of the so–called *Jacobi integral* is obtained as follows: let us multiply the first equation in (5) by $\dot{q}_1$ and the second by $\dot{q}_2$; over summation one obtains:

$$\dot{q}_1\ddot{q}_1 + \dot{q}_2\ddot{q}_2 = \dot{q}_1\overline{\Omega}_{q_1} + \dot{q}_2\overline{\Omega}_{q_2}\ .$$

Therefore one has $\frac{1}{2}\frac{d}{dt}(\dot{q}_1^2+\dot{q}_2^2) = \frac{d}{dt}(\overline{\Omega})$, from which it follows that

$$\dot{q}_1^2 + \dot{q}_2^2 = 2\overline{\Omega} - C' = 2\Omega - C\ .$$

We define the *Jacobi constant* as

$$C \equiv 2\Omega - (\dot q_1^2 + \dot q_2^2)\ .$$

Since $\dot q_1 = p_1 + q_2$ and $\dot q_2 = p_2 - q_1$, then $p_1 = \dot q_1 - q_2$, $p_2 = \dot q_2 + q_1$. Hence, it is useful to rewrite (3) as

$$\begin{aligned}\tilde H &= \frac{1}{2}(\dot q_1^2 + \dot q_2^2) - \frac{1}{2}(q_1^2 + q_2^2) - V(q_1, q_2)\\ &= \frac{1}{2}(\dot q_1^2 + \dot q_2^2) - \overline{\Omega}\ .\end{aligned}$$

Making use of the Jacobi integral one gets

$$\tilde H + \Omega = \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}(\dot q_1^2 + \dot q_2^2) = \frac{1}{2}\mu_1\mu_2 + \Omega - \frac{C}{2}\ ,$$

which provides the relation

$$\tilde H = \frac{\mu_1\mu_2 - C}{2}\ .$$

5 The Levi–Civita transformation

5.1 The regularization of the two–body problem

Recall that the Hamiltonian function describing the two–body problem is given (in normalized units of measure) by

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{(q_1^2 + q_2^2)^{\frac{1}{2}}}\ .$$

Let us consider a canonical transformation with generating function of the form

$$W(p_1, p_2, Q_1, Q_2) = p_1 f(Q_1, Q_2) + p_2 g(Q_1, Q_2)\ .$$

Denoting by i the imaginary unit, the *Levi–Civita transformation* is obtained setting

$$f + ig \equiv (Q_1 + iQ_2)^2 = Q_1^2 - Q_2^2 + i \cdot 2Q_1Q_2\ ,$$

namely

$$f(Q_1, Q_2) \equiv Q_1^2 - Q_2^2\ , \qquad\qquad g(Q_1, Q_2) \equiv 2Q_1Q_2\ .$$

The change of variables associated to the generating function W is given by

$$\begin{aligned}q_1 &= \frac{\partial W}{\partial p_1} = f(Q_1, Q_2) = Q_1^2 - Q_2^2\\ q_2 &= \frac{\partial W}{\partial p_2} = g(Q_1, Q_2) = 2Q_1Q_2\\ P_1 &= \frac{\partial W}{\partial Q_1} = p_1\frac{\partial f}{\partial Q_1} + p_2\frac{\partial g}{\partial Q_1} = 2p_1Q_1 + 2p_2Q_2\\ P_2 &= \frac{\partial W}{\partial Q_2} = p_1\frac{\partial f}{\partial Q_2} + p_2\frac{\partial g}{\partial Q_2} = -2p_1Q_2 + 2p_2Q_1\ .\end{aligned} \tag{6}$$

We refer to (q_1, q_2) as the *physical plane* and to (Q_1, Q_2) as the *parametric plane* with $q_1 + iq_2 = f + ig = (Q_1 + iQ_2)^2$. Using matrix notation, we remark that the Levi–Civita transformation can be written as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} Q_1^2 - Q_2^2 \\ 2Q_1Q_2 \end{pmatrix} = \begin{pmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A_0 \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} ,$$

where we defined the matrix $A_0 = \begin{pmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{pmatrix}$. It is immediate to check that the Levi–Civita transformation is characterized by the following properties:

(1) the matrix A_0 is orthogonal;

(2) the elements of A_0 are linear and homogeneous functions of Q_1, Q_2;

(3) [illegible] the Q–vector.

[illegible]nsformation is that the angles at the origin

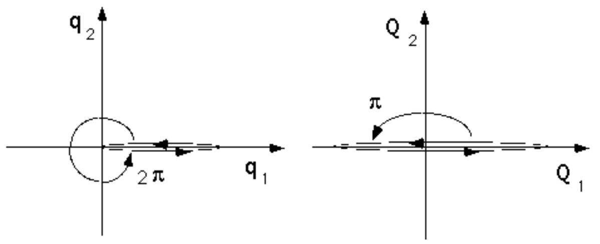

Figure 4: The Levi–Civita transformation doubles the angles at the origin.

