Dynamics of a Massive Piston in an Ideal Gas

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Abstract

We study a dynamical system consisting of a massive piston in a cubical container of large size L filled with an ideal gas. The piston has mass $M \sim L^2$ and undergoes elastic collisions with $N \sim L^3$ non-interacting gas particles of mass $m = 1$. We find that, under suitable initial conditions, there is, in the limit $L \to \infty$, a scaling regime with time and space scaled by L, in which the motion of the piston and the one particle distribution of the gas satisfy autonomous coupled equations (hydrodynamical equations), so that the mechanical trajectory of the piston converges, in probability, to the solution of the hydrodynamical equations for a certain period of time. We also discuss heuristically the dynamics of the system on longer intervals of time.

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1 Introduction

The evolution of a macroscopic system consisting of a gas in a container divided by a massive movable wall (piston) is an old problem in statistical physics with a colorful history. It was discussed by Landau and Lifshitz [LL] and later by Lebowitz [L1], Feynman [F], Kubo [Ku], see recent surveys by Lieb [Li], Gruber [G] and others [KBM].

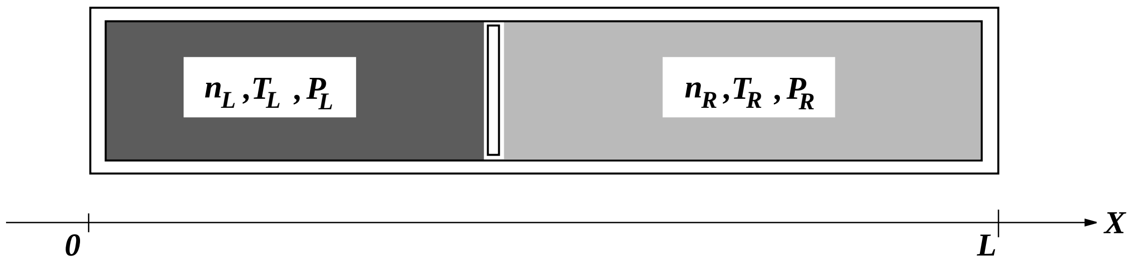

Figure 1: Piston in a cylinder filled with gas.

In its simplest form, the model consists of an isolated cylinder filled with gas and divided into two compartments by a large piston which is free to move along the axis of the cylinder, see Fig. 1. Initially, the piston is held fixed by a clamp and the gas in each compartment evolves independently and is in equilibrium[1]. We denote the density and temperature in the left and right compartments by n_L, T_L and n_R, T_R, respectively. The gas exerts pressure (= force per unit area) on the piston, which is given by equilibrium statistical mechanics as a function of density and temperature: $P_L = P(n_L, T_L)$ and $P_R = P(n_R, T_R)$ on the left and on the right, respectively. At time $t = 0$ the clamp is removed and the piston is released. Now one wants to describe the evolution of the system, especially its limit (final) state as $t \to \infty$.

Starting with $P_L \neq P_R$, the piston moves under the net pressure difference and compresses the gas whose pressure is lower, until its pressure builds up and it pushes the piston back. Depending on the initial values of n_L, T_L, n_R, T_R and the dynamical characteristics of the gases, the piston may follow a complicated trajectory, sloshing back and forth, but gradually it comes to rest at a place where the pressures are equalized on both sides: $P_L = P_R$. At that time one also expects that the gas in each compartment will again be in equilibrium.

[1]For an isolated system of N atoms and total energy E, the equilibrium distribution is defined in statistical mechanics by a uniform probability distribution $\rho_{\rm eq}$ on the energy surface $E =$ const in the phase space ($\rho_{\rm eq}$ is called a microcanonical ensemble, it remains invariant under the dynamics by the Liouville theorem). One says that the system (gas) is "in equilibrium" if its states are "typical" for the measure $\rho_{\rm eq}$. In this state, a macroscopic gas will have (approximately, for large N) a uniform spatial density and a Maxwellian velocity distribution. The latter is defined so that the x, y, z components of the velocity vectors are independent normal random variables $N(0, \sigma^2)$ with the same variance $\sigma^2 = k_B T/m$, where k_B is Boltzmann's constant, T the temperature of the gas (which is a function of E, see below), and m the mass of an atom.

