Universality of Critical Behaviour in a Class of Recurrent Random Walks

O.Hryniv
Statistical Laboratory, DPMMS,
University of Cambridge, Cambridge CB3 0WB, UK
o.hryniv@statslab.cam.ac.uk

Y.Velenik
Laboratoire de Mathématiques Raphaël Salem
UMR-CNRS 6085, Université de Rouen
F-76821 Mont Saint Aignan
Yvan.Velenik@univ-rouen.fr

October 15, 2003

Abstract

Let $X_0 = 0, X_1, X_2, \dots$ be an aperiodic random walk generated by a sequence $\xi_1, \xi_2, \dots$ of i.i.d. integer-valued random variables with common distribution $p(\cdot)$ having zero mean and finite variance. For an N-step trajectory $\mathbb{X} = (X_0, X_1, \dots, X_N)$ and a monotone convex function $V : \mathbb{R}^+ \to \mathbb{R}^+$ with $V(0) = 0$, define $\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N-1} V\big(|X_j|\big)$. Further, let $\mathcal{I}_{N,+}^{a,b}$ be the set of all non-negative paths $\mathbb{X}$ compatible with the boundary conditions $X_0 = a$, $X_N = b$. We discuss asymptotic properties of $\mathbb{X} \in \mathcal{I}_{N,+}^{a,b}$ w.r.t. the probability distribution

$$\mathrm{P}_{N,+,\lambda}^{a,b}(\mathbb{X}) = \big(Z_{N,+,\lambda}^{a,b}\big)^{-1} \exp\Big\{-\lambda\,\mathbb{V}(\mathbb{X})\Big\} \prod_{i=0}^{N-1} p\,(X_{i+1} - X_i)$$

as $N \to \infty$ and $\lambda \to 0$, $Z_{N,+,\lambda}^{a,b}$ being the corresponding normalization.

If $V(\,\cdot\,)$ grows not faster than polynomially at infinity, define $H(\lambda)$ to be the unique solution to the equation

$$\lambda H^2\, V(H) = 1\,.$$

Our main result reads that as $\lambda \to 0$, the typical height of $X_{[\alpha N]}$ scales as $H(\lambda)$ and the correlations along $\mathbb{X}$ decay exponentially on the scale $H(\lambda)^2$. Using a suitable blocking argument, we show that the distribution tails of the rescaled height decay exponentially with critical exponent $3/2$. In the particular case of linear potential $V(\,\cdot\,)$, the characteristic length $H(\lambda)$ is proportional to $\lambda^{-1/3}$ as $\lambda \to 0$.

1 Introduction

In this work, we are interested in the path-wise behaviour of a general class of random walks on the integers, whose path measure is submitted to a special form of exponential perturbation, the physical motivation of which is discussed at the end of this section. More precisely, to each $i \in \mathbb{Z}$, we associate an integer[1] non-negative value X_i and for any integer interval

$$\Delta_{l,r} = (l,r) \equiv \bigl\{ l+1, l+2, \ldots, r-1 \bigr\} \subset \mathbb{Z} \tag{1.1}$$

we denote by $\mathcal{I}_{\Delta_{l,r},+}$ the set of all such trajectories in $\Delta_{l,r}$:

$$\mathcal{I}_{\Delta_{l,r},+} = \mathcal{I}_{(l,r),+} = \Bigl\{ \mathbb{X} = (X_i)_{l<i<r} : X_i \geq 0 \Bigr\}.$$

Let $V : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex increasing continuous function with $V(0) = 0$ and a bounded growth at infinity:

There exists $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\alpha > 0$ we have

$$\limsup_{x\to\infty} \frac{V(\alpha x)}{V(x)} \leq f(\alpha) < \infty. \tag{1.2}$$

This property holds clearly for any (convex) polynomial function.

The probability of a trajectory $\mathbb{X} \in \mathcal{I}_{(l,r),+}$ is defined then via

$$\mathrm{P}^{a,b}_{(l,r),+,\lambda}(\mathbb{X}) = \bigl(Z^{a,b}_{(l,r),+,\lambda}\bigr)^{-1} \exp\Bigl\{ -\lambda \sum_{i=l+1}^{r-1} V\bigl(X_i\bigr) \Bigr\} \prod_{i=l}^{r-1} p(X_{i+1} - X_i)\,, \tag{1.3}$$

where the boundary conditions are given by $X_l = a$ and $X_r = b$, the parameter λ is some strictly positive real number and $p(\,\cdot\,)$ are the transition probabilities of a 1D integer-valued random walk with zero mean and finite second moment. We suppose that the random walk is *strictly aperiodic* in the sense that its n-step transition probabilities $p^n(\,\cdot\,)$ possess the following property:

there is $A > 0$ such that

$$\min\Bigl\{ p^n(-1), p^n(0), p^n(1) \Bigr\} > 0 \quad \textit{for all } n \geq A. \tag{1.4}$$

When the boundary conditions are chosen such that $a = b = 0$, we omit them from the notation. We will denote by $\mathrm{P}^{a,b}_{\Delta_{l,r},\lambda}$ the analogous probability measure without the positivity constraint, and by $Z^{a,b}_{\Delta_{l,r},\lambda}$ the associated partition function. The special case of $\Delta_{0,N} = (0,N)$ will be abbreviated to

$$\mathcal{I}_{N,+} = \Bigl\{ \mathbb{X} = (X_i)_{1\leq i\leq N-1} : X_i \geq 0 \Bigr\} \tag{1.5}$$

[1] Although in the sequel we'll discuss mainly integer-valued one-dimensional random walks, analogous results for real-valued walks can be obtained in a similar way.

and

$$\mathrm{P}^{a,b}_{N,+,\lambda}(\mathbb{X}) = \left(Z^{a,b}_{N,+,\lambda}\right)^{-1} \exp\Bigl\{-\lambda \sum_{i=1}^{N-1} V\bigl(X_i\bigr)\Bigr\} \prod_{i=0}^{N-1} p(X_{i+1}-X_i) \tag{1.6}$$

respectively.

When $\lambda = 0$ it is expected that, after a suitable rescaling, the law of the random walk converges to that of a Brownian excursion; at least, it is a direct corollary ot the results in [4] and [7] that, in the diffusive scaling, the large-N limit of the distribution $\mathrm{P}_{N,+,0}(\,\cdot\,)$ is non-trivial. In particular, the path delocalizes. When $\lambda > 0$, the behaviour changes drastically: We'll prove below that the path remains localized, and that the correlations between positions X_i and X_j of the random walk decay exponentially with their separation $|i-j|$. Our main goal is to investigate how delocalization occurs as λ decreases to 0. The corresponding critical behaviour can be analyzed in quite some details and, most interestingly, under very weak assumptions on the original random walk. In this way, it is possible to probe its degree of universality.

The physical motivation for the path measure considered in this work is the phenomenon of critical prewetting. Consider a vessel containing the thermodynamically stable gaseous phase of some substance. When the boundary of the vessel displays a sufficiently strong preference towards the thermodynamically unstable liquid phase, there may be creation of a microscopic film of liquid phase coating the walls. As the system is brought closer and closer to liquid/gas phase coexistence, the layer of unstable phase starts to grow. For systems with short-range interactions, two kind of behaviours are possible: either there is an infinite sequence of first-order (so-called layering) phase transitions, at which the thickness increases by one mono-layer, or the growth occurs continuously; this is the case of critical prewetting, and it is typical in two-dimensional systems, as those modelled in the present work. We are thus interested in quantifying the growth as a function of the distance to phase coexistence. A natural parameter is the difference between the free energy densities of the stable and unstable phases. Choosing $V(x) = |x|$, we see that the perturbation $\lambda \sum_i V(X_i)$ can be interpreted as the total excess free energy associated to the unstable layer, the parameter λ playing the role of the excess free energy density.

The problem of critical prewetting in continuous effective interface models in higher dimensions, as well as in the 2D Ising model, has been considered in [11]. The latter results are however restricted to a much smaller class of interactions, and take also a weaker form than those we obtain here. Notice however that the thickness of the layer of the unstable phase in the 2D Ising model also grows with exponent 1/3, showing (not surprisingly) that this model is in the same universality class as those considered in the present work; one can hope therefore that the finer estimates we obtain here have similar counterparts in the 2D Ising model. Results about critical prewetting for one-dimensional effective interface models have already been obtained in [1]; they are limited to the particular case of real-valued random walks with $V(x)=|x|$ and $p(x)\propto e^{-\beta|x|}$, which turn out to be exactly solvable. They are thus able to extract more precise information than those we present here, including prefactors. It is not clear however to what extent these finer properties also have universal significance.

Acknowledgements. It is a pleasure to thank H.Spohn for his interest in this work. O.H. also thanks L.C.G.Rogers, S.Molchanov and G.BenArous for stimulating discussions.

1.1 Main results

We recall that the potential function $V(\,\cdot\,)$ in (1.6) is assumed to be continuous and increasing from 0 to ∞ as x varies from 0 to ∞. Therefore, for any $\gamma>0$ there is a unique solution $H_\gamma>0$ to the equation

$$\lambda\, H^2\, V(2\gamma H)\,|_{H=H_\gamma}=1. \tag{1.7}$$

The scale $H_\gamma=H_\gamma(\lambda)$ will play an important role in our future considerations. As a simple corollary of the definition (1.7), note that $\gamma^{1/3}H_\gamma$ is a non-increasing function of γ; indeed, thanks to convexity and monotonicity of $V(\,\cdot\,)$, for any $0<\gamma_1<\gamma_2$ and any $0<a<1$ we get:

$$V(2\gamma_1 H_{\gamma_1})H_{\gamma_1}^2\equiv V(2\gamma_2 H_{\gamma_2})H_{\gamma_2}^2=\frac{1}{\lambda}\geq V\bigl(2a^3\gamma_2 H_{\gamma_2}/a\bigr)\bigl(H_{\gamma_2}/a\bigr)^2;$$

now take a satisfying $a^3\gamma_2=\gamma_1$ and recall monotonicity of the function $x\mapsto V(2\gamma_1 x)x^2$ to infer $aH_{\gamma_1}\geq H_{\gamma_2}$.

Our first result says that the "average height" of the interface in the limit of small λ is of order $H_1(\lambda)$.

Theorem 1.1 *Let $H=H_1(\lambda)$ be as defined in (1.7). There exist positive constants δ_0, λ_0, and C_1, C_2 such that the inequalities*

$$\mathrm{P}_{N,+,\lambda}\Bigl(\sum_{i=1}^N X_i\geq\delta^{-1}HN\Bigr)\leq\frac{1}{C_1}e^{-C_1\delta^{-1}H^{-2}N}\,, \tag{1.8}$$

$$\mathrm{P}_{N,+,\lambda}\Bigl(\sum_{i=1}^N X_i\leq\delta\, HN\Bigr)\leq\frac{1}{C_2}e^{-C_2\delta^{-2}H^{-2}N} \tag{1.9}$$

hold uniformly in $0<\lambda<\lambda_0$, $0<\delta\leq\delta_0$, and $N\geq H^2$.

Our next result describes the tails of the point-wise height distribution. Although it is formulated for the height in the middle of a typical interface, the result holds for all points from $[AH^2,N-AH^2]$, with any fixed $A>0$, ie., lying sufficiently deep in $[0,N]$; then, clearly, $c_i=c_i(A)\to 0$, $i=1,2$, as $A\to 0$.

Theorem 1.2 *Let $H = H_1(\lambda)$ be as defined in (1.7). There exist positive constants T_0, K, c_1 and c_2 such that for any $T \geq T_0$ and all $N \geq KH^2$ the inequalities*

$$\frac{1}{c_1}\exp\Big\{-c_1 T^{3/2}\Big\} \leq \mathrm{P}_{N,+,\lambda}\Big(X_{[N/2]} > TH_1\Big) \leq \frac{1}{c_2}\exp\Big\{-c_2 T^{3/2}\Big\} \qquad (1.10)$$

hold for all $\lambda \in (0, \lambda_0]$, where $\lambda_0 = \lambda_0(T) > 0$. In particular, these estimates are uniform on compact subsets of $[0, \infty)$.

Remark 1.3 *Tail estimates for small λ uniform in T large enough can be obtained taking into account the tail behaviour of the original random-walk. For example, a variant of the argument above, actually even simpler, proves that for a Gaussian random walk, i.e. $p(x) \propto e^{-c|x|^2}$, the tail behaviour proved above holds with the same exponent for all $\lambda \leq \lambda_0$ uniformly in large T. This behaviour is not universal however, as different behaviour of $p(\,\cdot\,)$ can give rise to completely different tails.*

Further, we describe the decay of correlations along the interface. Here again the horizontal scale H^2 plays an important role.

Theorem 1.4 *Let $H = H_1(\lambda)$ be as defined in (1.7). There exist positive constants C, c, and λ_0 such that for every $\lambda \in (0, \lambda_0]$ and all i, $j \in (0, N)$ we have*

$$\mathsf{Cov}(X_i, X_j) \leq CH^{5/2}\exp\big\{-c\,|i-j|\,H^{-2}\big\}.$$

Thinking of N as of time parameter, our system can be described as a Markov chain on the positive half-line with certain attraction to the origin. Being positive recurrent, its distribution μ_N^λ at time N approaches its stationary distribution π_λ exponentially fast on the horizontal time scale H^2:

Theorem 1.5 *Let $H = H_1(\lambda)$ be as defined in (1.7). There exist positive constants C, c, and λ_0 such that for every $\lambda \in (0, \lambda_0]$ we have*

$$\big\|\mu_N^\lambda - \pi_\lambda\big\|_{\mathsf{TV}} \leq CH^2\exp\big\{-cN\,H^{-2}\big\},$$

where $\|\cdot\|_{\mathsf{TV}}$ denotes the total variational distance between the probability measures.

Remark 1.6 *The reader might wonder whether the appearance of the exponents $1/3$, $2/3$ and $3/2$ in the case $V(x) = |x|$ hints at a relationship between the critical behaviour of the model considered here and the much-studied Tracy-Widom distribution. We do not have a precise answer to this question; however at a heuristic level, the appearance of the same critical exponents can be understood by noticing the similarities between our model and the multi-layer PNG model introduced in [10], whose relation with the Tracy-Widom distribution has been studied in the latter work.*

2 Proof of Theorem 1.1

2.1 A basic comparison of partition functions

Lemma 2.1 *For any fixed $\rho > 0$ and $H_1(\lambda)$ defined as in (1.7), put*

$$H = \rho H_1(\lambda). \tag{2.1}$$

Then there exist positive constants λ_0, c, and C such that, for any $0 < \lambda \le \lambda_0$, every $N \ge H^2$, and all boundary conditions $0 \le a, b \le H$, one has

$$c\, e^{-CNH^{-2}} e^{-\lambda V(2H)N} Z^{a,b}_{N,+,0} \le Z^{a,b}_{N,+,\lambda} \le Z^{a,b}_{N,+,0}\,. \tag{2.2}$$

Remark 2.2 *Observe that*

$$\frac{Z^{a,b}_{N,+,\lambda}}{Z^{a,b}_{N,+,0}} = \mathrm{E}^{a,b}_{N,+,0}\Big[\exp\Big\{-\lambda \sum_{j=1}^{N-1} V\big(X_j\big)\Big\}\Big] \tag{2.3}$$

and thus the lemma states that for any $\lambda > 0$ the exponential moment of the functional

$$\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N-1} V\big(X_j\big)$$

decays no faster than exponentially in N indicating that the typical value of $V\big(X_j\big)$ (equivalently, the height X_j of the interface) is "bounded on average". It is instructive to compare this property to the asymptotics

$$\mathrm{E}^{a,b}_{N,+,0}\Big[N^{-3/2} \sum_{j=1}^{N} X_j\Big] \to \mathsf{const} > 0$$

following from [4, 9].