Indeed, when the orbital eccentricity tends to 1, the ellipse degenerates into a straight line. In the physical plane, the position vector makes a sharp bend of angle 2π at the origin. In the parametric plane, the singularity is removed allowing the particle to pass through the origin; in this case, the position vector makes an angle π at the origin. More specifically, denoting by θ and ψ the angles formed by the position vector at the origin, one obtains

$$\tan\theta = \frac{q_2}{q_1} = \frac{2Q_1Q_2}{Q_1^2 - Q_2^2} = \frac{2Q_2/Q_1}{1 - (\frac{Q_2}{Q_1})^2} = \frac{2\tan\psi}{1 - \tan^2\psi} = \tan 2\psi$$

Coming back to the Levi–Civita transformation (6), we observe that

$$P = 2A_0^T\, p$$

and that the inverse of the matrix A_0 is provided by $A_0^{-1} = \frac{1}{\det A_0} A_0^T$. Let us define $D \equiv 4 \det A_0 = 4(Q_1^2 + Q_2^2) > 0$; then one obtains

$$P_1^2 + P_2^2 = D(p_1^2 + p_2^2) \ .$$

As a consequence, the new Hamiltonian becomes

$$\tilde{H} = \tilde{H}(P_1, P_2, Q_1, Q_2) = \frac{1}{2D}(P_1^2 + P_2^2) - \frac{1}{(f(Q_1, Q_2)^2 + g(Q_1, Q_2)^2)^{\frac{1}{2}}} \ .$$

The corresponding Hamilton's equations are

$$\begin{aligned}
\dot{Q}_1 &= \frac{P_1}{D} \\
\dot{Q}_2 &= \frac{P_2}{D} \\
\dot{P}_1 &= \frac{1}{2D}(P_1^2 + P_2^2)\frac{\partial D}{\partial Q_1} - \frac{1}{2}\frac{1}{(f^2+g^2)^{\frac{3}{2}}}\frac{\partial (f^2+g^2)}{\partial Q_1} \\
\dot{P}_2 &= \frac{1}{2D}(P_1^2 + P_2^2)\frac{\partial D}{\partial Q_2} - \frac{1}{2}\frac{1}{(f^2+g^2)^{\frac{3}{2}}}\frac{\partial (f^2+g^2)}{\partial Q_2} \ .
\end{aligned}$$

Let us introduce the *extended phase space* (see Appendix A) by defining a new variable T conjugated to time, such that the extended Hamiltonian becomes

$$\Gamma(P_1, P_2, T, Q_1, Q_2, t) = \frac{1}{2D}(P_1^2 + P_2^2) + T - \frac{1}{(f(Q_1, Q_2)^2 + g(Q_1, Q_2)^2)^{\frac{1}{2}}} \ .$$

Since $\dot{t} = \frac{\partial \Gamma}{\partial T} = 1$ and $\dot{T} = -\frac{\partial \Gamma}{\partial t} = 0$, one obtains that T is constant and in particular, along any solution, one has $T(t) \equiv T = -\tilde{H}$.

Next step consists in introducing a *fictitious (or regularized) time s* defined through the relation

$$dt = D(Q_1, Q_2)\ ds \qquad\qquad \text{or} \qquad\qquad \frac{d}{dt} = \frac{1}{D}\frac{d}{ds} \ .$$

Since $\dot{Q} = \frac{\partial \Gamma}{\partial P}$, it follows that

$$\dot{Q} = \frac{dQ}{dt} = \frac{dQ}{ds}\frac{ds}{dt} = \frac{1}{D}\frac{dQ}{ds} = \frac{\partial \Gamma}{\partial P} \ ;$$

the above relation implies that $\frac{dQ}{ds} = \frac{\partial \Gamma^*}{\partial P}$ with $\Gamma^* \equiv D\Gamma$. Similarly, we use $\dot{P} = -\frac{\partial \Gamma}{\partial Q}$ to obtain

$$\dot{P} = \frac{dP}{dt} = \frac{dP}{ds}\frac{ds}{dt} = \frac{1}{D}\frac{dP}{ds} = -\frac{\partial \Gamma}{\partial Q} \ ,$$

which provides

$$\frac{dP}{ds} = -\frac{\partial \Gamma^*}{\partial Q} \ ,$$

where $\Gamma^* \equiv D\Gamma$; in fact, we observe that

$$\frac{\partial \Gamma^*}{\partial Q} = \frac{\partial D}{\partial Q}\Gamma + D\frac{\partial \Gamma}{\partial Q} = D\frac{\partial \Gamma}{\partial Q} \ ,$$

being $\Gamma = 0$ along a solution. Finally, the new Hamiltonian Γ^* is given by

$$\Gamma^* \equiv D\Gamma = DT + \frac{1}{2}(P_1^2 + P_2^2) - \frac{D}{(f^2+g^2)^{\frac{1}{2}}} \ .$$

The associated Hamilton's equations ($j = 1, 2$) are

$$
\begin{aligned}
\frac{dQ_j}{ds} &= P_j \\
\frac{dP_j}{ds} &= -\frac{\partial}{\partial Q_j}\Big[DT - \frac{D}{(f^2+g^2)^{\frac{1}{2}}}\Big] \\
\frac{dt}{ds} &= D \\
\frac{dT}{ds} &= 0 \ .
\end{aligned}
$$

Notice that the singularity of the problem is associated to the term $\frac{D}{(f^2+g^2)^{\frac{1}{2}}}$, which is transformed as

$$
\begin{aligned}
\frac{D}{(f^2+g^2)^{\frac{1}{2}}} &= \frac{D}{r} = \frac{4(Q_1^2+Q_2^2)}{(Q_1^4+Q_2^4-2Q_1^2Q_2^2+4Q_1^2Q_2^2)^{\frac{1}{2}}} \\
&= \frac{4(Q_1^2+Q_2^2)}{(Q_1^2+Q_2^2)} = 4 \ ,
\end{aligned}
$$

where we used $f = Q_1^2 - Q_2^2$ and $g = 2Q_1Q_2$.