We observe, however, that the equality of pressures $P_L = P_R$ and the fact that the gas in each compartment separately is at equilibrium does not guarantee that $T_L = T_R$. In particular, for dilute gases (which we shall consider from now on), the pressure is related to the density and temperature by $P = nk_BT$, where k_B is Boltzmann's constant, and it is possible that $T_L < T_R$ while $n_L > n_R$, so that the gas in the left compartment is cooler but denser, and in the right one hotter but more dilute (or vice versa). The exact values of the temperatures T_L and T_R, at the time when there is no longer any pressure difference between the left and right and the piston comes to rest, depend on the initial conditions and other characteristics of the system, see [CPS, G], for example. Therefore, we have two possibilities now. If it happens that $T_L = T_R$ as well as $P_L = P_R$, then the system as a whole will be in equilibrium, and we say that it came to a *thermal equilibrium*. On the contrary, if $P_L = P_R$ but $T_L \neq T_R$, the system is said to be at *mechanical equilibrium*, or *quasi-equilibrium*.

One may now ask whether the mechanical equilibrium is stable in the sense that it can last forever (assuming that the whole system remains perfectly isolated from the outside world), or will the gases find ways to exchange energy through the piston and eventually bring the system to a thermal equilibrium? It was claimed in some textbooks, based on a simplistic interpretation of the laws of thermodynamics, see below, that indeed the mechanical equilibrium could persist "forever", cf. [G, GF] for some history.

On the other hand, Landau and Lifshitz [LL], Feynman [F] and many others argued intuitively that the system should converge from the mechanical equilibrium to a thermal equilibrium. They predicted that the cooler compartment should gradually heat up and the hotter one cool down, while the piston slowly moves from the cooler side to the hotter side, so that the pressure balance is maintained until the temperatures are equalized, and the piston makes its final stop.

The confusion about the evolution of the gas after the establishment of mechanical equilibrium is due to the following: Heat conduction through a wall is normally associated with the internal motion of the molecules of the wall colliding with those of the gas and thus exchanging momentum and energy. However, the piston and the walls in our idealized model are supposed to be rigid, solid, and structureless bodies and the gas atoms bounce off them elastically. This idealization is exactly the reason why the gases in the different compartments could be in equilibrium at different temperatures when the piston was clamped. The unclamped piston, on the other hand, interacts with gas atoms as a whole, i.e. as one huge and massive molecule. It then makes tiny microscopic movements (vibrations) induced by collisions with atoms on both sides. Hence, some microscopic exchange of momentum and energy does take place. But these microscopic vibrations of the piston are not part of macroscopic thermodynamics, in which the action of the piston on the gas in each compartment is regarded as an external mechanical force. Under this condition (and assuming that the piston has no entropy of its own) the second law of thermodynamics would predict that the entropy of the gas, as it goes from some initial equilibrium state to a final equilibrium state, could not decrease. When the gas in each compartment is in equilibrium, its thermodynamic entropy is known to be [LL, Ca]

$$S_i = N_i \left[-\log P_i + (1 + 3/2) \log T_i \right] + f(N_i), \qquad i = L, R$$

where $N_L = n_L V_L$ and $N_R = n_R V_R$ denote the number of atoms in the gases, and the explicit form of $f(N_i)$ is irrelevant for us, since its value does not change in time. Now, if our system does evolve from mechanical equilibrium to thermal equilibrium, keeping the pressure balance $P_L = P_R$ and the total kinetic energy $\frac{3}{2} k_B (N_L T_L + N_R T_R)$ fixed, then one can easily compute (we leave this as an exercise) that the pressure of the gases stays constant in time and the total entropy of the system $S = S_L + S_R$ grows until it reaches its maximum at the point of thermal equilibrium. At the same time, the entropy S_R decreases, while S_L increases, since T_R goes down, T_L goes up, and the pressure $P_R = P_L$ remains constant. This decrease of the entropy would, as already noted, violate the second law of thermodynamics, if the piston remained mechanical, see further discussions in [Li, CPS] and critical remarks in [G, GF].

Therefore, the evolution of the system beyond the mechanical equilibrium cannot be described by macroscopic thermodynamics (beyond the statement that any evolution in an isolated macroscopic system will not decrease the total entropy). The actual evolution is a result of microscopic energy transfer between the gases via collisions with the piston. This process is purely microscopic and, in a sense, counterintuitive, as we explain next. Under the collisions with gas atoms on both sides the piston vibrates, i.e. it jiggles back and forth. When the piston moves toward the hotter side, the atoms of the hotter gas bounce off the piston with an increased speed and so gain energy, while the atoms of the cooler gas collide with the piston and slow down, hence lose some energy. When the piston moves toward the cooler side it is vice versa. Since, on the average, the hotter gas must cool down and the cooler gas must heat up, one may conclude that the piston's movements toward the cooler side dominate. On the other hand, the piston has to slowly move toward the hotter side in order to maintain the pressure balance, see above, so its displacements in the direction of the hotter gas actually dominate. It is not quite clear how these seemingly opposite trends manage to coexist. Some physicists joke about a "conspiracy" between the microscopic vibrations of the piston and the incoming atoms of the gases [GF, GP]. In the words of Callen [Ca], "the movable adiabatic wall presents a unique problem with subtleties".