As a straightforward application of the lemma above we get the following simple but quite useful fact.

Corollary 2.3 *Under the conditions of Lemma 2.1, for any collection $\mathcal{A} \subset \mathcal{I}_{N,+}$ of trajectories we have:*

$$\mathrm{P}^{a,b}_{N,+,\lambda}\big(\mathcal{A}\big) \le \frac{1}{c} \exp\Big\{CNH^{-2} + \lambda V(2H)N\Big\} \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}\big). \tag{2.4}$$

Similarly, for any $\varepsilon > 0$, we get

$$\begin{aligned} &\mathrm{P}^{a,b}_{N,+,\lambda}\Big(\mathcal{A}\,; \sum_{i=1}^{N} X_i \ge \varepsilon HN\Big) \\ &\qquad \le \frac{1}{c} \exp\Big\{CNH^{-2} + \lambda V(2H)N - \lambda V(\varepsilon H)N\Big\} \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}\big) \end{aligned} \tag{2.5}$$

and for any constants A, $B > 0$,

$$\begin{aligned} \mathrm{P}^{a,b}_{N,+,\lambda}\Big(\mathcal{A}\,;\mathbb{V}(\mathbb{X}) \ge AV(BH)N\Big) \\ \le \frac{1}{c}\exp\Big\{CNH^{-2} + \lambda V(2H)N - \lambda AV(BH)N\Big\}\mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}\big). \end{aligned} \tag{2.6}$$

Proof. Indeed, (2.5) follows immediately from convexity of $V(\,\cdot\,)$, as then

$$\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N} V\big(X_j\big) \ge NV\Big(\sum_{j=1}^{N} X_j/N\Big) \ge NV\big(\varepsilon H\big).$$

Other inequalities are obvious. ■

Proof of Lemma 2.1. As the upper bound is obvious, one only needs to check the left inequality.

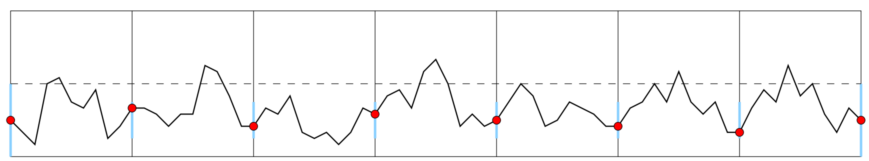

Figure 1: The event in the proof of Lemma 2.1.

We use a renormalisation argument. With H defined as in (2.1), take positive ε small enough to satisfy $\varepsilon \in (0, 1/4]$ and cut each trajectory of our RW into blocks of length[2] $\Delta = \varepsilon H^2$; this generates $n_\varepsilon = \big[N/\varepsilon H^2\big] \ge 4$ such blocks (if there is a shorter piece left, we attach it to the last block). Further, denote $n_1 = \Delta$, $n_2 = (n_\varepsilon - 1)\Delta$ and consider the events

$$\mathcal{A} = \Big\{{}^\forall j = 2, \dots, n_\varepsilon - 2 : \frac{1}{4}H \le X_{j\Delta} \le \frac{3}{4}H\Big\},$$

$$\mathcal{B} = \Big\{{}^\forall j = n_1 + 1, \dots, n_2 - 1 : 0 \le X_j \le 2H\Big\},$$

$$\mathcal{C}_1 = \Big\{{}^\forall j = 1, \dots, n_1 - 1 : 0 \le X_j \le 2H\Big\} \cap \Big\{\frac{1}{4}H \le X_{n_1} \le \frac{3}{4}H\Big\},$$

$$\mathcal{C}_2 = \Big\{{}^\forall j = n_2 + 1, \dots, N - 1 : 0 \le X_j \le 2H\Big\} \cap \Big\{\frac{1}{4}H \le X_{n_2} \le \frac{3}{4}H\Big\}.$$

Using (2.3), we immediately get

$$\frac{Z^{a,b}_{N,+,\lambda}}{Z^{a,b}_{N,+,0}} \ge \exp\Big\{-\lambda V\big(2H\big)N\Big\}\,\mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_1 \cap \mathcal{C}_2\big) \tag{2.7}$$

[2] both H and Δ are assumed to be integer

and it remains to bound below the probability of $\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}_1\cap\mathcal{C}_2$.

First, observe that in view of the Donsker invariance principle we have

$$\min_{H/4\le x\le 3H/4} \mathrm{P}^{x}_{\Delta,+}\bigl(H/4\le X_\Delta\le 3H/4\bigr)\ge C_1\,,$$

$$\min_{H/4\le c,d\le 3H/4} \mathrm{P}^{c,d}_{\Delta,+,0}\bigl(\forall j=1,\dots,\Delta-1: 0\le X_j\le 2H\bigr)\ge C_2\,.$$

By the Markov property we thus estimate

$$\mathrm{P}^{a,b}_{N,+,0}\bigl(\,\mathcal{A}\cap\mathcal{B}\mid\mathcal{C}_1\cap\mathcal{C}_2\,\bigr)\ge\bigl(C_1C_2\bigr)^{n_\varepsilon-2}\ge c\exp\bigl\{-Cn_\varepsilon\bigr\}$$

with some c, $C>0$. On the other hand, by (conditional) independence of $\mathcal{C}_1$ and $\mathcal{C}_2$ it is sufficient to bound below the probabilities $\mathrm{P}^{a,b}_{N,+,0}\bigl(\,\mathcal{C}_1\,\bigr)$ and $\mathrm{P}^{a,b}_{N,+,0}\bigl(\,\mathcal{C}_2\,\bigr)$. We shall estimate the former, the latter will follow in a similar way.

Combining the argument above with the conditional invariance principle due to Bolthausen [4] we get

$$\min_{a\in[H/4,H]\cup\{0\}} \mathrm{P}^{a}_{\Delta,+}\bigl(\,\mathcal{C}_1\,\bigr)\ge C_3>0 \tag{2.8}$$

with some constant $C_3=C_3(\varepsilon)$, uniformly in $\lambda>0$ small enough. Observe that the last bound holds also for any $\widetilde\Delta$, $\Delta/2\le\widetilde\Delta\le\Delta$, and perhaps smaller $C_3>0$. Consequently, it is enough to show that for some constant $C_4=C_4(\varepsilon)>0$ we have, uniformly in sufficiently small $\lambda>0$,

$$\min_{0<a<H/4} \mathrm{P}^{a}_{\Delta,+}\Bigl(\,\min\Bigl\{j\ge 0: X_j\in\bigl[H/4,H\bigr]\Bigr\}\le\frac{\Delta}{4}\,\Bigr)\ge C_4>0 \tag{2.9}$$

as then immediately

$$\min_{0<a<H/4} \mathrm{P}^{a}_{\Delta,+}\bigl(\,\mathcal{C}_1\,\bigr)\ge C_3\,C_4>0\,. \tag{2.10}$$

To check (2.9), we fix an arbitrary integer a, $0<a<H/4$, and consider two independent trajectories X^a and X^0 distributed according to $\mathrm{P}^{a}_{\Delta,+}(\,\cdot\,)$ and $\mathrm{P}^{0}_{\Delta,+}(\,\cdot\,)$ respectively. Let $\mathcal{D}=\mathcal{D}(k)$ denote the "crossing event" at k,

$$\mathcal{D}(k)=\Bigl\{\,{}^\forall j=0,1,\dots,k-1: X^a_j>X^0_j\qquad\text{and}\qquad X^a_k\le X^0_k\,\Bigr\},$$

and denote $\mathcal{D}_\Delta=\cup_{k=1}^{n_3}\mathcal{D}(k)$, $n_3=[\Delta/4]$. Our aim is to show that there exists a positive constant $C_5=C_5(\varepsilon)$ such that, uniformly in small enough $\lambda>0$, one has (here and below, $\mathcal{D}^c_\Delta$ stands for the complement of $\mathcal{D}_\Delta$)

$$\mathrm{P}^{a}_{\Delta,+}\bigl(\,\mathcal{C}_1\bigm|\mathcal{D}_\Delta\,\bigr)\ge C_5\,,\qquad \mathrm{P}^{a}_{\Delta,+}\bigl(\,\mathcal{C}_1\bigm|\mathcal{D}^c_\Delta\,\bigr)\ge C_5\,. \tag{2.11}$$

Then the target inequality

$$\mathrm{P}^{a,b}_{N,+,0}\bigl(\,\mathcal{C}_1\,\bigr)\ge C_6(\varepsilon)>0$$

follows immediately.

The key observation towards (2.11) is the following. For an integer $x \ge 0$, the jump distribution $p_x(\,\cdot\,)$ of our random walk from x is given by

$$p_x(k) = \frac{p(k)}{\mathrm{P}(\xi \ge -x)}\,\mathbb{1}_{\{k \ge -x\}}\,,$$

where ξ is a random variable with the unconstrained jump distribution $p(\,\cdot\,)$. Clearly, the mean e_x and the variance σ_x^2 of $p_x(\,\cdot\,)$ satisfy

$$\begin{aligned} e_x &= \sum_k k p_x(k) \searrow e_\infty \equiv 0 \qquad \text{as } x \nearrow \infty \\ \sigma_x^2 &= \sum_k k^2 p_x(k) - \bigl(e_x\bigr)^2 \le \frac{\sigma^2}{\mathrm{P}(\xi \ge -x)} \end{aligned} \tag{2.12}$$

with σ^2 denoting the variance of $p(\,\cdot\,)$.

Now, suppose that the crossing event $\mathcal{D}(k)$, $k \in \{1, 2, \dots, n_3\}$, takes place. In view of (2.12), with positive probability we have $X_k^0 - X_k^a \le M$, where the constant M is independent of λ. Thanks to the analogue of the aperiodicity property (1.4) for the distributions $p_x(\,\cdot\,)$ and $p_y(\,\cdot\,)$, where $x = X_k^a$ and $y = X_k^0$ (with the same lower bound (1.4) for all $x, y \ge 0$), two independent trajectories started at x and y meet with positive probability within AM steps. Thus, the first inequality in (2.11) follows immediately from the standard independent coupling and the properties of $\mathrm{P}^0_{\Delta,+}(\,\cdot\,)$ mentioned above.

If $\mathcal{D}^c_\Delta$ takes place, we have $X_j^a > X_j^0$ for all $j = 0, 1, \dots, n_3$. Consequently, with positive probability the stopping time

$$\tau = \min\bigl\{j \ge 0 : X_j^a \ge H/4\bigr\}$$

satisfies $\tau \le n_3$. Then the finite variance argument used above implies that, with positive probability, we have $H/4 \le X_\tau^a \le H$ and the second inequality in (2.11) follows from a straightforward generalization of (2.8). ∎

A literal repetition of the argument above implies also the following result:

Corollary 2.4 *For positive ρ and λ, put $H = 2\rho H_1(\lambda)$ and define the event*

$$\mathcal{B} = \mathcal{B}_{H,N} = \Bigl\{{}^\forall j = 1, \dots, N : 0 \le X_j \le 2H\Bigr\}\,. \tag{2.13}$$

Then, for any ρ, $\eta > 0$, there exist positive constants λ_0, c, and C such that for all $0 < \lambda \le \lambda_0$, $N \ge \eta H^2$ and $0 \le a, b \le H$, we have

$$\mathrm{P}^{a,b}_{N,+,0}\bigl(\mathcal{B}\bigr) \ge c \exp\bigl\{-CNH^{-2}\bigr\}. \tag{2.14}$$

Moreover, for any other event $\mathcal{A}$ with $\mathrm{P}^{a,b}_{N,+,0}\bigl(\mathcal{A} \mid \mathcal{B}\bigr) > 0$, we get

$$c\, e^{-CNH^{-2} - 2\lambda V(2H)N} \le \frac{\mathrm{P}^{a,b}_{N,+,\lambda}\bigl(\mathcal{A} \mid \mathcal{B}\bigr)}{\mathrm{P}^{a,b}_{N,+,0}\bigl(\mathcal{A} \mid \mathcal{B}\bigr)} \le c^{-1}\, e^{CNH^{-2} + 2\lambda V(2H)N}\,, \tag{2.15}$$

and thus, uniformly in bounded NH^{-2}, both conditional probabilities are positive simultaneously.

Proof. Since $V(\,\cdot\,)$ is non-negative and monotone, (2.13) implies that, for any event $\mathcal{A}$,

$$e^{-\lambda V(2H)N} Z^{a,b}_{N,+,0}\, \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}\mathcal{B}\big) \le Z^{a,b}_{N,+,\lambda} \mathrm{P}^{a,b}_{N,+,\lambda}\, \big(\mathcal{A}\mathcal{B}\big) \le Z^{a,b}_{N,+,0} \mathrm{P}^{a,b}_{N,+,0}\, \big(\mathcal{A}\mathcal{B}\big)\,.$$

The inequality (2.15) now follows immediately from Lemma 2.1. ∎

Remark 2.5 *Although the importance of the scale $H = H_1$ (see (1.7)) should be clear from the proofs above, it is instructive to give another motivation for the definition (1.7). Clearly, each interface under consideration can be naturally decomposed into elementary excursions above the wall. Without external field (ie., with $\lambda = 0$) each such excursion of horizontal length l^2 has typical height of order l. For $\lambda > 0$ its energetic price is of order at most $\lambda l^2 V(l)$ and thus the interaction with the field is negligible if $\lambda l^2 V(l) \ll 1$, that is if $l = o(H_1)$. In other words, the presence of the field λ is felt on the (vertical) scale H_1 or larger.*

2.2 The upper bound

The first half of Theorem 1.1, namely

$$\mathrm{P}_{N,+,\lambda}\Big[N^{-1}\sum_{i=1}^{N} X_i \ge \delta^{-1}H\Big] \le \frac{1}{c}\, e^{-C\delta^{-1}NH^{-2}},$$

(with $H = H_1(\lambda)$, see (1.7)) follows directly from (2.5) and the inequality

$$V(\delta^{-1}H) \ge (2\delta)^{-1}V(2H)$$

valid for any $H \ge 0$ and $0 < 2\delta \le 1$:

$$\mathrm{P}_{N,+,\lambda}\Big[N^{-1}\sum_{i=1}^{N} X_i \ge \delta^{-1}H\Big] \le \frac{1}{c}\, e^{CNH^{-2}-((2\delta)^{-1}-1)\lambda V(2H)N} \le \frac{1}{c}\, e^{-C_\delta NH^{-2}};$$

here $C_\delta = 1/(4\delta)$ and δ is chosen small enough to satisfy

$$0 < \delta \le \min\Big(1, \frac{1}{4(C+1)}\Big)\,.$$

∎

2.3 The lower bound

Our proof of the lower bound (with $H = H_1(\lambda)$, see (1.7)),

$$\mathrm{P}_{N,+,\lambda}\Big[N^{-1}\sum_{i=1}^{N} X_i \le \delta H\Big] \le c\, e^{-C\delta^{-2}NH^{-2}}, \tag{2.16}$$

is based upon a certain renormalisation procedure. Namely, take $\lambda > 0$ small enough and $\varepsilon > 0$ to be chosen later (assuming, without loss of generality, that $\varepsilon^2 \in (0, 1/12)$ and $\varepsilon^2 H^2$ is an integer number larger than 1) and split every trajectory of the random walk under consideration into pieces of length $4\,\varepsilon^2 H^2$ to be called blocks; clearly, there are exactly $n_\varepsilon = \big[N/(4\,\varepsilon^2 H^2)\big]$ such blocks (and, perhaps, an additional piece of shorter length). Next, we split each block into four equal parts and use $\mathcal{I}_m$, $m = 1, \dots, 4n_\varepsilon$ to denote all obtained sub-blocks. Further, we fix a small enough $\rho > 0$ and say that the trajectory under consideration is *ρ-high* in the k-th block, if

$$\max_{j\in\mathcal{I}_{4k+2}} X_j > \rho\,\varepsilon H, \qquad \max_{j\in\mathcal{I}_{4k+4}} X_j > \rho\,\varepsilon H\,.$$

The main idea behind the argument below is as follows: for $\rho > 0$ small enough, the number of ρ-high blocks in a typical trajectory is of order n_ε; however, a typical contribution of a ρ-high block to the total area is of order at least $\rho\,\varepsilon^3 H^3$; as a result, the typical area is bounded below by a quantity of order at least $\rho\,\varepsilon^3 H^3 n_\varepsilon \asymp \rho\,\varepsilon HN$ and thus, for $\delta > 0$ small enough, the event

$$\mathcal{A}_\delta \equiv \Big\{\sum_{i=1}^{N} X_i \le \delta HN\Big\}$$

falls into the large deviation region for the distribution under consideration. The target inequality (2.16) gives a quantitative estimate for this to happen.