Denoting by a prime the derivative with respect to s and recalling that $T = -\tilde{H}$, the equations of motion are ($j = 1, 2$):

$$
\begin{aligned}
Q_j' &= P_j \\
P_j' &= -T\frac{\partial D}{\partial Q_j} = -T \cdot 8Q_j = 8\tilde{H}Q_j
\end{aligned}
$$

Therefore, one gets the second order differential equation

$$
Q_j'' = 8\tilde{H}Q_j \qquad\qquad (j = 1, 2) \ . \tag{7}
$$

Notice that if $\tilde{H} < 0$ (corresponding to an elliptic orbit), one obtains the equation of an harmonic oscillator.

Remark: In relation to the remark of section 3, we observe that the equation (7) describes pure harmonic oscillations with fixed frequency; therefore, one gets regular differential equations and the solution is **stable.**

We conclude this section by showing the relation between the fictitious time and the eccentric anomaly. To this end, setting $\omega^2 = -8\tilde{H}$, let us write equation (7) as

$$
Q_1'' + \omega^2 Q_1 = 0 \ , \qquad Q_2'' + \omega^2 Q_2 = 0 \ .
$$

Assuming the initial conditions

$$
\begin{aligned}
Q_1(0) &= \alpha \qquad\qquad Q_2(0) = 0 \\
Q_1'(0) &= 0 \qquad\qquad Q_2'(0) = \beta\omega \ ,
\end{aligned}
$$

the solution is given by

$$Q_1(s) = \alpha\cos(\omega s) \ , \qquad Q_2(s) = \beta\sin(\omega s) \ .$$

Defining $E = 2\omega s$, we get $Q_1(E) = \alpha\cos(\frac{E}{2})$, $Q_2(E) = \beta\sin(\frac{E}{2})$. Using the transformation $(Q_1, Q_2) \to (q_1, q_2)$, we get

$$\begin{aligned} q_1 &= Q_1^2 - Q_2^2 = -\frac{\beta^2-\alpha^2}{2} + \frac{\beta^2+\alpha^2}{2}\cos E \\ q_2 &= 2Q_1Q_2 = \alpha\beta\sin E \ , \end{aligned}$$

which describes an ellipse with center in $(-\frac{\beta^2-\alpha^2}{2}, 0)$. Moreover, the major semiaxis is $a = \frac{\beta^2+\alpha^2}{2}$ and the distance of the focus from the center of the ellipse is given by $ae = \frac{\beta^2-\alpha^2}{2}$. Therefore we obtain

$$\begin{aligned} r &= \sqrt{q_1^2+q_2^2} = Q_1^2 + Q_2^2 \\ &= \frac{\beta^2+\alpha^2}{2} - \frac{\beta^2-\alpha^2}{2}\cos E = a(1 - e\cos E) \ , \end{aligned}$$

which corresponds to the standard Keplerian relation between the radial distance and the eccentric anomaly.

5.2 The regularization of the planar, circular, restricted three–body problem

In a synodic reference frame, the Hamiltonian of the planar, circular, restricted three–body problem is given by

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) + q_2p_1 - q_1p_2 - V(q_1, q_2) \ ,$$

where $V(q_1, q_2) = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$ and

$$r_1 = [(q_1 - \mu_2)^2 + q_2^2]^{\frac{1}{2}} \ , \qquad r_2 = [(q_1 + \mu_1)^2 + q_2^2]^{\frac{1}{2}} \ .$$

To regularize collisions with $\overline{P}_1$, we consider again a generating function of the form

$$W(p_1, p_2, Q_1, Q_2) = p_1 f(Q_1, Q_2) + p_2 g(Q_1, Q_2) \ ,$$

with $f(Q_1, Q_2) = Q_1^2 - Q_2^2 + \mu_2$, $g(Q_1, Q_2) = 2Q_1Q_2$ and characteristic equations

$$\begin{aligned} q_1 &= \frac{\partial W}{\partial p_1} = f(Q_1, Q_2) \\ q_2 &= \frac{\partial W}{\partial p_2} = g(Q_1, Q_2) \\ P_1 &= \frac{\partial W}{\partial Q_1} = p_1\frac{\partial f}{\partial Q_1} + p_2\frac{\partial g}{\partial Q_1} \\ P_2 &= \frac{\partial W}{\partial Q_2} = p_1\frac{\partial f}{\partial Q_2} + p_2\frac{\partial g}{\partial Q_2} \ . \end{aligned}$$

Remark: To regularize collisions with $\overline{P}_2$, it suffices to substitute f with $f(Q_1,Q_2) = Q_1^2 - Q_2^2 - \mu_1$.

The above change of coordinates transforms $p_1^2+p_2^2$ into $\frac{1}{D}(P_1^2+P_2^2)$, while the term $q_2p_1-p_2q_1$ becomes

$$q_2p_1 - p_2q_1 = \frac{1}{2D}\ [P_1\frac{\partial}{\partial Q_2}(f^2+g^2) - P_2\frac{\partial}{\partial Q_1}(f^2+g^2)]\ .$$

Therefore, the transformed Hamiltonian is given by

$$\tilde{H}(P_1,P_2,Q_1,Q_2) = \frac{1}{2D}[P_1^2+P_2^2+P_1\frac{\partial}{\partial Q_2}(f^2+g^2) - P_2\frac{\partial}{\partial Q_1}(f^2+g^2)] - \tilde{V}(Q_1,Q_2)\ ,$$

where $\tilde{V}$ corresponds to V with q_1 replaced by $f(Q_1,Q_2)$ and q_2 replaced by $g(Q_1,Q_2)$. The Hamiltonian in the extended phase space becomes

$$\Gamma = T + \frac{1}{2D}[P_1^2+P_2^2+P_1\frac{\partial}{\partial Q_2}(f^2+g^2) - P_2\frac{\partial}{\partial Q_1}(f^2+g^2)] - \tilde{V}(Q_1,Q_2)\ .$$