In order to understand the mechanism of the heat transfer across the piston at mechanical equilibrium ($P_L = P_R$), one usually considers the simplest gas of noninteracting particles, that is an *ideal gas*. As early as in 1959, Lebowitz [L1] studied a piston interacting with two infinite reservoirs filled with ideal gases held at different temperatures $T_L \neq T_R$. His piston also interacted with an external potential, e.g. a spring, and it could therefore come to a stationary nonequilibrium state under the influence of the infinite reservoirs. He used an approximation by a Markov process and found the distribution of the piston velocity to be Maxwellian corresponding to some intermediate temperature $T \in (T_L, T_R)$, which led to a systematic heat transfer between the gases. Recently, Gruber and others [GF, PG, GP] used kinetic theory to study a freely movable piston of mass M interacting with two infinite ideal gases of atoms of mass $m \ll M$ at equal pressures but different temperatures. They use the expansion of the Boltzmann equation in $\varepsilon = m/M$ to show that a macroscopic heat flux across the piston does occur whenever $T_L \neq T_R$, hence the system gradually approaches thermal equilibrium. They also found a stationary distribution of the piston velocity, whose average value is given by

$$\langle V \rangle = \frac{\sqrt{2\pi m}\,(\sqrt{k_B T_R} - \sqrt{k_B T_L}\,)}{4M} + o\left(\frac{m}{M}\right) \tag{1.1}$$

(it is independent of the gas densities). We note that if $T_L < T_R$, then $\langle V \rangle > 0$, confirming our previous observation that the piston moves from the cooler side to the hotter side.

Equation (1.1) shows that the average velocity of the piston is different from zero, albeit just of order $O(m/M)$, despite the perfect pressure balance $P_L = P_R$. We note, however, that for a macroscopic-size piston the ratio m/M is so small that the time it takes the piston to cover any noticeable distance is much longer than the age of the universe [GF], so such a phenomenon cannot be observed experimentally.

We conclude that the evolution of the system proceeds in two different stages. The first one is the convergence to a mechanical equilibrium, which is relatively fast and can, in principle, be computed on the basis of macroscopic equations. The second stage is the transition of the system from mechanical equilibrium to thermal equilibrium. This process is very slow and much less understood.

In addition, for the ideal gas (in which the atoms do not interact) a new problem arises. At time $t = 0$, before the piston is released, the gas atoms move independently of each other – every atom bounces off the walls and the clamped piston, without exchanging momentum or energy with other atoms. Therefore, the velocity distribution does not have to be Maxwellian. A stationary state of the ideal gas can be described by any Poisson process with a uniform spacial density and a symmetric velocity distribution. For example, half of the atoms may move toward the piston with unit velocity $v = 1$ and the other half – in the opposite direction with velocity $v = -1$, and this state will be stationary. However, once the piston is released, the atoms start interacting with each other, indirectly, via collisions with the piston. This provides a way to exchange momentum and energy between the atoms. One can expect that these interactions will lead, ultimately, to a true thermal equilibration, when the velocity distribution becomes Maxwellian, as we explain in Section 5. This process, however, may take even longer than the equilibration of the mean kinetic energies described above.

To consider this new process in its "pure" form, we assume that initially the system is already in a homogeneous state – the gas density is constant across the entire cylinder and the velocity distribution is the same in both compartments (but different from Maxwellian). Then there seem to be no forces of any kind that would drive the piston anywhere. In particular, when the piston is initially placed in the middle of the cylinder, then by symmetry there should be no reason for it to move either way! On the other hand, the system is not in equilibrium until the velocity distribution becomes Maxwellian, hence it should find ways to evolve toward equilibrium, thus changing its macroscopic state. We discuss this further in Section 5.

One can also consider a simpler case when that the container is infinitely long on both sides of the piston and the ideal gases have infinite number of atoms, as in [L1, GP, PG]. In that case the problem reduces to the classical Rayleigh gas – a big massive particle submerged in an ideal gas. In particular, our piston in an infinite cylinder becomes a one-dimensional Rayleigh gas, which we describe in some detail.