To start, we use (2.4) to remove the external field λ,

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\big[\,\mathcal{A}_\delta\,\big] &\le c^{-1}\exp\Big\{CNH^{-2} + \lambda V(2H)N\Big\}\mathrm{P}_{N,+,0}\big[\,\mathcal{A}_\delta\,\big] \\ &= c^{-1}\exp\Big\{(C+1)NH^{-2}\Big\}\mathrm{P}_{N,+,0}\big[\,\mathcal{A}_\delta\,\big]\,.\end{aligned}$$

Now, conditionally on the configuration $\mathbb{X}$ in the blocks $\mathcal{I}_{2m+1}$, $m = 0, \dots, n_\varepsilon - 1$, the events

$$\Big\{\max_{j\in\mathcal{I}_{2m}} X_j > \rho\,\varepsilon H\Big\}$$

are mutually independent. Moreover, a straightforward generalization of the argument used to estimate below the probability of $\mathcal{C}_1$ in Lemma 2.1 shows that

$$a_\rho \equiv \sup_m \sup_{0\le a,b\le\rho\,\varepsilon H} \mathrm{P}^{a,b}_{\mathcal{I}_{2m},+,0}\Big(\max_{j\in\mathcal{I}_{2m}} X_j \le \rho\varepsilon H\Big) \le 1-\eta$$

with some $\eta > 0$, uniformly in $0 < \rho \le \rho_0$ and all $\varepsilon H = \varepsilon H_1(\lambda)$ large enough, implying that the events

$$\Big\{ \; k\text{-th block is } \rho\text{-high} \; \Big\} ,$$

which are also conditionally independent for fixed configuration in $\mathcal{I}_{4k+1}$, $k = 0, \ldots, n_\varepsilon - 1$, occur with probability at least $(1 - a_\rho)^2 \ge \eta^2$. As a result, the number n_ρ of ρ-high blocks for a typical trajectory is not less than $\eta^2 n_\varepsilon/2$. More precisely, since the events under consideration are independent for individual blocks (conditionally on every fixed configuration in between), the standard large deviation bound implies

$$\mathrm{P}_{N,+,0}\Big(n_\rho < \frac{\eta^2}{2} n_\varepsilon \Big) \le \exp\big\{ -c n_\varepsilon \big\} = \exp\big\{ -c' \varepsilon^{-2} N H^{-2} \big\}$$

with some $c' = c'(\rho) > 0$ not depending on ε. Thus, taking $\varepsilon > 0$ small enough, we obtain

$$\mathrm{P}_{N,+,\lambda}\Big(n_\rho < \frac{n_\varepsilon}{4} \Big) \le \exp\big\{ -c_1 \varepsilon^{-2} N H^{-2} \big\}.$$

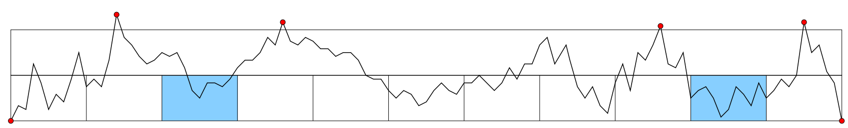

Figure 2: Two ρ-high blocks with oscillation

From now on, we shall restrict ourselves to the trajectories containing at least $\eta^2 n_\varepsilon/2$ blocks that are ρ-high. We shall say that a ρ-high block oscillates if

$$\min_{j \in \mathcal{I}_{4k+3}} X_j < \frac{\rho\,\varepsilon}{2} H$$

and observe that each ρ-high block without oscillation contributes an amount at least $\rho\,\varepsilon^3 H^3/2$ to the total area. Our final step of the proof consists in evaluating the typical amount of oscillating blocks.

Define

$$l_k = \min\Big\{ j \in \mathcal{I}_{4k+2} : X_j > \rho\,\varepsilon H \Big\}, \quad r_k = \max\Big\{ j \in \mathcal{I}_{4k+4} : X_j > \rho\,\varepsilon H \Big\}$$

and put $a_k = X_{l_k}$, $b_k = X_{r_k}$, $L = 4k\varepsilon^2 H^2$, $R = 4(k+1)\varepsilon^2 H^2$. Then using essentially the same arguments as in the proof of Lemma 2.1, we deduce that

$$\min_{a,b} \mathrm{P}^{a,b}_{(L,R),+,0}\big(\; k\text{-th block does not oscillate} \; \big)$$
$$\ge \min \mathrm{P}^{a_k,b_k}_{r_k - l_k,+,0}\Big(\min_j X_j > \rho\,\varepsilon H/2 \Big) \ge 1 - \bar{a}_\rho \, ,$$

where the bound $\bar{a}_\rho < 1$ holds for any fixed ρ small enough. Arguing as before, we deduce that the number $\bar{n}_{\mathsf{osc}}$ of blocks without oscillation satisfies the estimate

$$\mathrm{P}_{N,+,0}\Big(\,\bar{n}_{\mathsf{osc}} < \frac{1-\bar{a}_\rho}{2} n_\rho\Big) \le \exp\big\{\,-c'' n_\rho\,\big\} = \exp\big\{\,-c''' \varepsilon^{-2} N H^{-2}\,\big\}$$

with some $c''' = c'''(\rho) > 0$ not depending on ε. Again, taking $\varepsilon > 0$ small enough, we obtain

$$\mathrm{P}_{N,+,\lambda}\Big(\,\bar{n}_{\mathsf{osc}} < \frac{1-\bar{a}_\rho}{2} n_\rho\Big) \le \exp\big\{\,-c_2 \varepsilon^{-2} N H^{-2}\,\big\}.$$

However, on the complementary event we get

$$\bar{n}_{\mathsf{osc}} \ge \frac{1-\bar{a}_\rho}{2}\, n_\rho \ge \frac{1-\bar{a}_\rho}{4}\, \eta^2\, n_\varepsilon$$

for all $\rho > 0$ small enough and thus the inequality

$$\sum_{i=1}^{N} X_i \ge \frac{\rho\, \varepsilon^3 H^3}{2} \bar{n}_{\mathsf{osc}} > \frac{\rho(1-\bar{a}_\rho)}{64}\, \eta^2 \varepsilon H N$$

renders the event $\mathcal{A}_\delta$ impossible for $\delta = \rho(1-\bar{a}_\rho)\eta^2\varepsilon/64$. As a result, for such $\delta > 0$ we get

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\big[\,\mathcal{A}_\delta\,\big] &\le \mathrm{P}_{N,+,\lambda}\Big(\,n_\rho < \frac{\eta^2 n_\varepsilon}{2}\Big) + \mathrm{P}_{N,+,\lambda}\Big(\,\bar{n}_{\mathsf{osc}} < \frac{1-\bar{a}_\rho}{2} n_\rho\Big) \\ &\le 2\exp\big\{\,-c_3 \delta^{-2} N H^{-2}\,\big\}\end{aligned}$$

with some $c_3 = c_3(\rho) > 0$. ■

Remark 2.6 *Obviously, the obtained lower L^1-bound on the total area implies immediately a simple lower bound for the height of the maximum of interfaces:*

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\Big(\max_{1\le k\le N} X_k \le \delta H_1\Big) &\le \mathrm{P}_{N,+,\lambda}\Big(N^{-1}\sum_{i=1}^{N} X_i \le \delta H_1\Big) \\ &\le c\,\exp\Big\{-C\delta^{-2}\lambda^{2/3}\,N\Big\}.\end{aligned} \tag{2.17}$$

We shall obtain a complementary bound after a more detailed analysis of the interfaces.

3 Proof of Theorem 1.2

We treat the lower and the upper bounds in (1.10) separately, the latter being based upon the following apriori estimates.

3.1 Two refinements of the basic comparison lemma

The following version of Lemma 2.1 gives a better bound than (2.2) for large values of ρ, $\rho \geq \rho_0(\eta) > 0$. With H defined as in (2.1) and $\eta \in (0, 1/2)$, we put[3]

$$\widetilde{H} = (1-2\eta)\rho H_1(\lambda), \qquad \widetilde{\Delta} = \varepsilon \widetilde{H}^2. \tag{3.1}$$

Lemma 3.1 *Let ρ, H, c and C be as in Lemma 2.1. There exists $\lambda_0 > 0$ such that for any $\eta \in (0, 1/2)$ and $\zeta \in (0, 1/2)$ there is a constant $\tilde{c} > 0$ such that for any $0 < \lambda \leq \lambda_0$, every $N \geq H^2/(2\zeta)$, and all boundary conditions $0 \leq a, b \leq H$, one has*

$$\tilde{c}c \exp\Bigl\{-\frac{CN}{\widetilde{H}^2} - \lambda\Bigl[\zeta V(2H) + (1-\zeta)V(2\widetilde{H})\Bigr]N\Bigr\} \leq \frac{Z^{a,b}_{N,+,\lambda}}{Z^{a,b}_{N,+,0}} \leq 1\,. \tag{3.2}$$

Proof. Let $H = \rho H_1(\lambda)$, $\varepsilon \in (0, 1/4]$ and $\Delta = \varepsilon H^2$ be as in the proof of Lemma 2.1. Similarly, for $\eta \in (0, 1/2)$ we define

$$\tilde{n}_\varepsilon = \Bigl[\frac{N - 2\varepsilon H^2}{\widetilde{\Delta}}\Bigr] \equiv \Bigl[\frac{N - 2\varepsilon H^2}{\varepsilon(1-2\eta)^2\, H^2}\Bigr] \geq 2\,. \tag{3.3}$$

Further, let

$$J_0 = \Bigl\{\Delta + j\widetilde{\Delta} : j = 0, 1, \dots, \tilde{n}_\varepsilon - 1\Bigr\} \cup \{N - \Delta\}$$

$$J_1 = \Bigl\{\Delta, \Delta + 1, \dots, N - \Delta\Bigr\}, \qquad J_2 = \Bigl\{1, \dots, N\Bigr\} \setminus J_1$$

and introduce the events (see Fig. 3):

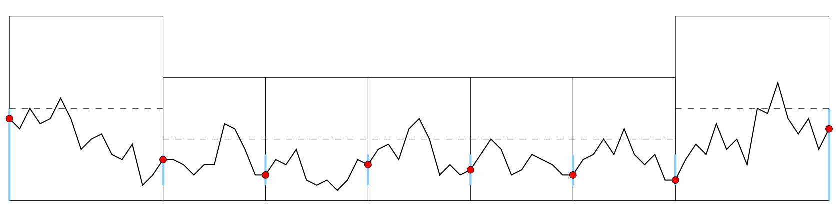

Figure 3: Renormalization scheme in Lemma 3.1.

$$\mathcal{A} = \Bigl\{{}^\forall j \in J_0 : \frac{1}{4}\widetilde{H} \leq X_j \leq \frac{3}{4}\widetilde{H}\Bigr\},$$

$$\mathcal{B} = \Bigl\{{}^\forall j \in J_1 : 0 \leq X_j \leq 2\widetilde{H}\Bigr\} \cap \Bigl\{{}^\forall j \in J_2 : 0 \leq X_j \leq 2H\Bigr\}.$$

[3] assuming $\widetilde{H}$ and $\widetilde{\Delta}$ to be integer.

For trajectories belonging to $\mathcal{A}\cap\mathcal{B}$ we have

$$\mathbb{V}(\mathbb{X}) = \sum_{j=1}^{N-1} V\bigl(X_j\bigr) \le 2\varepsilon H^2 V(2H) + (N-2\varepsilon H^2)V(2\widetilde{H})$$

and therefore, denoting $\zeta = 2\varepsilon H^2/N$ we get (cf. (2.7))

$$\frac{Z^{a,b}_{N,+,\lambda}}{Z^{a,b}_{N,+,0}} \ge \exp\Bigl\{-\lambda\Bigl[\zeta V(2H) + (1-\zeta)V(2\widetilde{H})\Bigr]N\Bigr\}\,\mathrm{P}^{a,b}_{N,+,0}\bigl(\mathcal{A}\cap\mathcal{B}\bigr).$$

Moreover, using the scaling assumption (3.3) and arguing as in the proof of Lemma 2.1 we get

$$\mathrm{P}^{a,b}_{N,+,0}\bigl(\mathcal{A}\cap\mathcal{B}\bigr) \ge ce^{-2\widetilde{C}\tilde{n}_\varepsilon} \ge c\,e^{-CN\widetilde{H}^{-2}}$$

with perhaps slightly smaller constant $\lambda_0 > 0$. ∎

Next, we present a short-droplet analogue of the previous lemma.

Lemma 3.2 *Let $H = \rho H_1(\lambda)$ and $N \le KH_1^2(\lambda)$. There exist positive constants ζ, λ_0 and C such that uniformly in $K/\rho^2 < \zeta$, in $\lambda \in (0,\lambda_0]$ and all boundary conditions $0 \le a, b \le H$ one has*

$$CZ^{a,b}_{N,+,0} \le Z^{a,b}_{N,+,\lambda} \le Z^{a,b}_{N,+,0}\,.$$

Our argument is based upon the following small droplet bound to be verified in Appendix A below.

Lemma 3.3 *Let $S_0 = 0$, $S_k = \xi_1 + \cdots + \xi_k$, $k \ge 1$, be the random walk generated by a sequence $\xi_1, \xi_2, \ldots$ of i.i.d. random variables such that $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 = \sigma^2 < \infty$. Let $D > 0$ be an arbitrary constant and, for any $m \ge 1$, let d_m satisfy $\mathbf{P}(S_m = d_m) > 0$ and $|d_m| \le D$. Then there exists $\zeta > 0$ such that*

$$\mathbf{P}\bigl(\max_{0<k<m} S_k > M \mid S_m = d_m\bigr) \le \frac{1}{3}, \qquad \text{as } M \to \infty, \tag{3.4}$$

uniformly in $m/M^2 \le \zeta$.