Next, we introduce the fictitious time $dt\ =\ D\ ds$, obtaining the Hamiltonian

$$\Gamma^* \equiv D\Gamma = DT + \frac{1}{2}[P_1^2+P_2^2+P_1\frac{\partial}{\partial Q_2}(f^2+g^2) - P_2\frac{\partial}{\partial Q_1}(f^2+g^2)] - D\tilde{V}(Q_1,Q_2)\ .$$

In the following, it will be useful to define the function $\Phi(Q_1,Q_2)$ as

$$\Phi(Q_1,Q_2) \equiv f(Q_1,Q_2) + ig(Q_1,Q_2)\ .$$

From equation (4) one gets $\overline{\Omega} = \frac{1}{2}(q_1^2+q_2^2) + V = \Omega - \frac{1}{2}\mu_1\mu_2$ and $\frac{1}{2}(f^2+g^2) + \tilde{V} = \Omega - \frac{1}{2}\mu_1\mu_2$ (with $q_1 = f$ and $q_2 = g$). Observing that $|\Phi|^2 = f^2+g^2$, one has

$$\frac{1}{2}|\Phi|^2 + \tilde{V} = \Omega - \frac{1}{2}\mu_1\mu_2\ .$$

From the definition of the Jacobi integral, we obtain

$$\tilde{H} = -T = \frac{\mu_1\mu_2 - C}{2}\ ,$$

namely

$$\frac{1}{2}|\Phi|^2 - T + \tilde{V} = \Omega - \frac{C}{2}\ .$$

Since $D\tilde{V} = D(\Omega - \frac{C}{2}) - \frac{1}{2}D|\Phi|^2 + DT$, we notice that the critical term is just $D(\Omega - \frac{C}{2})$. In order to achieve the desired regularization, let us define the complex physical and parametric coordinates as $z = q_1 + iq_2$ and $w = Q_1 + iQ_2$. The physical coordinates of the primaries are $z_1 = \mu_2$ and $z_2 = -\mu_1$. The regularizing transformation at $\overline{P}_1$ can be written as $z = w^2 + \mu_2$, while the transformation $z = w^2 - \mu_1$ regularizes the singularity at $\overline{P}_2$. By means of the first transformation, the primary $\overline{P}_1$ is moved to the origin of the w–plane, while $\overline{P}_2$ has coordinates $w_{1,2} = \pm i$.
Since $r_1 = |w|^2$, $r_2 = |1+w^2|$ and since

$$\mu_1 r_1^2 + \mu_2 r_2^2 = \mu_1(z-\mu_2)^2 + \mu_2(z+\mu_1)^2 = z^2 + \mu_1\mu_2\ ,$$

we obtain that

$$\begin{aligned} U &= \Omega - \frac{C}{2} = \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}(q_1^2 + q_2^2) + V - \frac{C}{2} \\ &= \frac{1}{2}(\mu_1 r_1^2 + \mu_2 r_2^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} - \frac{C}{2} \\ &= \frac{1}{2}[\mu_1 |w|^4 + \mu_2 |1 + w^2|^2] + \frac{\mu_1}{|w|^2} + \frac{\mu_2}{|1 + w^2|} - \frac{C}{2} \ . \end{aligned}$$

Observing that $D = 4(Q_1^2 + Q_2^2) = 4|w|^2$, we find that the term $DU = D(\Omega - \frac{C}{2})$ does not contain singularities at $\overline{P}_1$; in fact, we get

$$DU = D(\Omega - \frac{C}{2}) = 2|w|^2 \ [\mu_1 |w|^4 + \mu_2 |1 + w^2|^2] + 4\mu_1 + \frac{4\mu_2 |w|^2}{|1 + w^2|} - 2C|w|^2 \ ,$$

which is regular as far as $w \neq \pm i$, corresponding to the location of the other primary $\overline{P}_2$.

Remark: The expression of the velocity in terms of the fictitious time is obtained as follows. In the physical space the Jacobi integral is $|\dot{z}|^2 = 2U$, while in the parametric space it takes the form

$$|w'|^2 = 8|w|^2 U \ ,$$

where we used $D = 4|w|^2$, $z = w^2 + \mu_2$, $\dot{z} = 2w\dot{w} = \frac{2}{D}ww'$, namely $|w'|^2 = \frac{D^2}{4|w|^2}|\dot{z}|^2$. Therefore, we have:

$$|w'|^2 = 8\mu_1 + |w|^2 \Big[\frac{8\mu_2}{|1 + w|^2} + 4\mu_1 |w|^4 + 4\mu_2 |1 + w^2|^2 - 4C\Big] \ .$$

From the previous relation, we conclude that
i) in $\overline{P}_1$ one has $r_1 = 0$, namely $w = 0$, while $|w'|^2 = 8\mu_1$ and the velocity is *finite;*
ii) in $\overline{P}_2$ one has $r_2 = 0$, namely $w = \pm i$, while $|w'|^2 = \infty$ and the velocity is *infinite.*

6 The Kustaanheimo–Stiefel transformation

In this section we outline the procedure which allows to regularize the singularities in the *spatial* case.