Let a heavy tagged particle (called molecule) of mass M move on a line under elastic collisions with atoms of mass m of an ideal gas with a uniform density n and some velocity distribution $f(v)\,dv$. Denote by $X(t)$ and $V(t) = \dot{X}(t)$ the position and velocity of the molecule at time t. Even though $f(v)$ need not be Maxwellian, the velocity function $V(t)$ and the coordinate function $X(t)$ can be approximated by certain Gaussian stochastic processes:

Theorem 1.1 (Holley [H]) *Let the density $f(v)$ be symmetric $f(v) = f(-v)$ and have a finite fourth moment $\int v^4 f(v)\,dv < \infty$. Then for every finite $t_0 < \infty$, the function $V(t)\sqrt{M}$ on the interval $[0, t_0]$ converges, in distribution, as $M, n \to \infty$ and $M/n \to \text{const}$, to an Ornstein-Uhlenbeck velocity process $\mathcal{V}_t$, while $X(t)\sqrt{M}$ converges to an Ornstein-Uhlenbeck position process $\mathcal{X}_t$.*

An Ornstein-Uhlenbeck process $(\mathcal{X}_t, \mathcal{V}_t)$ is defined by [Ne]

$$d\mathcal{X}_t = \mathcal{V}_t\,dt, \qquad d\mathcal{V}_t = -a\mathcal{V}_t\,dt + \sqrt{D}\,d\mathcal{W}_t$$

where $a > 0$, $D > 0$ are constants and $\mathcal{W}_t$ a Wiener process. The Ornstein-Uhlenbeck position process $\mathcal{X}_t$ converges in an appropriate limit (e.g. $a \to \infty$, $a^2/D = \text{const}$) to a Wiener process.

We note that the typical velocity of the molecule $V(t)$ is of order $O(1/\sqrt{M})$, which agrees with the equipartition of energy in the system requiring that average energies of all particles be equal, i.e. $M\langle V^2\rangle = m\langle v^2\rangle = m\int v^2 f(v)\,dv$.

Dürr et al [DGL] extended the above theorem to arbitrary dimension and to asymmetric velocity distributions. The main technical difficulty in the proof of this theorem comes from the so called *recollisions*, which occur when an atom collides with the molecule more than once. Recollisions result in intricate autocorrelations in the process $(X(t), V(t))$, which otherwise would be Markovian. The proof essentially consists in estimating the undesirable effect of recollisions and showing that it vanishes in the limit $M \to \infty$.

When the gas is confined in a finite cylinder, though, the effect of recollisions becomes crucial. All atoms will travel to the walls, bounce off it and come back to the piston for more and more collisions. The induced autocorrelations will build up. There is no standard techniques available to estimate (let alone eliminate) the effect of recollisions in general, but we make partial progress in this direction, see Section 4.

In summary, the piston problem raises serious mathematical questions and even leads to confusions in the physical theories. The "notorious piston", as it is known among physicists, again attracted much attention recently due to a series of papers [GF, GP, PG, LPS] where a more extensive mathematical apparatus was developed. At the same time, many new numerical experiments led to better theoretical understanding of the underlying dynamics but also raised some new questions. We emphasize that very few rigorous results are available, even for ideal gases, apart from the Rayleigh-type stochastic approximations in the infinite cylinder mentioned above.

We study the piston in a finite cylinder filled with ideal gases. Since our gases are ideal, we will not need to assume that the velocity distribution of atoms is Maxwellian. Since autocorrelations induced by recollisions present a major difficulty, we specify the initial state in such a way, that during a certain interval of time each gas atom collides with the piston at most twice. The main goal of our work is to describe rigorously the dynamics of the piston during that time interval. We show that, in an appropriate limit, the evolution of the piston and the gas converges to a deterministic process, which satisfies a certain closed system of differential equations. The assumptions that we make here simplify technical considerations but by no means reduce the problem to a triviality. In fact, many intriguing questions still remain open in our context, and we discuss them in the last two sections of the paper.

Precise statement of problem and main results. Consider a cubical domain Λ_L of size L separated into two parts by a movable wall (piston). Each part of Λ_L contains a gas of noninteracting particles of mass $m = 1$. The particles collide with the outer (fixed) walls of Λ_L and with the moving piston elastically. The piston has mass $M = M_L$ and moves along the x-axis under the collisions with the gas particles on both sides. The size L of the cube is a large parameter of our model, and we are interested in the behavior as $L \to \infty$. We will assume that M_L is proportional to the area of the piston, i.e. $M_L \sim L^2$, and the number of gas particles N is proportional to the volume of the cube Λ_L, i.e. $N \sim L^3$, while the particle velocities remain of order one.