Proof of Lemma 3.2. As in Lemma 2.1, our argument is based on the bound (recall (2.7))

$$\frac{Z^{a,b}_{N,+,\lambda}}{Z^{a,b}_{N,+,0}} \ge \exp\Bigl\{-\lambda V\bigl(5H\bigr)N\Bigr\}\,\mathrm{P}^{a,b}_{N,+,0}\bigl(\mathcal{B}\bigr)\,,$$

where

$$\mathcal{B} = \Bigl\{\mathbb{X} \in \mathcal{I}_{N,+} : \max X_j \le 5H\Bigr\};$$

because of (1.7), (1.2) and the condition $N \le K H_1^2(\lambda)$, it remains to verify that the last probability is uniformly positive. Let $\mathbb{X}$ be an arbitrary trajectory from $\mathcal{I}_{N,+}$ (recall (1.5)). Then, either it belongs to the set

$$\mathcal{A}_1 = \Big\{ \mathbb{X} : {}^\forall j = 1, \dots, N-1 : 0 \le X_j \le H \Big\}$$

or there exists a non-empty set $[\mathfrak{l}', \mathfrak{r}'] \subset (0, N)$ such that

$$\mathfrak{l}' = \min\big\{ j > 0 : X_j > H \big\}, \quad \mathfrak{r}' = \max\big\{ j < N : X_j > H \big\}. \tag{3.5}$$

We shall write $\Delta' = (\mathfrak{l}', \mathfrak{r}')$ and $|\Delta'| = \mathfrak{r}' - \mathfrak{l}'$. Fix any $L \in (0, H)$ and denote

$$\mathcal{A}_2 = \Big\{ \mathbb{X} : X_{\mathfrak{l}'} \le H + L, X_{\mathfrak{r}'} \le H + L \Big\}. \tag{3.6}$$

According to the bounded variance estimate (2.12) and the Chebyshev inequality, one immediately gets

$$\mathrm{P}^{a,b}_{N,+,0}\big(\mathbb{X} \notin \mathcal{A}_1 \cup \mathcal{A}_2 \big) \le 2\tilde\sigma^2 L^{-2}, \tag{3.7}$$

where

$$\tilde\sigma^2 = \max\Big(\frac{\sigma^2}{\mathrm{P}(\xi \ge 0)}, \frac{\sigma^2}{\mathrm{P}(\xi \le 0)} \Big).$$

Taking L sufficiently large to have $\mathrm{P}^{a,b}_{N,+,0}\big(\mathbb{X} \in \mathcal{A}_1 \cup \mathcal{A}_2 \big) \ge 1/2$, we shall restrict ourselves to trajectories $\mathbb{X}$ belonging to $\mathcal{A}_1 \cup \mathcal{A}_2$ only (see Fig. 4).

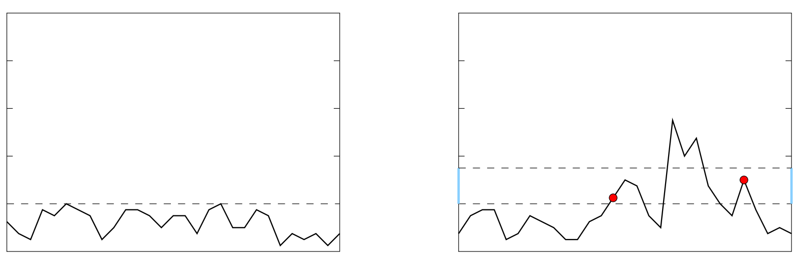

Figure 4: Trajectories from $\mathcal{A}_1$ (left) and $\mathcal{A}_2$ (right); black dots correspond to the decomposition in (3.9).

As $\mathcal{A}_1 \subset \mathcal{B}$, it remains to show that

$$\mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{B} \mid \mathcal{A}_2 \big) \ge c \tag{3.8}$$

for some constant $c > 0$. Indeed, once (3.8) is verified, we immediately get

$$\begin{aligned} \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{B} \big) &\ge \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{B} \mid \mathcal{A}_2 \big) \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}_2 \big) + \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}_1 \big) \\ &\ge c\, \mathrm{P}^{a,b}_{N,+,0}\big(\mathcal{A}_1 \cup \mathcal{A}_2 \big) \ge c/2 \end{aligned}$$

and therefore

$$\frac{Z^{a,b}_{N,+,\lambda}}{Z^{a,b}_{N,+,0}} \geq \exp\Big\{-\lambda V\big(5H\big)N\Big\}\, c/2 \geq \tilde{c}$$

uniformly in such N and $\{a,b\} \subset [0,H]$.

To prove (3.8), we rewrite

$$\begin{aligned}\mathrm{P}^{a,b}_{N,+,0}\big(\,\overline{\mathcal{B}} \mid \mathcal{A}_2\,\big) = \sum_{l',r'} \mathrm{P}^{a,b}_{N,+,0}\big(\mathfrak{l}' = l', \mathfrak{r}' = r' \mid \mathcal{A}_2\big) & \\ \times \sum_{H \leq a',b' \leq H+L} \mathrm{P}^{\,a',b'}_{(l',r'),+,0}\big(\,\overline{\mathcal{B}}\,\big) & \qquad (3.9) \\ \times\, \mathrm{P}^{a,b}_{N,+,0}\big(X_{l'} = a', X_{r'} = b' \mid \mathfrak{l}' = l', \mathfrak{r}' = r'\big)\,. &\end{aligned}$$

However,

$$\mathrm{P}^{\,a',b'}_{(l',r'),+,0}\big(\,\overline{\mathcal{B}}\,\big) \leq \frac{\mathrm{P}_{|\Delta'|}(\max S_j > 3H \mid S_{|\Delta'|} = b'-a')}{1 - \mathrm{P}_{|\Delta'|}(\min S_j < -H \mid S_{|\Delta'|} = b'-a')}\,, \qquad (3.10)$$

where $\mathrm{P}_{|\Delta'|}$ refers to the distribution of $|\Delta'|$-step unconstrained random walk with the step distribution $p(\,\cdot\,)$, recall (1.3). Finally, using the small droplet bound (3.4) and taking $|\Delta'|/H^2 \leq N/H^2$ sufficiently small, we can make the RHS above smaller than $1-c$. This finishes the proof. ■

3.2 The upper bound

We turn now to the proof of the upper bound in Theorem 1.2. Recall that due to the assumption (1.2) the function $V(\,\cdot\,)$ does not grow too fast at infinity.

For $\rho > 0$ and $H_1 = H_1(\lambda)$, our canonical scale from (1.7), define (cf. (1.1))

$$\begin{aligned}\mathfrak{l}_\rho(M) = \max\Big\{j < M : X_j \leq \rho H_1\Big\}, \quad \mathfrak{r}_\rho(M) = \min\Big\{j > M : X_j \leq \rho H_1\Big\}, \\ \Delta_\rho(M) = \big\{\mathfrak{l}_\rho + 1, \mathfrak{l}_\rho + 2, \ldots, \mathfrak{r}_\rho - 1\big\}, \qquad |\Delta_\rho| = \mathfrak{r}_\rho - \mathfrak{l}_\rho - 1\end{aligned} \qquad (3.11)$$

and, for any integer interval Δ,

$$\mathcal{A}_\rho(\Delta) = \Big\{{}^\forall j \in \Delta, X_j > \rho H_1\Big\}\,. \qquad (3.12)$$

Then, with Δ and T being (large) natural numbers to be chosen later, we get (for $N_2 = [N/2]$ and $\Delta_\rho = \Delta_\rho(N_2)$)

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\Big(X_{N_2} > TH_1\Big) \leq \mathrm{P}_{N,+,\lambda}\Big(X_{N_2} > TH_1 \mid |\Delta_\rho| < \Delta\Big) & \\ + \mathrm{P}_{N,+,\lambda}\big(|\Delta_\rho| \geq \Delta\big)\,. &\end{aligned} \qquad (3.13)$$

To estimate the length of the droplet, rewrite

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\big(|\Delta_\rho|\ge\Delta\big)&=\sum_{l,r}\mathrm{P}_{N,+,\lambda}\big(\Delta_\rho=(l,r)\big)\\&=\sum_{l,r}\sum_{0\le a,b\le\rho H_1}\mathrm{P}_{N,+,\lambda}\big(X_l=a,X_r=b\big)\\&\qquad\qquad\times\mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_\rho(l,r)\big)\,,\end{aligned}\tag{3.14}$$

where the first summation goes over all l, r satisfying

$$0\le l<N_2<r\le N,\qquad r-l-1\ge\Delta\,.\tag{3.15}$$

Next, by convexity of $V(\,\cdot\,)$ and the bounded growth assumption (1.2),

$$V(H)\equiv V(\rho H_1)\ge\frac{\rho}{2}\,V(2H_1)=\frac{\rho}{2\lambda H_1^2},$$

$$\zeta V(2H)+(1-\zeta)V(2\widetilde{H})\le\Big[\zeta f(2)+2(1-\zeta)(1-2\eta)\Big]V(H)\le\frac{1}{2}V(H)$$

where $\widetilde{H}$ is as in (3.1) and the constants ζ, η are chosen via

$$\zeta=\frac{1}{4f(2)},\qquad\eta=\frac{7}{16}.$$

Further, applying Lemma 3.1, we obtain (cf. Corollary 2.3)

$$\begin{aligned}\mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_\rho(l,r)\big)&\le C_1\exp\Big\{\Big[\frac{C}{\rho^2(1-2\eta)^2}-\frac{\lambda V(H)H_1^2}{2}\Big]\frac{r-l}{H_1^2}\Big\}\\&\qquad\times\mathrm{P}^{\,a,b}_{(l,r),+,0}\big(\mathcal{A}_\rho(l,r)\big)\\&\le C_1\exp\Big\{-\Big[1-\frac{256C}{\rho^3}\Big]\frac{\rho\Delta}{4H_1^2}\Big\}\,\mathrm{P}^{\,a,b}_{(l,r),+,0}\big(\mathcal{A}_\rho(l,r)\big).\end{aligned}\tag{3.16}$$

With Δ, $\alpha\in(0,1)$ and $T>0$ satisfying

$$\Delta=\sqrt{T}H_1^2,\qquad\alpha T=\rho\ge\rho_0=8\cdot C^{1/3}\,,\tag{3.17}$$

where C denotes the same constant as in Lemma 3.1, the last bound reads

$$\mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_\rho(l,r)\big)\le C_1\exp\Big\{-\frac{\alpha}{8}\,T^{3/2}\Big\}\,\mathrm{P}^{\,a,b}_{(l,r),+,0}\big(\mathcal{A}_\rho(l,r)\big)\,.$$

Inserting it into (3.14) we immediately get

$$\mathrm{P}_{N,+,\lambda}\big(|\Delta_\rho|\ge\Delta\big)\le C_1\exp\Big\{-\frac{\alpha}{8}\,T^{3/2}\Big\}\,.\tag{3.18}$$

It remains to estimate the first term in (3.13). Conditioning on the endpoints of the droplet of interest, we decompose

$$
\begin{aligned}
&\mathrm{P}_{N,+,\lambda}\big(X_{N_2} > TH_1 \mid |\Delta_\rho| < \Delta\big) \\
&\qquad \le \sum_{l,r} \mathrm{P}_{N,+,\lambda}\big(\mathfrak{l}_\rho = l, \mathfrak{r}_\rho = r \mid |\Delta_\rho| < \Delta\big) \\
&\qquad\qquad \times \sum_{0\le a,b\le H} \mathrm{P}_{N,+,\lambda}\big(X_l = a, X_r = b \mid \mathfrak{l}_\rho = l, \mathfrak{r}_\rho = r\big) \\
&\qquad\qquad \times \mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(X_{N_2} > TH_1 \mid \mathcal{A}_\rho(l,r)\big)\,,
\end{aligned}
\tag{3.19}
$$

where the first summation goes over all l, r satisfying (cf. (3.15))

$$
0 \le l < N_2 < r \le N, \qquad r - l - 1 < \Delta\,.
$$

To finish the proof of the lemma it remains to establish the following inequality

$$
\mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(X_{N_2} > TH_1 \mid \mathcal{A}_\rho(l,r)\big) \le \frac{1}{C_2} e^{-C_2 T^{3/2}}\,. \tag{3.20}
$$

Notice that taking T_0 large enough, we can achieve the bound

$$
\frac{\Delta}{(\rho H_1)^2} \le \frac{\sqrt{T} H_1^2}{(\alpha T H_1)^2} = \frac{1}{\alpha^2 T_0^{3/2}} \le \zeta
$$

for all $T \ge T_0$, where ζ is the same constant as in Lemma 3.2, and thus can remove the field λ from our further considerations.

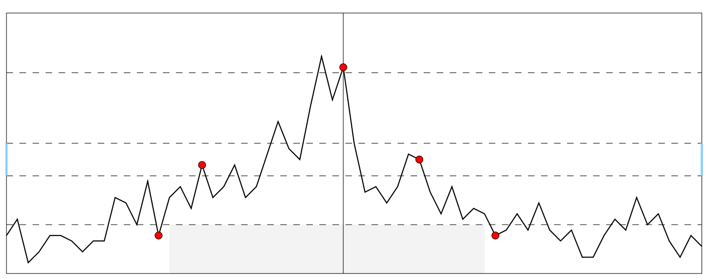

Figure 5: Decompositions (3.19) and (3.9′)

To prove (3.20), we shall proceed as in the proof of Lemma 3.2. Namely, defining $\Delta' \subset (l,r)$ and $\mathcal{A}_2$ similarly to (3.5) and (3.6), (see Fig. 5), we bound above

$$
\mathrm{P}^{\,a,b}_{(l,r),+,0}\big(X_{N_2} > TH_1 \mid \mathcal{A}_\rho(l,r)\big) \le \mathrm{P}^{\,a,b}_{(l,r),+,0}\big(\overline{\mathcal{A}_2}\big) + \mathrm{P}^{\,a,b}_{(l,r),+,0}\big(X_{N_2} > TH_1 \mid \mathcal{A}_2\big)
$$

and decompose (cf. (3.9))

$$\mathrm{P}^{a,b}_{(l,r),+,0}\big(X_{N_2} > TH_1 \mid \mathcal{A}_2 \big) = \sum_{l\le l'\le r'\le r} \mathrm{P}^{a,b}_{(l,r),+,0}\big(\mathfrak{l}' = l', \mathfrak{r}' = r' \mid \mathcal{A}_2\big)$$

$$\times \sum_{H\le a',b'\le H+L} \mathrm{P}^{a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big) \qquad (3.9')$$

$$\times\, \mathrm{P}^{a',b'}_{(l',r'),+,0}\big(X_{l'} = a', X_{r'} = b' \mid \mathfrak{l}' = l', \mathfrak{r}' = r'\big)\,.$$

We shall estimate the probability $\mathrm{P}^{a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big)$ depending on the length of the interval Δ'. First, we observe that for any $L > 0$ the Donsker invariance principle gives (recall (3.10))

$$\mathrm{P}^{a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big) \le \frac{\mathrm{P}_{|\Delta'|}(\max S_j > (T-2\rho)H_1 \mid S_{|\Delta'|} = b'-a')}{1-\mathrm{P}_{|\Delta'|}(\min S_j < -\rho H_1 \mid S_{|\Delta'|} = b'-a')}$$

$$\le C_3 \exp\Big\{ -C_4 \frac{(T-2\rho)^2}{\sqrt{T}} \Big\} = C_3 e^{-C_4(1-2\alpha)^2 T^{3/2}}$$

uniformly in $H_1^{7/6} \le |\Delta'| \le \Delta = \sqrt{T}H_1^2$, in $\{a',b'\} \subset [H, H+L]$ and all λ small enough.