6.1 The equations of motion and the Hamiltonian

In the framework of the circular, restricted three–body problem, let us consider the motion in the 3–dimensional space of the three bodies S, $\overline{P}_1$ and $\overline{P}_2$. The primaries move in the q_1q_2–plane around their common center of mass, while in the synodic frame their coordinates become $\overline{P}_1(\mu_2, 0, 0)$, $\overline{P}_2(-\mu_1, 0, 0)$. Assume that the q_1q_2–plane rotates with unit angular velocity about the vertical axis. Then, the Hamiltonian function is given by

$$H(p_1, p_2, p_3, q_1, q_2, q_3) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_2 p_1 - q_1 p_2 - V(q_1, q_2, q_3) \ ,$$

where p_1, p_2, p_3 are the momenta conjugated to the coordinates q_1, q_2, q_3. The equations of motion of S are provided by the differential equations

$$\begin{aligned} \ddot{q}_1 - 2\dot{q}_2 &= \Omega_{q_1} \\ \ddot{q}_2 + 2\dot{q}_1 &= \Omega_{q_2} \\ \ddot{q}_3 &= \Omega_{q_3} \ , \end{aligned}$$

where

$$\Omega = \frac{1}{2}(q_1^2 + q_2^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}\mu_1\mu_2 \ ,$$

with $r_1^2 \equiv (q_1 - \mu_2)^2 + q_2^2 + q_3^2$ and $r_2^2 \equiv (q_1 + \mu_1)^2 + q_2^2 + q_3^2$. More explicitely, the equations of motion are

$$\begin{aligned} \ddot{q}_1 - 2\dot{q}_2 &= q_1 - \frac{\mu_1}{r_1^3}(q_1 - \mu_2) - \frac{\mu_2}{r_2^3}(q_1 + \mu_1) \\ \ddot{q}_2 + 2\dot{q}_1 &= q_2 - \frac{\mu_1}{r_1^3}q_2 - \frac{\mu_2}{r_2^3}q_2 \\ \ddot{q}_3 &= -\frac{\mu_1}{r_1^3}q_3 - \frac{\mu_2}{r_2^3}q_3 \ . \end{aligned}$$

6.2 The KS–transformation

As in the Levi–Civita transformation, we define the fictitious time s as

$$dt = D\ ds \ ,$$

for some factor D to be defined later. The relation between the second derivatives with respect to t and s is given by

$$\begin{aligned} \frac{d^2}{dt^2} &= \frac{d}{dt}(\frac{1}{D}\frac{d}{ds}) = \frac{1}{D}\frac{d}{ds}(\frac{1}{D}\frac{d}{ds}) \\ &= \frac{1}{D^3}(D\frac{d^2}{ds^2} - \frac{dD}{ds}\frac{d}{ds}) = \frac{1}{D^2}\frac{d^2}{ds^2} - \frac{1}{D^3}\frac{dD}{ds}\frac{d}{ds} \ . \end{aligned}$$

In terms of the fictitious time, the equations of motion are

$$\begin{aligned} Dq_1'' - D'q_1' - 2D^2q_2' &= D^3\Omega_{q_1} \\ Dq_2'' - D'q_2' + 2D^2q_1' &= D^3\Omega_{q_2} \\ Dq_3'' &= D^3\Omega_{q_3} \ , \end{aligned} \tag{8}$$

where the singular terms are contained in the right hand sides of the previous equations. Notice that Ω_{q_1}, Ω_{q_2}, $\Omega_{q_3} \sim O(\frac{1}{r_1^3})$.

Remark: We recall that in the planar case the Levi–Civita transformation is given by

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} Q_1^2 - Q_2^2 \\ 2Q_1Q_2 \end{pmatrix} \ ,$$

where every element of the matrix $A_0(Q) \equiv \begin{pmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{pmatrix}$ is linear in Q_1, Q_2, with the matrix $A_0(Q)$ being orthogonal.

In order to achieve the regularization in space, we start by investigating the existence of a generalization $A(Q)$ of the matrix $A_0(Q)$ in $\mathbf{R}^n$, with the following properties:

$i)$ the elements of $A(Q)$ must be linear homogeneous functions of the Q_i;
$ii)$ the matrix $A(Q)$ must be orthogonal, namely
  $a)$ the scalar product of different rows must vanish;
  $b)$ each row must have norm $Q_1^2 + ... + Q_n^2$.

A result by A. Hurwitz ([5]) proves that such matrix exists only within spaces of dimensions $n = 1, 2, 4$ or 8. Therefore, it becomes necessary to map the 3–dimensional physical space into a 4–dimensional parametric space by defining the matrix

$$A(Q) = \begin{pmatrix} Q_1 & -Q_2 & -Q_3 & Q_4 \\ Q_2 & Q_1 & -Q_4 & -Q_3 \\ Q_3 & Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{pmatrix} .$$

Consistently, we will extend the physical space by setting the fourth component equal to zero: $(q_1, q_2, q_3, 0)$.