The position of the piston at time t is specified by a single coordinate $X = X_L(t)$, $0 \leq X \leq L$, its velocity is then given by $V = V_L(t) = \dot{X}_L(t)$. Since the components of the particle velocities perpendicular to the x-axis play no role in the dynamics, we may assume that each particle has only one coordinate, x, and one component of velocity, v, directed along the x-axis.

When a particle with velocity v collides with the piston with velocity V, their velocities after the collision, v' and V', respectively, are given by

$$V' = (1 - \varepsilon)V + \varepsilon v \tag{1.2}$$

$$v' = -(1 - \varepsilon)v + (2 - \varepsilon)V \tag{1.3}$$

where $\varepsilon = 2m/(M + m)$. We assume that $M + m = 2mL^2/a$, where $a > 0$ is a constant, so that

$$\varepsilon = \frac{2m}{M + m} = \frac{a}{L^2} \tag{1.4}$$

When a particle collides with a wall at $x = 0$ or $x = L$, its velocity just changes sign.

The evolution of the system is then completely deterministic, but one needs to specify the initial conditions. We shall assume that the piston starts at the midpoint $X_L(0) =$

$L/2$ with zero velocity $V_L(0) = 0$ (see also Section 2). The initial configuration of gas particles and their velocities is chosen at random as a realization of a (two-dimensional) Poisson process on the (x, v)-plane (restricted to $0 \le x \le L$) with density $L^2 p_L(x, v)$, where $p_L(x, v)$ is a function satisfying certain conditions, see below, and the factor of L^2 is the cross-sectional area of the container. This means that for any domain $D \subset [0, L] \times \mathbb{R}^1$ the number N_D of gas particles $(x, v) \in D$ at time $t = 0$ has a Poisson distribution with parameter

$$\lambda_D = L^2 \iint_D p_L(x, v)\, dx\, dv$$

For any two nonoverlapping domains, say $D_1 \cap D_2 = \emptyset$, the corresponding numbers N_{D_1} and N_{D_2} are statistically independent. We remark that the total number of gas particles N is a Poisson random variable, too. The total energy and the total initial momentum are random as well.

Let Ω_L denote the space of all possible configurations of gas particles in Λ_L (i.e., all countable subsets of $[0, L] \times \mathbb{R}^1$). For each realization[2] $\omega \in \Omega_L$ the deterministic piston trajectory will be denoted by $X_L(t, \omega)$ and its velocity by $V_L(t, \omega)$.

The above model is a mechanical system whose dynamical characteristics $X_L(t, \omega)$ and $V_L(t, \omega)$ depend on the large parameter L and, for each L, are random (depend on ω).

In order to obtain a deterministic description of the dynamics of the piston one needs to take a limit as $L \to \infty$ and simultaneously rescale space and time. We introduce new space and time coordinates by

$$y = x/L \quad \text{and} \quad \tau = t/L. \tag{1.5}$$

which corresponds to Euler scaling for the hydrodynamical limit transition. We call y and τ the *macroscopic* ("slow") variables, as opposed to the original *microscopic* ("fast") x and t. Now let

$$Y_L(\tau, \omega) = X_L(\tau L, \omega)/L, \qquad W_L(\tau, \omega) = V_L(\tau L, \omega) \tag{1.6}$$

denote the position and velocity of the piston in the macroscopic context. The initial conditions are then $Y_L(0) = X_L(0)/L = 0.5$ and $W(0) = V(0) = 0$.

It is now very natural to assume that the initial density $p_L(x, v)$ agrees with our rescaling:

$$p_L(x, v) = \pi_0(x/L, v) \tag{1.7}$$

where the function $\pi_0(y, v)$ is independent of L. Without loss of generality, we can assume that π_0 is normalized so that

$$\int_0^1 \int_{-\infty}^{\infty} \pi_0(y, v)\, dv\, dy = 1$$

[2] Technically, it is possible that two or more particles collide with the piston simultaneously, and then the dynamics will no longer be defined, but multiple collisions are known to occur with probability zero [H], so we will ignore such anomalies.

Then the mean number of particles in the entire container Λ_L is exactly equal to L^3:

$$E(N) = \iint L^2 p_L(x,v)\,dv\,dx = L^3$$

where $E(\cdot)$ is the expected value.

Furthermore, we assume that the function $\pi_0(y,v)$ satisfies several technical requirements stated below. The meaning and purpose of these assumptions will become clear later.