On the other hand, for $|\Delta'| \le H_1^{7/6}$, the conditional Chebyshev inequality for the maximum (see Lemma A.1 in the appendix below) gives

$$\mathrm{P}^{a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big) \le \frac{\mathsf{const}_L |\Delta'|^{3/2}(T-2\rho)^{-2}H_1^{-2}}{1-\mathsf{const}_L|\Delta'|\rho^{-2}H_1^{-2}} \le C_5 \frac{\mathsf{const}_L}{T^2 H_1^{1/4}}\,.$$

Now, combining the last two estimates with the bound (3.7), we obtain:

$$\mathrm{P}^{a,b}_{(l,r),+,0}\big(X_{N_2} > TH_1 \mid \mathcal{A}_2 \big) \le \frac{2\tilde{\sigma}^2}{L^2} + C_3 e^{-C_4(1-2\alpha)^2T^{3/2}} + C_5 \frac{\mathsf{const}_L}{T^2 H_1^{1/4}}\,. \quad (3.21)$$

To finish the proof, we first take $L = \exp\Big\{ \frac{C_4}{2}(1-2\alpha)^2 T^{3/2} \Big\}$ and then λ sufficiently small to make the last term smaller than the second. With this choice (3.21) reduces to (3.20). ∎

An obvious generalization of the argument above gives also the following bound.

Corollary 3.4 *Let ρ and ε denote some (small) positive constants and let the integers l_0, r_0, $M \in [0,N]$ be such that*

$$l_0 < M < r_0 \qquad \text{and} \qquad \min\big(|M-l_0|, |M-r_0|\big) \ge \varepsilon H_1^2$$

with $H_1 = H_1(\lambda)$ being our canonical scale from (1.7). Then there exists $c_1 > 0$ such that for any $T > 0$ large enough and all a_0, $b_0 \in [0, \rho H_1]$ the inequality

$$\mathrm{P}^{a_0,b_0}_{(l_0,r_0),+,\lambda}\big(X_M > TH_1 \,\big|\, \mathcal{A}_\rho(l_0,r_0)\big) \le e^{-c_1 T^{3/2}}$$

holds for all $\lambda \in (0,\lambda_0]$, where $\lambda_0 = \lambda_0(T) > 0$.

As a straightforward modification of the proofs above one can show existence of moments of X_{N_2} to be used below.

Corollary 3.5 *There exist positive constants K and λ_0 such that for all p, $1 < p < 21/8$, we have*

$$\mathrm{P}_{N,+,\lambda}\big(X_{N_2} \big)^{2p} \leq C(p)\, H_1^{2p+1}\,,$$

uniformly in $\lambda \in (0, \lambda_0]$ and $N \geq K H_1^2$.

Proof. Using the decomposition (3.14) with $\rho = T/4$ and $\Delta = H_1^2 T^{-9/10}$ in (3.17), we get the following analogue of (3.18):

$$\mathrm{P}_{N,+,\lambda}\big(|\Delta_\rho| \geq \Delta\big) \leq C_1 \exp\Big\{-\frac{1}{32}\, T^{1/10}\Big\}\,.$$

Next, we use the decomposition (3.9′) and bound above the height of the inner droplet

$$\mathrm{P}^{\,a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big)\,.$$

As in the proof of Lemma A.3 two cases to be considered separately, $|r'-l'| \leq m_0$ and $m_0 \leq |r'-l'| \leq \Delta$. Clearly, w.l.o.g. we may and shall assume that m_0 is chosen large enough to satisfy (cf. (A.2))

$$\mathbf{P}\big(S_m = d_m\big) \geq \frac{1}{2e\sqrt{2\pi\sigma^2 m}}$$

for all $m \geq m_0$.

Let $|r'-l'| \leq m_0$. Combining (A.6) and (A.3), we get

$$\mathrm{P}^{\,a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big) \leq \frac{4\sigma^4 {m_0}^4}{p(m_0, D)}\big(TH_1\big)^{-4}\,.$$

On the other hand, for (l', r') satisfying $m_0 \leq |r'-l'| \leq \delta$, we apply Lemma A.2 to obtain

$$\mathrm{P}^{\,a',b'}_{(l',r'),+,0}\big(X_{N_2} > TH_1 \big) \leq \frac{C\sigma|r'-l'|^{5/2}}{(TH_1)^4}$$

with a numeric constant $C \leq 18$. As a result, for any $T > 0$ we get

$$\mathrm{P}_{N,+,\lambda}\big(X_{N_2} > TH_1 \big) \leq C_1 \exp\Big\{-\frac{1}{32}\, T^{1/10}\Big\} + \frac{C_2(m_0, D)}{(TH_1)^4} + \frac{C_3 H_1}{T^{25/4}}\,,$$

and therefore, for $1 < p < 21/8$,

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\big(X_{N_2} \big)^{2p} &\leq H_1^{2p}\Big(1 + \sum_{T>0} (T+1)^{2p}\mathrm{P}_{N,+,\lambda}\big[TH_1 \leq X_i < (T+1)H_1\big]\Big)\\ &\leq C_4(p)\, H_1^{2p+1}\,.\end{aligned}$$

■

3.3 The lower bound

Fix a (big) positive T and an integer $\Delta = \sqrt{T}H_1^2$ and denote (recall the notation $N_2 = [N/2]$)

$$l = N_2 - \Delta, \qquad r = N_2 + \Delta\,.$$

It follows from the argument in Sect. 3.2 that for some constant $T_0 > 0$ not depending of $\lambda > 0$ we have

$$\mathrm{P}_{N,+,\lambda}\big(\{X_l, X_r\} \subset [0, T_0 H_1]\big) \geq \frac{1}{2} \tag{3.22}$$

(recall the running assumption that we omit the boundary conditions $a = b = 0$ from the notation). We thus rewrite

$$\begin{aligned}\mathrm{P}_{N,+,\lambda}\big(X_{N_2} > TH_1\big) \geq \sum_{0\leq a,b\leq T_0 H} & \mathrm{P}_{N,+,\lambda}\big(X_l = a, X_r = b\big) \\ & \times \mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(X_{N_2} > TH_1\big)\,.\end{aligned} \tag{3.23}$$

Now, define

$$l' = l + H^2, \qquad r' = r - H^2, \qquad X_{l'} = a', \qquad X_{r'} = b'$$

and estimate

$$\begin{aligned}\mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(X_{N_2} > TH_1\big) \geq \sum_{H/4\leq a',b'\leq 3H/4} & \mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(X_{l'} = a', X_{r'} = b'\big) \\ & \times \mathrm{P}^{\,a',b'}_{(l',r'),+,\lambda}\big(X_{N_2} > TH_1\big)\,.\end{aligned} \tag{3.24}$$

By the Donsker invariance principle and the estimates for the maximum of the Brownian bridge, the last factor is bounded below by

$$\frac{1}{c}\exp\big\{-c(TH_1)^2/(r'-l')\big\} \geq \frac{1}{c'}\exp\big\{-c'T^{3/2}\big\}$$

uniformly in a' and b' under consideration, provided only $\lambda > 0$ is small enough. Next, a literal repetition of the proof of (2.8) and (2.10) combined with the estimate (3.24) gives

$$\mathrm{P}^{\,a,b}_{(l,r),+,\lambda}\big(X_{N_2} > TH_1\big) \geq \frac{1}{c''}\exp\big\{-c''T^{3/2}\big\} \tag{3.25}$$

uniformly in a, b from $[0, T_0 H_1]$ provided only $\lambda > 0$ is sufficiently small. The lower bound in (1.10) now follows from (3.22), (3.23), and (3.25). ∎

4 Refined asymptotics

4.1 Quasirenewal structure

The importance of the scale $H_1(\lambda)$ demonstrated in the proofs of the previous sections is even more pronounced in the study of the refined behaviour of the interfaces under consideration. The aim of this section is to describe certain intrinsic renewal-type structure of the random walks distributed via (1.5)–(1.4) that manifests itself in the diffusing scaling (i.e., $H_1(\lambda)^2$ in the horizontal direction and $H_1(\lambda)$ in the vertical one).

For any $\rho > 0$ and $\lambda > 0$, let S_ρ denote the horizontal strip of width $4\rho H_1$,

$$S_\rho = \Big\{(x,y) \in \mathbb{Z}^2 : y \in [0, 4\rho H_1]\Big\}.$$

In this section we shall establish certain quasirenewal property stating roughly that for all $\lambda > 0$ small enough the "density" of visits of the RW under consideration to the strip S_ρ is "positive on the scale $H_1(\lambda)^2$".

More precisely, for positive real ε, ρ, λ and integer $K > 0$ we split our trajectories into K-blocks $\mathcal{I}_m$ of length $K\varepsilon H_1(\lambda)^2$ (assuming w.l.o.g. $\varepsilon H_1(\lambda)^2$ to be integer) and introduce the random variables

$$Y_m = Y_m^\rho = \mathbb{1}_{\{\min_{j\in\mathcal{I}_m} X_j > 2\rho H_1\}} \,. \tag{4.1}$$

Let $n_K = \big[N/(K\varepsilon H_1(\lambda)^2)\big]$ be the total number of such blocks and let n_Y be the total number of K-blocks labelled by ones:

$$n_Y = \big|\{m : Y_m = 1\}\big| \,.$$

Our first observation is that with high probability the total length $n_Y K\varepsilon H_1(\lambda)^2$ of such K-blocks can not be large:

Lemma 4.1 *Let $f(\cdot)$ be defined as in* (1.2)*, ε be a positive constant, and α satisfy $\alpha \in (0,1)$. For any $\rho > 0$ there exist positive constants λ_0, and K_0, depending on α, ε, and ρ only, such that for any $\lambda \in (0,\lambda_0)$, any $K \geq K_0$ and any $N \geq 3K_0\varepsilon H_1(\lambda)^2$ we have*

$$\mathrm{P}^{a,b}_{N,+,\lambda}\big(n_Y \geq \alpha n_K\big) \leq \exp\Big\{-\frac{\alpha}{8f(2/\rho)} N H_1(\lambda)^{-2}\Big\} \tag{4.2}$$

uniformly in a, $b \in (0, \rho H_1)$.

Proof of Lemma 4.1. We use the blocking procedure described at the beginning of this section where, given $\rho > 0$ and $\varepsilon > 0$, the constant $K_0 \geq 8$ is chosen large enough, see (4.8) below.

The Y-labels defined in (4.1) with any $K \geq K_0$ introduce a $0-1$ encoding of each trajectory; using this encoding, we split the K-blocks labelled by ones into maximal "connected components" to be called K-clusters. Two neighbouring K-clusters are called connected if they are separated by exactly one K-block labelled by a zero. A K-cluster that is not connected to its neighbours is called isolated. Our next goal is to show that for any collection of K-clusters consisting of n_Y K-blocks there is a sub-collection of isolated K-clusters of total length at least $\lceil n_Y/4 \rceil$. As soon as this is done, a simple reduction argument will imply the target estimate (4.2).

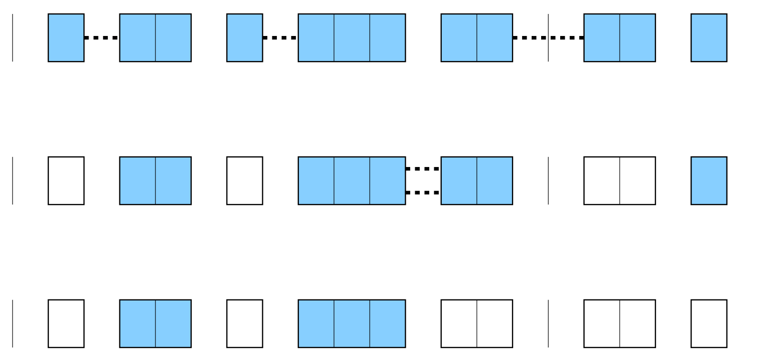

Figure 6: Two-step selection procedure: from 7 clusters of total length 12 choose 2 clusters of total length 5; selected K-clusters are shadowed

Our selection procedure consists of two steps. First, we split all K-clusters into subsequent pairs of neighbouring clusters and from each such pair we choose the longest cluster (or the left one if they are of equal length). Observe that each chosen cluster is either isolated or belongs to a pair of connected K-clusters. Next, we split all isolated K-clusters into subsequent pairs[4] and from each pair (connected or isolated) we choose the longest cluster (or the left one if they are of equal length). The obtained collection (together with the temporarily neglected K-cluster, if there was one) consists of isolated clusters which altogether contain at least $n_Y/4$ of K-blocks, see Fig. 6.

Our second step relies upon a finer renormalisation, this time on the integer scale $\varepsilon H_1(\lambda)^2$. We split our trajectory into $n_\varepsilon = \lceil N/\varepsilon H_1(\lambda)^2 \rceil$ blocks $\mathcal{J}_l$ of length $\varepsilon H_1(\lambda)^2$ each and similarly to (4.1) introduce the labels

$$Z_l = Z_l^\rho = \mathbb{I}_{\{\min_{j \in \mathcal{J}_l} X_j > 2\rho H_1\}} .$$

Of course, the natural (inclusion) correspondence between ε-blocks $\mathcal{J}_l$ and K-blocks $\mathcal{I}_m$,

$$\mathcal{J}_l \subset \mathcal{I}_m ,$$

has the following property: if $Y_m = 1$ and $\mathcal{J}_l$ corresponds to $\mathcal{I}_m$, then $Z_l = 1$. As before, we split all ε-blocks labelled by ones into maximal connected components to be called ε-clusters. Clearly, as subsets of $\{1, \dots, N\}$, every K-cluster is included in the corresponding ε-cluster. Let $\mathcal{E}$ be the collection of ε-clusters

[4] temporarily neglecting the very last K-cluster, if their total number is odd

that correspond to the isolated K-clusters selected by applying a procedure as in Fig. 6. The following two properties of the collection $\mathcal{E}$ will be important for our future application: 1) every ε-cluster from $\mathcal{E}$ is bounded by two boundary ε-blocks labelled by zeroes; moreover, for different ε-clusters the boundary blocks are different; 2) the total length of ε-clusters from $\mathcal{E}$ is at least $Kn_Y\varepsilon H_1(\lambda)^2/4$.

Our next aim is to establish certain one-droplet estimate from which the target bound (4.2) will follow immediately. Consider any ε-cluster from $\mathcal{E}$ and denote its extremal ε-blocks (the first and the last one) by $\mathcal{J}_{m_1}$ and $\mathcal{J}_{m_2}$ respectively. Clearly, the length of this ε-cluster is $n_\varepsilon^0\varepsilon H_1^2 \equiv (m_2-m_1+1)\varepsilon H_1^2$, $n_\varepsilon^0 \geq K$. Further, define

$$l = \max\Bigl\{j \in \mathcal{I}_{m_1-1} : X_j < 2\rho H_1\Bigr\}, \qquad X_l = a,$$
$$r = \min\Bigl\{j \in \mathcal{I}_{m_2+1} : X_j < 2\rho H_1\Bigr\}, \qquad X_r = b.$$

Similarly, let

$$l_0 = m_1\varepsilon H_1^2, \quad X_{l_0} = a_0, \quad r_0 = (m_2-1)\varepsilon H_1^2, \quad X_{r_0} = b_0\,. \tag{4.3}$$

Using the notation $\mathcal{A}_2 = \mathcal{A}_{2\rho}(l_0,r_0)$ (recall (3.12)), one can bound above the partition function corresponding to this droplet by

$$Z^{a,b}_{(l,r),+,\lambda}\bigl(\mathcal{A}_2\bigr) = \sum_{a_0,b_0\geq 2\rho H_1} Z^{a,a_0}_{(l,l_0),+,\lambda} Z^{a_0,b_0}_{(l_0,r_0),+,\lambda}\bigl(\mathcal{A}_2\bigr) Z^{b_0,b}_{(r_0,r),+,\lambda}\,. \tag{4.4}$$

Clearly, the target inequality (4.2) follows immediately from the one-droplet bound

$$\sup_{\rho H_1\leq a_0,b_0\leq 2\rho H_1} \frac{Z^{a_0,b_0}_{(l_0,r_0),+,\lambda}\bigl(\mathcal{A}_2\bigr)}{Z^{a_0,b_0}_{(l_0,r_0),+,\lambda}} \leq \exp\Bigl\{-\frac{\varepsilon n_\varepsilon^0}{2f(2/\rho)}\Bigr\} \tag{4.5}$$

the lower bound on the total length of ε-clusters from $\mathcal{E}$, provided only

$$K\varepsilon\alpha > 4f(2/\rho)\log 2\,,$$

to suppress the total number of $0-1$ encodings (that is bounded above by 2^{n_K}).