Considering a collision with the primary $\overline{P}_1$, the Kustaanheimo–Stiefel (KS) regularization is defined as follows. Let

$$\begin{aligned} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \end{pmatrix} &= A(Q) \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} + \begin{pmatrix} \mu_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} Q_1 & -Q_2 & -Q_3 & Q_4 \\ Q_2 & Q_1 & -Q_4 & -Q_3 \\ Q_3 & Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} + \begin{pmatrix} \mu_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \end{aligned}$$

namely

$$\begin{aligned} q_1 &= Q_1^2 - Q_2^2 - Q_3^2 + Q_4^2 + \mu_2 \\ q_2 &= 2Q_1Q_2 - 2Q_3Q_4 \\ q_3 &= 2Q_1Q_3 + 2Q_2Q_4 \ . \end{aligned}$$

Remarks:
1) Whenever $Q_3 = Q_4 = 0$, the KS–transformation reduces to the planar Levi–Civita transformation.
2) The norms of each row (or column) of the matrix A are equal to $|Q|^2 \equiv Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2$.
3) The regularization with $\overline{P}_2$ is obtained by replacing the constant vector $(\mu_2, 0, 0, 0)$ with $(-\mu_1, 0, 0, 0)$.
4) The matrix A is orthogonal: $A^T(Q)A(Q) = (Q, Q) \cdot Id$. Therefore, denoting by $q \equiv (q_1 - \mu_2, q_2, q_3, 0)$, one obtains

$$\begin{aligned} r_1^2 &= (q, q) = q^T q = Q^T A^T(Q) A(Q) Q \\ &= Q^T Q (Q, Q) = (Q, Q)^2 \ , \end{aligned}$$

namely $r_1 = (Q, Q) = |Q|^2 = Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2$.
5) A trivial computation shows that $A(Q)' = A(Q')$. As a consequence,

$$q' = A(Q')Q + A(Q)Q' = 2A(Q)Q' ,$$

which yields

$$\begin{pmatrix} q_1' \\ q_2' \\ q_3' \\ 0 \end{pmatrix} = 2A(Q)Q' = 2 \begin{pmatrix} Q_1Q_1' - Q_2Q_2' - Q_3Q_3' + Q_4Q_4' \\ Q_2Q_1' + Q_1Q_2' - Q_4Q_3' - Q_3Q_4' \\ Q_3Q_1' + Q_1Q_3' + Q_4Q_2' + Q_2Q_4' \\ Q_4Q_1' - Q_3Q_2' + Q_2Q_3' - Q_1Q_4' \end{pmatrix} .$$

The last equation is known as the *bilinear relation:*

$$Q_4Q_1' - Q_3Q_2' + Q_2Q_3' - Q_1Q_4' = 0 .$$

In order to prove the canonicity of the transformation induced by the KS–procedure, it is necessary to choose the initail conditions in order that the bilinear equation is satisfied.
6) The second derivative with respect to the fictitious time of the physical coordinates is given by

$$q'' = 2A(Q)Q'' + 2A(Q')Q' .$$

In order to regularize the equations of motion, it is convenient to select the scaling factor D as

$$D \equiv 4r_1 = 4(Q, Q) = 4(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) .$$

The regularization is finally obtained mimicking the Levi–Civita procedure. More specifically, the scheme is the following: express the coordinates and their first and second derivatives in terms of Q, Q', Q''; recall that the singular part of the equations (8) is given by $D^3\Omega_{q_1}$, $D^3\Omega_{q2}$, $D^3\Omega_{q3}$. In a qualitative way, we proceed to remark that due to the fact that $\Omega_{q_1} \propto \frac{1}{r_1^3}$ and that $D \propto r_1$, one obtains that $D^3\Omega_{q_1} = O(1)$. Therefore we achieved the regularization of the singularity in $\overline{P}_1$. We refer the reader to [5] for complete details.

7 Birkhoff transformation

Let us consider two bodies $\overline{P}_1$, $\overline{P}_2$ with masses $1 - \mu$, μ, moving on circular orbits around the barycenter O. Let us normalize to unity their distance. In the framework of the circular, restricted three–body problem, let us consider the motion of a third body S, moving in the same plane of the primaries. Let its coordinates be (x_1, x_2) in the synodic reference frame.
The Hamiltonian function governing the motion of S is given by

$$H(y_1, y_2, x_1, x_2) = \frac{1}{2}(y_1^2 + y_2^2) + x_2y_1 - x_1y_2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} ,$$

where

$$r_1 = \sqrt{(x_1 + \mu)^2 + x_2^2} , \qquad r_1 = \sqrt{(x_1 - 1 + \mu)^2 + x_2^2} .$$

We shift the origin of the reference frame to the midpoint between $\overline{P}_1$ and $\overline{P}_2$, by means of the complex transformation

$$q_1 + iq_2 = x_1 + ix_2 - \frac{1}{2} + \mu . \qquad (9)$$

Therefore, Figure 6).

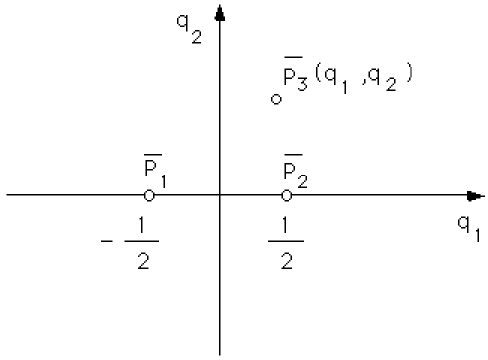

Figure 5: The coordinates of the three bodies after the transformation (9).