(P1) *Smoothness.* $\pi_0(y,v)$ is a piecewise C^1 function with uniformly bounded partial derivatives, i.e. $|\partial\pi_0/\partial y| \le D_1$ and $|\partial\pi_0/\partial v| \le D_1$ for some $D_1 > 0$.

(P2) *Discontinuity lines.* $\pi_0(y,v)$ may be discontinuous on the line $y = Y_L(0)$ (i.e., "on the piston"). In addition, it may have a finite number ($\le K_1$) of other discontinuity lines in the (y,v)-plane with strictly positive slopes (each line is given by an equation $v = f(y)$ where $f(y)$ is C^1 and $0 < c_1 < f'(y) < c_2 < \infty$).

(P3) *Density bounds.* Let

$$\pi_0(y,v) > \pi_{\min} > 0 \qquad \text{for} \quad v_1 < |v| < v_2 \tag{1.8}$$

for some $0 < v_1 < v_2 < \infty$, and

$$\sup_{y,v} \pi_0(y,v) = \pi_{\max} < \infty \tag{1.9}$$

The requirements (1.8) and (1.9) basically mean that $\pi_0(y,v)$ takes values of order one.

(P4) *Velocity "cutoff".* Let

$$\pi_0(y,v) = 0, \quad \text{if} \quad |v| \le v_{\min} \quad \text{or} \quad |v| \ge v_{\max} \tag{1.10}$$

with some $0 < v_{\min} < v_{\max} < \infty$. This means that the speed of gas particles is bounded from above by $v_{\max}$ and from below by $v_{\min}$.

(P5) *Approximate pressure balance.* $\pi_0(y,v)$ must be nearly symmetric about the piston, i.e.

$$|\pi_0(y,v) - \pi_0(1-y,-v)| < \varepsilon_0 \tag{1.11}$$

for all $0 < y < 1$ and some sufficiently small $\varepsilon_0 > 0$.

The requirements (P4) and (P5) are crucial. We will see that they are made to ensure that the speed of the piston $|V_L(t,\omega)|$ will be smaller than the minimum speed of the gas particles, with probability close to one, for times $t = O(L)$. Such assumptions were first made in [LPS].

We think of D_1, K_1, c_1, c_2, v_1, v_2, $v_{\min}$, $v_{\max}$, $\pi_{\min}$ and $\pi_{\max}$ in (P1)–(P4) as fixed (global) constants and ε_0 in (P5) as an adjustable small parameter. We will assume throughout the paper that ε_0 is small enough, meaning that

$$\varepsilon_0 < \bar{\varepsilon}_0(D_1, K_1, c_1, c_2, v_1, v_2, v_{\min}, v_{\max}, \pi_{\min}, \pi_{\max})$$

It is important to note that the hydrodynamic limit does *not* require that $\varepsilon_0 \to 0$. The parameter ε_0 stays positive and fixed as $L \to \infty$.

Now we state our main result:

Theorem 1.2 *There is an L-independent function $Y(\tau)$ defined for all $\tau \geq 0$ and a positive $\tau_* \approx 2/v_{\max}$ (actually, $\tau_* \to 2/v_{\max}$ as $\varepsilon_0 \to 0$), such that*

$$\sup_{0 \leq \tau \leq \tau_*} |Y_L(\tau, \omega) - Y(\tau)| \to 0 \tag{1.12}$$

and

$$\sup_{0 \leq \tau \leq \tau_*} |W_L(\tau, \omega) - W(\tau)| \to 0 \tag{1.13}$$

in probability, as $L \to \infty$. Here $W(\tau) = \dot{Y}(\tau)$.

This theorem establishes the convergence in probability of the random functions $Y_L(\tau, \omega), W(\tau, \omega)$ characterizing the mechanical evolution of the piston to the deterministic functions $Y(\tau), W(\tau)$, in the hydrodynamical limit $L \to \infty$.

The functions $Y(\tau)$ and $W(\tau)$ satisfy certain (Euler-type) differential equations stated in the next section. Those equations have solutions for all $\tau \geq 0$, but we can only guarantee the convergence (1.12) and (1.13) for $\tau < \tau_*$. What happens for $\tau > \tau_*$, especially as $\tau \to \infty$, remains an open problem. Some numerical results and heuristic observations in this direction are presented in Section 5.

Remarks. The function $Y(\tau)$ is at least C^1 and, furthermore, piecewise C^2. On the interval $(0, \tau_*)$, its first derivative $W = \dot{Y}$ (velocity) and its second derivative $A = \ddot{Y}$ (acceleration) remain ε_0-small: $\sup_\tau |W(\tau)| \leq \text{const} \cdot \varepsilon_0$ and $\sup_\tau |A(\tau)| \leq \text{const} \cdot \varepsilon_0$, see the next section.