Our proof of (4.5) will be based upon the decomposition (4.4) and the following two facts:

F1) there is $\lambda_0 = \lambda_0(\varepsilon,\rho,...) > 0$ for which: for any $\eta > 0$ there exists $T > 0$ such that uniformly in $\lambda \in (0,\lambda_0]$ one has (recall (4.3))

$$Z^{a,b}_{(l,r),+,\lambda}\bigl(\mathcal{A}_2, a_0 \leq 2TH_1, b_0 \leq 2TH_1\bigr) \geq (1-\eta) Z^{a,b}_{(l,r),+,\lambda}\bigl(\mathcal{A}_2\bigr) \tag{4.6}$$

F2) for ε, ρ, T and λ_0 as above there is a finite constant $M > 0$ such that, uniformly in $\lambda \in (0,\lambda_0]$,

$$\sup \frac{Z^{a,a_0}_{\Delta,+,\lambda}}{Z^{a,a_0'}_{\Delta,+,\lambda}} \leq M \tag{4.7}$$

with supremum taken over $\varepsilon H_1^2 \le \Delta \le 2\varepsilon H_1^2$, $\rho H_1 \le a \le 2\rho H_1$, *and* $2\rho H_1 \le a_0 \le (2T+\rho)H_1$, $a_0' \equiv a_0 - \rho H_1$; *a similar estimate (with the same constant* M*) holds for the ratio* $Z^{a_0,a}_{\Delta,+,\lambda}/Z^{a_0',a}_{\Delta,+,\lambda}$.

The inequality (4.5) follows easily from (4.6) and (4.7). Indeed, combining (4.4) and (4.6) we get

$$Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2\big) \le \frac{1}{1-\eta} Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2, a_0 \le 2TH_1, b_0 \le 2TH_1\big)$$
$$\le \frac{1}{1-\eta} \sum Z^{a,a_0}_{(l,l_0),+,\lambda} Z^{a_0,b_0}_{(l_0,r_0),+,\lambda}\big(\mathcal{A}_2\big) Z^{b_0,b}_{(r_0,r),+,\lambda}$$

with the sum running over $2\rho H_1 \le a_0, b_0 \le (2T+\rho)H_1$. Further, denoting $\mathcal{A}_1 = \mathcal{A}_\rho(l_0,r_0)$, taking $\eta = 1/2$, and using the estimate (4.7), the convexity of the function $V(\,\cdot\,)$ and the reduction of the central part of the droplet as in Fig. 7, we bound the last expression by

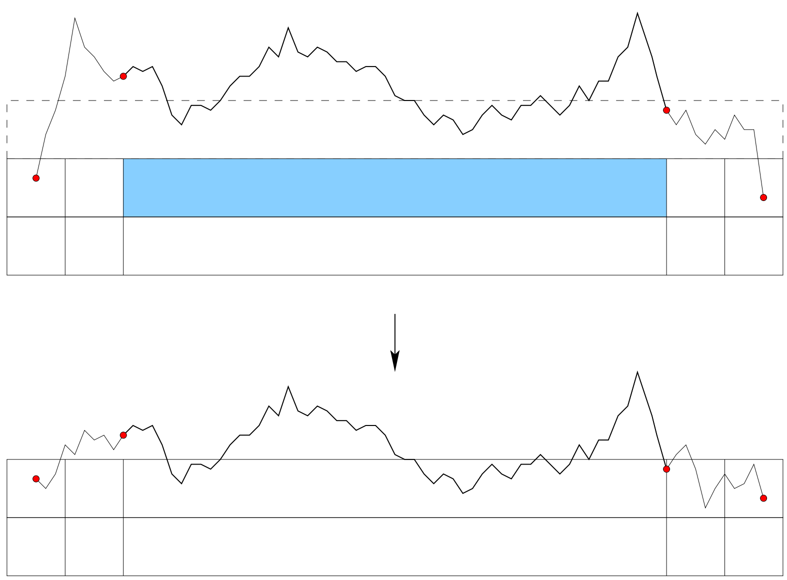

Figure 7: Reduction of an ε-cluster

$$2M^2 e^{-\lambda n^0_\varepsilon \varepsilon H_1^2 V(\rho H_1)} \sum Z^{a,a_0}_{(l,l_0),+,\lambda} Z^{a_0,b_0}_{(l_0,r_0),+,\lambda}\big(\mathcal{A}_1\big) Z^{b_0,b}_{(r_0,r),+,\lambda}$$
$$\le 2M^2 e^{-\varepsilon(n^0_\varepsilon - 2)/f(2/\rho)} Z^{a,b}_{(l,r),+,\lambda}$$

with sum running over $\rho H_1 \le a_0, b_0 \le 2TH_1$. Finally, taking $K_0 \ge 8$ such that

$$2M^2 \le \exp\big\{K_0\varepsilon/4f(2/\rho)\big\} \tag{4.8}$$

we immediately get

$$Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2\big) \le \exp\Big\{-\frac{n^0_\varepsilon \varepsilon}{2f(2/\rho)}\Big\}\, Z^{a,b}_{(l,r),+,\lambda}$$

and thus (4.5). It remains to verify (4.6) and (4.7).

The proof of (4.6) follows the argument of Sect. 3.2. Clearly, it is enough to show that for some constant $c_1 > 0$ one has

$$Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2, a_0 > 2TH_1\big) \le \exp\big\{-c_1 T^{3/2}\big\} Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2\big)\,.$$

Using the definitions (3.11) and (3.12), the partition function in the LHS of the previous display is bounded above by

$$\sum Z^{a,a'}_{(l,l'),+,\lambda} Z^{a',b'}_{(l',r'),+,\lambda}\big(\mathcal{A}_{2\rho}(l',r'), a_0 > 2TH_1\big) Z^{b',b}_{(r',r),+,\lambda}\,,$$

where the sum runs over $0 \le a', b' \le 2\rho H_1$ and l', r' such that (recall (3.11))

$$(l_0, r_0) \subset (l', r') \subseteq (l, r) \qquad \text{and} \qquad \Delta_{2\rho} = (l', r')\,.$$

Using Corollary 3.4 the sum above can be further majorated by

$$e^{-c_1 T^{3/2}} \sum Z^{a,a'}_{(l,l'),+,\lambda} Z^{a',b'}_{(l',r'),+,\lambda}\big(\mathcal{A}_{2\rho}(l',r')\big) Z^{b',b}_{(r',r),+,\lambda} \le e^{-c_1 T^{3/2}}\, Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2\big)\,.$$

As a result,

$$Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2, a_0 \le 2TH_1, b_0 \le 2TH_1\big) \ge \big(1 - 2e^{-c_1 T^{3/2}}\big) Z^{a,b}_{(l,r),+,\lambda}\big(\mathcal{A}_2\big)$$

and it remains to choose T large enough. The estimate (4.6) follows.

Finally, we check (4.7). First, applying an obvious extension of Lemma 2.1, we remove the field λ (as above, we put $b' = b - \rho H_1$): for some $M_1 = M_1(\varepsilon, \rho, T)$,

$$\frac{Z^{a,b}_{\Delta,+,\lambda}}{Z^{a,b'}_{\Delta,+,\lambda}} \le M_1 \frac{Z^{a,b}_{\Delta,+,0}}{Z^{a,b'}_{\Delta,+,0}} = M_1 \frac{\mathrm{P}^{\,a}_{\Delta,+,0}\big(X_\Delta = b\big)}{\mathrm{P}^{\,a}_{\Delta,+,0}\big(X_\Delta = b'\big)} \tag{4.9}$$

uniformly in a, b, Δ under consideration and all $\lambda > 0$ small enough. Here and below, $\mathrm{P}^{\,a}_{\Delta,+,0}(\cdot)$ denotes the probability distribution of the Δ-step random walk starting from a with transition probabilities $p(\cdot)$ restricted to the set $\mathcal{I}_{\Delta,+}$ of non-negative trajectories (recall (1.5)).

Now, denoting by ≥ 0 the wall constraint $\mathbb{X} \in \mathcal{I}_{\Delta,+}$, we rewrite the last ratio as

$$\frac{\mathrm{P}^a_\Delta\big(X_\Delta = b \mid \ge 0\big)}{\mathrm{P}^a_\Delta\big(X_\Delta = b' \mid \ge 0\big)} = \frac{\mathrm{P}^a_\Delta\big(X_\Delta = b, \ge 0\big)}{\mathrm{P}^a_\Delta\big(X_\Delta = b', \ge 0\big)} \tag{4.10}$$

and observe that uniformly in a, Δ under consideration the P^a_Δ-probability of the event ≥ 0 is uniformly positive. Thus, applying the standard argument (see, eg., [2, §11], [3, §9]) one deduces that, uniformly in a, b, Δ, and λ under consideration, the ratio in (4.10) is bounded above by a positive constant $M_2 = M_2(\varepsilon, \rho, \Delta, T)$. The estimate(4.7) follows from (4.9) and (4.10).

The proof of the lemma is finished. ■

Next, we fix ρ, K, and ε as in the proof above, use the K-blocks decomposition and introduce the labels (cf. (4.1))

$$U_m = U_m^\rho(X) = \mathbb{1}_{\{\max_{j\in\mathcal{I}_m} X_j < 4\rho H_1\}} \,. \tag{4.11}$$

Denote $n_U = \bigl|\{m : U_m = 1\}\bigr|$.

Lemma 4.2 *For any $\rho > 0$ there exist positive constants λ_0, γ_0, K_0, c, and C such that for any $\lambda \in (0, \lambda_0)$, $\gamma \in (0, \gamma_0)$, $K \geq K_0$ and all $N \geq 3KH_1(\lambda)^2$ we have*

$$\mathrm{P}^{a,b}_{N,+,\lambda}\bigl(n_U \leq \gamma n_K\bigr) \leq C \exp\Bigl\{-cNH_1(\lambda)^{-2}\Bigr\} \tag{4.12}$$

uniformly in a, $b \in \bigl(0, 2\rho H_1(\lambda)\bigr)$.

Proof. Our argument is similar to that of Sect 2.3. First, taking $K = 3K_0$ in Lemma 4.1, we split all K-blocks into triples of consecutive blocks (neglecting the non-complete last triple if there is one) and call an index m regular if

$$Y_{3m+1} = Y_{3m+3} = 0\,.$$

Using the previous Lemma with $\alpha = 1/9$, we deduce that with probability not smaller than $1 - \exp\bigl\{-N/(72 f(2/\rho) H_1(\lambda)^2)\bigr\}$ there are at least $2n_K/9$ regular indices m.

For each such m we define

$$l = \max\Bigl\{j \in \mathcal{I}_{3m+1} : X_j < 2\rho H_1(\lambda)\Bigr\}, \qquad X_l = a,$$
$$r = \min\Bigl\{j \in \mathcal{I}_{3m+3} : X_j < 2\rho H_1(\lambda)\Bigr\}, \qquad X_r = b.$$

Now, $K\varepsilon H_1(\lambda)^2 = 3K_0\varepsilon H_1(\lambda)^2 \leq r - l \leq 3K\varepsilon H_1(\lambda)^2$ and thus, using Corollary 2.4 we get

$$\begin{aligned} \mathrm{P}^{a,b}_{(l,r),+,\lambda}\bigl(U_{3m+2} = 1\bigr) &\geq c e^{-C(r-l)/(2\rho H_1(\lambda))^2} \\ &\geq c e^{-3CK\varepsilon/(4\rho^2)} =: p_{\rho,K\varepsilon} > 0\,. \end{aligned} \tag{4.13}$$

Therefore, on average there are at least $2p_{\rho,K\varepsilon} n_K/9$ indices m whose labels satisfy $U_{3m+2} = 1$. By a standard large deviation bound we get

$$\mathrm{P}_{N,+,\lambda}\Bigl(n_U < \frac{p_{\rho,K\varepsilon}}{9} n_K\Bigr) \leq C e^{-cn_K}$$

and the lemma is proved. ■

4.2 Coupling

With $\rho > 0$ fixed as above and $\lambda > 0$ denote (recall (1.7))

$$H = H(\rho, \lambda) := 4\rho H_1(\lambda). \tag{4.14}$$

Let X, Y be two independent trajectories of our process and let $\mathbb{P}^{a_x,b_x,a_y,b_y}_{N,+,\lambda}(\,\cdot\,,\,\cdot\,)$ denote their joint distribution (recall (1.5)–(1.6)),

$$\mathbb{P}^{a_x,b_x,a_y,b_y}_{N,+,\lambda}(\,\cdot\,,\,\cdot\,) = \mathrm{P}^{a_x,b_x}_{N,+,\lambda}(\,\cdot\,) \otimes \mathrm{P}^{a_y,b_y}_{N,+,\lambda}(\,\cdot\,)$$

with the shorthand notation $\mathbb{P}_{N,+,\lambda}(\,\cdot\,)$ if $a_x = b_x = a_y = b_y = 0$. Everywhere in this section we shall consider only boundary conditions satisfying $0 \le a_x, b_x, a_y, b_y \le H$. For a set of indices $A \subseteq [0, N] \cap \mathbb{Z}$, let

$$\mathcal{N}_A = \mathcal{N}_A(X, Y) = \Big\{ {}^\forall j \in A, X_j \neq Y_j \Big\}$$

be the event "trajectories X and Y do not intersect within the set A". Our main observation is that, with probability going to one, any two independent trajectories of our RW meet within a time interval of order at most $H^2 = O\big(H_1(\lambda)^2\big)$:

Lemma 4.3 *There exist positive constants λ_0, C, c, and ρ_0 such that the inequality*

$$\mathbb{P}_{N,+,\lambda}\Big(\mathcal{N}_{(0,N)}\Big) \le C e^{-cN/H^2} \tag{4.15}$$

holds uniformly in $0 < \lambda \le \lambda_0$, $0 < \rho \le \rho_0$ and $N \ge H^2$.