Let us write the change of coordinates (9) as

$$\begin{aligned} p_1 &= y_1 \qquad q_1 = x_1 - \frac{1}{2} + \mu \\ p_2 &= y_2 \qquad q_2 = x_2 \ ; \end{aligned}$$

we obtain the Hamiltonian function

$$H_1(p_1, p_2, q_1, q_2) = \frac{1}{2}({p_1}^2 + {p_2}^2) + q_2 p_1 - (q_1 + \frac{1}{2} - \mu)p_2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \ ,$$

where

$$r_1 = \sqrt{(q_1 + \frac{1}{2})^2 + {q_2}^2} \ , \qquad r_2 = \sqrt{(q_1 - \frac{1}{2})^2 + {q_2}^2} \ .$$

We remark that the singularities are now located at $\overline{P}_1(-\frac{1}{2}, 0)$, $\overline{P}_2(\frac{1}{2}, 0)$. The aim of the Birkhoff transformation will be to regularize simultaneously both collisions with $\overline{P}_1$ and $\overline{P}_2$. To this end, let us write the equations of motion as

$$\ddot{q} + 2i\dot{q} = \nabla_q U(q) \ , \tag{10}$$

where $q = q_1 + iq_2$. Denoting by C the Jacobi constant, one has

$$U(q) = \Omega(q) - \frac{C}{2} \ ,$$

where

$$\begin{aligned} \Omega(q) &= \frac{1}{2}[(1-\mu){r_1}^2 + \mu {r_2}^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \\ &= \frac{1}{2}[(1-\mu){r_1}^2 + \mu {r_2}^2] + \Omega_c(q) \ . \end{aligned}$$

Let us define $\Omega_c(q)$ as the *critical part* given by the expression

$$\Omega_c(q) \equiv \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

and let us write the Jacobi integral as

$$|\dot{q}|^2 = 2U(q) = 2\Omega(q) - C \ .$$

In order to determine the regularizing transformation, we perform a change of variables setting the complex parametric coordinates as $w = Q_1 + iQ_2$ and defining

$$q = h(w) = \alpha w + \frac{\beta}{w} \ ;$$

the unknown expressions for α and β must be determined in order to achieve the desired regularization.

We start by implementing a time transformation from the ordinary time t to a fictitious time s by means of the expression

$$\frac{dt}{ds} = g(w) \equiv |k(w)|^2 = k(w)\overline{k(w)} \ ,$$

where the function $g(w)$, or equivalently $k(w)$, must be suitably determined. One easily finds the following relations:

$$\begin{aligned} \dot{q} &= \frac{dq}{dt} = \frac{dh(w)}{dw}\frac{dw}{ds}\frac{ds}{dt} = h'(w)w'\dot{s} \\ \ddot{q} &= h'(w)w'\ddot{s} + (h''(w)w'^2 + h'(w)w'')\dot{s}^2 \\ \nabla_w U &= \overline{h'}\ \nabla_q U \ . \end{aligned}$$

As a consequence, the equations of motion (10) become

$$w'' + 2i\frac{w'}{\dot{s}} + w'\frac{\ddot{s}}{\dot{s}^2} + w'^2\frac{h''}{h'} = \nabla_w U \ \frac{1}{|h'|^2\dot{s}^2} \ .$$

Using the relations $\dot{s} = \frac{1}{g} = \frac{1}{k\bar{k}}$, $\ddot{s} = -\frac{\dot{g}}{g^2}$, one finds that $\frac{\ddot{s}}{\dot{s}^2} = -\dot{g}$. Moreover, from

$$-\dot{g} = \Big[k\frac{d\bar{k}}{dw}\frac{d\bar{w}}{ds} + \bar{k}\frac{dk}{dw}\frac{dw}{ds}\Big]\dot{s} = -\Big[\frac{\bar{k}'\bar{w}'}{\bar{k}} + \frac{k'w'}{k}\Big] \ ,$$

one obtains

$$w'' + 2ik\bar{k}w' - \frac{|w'|^2}{\bar{k}}\bar{k}' + w'^2(\frac{h''}{h'} - \frac{k'}{k}) = \frac{|k|^4}{|h'|^2}\ \nabla_w U \ .$$

Finally, we make use of the energy integral to obtain

$$|\dot{q}|^2 = 2\Omega(q) - C = 2U(q) = |h'|^2\ |w'|^2\frac{1}{|k|^4} \ ,$$

which implies that

$$|w'|^2 \ = 2U\ \frac{|k|^4}{|h'|^2} \ .$$

Since

$$\frac{h''}{h'} - \frac{k'}{k} = \frac{d}{dw}\left(\log\ \frac{h'}{k}\right) \ ,$$

we obtain

$$w'' + 2ik\bar{k}w' + |w'|^2 \ \frac{d}{dw}\Big(\log\frac{h'}{k}\Big) = \frac{|k|^4}{|h'|^2}\ \Big[2U\ \frac{d\log\bar{k}}{dw} + \nabla_w U\Big]\ .$$

A suitable choice for the functions k and h is provided by the relation

$$k \ = \ h'\ ,$$

from which it follows that (10) becomes

$$w'' + 2i|h'|^2 w' \ = \ \nabla_w\Big(|h'|^2 U\Big)\ .$$

Concerning the choice of α and β, we require that

i) the transformation involving h must eliminate both singularities;

ii) $\overline{P}_1$, $\overline{P}_2$ must stay fixed.