We will also estimate the speed of convergence in (1.12) and (1.13). Precisely, we show that there is a $\tau_0 > 0$ ($\tau_0 \approx 1/v_{\max}$) such that

$$|Y_L(\tau, \omega) - Y(\tau)| = O(\ln L / L)$$

for $0 < \tau < \tau_0$ and

$$|Y_L(\tau, \omega) - Y(\tau)| = O(\ln L / L^{1/7})$$

for $\tau_0 < \tau < \tau_*$. The same bounds are valid for $|W_L(\tau, \omega) - W(\tau)|$, see Sections 3 and 4. These estimates hold with "overwhelming" probability, specifically they hold for all $\omega \in \Omega_L^* \subset \Omega_L$ such that $P(\Omega_L^*) = 1 - O(L^{-\ln L})$.

2 Hydrodynamical equations

The equations describing the deterministic function $Y(\tau)$ involve another deterministic function – the scaled density of the gas $\pi(y, v, \tau)$. Initially, $\pi(y, v, 0) = \pi_0(y, v)$, and for $\tau > 0$ the density $\pi(y, v, \tau)$ evolves according to the following rules.

(H1) *Free motion.* Inside the container the density satisfies the standard continuity equation for a noninteracting particle system without external forces:

$$\left(\frac{\partial}{\partial \tau} + v\,\frac{\partial}{\partial y}\right)\pi(y, v, \tau) = 0 \tag{2.1}$$

for all y except $y = 0$, $y = 1$ and $y = Y(\tau)$.

Equation (2.1) has a simple solution

$$\pi(y, v, \tau) = \pi(y - vs, v, \tau - s) \tag{2.2}$$

for $0 < s < \tau$ such that $y - vr \notin \{0, Y(\tau - r), 1\}$ for all $r \in (0, s)$. Equation (2.2) has one advantage over (2.1): it applies to all points (y, v), including those where the function π is not differentiable.

(H2) *Collisions with the walls.* At the walls $y = 0$ and $y = 1$ we have

$$\pi(0, v, \tau) = \pi(0, -v, \tau) \tag{2.3}$$

$$\pi(1, v, \tau) = \pi(1, -v, \tau) \tag{2.4}$$

(H3) *Collisions with the piston.* At the piston $y = Y(\tau)$ we have

$$\begin{aligned} \pi(Y(\tau) - 0, v, \tau) &= \pi(Y(\tau) - 0, 2W(\tau) - v, \tau) \quad \text{for } v < W(\tau) \\ \pi(Y(\tau) + 0, v, \tau) &= \pi(Y(\tau) + 0, 2W(\tau) - v, \tau) \quad \text{for } v > W(\tau) \end{aligned} \tag{2.5}$$

where v represents the velocity after the collision and $2W(\tau) - v$ that before the collision; here

$$W(\tau) = \frac{d}{d\tau} Y(\tau) \tag{2.6}$$

is the (deterministic) velocity of the piston.

It remains to describe the evolution of $W(\tau)$. Suppose the piston's position at time τ is Y and its velocity W. The piston is affected by the particles (y, v) hitting it from the right (such that $y = Y + 0$ and $v < W$) and from the left (such that $y = Y - 0$ and $v > W$).

(H4) *Piston's velocity.* The velocity $W = W(\tau)$ of the piston must satisfy the equation

$$\int_W^\infty (v-W)^2 \pi(Y-0,v,\tau)\,dv = \int_{-\infty}^W (v-W)^2 \pi(Y+0,v,\tau)\,dv \tag{2.7}$$

see also an additional requirement (H4') below.

In physical terms, (2.7) is a pressure balance: the piston "chooses" velocity W so that the pressure of the incoming particles balances out. Equation (2.7) is instrumental for our deterministic approximation of the piston dynamics.

One can combine the two integrals in (2.7) into one by introducing the density of the particles colliding with the piston ("density on the piston") by

$$q(v,\tau;Y,W) = \begin{cases} \pi(Y+0,v,\tau) & \text{if } v < W \\ \pi(Y-0,v,\tau) & \text{if } v > W \end{cases} \tag{2.8}$$

Then (2.7) can be rewritten as

$$\int_{-\infty}^\infty (v-W(\tau))^2 \,\mathrm{sgn}(v-W(\tau))\, q(v,\tau;Y(\tau),W(\tau))\,dv = 0$$

We also remark that for $\tau > 0$, when (2.5) holds,

$$W(\tau) = \frac{\int v\pi(Y-0,v,\tau)\,dv}{\int \pi(Y-0,v,\tau)\,dv} = \frac{\int v\pi(Y+0,v,\tau)\,dv}{\int \pi(Y+0,v,\tau)\,dv}$$

i.e. the piston's velocity is the average of the nearby particle velocities on each side.