Proof. Consider the decomposition into K-blocks $\mathcal{I}_m$ of length $K\varepsilon H_1(\lambda)^2$ described in the previous section and denote

$$n_U^2 = \Big| \Big\{ m : U_m(X) = U_m(Y) = 1 \Big\} \Big|$$

with labels $U_m(\,\cdot\,)$ defined as in (4.11). Following the proof of Lemma 4.2 with $\alpha = 1/9$ we deduce that with probability not smaller than

$$1 - 2\exp\Big\{ -\frac{N}{72 f(2/\rho) H_1(\lambda)^2} \Big\}$$

there are on average at least $\big(1/3 - 2\alpha\big) n_K = n_K/9$ regular indices m that are common for both X and Y. As a result (recall (4.13)),

$$\mathrm{P}_{N,+,\lambda}\Big(n_U^2 < \frac{(p_{\rho,K})^2}{18} n_K \Big) \le C_1 e^{-c_1 n_K} \le C_1 e^{-c_2 N/H^2}\,, \tag{4.16}$$

that is, with high probability there is a positive fraction of blocks $\mathcal{I}_m$ for which the event

$$D_m = \Big\{ U_m(X) = U_m(Y) = 1 \Big\} = \Big\{ {}^\forall j \in \mathcal{I}_m, 0 \le X_j, Y_j < 4\rho H_1 \Big\} \tag{4.17}$$

is realized. By taking each second such K-block we construct a disjoint collection $\mathcal{K}$ of K-blocks possessing property (4.17). The collection $\mathcal{K}$ has the following important properties to be used in the sequel:

1) with probability at least $1-C_1\exp\{-c_2NH^{-2}\}$*, there are no less than* $p_{\rho,K}^2 n_K/36$ *blocks in* $\mathcal{K}$*;*
2) conditioned on $\{U_m\}$ *and on the configuration in the complement of* K*-blocks from* $\mathcal{K}$*, distributions inside individual* K*-blocks are independent.*

Another important ingredient of our argument is the following observation:

Lemma 4.4 *Let* $H = H(\rho,\lambda)$ *be as in (4.14) and let* $D = D_N(X)\cap D_N(Y)$*, where*

$$D_N(Z) = \Big\{0 \le Z_j \le H, {}^\forall j = 1,\dots,N-1\Big\}. \qquad (4.18)$$

Then there exist positive constants λ_0 *and* c_3 *such that the estimate*

$$\max_{a_x,b_x,a_y,b_y} \mathbb{P}_{N,+,0}^{a_x,b_x,a_y,b_y}\Big(\mathcal{N}_{(0,N)} \mid D\Big) \le e^{-c_3NH^{-2}}, \qquad (4.19)$$

holds uniformly in $0<\lambda\le\lambda_0$*, in* $N\ge H^2$ *and in boundary conditions* $0\le a_x,b_x,a_y,b_y \le H$*.*

We postpone the proof of Lemma 4.4 till the end of this section and deduce our main estimate (4.15) first. Combining Corollary 2.4 with the inequality (4.19) and using the estimate

$$\lambda V(4\rho H_1)N \le \frac{2\rho N}{H_1^2}\lambda V(2H_1)H_1^2 = \frac{32\rho^3 N}{H^2}$$

we obtain the uniform bound (similarly as in (2.15))

$$\max_{a_x,b_x,a_y,b_y} \mathbb{P}_{N,+,\lambda}^{a_x,b_x,a_y,b_y}\Big(\mathcal{N}_{(0,N)} \mid D\Big) \le e^{-(c_3-64\rho^3)NH^{-2}} \le e^{-c_3N/2H^2} \qquad (4.20)$$

provided only $0<\rho\le\rho_0$ with $128\rho_0^2 < c_3$. Now, using the bound

$$\mathbb{P}_{N,+,\lambda}\Big(\mathcal{N}_{(0,N)}\Big) \le \mathbb{P}_{N,+,\lambda}\Big(n_U^2 < \frac{(p_{\rho,K})^2}{18}n_K\Big) + \mathbb{P}_{N,+,\lambda}\Big(\mathcal{N}_{(0,N)} \mid |\mathcal{K}| \ge \frac{(p_{\rho,K})^2}{36}n_K\Big),$$

"freezing" the joint configuration $(X,Y)_{\mathcal{K}^c}$ in all blocks that do not belong to the collection $\mathcal{K}$ and using the estimate (4.20) for all blocks from $\mathcal{K}$, we bound the last term by

$$\begin{aligned}\max_{(X,Y)_{\mathcal{K}^c}} \mathbb{P}_{N,+,\lambda}\Big(\mathcal{N}_{(0,N)} \mid (X,Y)_{\mathcal{K}^c}, |\mathcal{K}| \ge \frac{(p_{\rho,K})^2}{36}n_K\Big) \\ \le \exp\Big\{-\frac{c_3|\mathcal{I}_m|}{2H^2}\cdot\frac{(p_{\rho,K})^2}{36}n_K\Big\} \le e^{-c_4NH^{-2}}.\end{aligned} \qquad (4.21)$$

Averaging this inequality over $(X,Y)_{\mathcal{K}^c}$ and combining the result with (4.16), we obtain the target estimate (4.15). ■

Proof of Lemma 4.4. We use again a blocking argument. Fix two positive constants ε and Δ such that $\varepsilon \leq 1$, $\Delta \leq 1/8$ and assume w.l.o.g. that εH^2 is integer. Split the interval $[0, N]$ into blocks $\mathcal{I}_m$ of length $4\varepsilon H^2$, $m = 1, 2, \ldots, \lfloor N/(4\varepsilon H^2)\rfloor =: n_\varepsilon$, and for each such block introduce the "crossing event"

$$\mathcal{A}_m(X, Y) = \mathcal{A}_m^{\nearrow}(X) \cap \mathcal{A}_m^{\searrow}(Y),$$

where (see Fig. 8)

$$\begin{aligned} \mathcal{A}_m^{\nearrow}(Z) &= D_m(Z) \cap \widehat{D}_m^o(Z) \cap \mathcal{B}_m^{\nearrow}(Z), \\ \mathcal{A}_m^{\searrow}(Z) &= D_m(Z) \cap \widehat{D}_m^o(Z) \cap \mathcal{B}_m^{\searrow}(Z) \end{aligned} \tag{4.22}$$

with $D_m = D_{\mathcal{I}_m}$ defined as in (4.17), $\widehat{D}_m^o = \widehat{D}_{\mathcal{I}_m^o}$ given by

$$\widehat{D}_J(Z) = \Big\{\frac{1}{4}H \leq Z_j \leq \frac{3}{4}H, \forall j \in J\Big\}, \qquad \forall J \subseteq [0, N],$$
$$\mathcal{I}_m^o = \Big\{j \in \mathcal{I}_m : (4m-3)\varepsilon H^2 < j < (4m-1)\varepsilon H^2\Big\}$$

and, finally,

$$\mathcal{B}_m^{\nearrow}(Z) = \Big\{Z_{(4m-3)\varepsilon H^2} \in C_\Delta^-, Z_{(4m-1)\varepsilon H^2} \in C_\Delta^+\Big\},$$
$$\mathcal{B}_m^{\searrow}(Z) = \Big\{Z_{(4m-3)\varepsilon H^2} \in C_\Delta^+, Z_{(4m-1)\varepsilon H^2} \in C_\Delta^-\Big\}$$

with

$$C_\Delta^+ = \Big[\frac{1}{2}H, \frac{1+\Delta}{2}H\Big], \qquad C_\Delta^- = \Big[\frac{1-\Delta}{2}H, \frac{1}{2}H\Big].$$

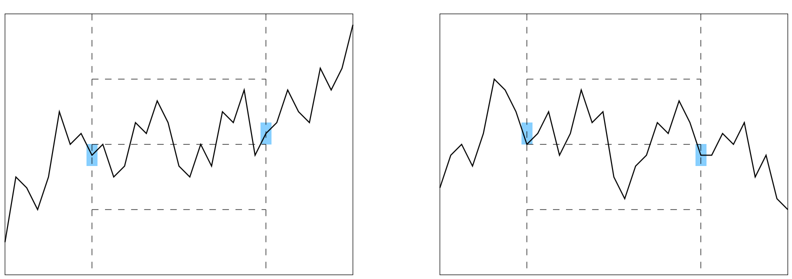

Figure 8: Crossing events $\mathcal{A}_m^{\nearrow}$ (left) and $\mathcal{A}_m^{\searrow}$ (right)

Our argument below is based upon the following three facts:

F1) conditioned on the "boundary" values

$$\Big\{(X_j, Y_j), j = 4m\varepsilon H^2, m = 1, 2, \ldots, n_\varepsilon\Big\},$$

the trajectories X and Y behave independently in different blocks;

F2) for all $\mathcal{I}_m$, uniformly in "boundary" values, the crossing event $\mathcal{A}_m(X,Y)$ occurs with positive probability:

$$\min_{a_x,b_x,a_y,b_y} \mathbb{P}^{a_x,b_x,a_y,b_y}_{\mathcal{I}_m,+,0}\Bigl(\mathcal{A}_m(X,Y) \mid D_m\Bigr) \geq p_1 > 0 \tag{4.23}$$

F3) conditioned on $\mathcal{A}_m(X,Y)$, the trajectories X and Y intersect in $\mathcal{I}^o_m$ with positive probability:

$$\min_{a_x,b_x,a_y,b_y} \mathbb{P}^{a_x,b_x,a_y,b_y}_{\mathcal{I}_m,+,0}\Bigl(\overline{\mathcal{N}}_{\mathcal{I}^o_m} \mid \mathcal{A}_m(X,Y)\Bigr) \geq p_2 > 0 \tag{4.24}$$

where $\overline{\mathcal{N}}_{\mathcal{I}^o_m}$ denotes the intersection event in the central part of the block $\mathcal{I}_m$:

$$\overline{\mathcal{N}}_{\mathcal{I}^o_m} = \Bigl\{\exists j \in \mathcal{I}^o_m : X_j = Y_j \in \Bigl[\frac{1}{8}H, \frac{7}{8}H\Bigr]\Bigr\}.$$

Indeed, let $\mathcal{A}$ be the collection of indices m for which the event $\mathcal{A}_m(X,Y)$ occurs. Denoting by n_{cross} the cardinality of $|\mathcal{A}|$, we use the Markov property and the standard large deviation bound to get

$$\max_{a_x,b_x,a_y,b_y} \mathbb{P}^{a_x,b_x,a_y,b_y}_{N,+,0}\Bigl(n_{\mathsf{cross}} < \frac{p_1}{2} n_\varepsilon \mid D_m\Bigr) \leq e^{-c_5 n_\varepsilon} \tag{4.25}$$

with some $c_5 > 0$; thus, it remains to consider the case $n_{\mathsf{cross}} = |\mathcal{A}| \geq p_1 n_\varepsilon/2$. Now, the bound (4.24) and the conditional independence of blocks $\mathcal{I}_m$ imply that

$$\begin{aligned}\mathbb{P}^{a_x,b_x,a_y,b_y}_{N,+,0}\Bigl(\mathcal{N}_{(0,N)} \mid \mathcal{A}, D_m\Bigr) &\leq \prod_{m\in\mathcal{A}} \mathbb{P}^{a^m_x,b^m_x,a^m_y,b^m_y}_{\mathcal{I}_m,+,0}\Bigl(\mathcal{N}_{\mathcal{I}^o_m} \mid \mathcal{A}_m(X,Y), D_m\Bigr)\\ &\leq (1-p_2)^{|\mathcal{A}|}\end{aligned}$$

uniformly in all such $\mathcal{A}$ and boundary conditions $a^m_x, b^m_x, a^m_y, b^m_y$ thus giving the uniform upper bound

$$(1-p_2)^{p_1 n_\varepsilon/2} \leq e^{-c_6 n_\varepsilon}\,. \tag{4.26}$$

Finally, (4.19) follows immediately from (4.25) and (4.26).

Thus, it remains to check the properties F1)–F3). The Markov property F1) being obvious, we need only to prove the inequalities (4.23) and (4.24).

To check (4.23) we proceed as in the proof of Lemma 2.1. By conditional independence, it is enough to show that for some $p_3 > 0$ one has

$$\begin{aligned}&\min_{a_x,b_x} \mathrm{P}^{a_x,b_x}_{\mathcal{I}_m,+,0}\bigl(\mathcal{A}^{\nearrow}_m(X) \mid D_m\bigr) \geq p_3,\\ &\min_{a_y,b_y} \mathrm{P}^{a_y,b_y}_{\mathcal{I}_m,+,0}\bigl(\mathcal{A}^{\searrow}_m(Y) \mid D_m\bigr) \geq p_3.\end{aligned}$$

We shall verify the first of these inequalities, the second follows from analogous considerations.

First, for any boundary conditions a, b we rewrite (recall (4.22))

$$\begin{aligned}\mathrm{P}^{a,b}_{\mathcal{I}_m,+,0}\big(\mathcal{A}^{\nearrow}_m \mid D_m\big) &= \mathrm{P}^{a,b}_{\mathcal{I}_m,+,0}\big(\widehat{D}^o_m \cap \mathcal{B}^{\nearrow}_m \mid D_m\big)\\ &= \mathrm{P}^{a,b}_{\mathcal{I}_m,+,0}\big(\widehat{D}^o_m \mid \mathcal{B}^{\nearrow}_m \cap D_m\big)\cdot \mathrm{P}^{a,b}_{\mathcal{I}_m,+,0}\big(\mathcal{B}^{\nearrow}_m \mid D_m\big).\end{aligned}$$

Next, using the Markov property and the usual invariance principle (for unconditioned RWs), we get

$$\begin{aligned}\mathrm{P}^{a,b}_{\mathcal{I}_m,+,0}\big(\widehat{D}^o_m \mid \mathcal{B}^{\nearrow}_m \cap D_m\big) &\geq \min_{x\in C^-_\Delta, y\in C^+_\Delta}\Big[\mathrm{P}^{x,y}_{2\varepsilon H^2}\big(\widehat{D}^o_m\big) - \mathrm{P}^{x,y}_{2\varepsilon H^2}\big(\overline{D}_{\mathcal{I}^o_m}\big)\Big]\\ &\geq \min_{x\in C^-_\Delta, y\in C^+_\Delta}\mathrm{P}^{x,y}_{2\varepsilon H^2}\Big(\max|X_j - x| \leq (1-2\Delta)H/4\Big)\\ &\quad - \max_{x\in C^-_\Delta, y\in C^+_\Delta}\mathrm{P}^{x,y}_{2\varepsilon H^2}\Big(\max|X_j - x| > (1-\Delta)H/2\Big) \geq p_4\end{aligned}$$

where, $\overline{D}_{\mathcal{I}^0_m}$ denotes the complement of the event $D_{\mathcal{I}^o_m}$ (cf. (4.17)),

$$D_{\mathcal{I}^o_m} = \Big\{{}^{\forall} j \in \mathcal{I}^o_m, 0 \leq X_j, Y_j < 4\rho H_1\Big\},$$

and for fixed $\Delta \in (0, 1/8]$, the bound $p_4 = p_4(\varepsilon, \Delta)$ is positive uniformly in all λ and ε small enough. On the other hand, a slight modification of the proof of Lemma 2.1 implies the inequality

$$\mathrm{P}^{a,b}_{\mathcal{I}_m,+,0}\big(\mathcal{B}^{\nearrow}_m \mid D_m\big) \geq p_5$$

with some $p_5 = p_5(\varepsilon, \Delta) > 0$ uniformly in all λ small enough. The estimate (4.23) follows.

Finally, we verify the inequality (4.24). Morally, our argument is based upon the following observation: on the event $\mathcal{A}_m(X, Y)$ there is $j \in \mathcal{I}^o_m$ satisfying

$$X_j - Y_j \leq 0 \leq X_{j+1} - Y_{j+1}; \tag{4.27}$$

since X and Y are independent processes whose jumps have the same distribution of finite variance $\sigma^2 > 0$, the bound

$$\min\Big\{|X_j - Y_j|, |X_{j+1} - Y_{j+1}|\Big\} \leq R\sigma \tag{4.28}$$

holds with positive probability provided only the absolute constant $R > 0$ is chosen large enough; finally, thanks to the aperiodicity property (1.4), conditioned on the event (4.28), the trajectories X and Y meet with positive probability within $A\lceil R\sigma\rceil$ steps.