In order to meet the above requirements, we proceed as follows. Concerning statement *i*), we consider the singular term $\Omega_c(w)\ |h'(w)|^2$, where

$$\Omega_c(w) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} = \frac{1-\mu}{|\alpha w + \frac{\beta}{w} + \frac{1}{2}|} + \frac{\mu}{|\alpha w + \frac{\beta}{w} - \frac{1}{2}|}$$

and

$$|h'(w)|^2 = \frac{|\alpha w^2 - \beta|^2}{|w|^4}\ ,$$

namely

$$\Omega_c(w)\ |h'(w)|^2 = \frac{1}{|w|^3}\ \Big[\frac{(1-\mu)|\alpha w^2-\beta|^2}{|\alpha w^2+\beta+\frac{w}{2}|} + \frac{\mu|\alpha w^2-\beta|^2}{|\alpha w^2+\beta-\frac{w}{2}|}\Big]\ .$$

We remark that the singularity at $q = \frac{1}{2}$ corresponds to the solutions of $|\alpha w^2 + \beta - \frac{w}{2}| = 0$, which are given by

$$w_{1,2} = \frac{1}{4\alpha}\ \Big[1 \pm \sqrt{1-16\alpha\beta}\Big]\ .$$

Therefore, the roots of the numerator $|\alpha w^2 - \beta| = 0$ must coincide with $w_{1,2}$:

$$\frac{1}{4\alpha}\ \Big[1 \pm \sqrt{1-16\alpha\beta}\Big] = \pm\sqrt{\frac{\beta}{\alpha}}\ ,$$

i.e.

$$\alpha\beta\ (1-16\alpha\beta) = 0\ .$$

Since α and β are different from zero, it follows that

$$16\alpha\beta = 1\ ,$$

which implies that

$$w_{1,2} = \frac{1}{4\alpha}\ .$$

Concerning statement *ii*), since $\overline{P}_2(\frac{1}{2}, 0)$ is transformed to $\overline{P}_2(\frac{1}{4\alpha}, 0)$, one needs to require that $\frac{1}{4\alpha} = \frac{1}{2}$. Therefore, one finds

$$\alpha \ = \ \frac{1}{2} \qquad \& \qquad \beta = \frac{1}{8}\ .$$

In order to regularize the singularity at $\overline{P}_1$, one needs to repeat the above procedure, which leads to exactly the same results, namely $\alpha = \frac{1}{2}$, $\beta = \frac{1}{8}$.
Notice that the equations of motion contain also the singular term $\frac{1}{|w|^3}$; however the singularity $w = 0$ corresponds to $q \to \infty$, which does not have a physical meaning.

In summary, the regularization steps performed to achieve Birkhoff transformation are the following. Let

$$q = \frac{1}{2}(w + \frac{1}{4w}) \ , \qquad w = Q_1 + iQ_2 \ ,$$

namely

$$\begin{aligned} q_1 &= \frac{1}{2}\ (Q_1 + \frac{Q_1}{4(Q_1^2 + Q_2^2)}) \ \equiv \ f(Q_1, Q_2) \\ q_2 &= \frac{1}{2}\ (Q_2 - \frac{Q_2}{4(Q_1^2 + Q_2^2)}) \ \equiv \ g(Q_1, Q_2) \ . \end{aligned}$$

We perform a change of coordinates with generating function

$$W(p_1, p_2, Q_1, Q_2) = p_1 f(Q_1, Q_2) + p_2 g(Q_1, Q_2) \ .$$

Let us define the matrix A as

$$A = \begin{pmatrix} f_{Q_1} & g_{Q_1} \\ f_{Q_2} & g_{Q_2} \end{pmatrix} \ ;$$

whose determinant takes the form

$$\det(A) = \frac{1}{64(Q_1^2 + Q_2^2)^2}\ \left[16(Q_1^2 + Q_2^2)^2 + 1 + 8(Q_2^2 - Q_1^2)\right] \ .$$

We introduce a fictitious time as

$$dt \ = \ D\ ds \ , \qquad D = \det(A) \ .$$

The regularization is finally obtained considering the equations of motion in the extended phase space.

Appendix A: The extended phase space.

We discuss the introduction of the *extended phase space*. If the Hamiltonian function $\tilde{H} = \tilde{H}(P, Q, t)$ depends explicitely on the time, one can introduce a time–independent Hamiltonian, defined as

$$\Gamma = \Gamma(P, Q, T, t) = \tilde{H}(P, Q, t) + T \ ,$$

where T is conjugated to t. We show that Γ is identically zero along any solution.

In fact, if we select the initial conditions such that $T(0) = -\tilde{H}(P(0), Q(0), 0)$, one obtains that $T(t) = -\tilde{H}(t)$ along a solution for any t. To this end, we first notice that $\frac{d\tilde{H}}{dt} = \frac{\partial \tilde{H}}{\partial t}$, since by

Hamilton's equations:

$$\begin{aligned}\frac{d\tilde{H}}{dt} = \frac{d\tilde{H}(P,Q,t)}{dt} &= \frac{\partial\tilde{H}}{\partial t} + \frac{\partial\tilde{H}}{\partial Q}\frac{dQ}{dt} + \frac{\partial\tilde{H}}{\partial P}\frac{dP}{dt} \\ &= \frac{\partial\tilde{H}}{\partial t} + \frac{\partial\tilde{H}}{\partial Q}\frac{\partial\tilde{H}}{\partial P} - \frac{\partial\tilde{H}}{\partial P}\frac{\partial\tilde{H}}{\partial Q} = \frac{\partial\tilde{H}}{\partial t}\ .\end{aligned}$$

Therefore, by Hamilton's equations one gets

$$\frac{dT}{dt} = -\frac{\partial\tilde{H}}{\partial t} = -\frac{d\tilde{H}}{dt}\ ,$$

from which one obtains that

$$\begin{aligned}T(t) &= T(0) + \int_0^t \frac{dT(s)}{ds}ds = T(0) - \int_0^t \frac{d\tilde{H}(s)}{ds}ds \\ &= -\tilde{H}(0) - \int_0^t \frac{d\tilde{H}(s)}{ds}ds = -\tilde{H}(t)\ .\end{aligned}$$

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