We sketch the main steps of the argument. In view of the Markov property, it is sufficient to show that

$$\min_{x\in C^-_\Delta, y\in C^+_\Delta}\mathrm{P}^{x,y}_{\mathcal{I}^o_m}\Big({}^{\exists} j \in \mathcal{I}^o_m,\ |X_j - Y_j| \leq R\sigma \mid \widehat{D}^o_m\Big)$$

is uniformly positive for all λ small enough. Clearly, the minimum above is bounded below by the expression

$$\mathsf{Pr}\Big(\big|\xi-\eta\big|\le R\sigma\Big)-2\max \mathrm{P}^{x,y}_{\mathcal{I}^o_m}\Big(\max_{j\in\mathcal{I}^o_m}|X_j-x|>\frac{1-2\Delta}{4}H\Big),$$

where ξ and η are i.i.d. r.v. with the basic distribution $p(\,\cdot\,)$ and max is taken over all $x\in C^-_\Delta$, $y\in C^+_\Delta$. As $R\to\infty$, the first term approaches 1, whereas the second vanishes asymptotically as $\varepsilon\to 0$, uniformly in $\Delta\in(0,1/8]$ and in all $\lambda>0$ small enough. Thus, for some $p_6=p_6(\varepsilon,\Delta,R)>0$ we get

$$\min_{a^m_x,b^m_x,a^m_y,b^m_y}\mathbb{P}^{a^m_x,b^m_x,a^m_y,b^m_y}_{\mathcal{I}_m,+,0}\Big(\exists j\in\mathcal{I}^o_m,\ |X_j-Y_j|\le R\sigma\mid\widehat{D}^o_m\Big)\ge p_6$$

and thanks to the aperiodicity property (1.4) the trajectories X and Y have a positive probability to meet within the time interval $J_0=[j_0,j_0+A\lceil R\sigma\rceil]$:

$$\mathbb{P}^{a^m_x,b^m_x,a^m_y,b^m_y}_{\mathcal{I}_m,+,0}\Big(\exists j\in J_0,\ X_j=Y_j\in[0,H]\ \Big|\ \mathcal{A}_m(X,Y),|X_{j_0}-Y_{j_0}|\le R\sigma\Big)\ge p_7$$

with some $p_7>0$, uniformly in boundary conditions $0\le a^m_x,b^m_x,a^m_y,b^m_y\le H$ and in positive λ small enough. This implies the estimate (4.24).

The proof of the lemma is complete. ∎

4.3 Relaxation to equilibrium

It is an immediate corollary of (4.15): just consider the initial RW and another one started at equilibrium. By the coupling inequality [8, pg. 12] the total variance distance between the distribution of our RW after N steps and the equilibrium measure is bounded above by the LHS of (4.15).

4.4 Inverse correlation length

Let X be our RW and Y its independent copy; we have:

$$\mathsf{Cov}(X_i,X_j)=\frac{1}{2}\mathrm{E}_{+,\lambda}\big[(X_i-Y_i)(X_j-Y_j)\big]\,, \tag{4.29}$$

where $\mathrm{E}_{+,\lambda}$ is the expectation w.r.t. the limiting measure and Cov is the corresponding covariance; denote by A the event that both RW's X and Y intersect between i and j. According to the above, the probability of the complement $\bar{A}$ of A is bounded above by the RHS of (4.15):

$$\mathrm{P}_{+,\lambda}(\bar{A})\le C\exp\big\{-c\,|i-j|\,H_1^{-2}\big\}.$$

Moreover, by symmetry of the RHS of (4.29) on the event A, we have

$$\mathrm{E}_{+,\lambda}\big[(X_i-Y_i)(X_j-Y_j)\mathbb{1}_A\big]=0.$$

Consequently, for any $p > 1$,

$$\begin{aligned} 2\mathsf{Cov}(X_i, X_j) &= \mathrm{E}_{+,\lambda}\big[(X_i - Y_i)(X_j - Y_j)\mathbb{1}_{\bar{A}}\big] \\ &\le \mathrm{E}_{+,\lambda}\big[(X_i X_j + Y_i Y_j)\mathbb{1}_{\bar{A}}\big] \\ &\le 2\big(\mathrm{E}_{+,\lambda}\big[(X_i X_j)^p\big]\big)^{1/p}\big(\mathrm{P}_{+,\lambda}(\bar{A})\big)^{(p-1)/p}. \end{aligned}$$

However, for any p, $1 < p < 21/8$ we get (recall Corollary 3.5)

$$\mathrm{E}_{+,\lambda}\big(X_{N_2} \big)^{2p} \le C(p)\, H_1^{2p+1}\,,$$

and thus, by the Cauchy-Schwarz inequality,

$$\mathrm{E}_{+,\lambda}\big[(X_i X_j)^p\big] \le \Big[\mathrm{E}_{+,\lambda}\big((X_i)^{2p}\big)\mathrm{E}_{+,\lambda}\big((X_j)^{2p}\big)\Big]^{1/2} \le C H_1^{2p+1}$$

leading to

$$\mathsf{Cov}(X_i, X_j) \le C H_1(\lambda)^{2+1/p} \exp\big\{-c\,|i-j|\,H_1^{-2}\big\}.$$

Finally, take $p = 2$. ∎

A Small droplet bound

Our aim here is to prove the small droplet bound—Lemma 3.3. The key step of our argument will be based upon the following, having an independent interest, conditional Chebyshev inequality for maximum.

Lemma A.1 *Let $S_0 = 0$, $S_k = \xi_1 + \cdots + \xi_k$, $k \ge 1$, be the random walk generated by a sequence $\xi_1, \xi_2, \dots$ of i.i.d. random variables such that $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 = \sigma^2 < \infty$. Let $D > 0$ be an arbitrary constant and, for any $m \ge 1$, let d_m satisfy $\mathbf{P}(S_m = d_m) > 0$ and $|d_m| \le D$. Then there exists a positive constant $c = c(D)$ such that the inequality*

$$\mathbf{P}\big(\max_{0<k<m} S_k > M \mid S_m = d_m\big) \le c\,\frac{m^{3/2}}{M^2} \tag{A.1}$$

holds for all $m \ge 2$.

Proof. Since $D > 0$ is a finite constant, the local limit theorem [6] implies that for some $c_1 = c_1(D) > 0$

$$\mathbf{P}\big(S_m = d_m\big) \ge \frac{c_1}{\sqrt{2\pi\sigma^2 m}} \exp\Big\{-\frac{(d_m)^2}{2\sigma^2 m}\Big\} \tag{A.2}$$

uniformly in $m \ge 1$ and $|d_m| \le D$.

On the other hand, by the Etemadi (see, eg, [3, pg. 256]) and Chebyshev inequalities,

$$\mathbf{P}\big(\max_{0<k<m} S_k > M\big) \le 3 \max_{0<k<m} \mathbf{P}\big(S_k > M/3\big) \le 3 \max_{0<k<m} \frac{\sigma^2 k}{(M/3)^2} = \frac{27\sigma^2 m}{M^2}\,.$$

The target bound (A.1) follows immediately from the last two displays and the assumption $|d_m| \le D$. ■

Proof of Lemma 3.3. Let first $m \le M^{7/6}$. Then, the Conditional Chebyshev inequality (A.1) gives

$$\mathbf{P}\big(\max_{0<k<m} S_k > M \mid S_m = d_m\big) \le C_1 \frac{m^{3/2}}{M^2} \le \frac{C_1}{M^{1/4}}\,.$$

Let now m satisfy $M^{7/6} \le m \le \zeta M^2$ (and thus $m \to \infty$). Since $D > 0$ is finite, it follows from the main result in [7] that

$$\mathbf{P}\big(\max_{0<k<m} S_k > M \mid S_m = d_m\big) \le C_2 \exp\Big\{-C_3 \frac{M^2}{m}\Big\} \le C_2 \exp\Big\{-\frac{C_3}{\zeta}\Big\}$$

if only M is large enough, $M \ge M_0$. The small droplet bound (3.4) follows, provided $\zeta > 0$ is chosen small enough. ■

Next, we present a simple one-point analogue of Lemma A.1.

Lemma A.2 *Under the conditions of Lemma A.1, there is a positive constant $\bar{c}$ depending on D and the distribution of ξ only, such that*

$$\max_{0<k<m} \mathbf{P}\big(S_k > M + D \mid S_m = d_m\big) \le \bar{c}\, \frac{m^{5/2}}{M^4}$$

for all $m \ge 2$.

Proof. Using the independence of jumps and the Chebyshev inequality, we get

$$\begin{aligned} \mathbf{P}\big(S_k > M + D, S_m = d_m\big) &\le \mathbf{P}\big(S_k > M\big)\mathbf{P}\big(S_m - S_k < -M\big) \\ &\le \frac{k(m-k)}{M^4}\sigma^4 \le \frac{m^2}{4M^4}\sigma^4\,. \end{aligned} \tag{A.3}$$

Combining this estimate with the lower bound (A.2), we deduce the result. ■

Finally, we present a stronger version of the previous claim.

Lemma A.3 *Under the conditions of Lemma A.1, there is a positive constant $\tilde{c}$ depending on D and the distribution of ξ only, such that*

$$\max_{0<k<m} \mathbf{P}\big(S_k > M + D \mid S_m = d_m\big) \le \tilde{c}\, \frac{m}{M^2}\,.$$

Proof. We start by observing that if ξ_1, ξ_2, ..., are i.i.d. random variables and d_m is chosen such that $\mathbf{P}(S_m = d_m) > 0$, then the variables η_j defined via $\eta_j = (\xi_j \mid S_m = d_m)$ are exchangeable. As a result [2, §24], for any k, $1 \le k \le m$,

$$\begin{aligned} \mathbf{E}\big(\, S_k \mid S_m = d_m \,\big) &= k\, \mathbf{E}\big(\, \xi_1 \mid S_m = d_m \,\big) = \frac{ka}{n}\,, \\ \mathbf{Var}\big(\, S_k \mid S_m = d_m \,\big) &= \frac{k(m-k)}{m-1}\, \mathbf{Var}\big(\, \xi_1 \mid S_m = d_m \,\big)\,. \end{aligned} \tag{A.4}$$

Our next observation formalizes an intuitively obvious fact that for large m the variable ξ_k becomes asymptotically independent of S_m and thus the variances of η_1 and ξ_1 are close to each other. We shall restrict ourselves to the case of integer-valued variables ξ having zero mean and the variance $\mathbf{E}\,\xi^2 = \sigma^2$.

For any finite $D > 0$ there exists m_0, depending only on D and the distribution of ξ, such that the inequality

$$\mathbf{E}\big({\xi_1}^2 \mid S_m = d\big) \le 4\mathbf{E}\,{\xi_1}^2 = 4\sigma^2 \tag{A.5}$$

holds uniformly in $m \ge m_0$ and $|d| \le D$.

To check (A.5), we observe that the characteristic function of $\eta = (\xi \mid S_m = d)$ equals

$$\mathbf{E}\big(e^{is\xi} \mid S_m = d\big) = \frac{\int \phi(t+s)\phi^{m-1}(t)e^{-itd}\,dt}{\int \phi^m(t)e^{-itd}\,dt}\,,$$

where $\phi(t)$ is the unconditional characteristic function of ξ, $\phi(t) = \mathbf{E}e^{it\xi}$, and the integration goes over an interval of periodicity of $\phi(t)$. Consequently,

$$\mathbf{E}\big(\,\xi^2 \mid S_m = d\,\big) = \frac{-\int \phi''(t)\phi^{m-1}(t)e^{-itd}\,dt}{\int \phi^m(t)e^{-itd}\,dt}\,.$$

According to (A.2), we have

$$\int \phi^m(t)e^{-itd}\,dt = \mathbf{P}(S_m = d) \ge \frac{1}{2\sqrt{2\pi\sigma^2 m}}\exp\Big\{-\frac{d^2}{2\sigma^2 m}\Big\}$$

uniformly in $|d| \le D$ and all $m \ge m_1$ with m_1 large enough. Analogously, applying the standard Laplace method to the integral in the numerator (see, eg, [5, 12]), we get

$$\Big|\int \phi''(t)\phi^{m-1}(t)e^{-itd}\,dt\Big| \le \frac{2}{\sqrt{2\pi\sigma^2 m}}\exp\Big\{-\frac{d^2}{2\sigma^2 m}\Big\}\,\mathbf{E}\,\xi^2\,,$$

uniformly in $|d| \le D$ and all $m \ge m_2$ with m_2 large enough. The bound (A.5) follows from the last two displays.

Next, we combine (A.4) and (A.5) to deduce that, uniformly in $|d| \le D$ and all $m \ge m_0$ with m_0 large enough, the inequality

$$\mathbf{E}\big(\,{S_k}^2 \mid S_m = d\,\big) \le \Big(\frac{kd}{m}\Big)^2 + \frac{k(m-k)}{m-1}\,4\mathbf{E}\,\xi^2 \le d^2 + \frac{m^2}{m-1}\mathbf{E}\,\xi^2$$

holds for all $k \in \big\{1,\dots,m\big\}$. By Chebyshev,

$$\mathbf{P}\big(S_k > M + D \mid S_m = d_m\big) \le \frac{C_1 m}{M^2}$$

for all such m.

It remains to consider $m \le m_0$. Denoting

$$p(m_0, D) = \min_{m \le m_0, |d| \le D} \Big\{ \mathbf{P}(\, S_m = d\,) : \mathbf{P}(\, S_m = d\,) > 0 \Big\} > 0\,, \tag{A.6}$$

we immediately get, via Chebyshev,

$$\mathbf{P}\big(S_k > M + D \mid S_m = d_m\big) \le \frac{\mathbf{P}(S_k > M + D)}{p(m_0, D)} \le \frac{C_2 m}{M^2}\,.$$

The proof is finished. ∎

References

[1] Abraham, D.B. and Smith, E.R.: *An exactly Solved Model with a Wetting Transition.* J. Statist. Phys. **43** (1986), nos. 3/4, 621–643

[2] Billingsley, P.: *Convergence of probability measures.* Wiley, 1968

[3] Billingsley, P.: *Convergence of probability measures, 2nd edition.* Wiley, 1999

[4] Bolthausen, E.: *On a functional central limit theorem for random walks conditioned to stay positive.* Ann. Probab. **4** (1976), no. 3, 480–485.

[5] de Bruijn, N. G. *Asymptotic methods in analysis.* North-Holland, 1958

[6] Gnedenko, B.V.: *The theory of probability.* Chelsea, 1962

[7] Kaigh, W.D.: *An invariance principle for random walk conditioned by a late return to zero.* Ann. Probab. **4** (1976), no. 1, 115–121.

[8] Lindvall, T.: *Lectures on the coupling method.* Wiley, 1992

[9] Louchard, G.: *Kac's formula, Levy's local time and Brownian excursion.* J. Appl. Probab. 21 (1984), no. 3, 479–499.

[10] Prähofer, M. and Spohn, H.: *Scale Invariance of the PNG Droplet and the Airy Process.* J. Statist. Phys.**108** (2002), nos. 5/6, 1071–1106

[11] Velenik, Y.: Entropic Repulsion of an Interface in an External Field. Preprint (2003)

[12] Widder, D.V.: The Laplace Transform. Princeton University Press, Princeton, N. J., 1941. 406 pp