The system of (hydrodynamical) equations given in (H1)–(H4) is closed and, given appropriate initial conditions, should completely determine the functions $Y(\tau)$, $W(\tau)$ and $\pi(y,v,\tau)$ for $\tau > 0$, as we will see shortly.

To specify the initial conditions, we set $\pi(y,v,0) = \pi_0(y,v)$ and $Y(0) = 0.5$. The initial velocity $W(0)$ does not have to be specified, it comes "for free" as the solution of the equation (2.7) at time $\tau = 0$. It is easy to check that the initial speed $|W(0)|$ will be smaller than $v_{\min}$, in fact $W(0) \to 0$ as $\varepsilon_0 \to 0$ in (P5).

We first determine conditions under which equation (2.7) has a solution W. Let

$$v_{\sup}^-(\tau) = \sup\{v :\, \pi(Y-0,v,\tau) > 0\}$$

(with the convention that the supremum of an empty set is $-\infty$) and

$$v_{\inf}^+(\tau) = \inf\{v :\, \pi(Y+0,v,\tau) > 0\}$$

(similarly, the infimum of an empty set must be set to $+\infty$).

Lemma 2.1 *We have three cases:*

(a) *If $v_{\sup}^- > v_{\inf}^+$ or $v_{\sup}^- = v_{\inf}^+ \in \mathbb{R}$, then (2.7) has a unique solution $W \in [v_{\inf}^+, v_{\sup}^-]$.*

(b) *If* $v^-_{\rm sup} < v^+_{\rm inf}$, *then the solutions of (2.7) occupy the entire interval* $[v^-_{\rm sup}, v^+_{\rm inf}]$.

(c) *If* $v^-_{\rm sup} = v^+_{\rm inf} = \infty$ *or* $v^-_{\rm sup} = v^+_{\rm inf} = -\infty$, *then (2.7) has no real solutions.*

Proof. In the case (a), the difference between the left hand side and the right hand side of (2.7) is a continuous and strictly monotonically decreasing function of W. For $W < v^+_{\rm inf}$ it is positive, and for $W > v^-_{\rm sup}$ negative. The rest of the proof goes by direct inspection. □

It is easy to show (we do not elaborate) that under our assumptions (P1)–(P4) for every $\tau > 0$ the density $\pi(y, v, \tau)$ has a compact support on the y, v plane, i.e. $\pi(y, v, \tau) \equiv 0$ for all $|v| > v_{\rm max}(\tau)$. Therefore, the "no solution" case (c) never occurs. The multiple solution case (b) is very unlikely, but not impossible. If that happens, the velocity $W(\tau)$ must be defined uniquely by an additional requirement:

(H4') If $W(\tau - 0) \in [v^-_{\rm sup}, v^+_{\rm inf}]$, we define $W(\tau)$ by continuity, $W(\tau) = W(\tau - 0)$. If $W(\tau - 0) < v^-_{\rm sup}$ or $W(\tau - 0) > v^+_{\rm inf}$, we set $W(\tau) = v^-_{\rm sup}$ or $W = v^+_{\rm inf}$, respectively.

This completes the definition of $W(\tau)$ started by (H4).

For generic piecewise smooth densities $\pi(y, v, \tau)$, the velocity $W(\tau)$ is continuous, but in some cases the continuity of $W(\tau)$ might be broken. The following simple lemma will be helpful, though:

Lemma 2.2 *Suppose that for every* $\tau \in [a, b]$ *the density* $\pi(y, v, \tau)$ *is piecewise* C^1 *and has a finite number of* C^1 *smooth discontinuity lines on the* y, v *plane with positive slopes, as we require of* $\pi_0(y, v)$ *in Section 1. Then* $W(\tau)$ *will be continuous and piecewise differentiable on the interval* $[a, b]$.

We now pause to make a few remarks. The piston mass is never used in our equations, because its macroscopic mass is zero. Indeed, for the mechanical system described in Section 1, the piston mass is $\sim L^2$, while the total mass of the gas particles is $\sim L^3$, hence the relative mass of the piston vanishes as $L \to \infty$. Consider now the total (macroscopic) mass of the gas

$$\mathcal{M}_{\rm tot}(\tau) = \int_0^1 \int \pi(y, v, \tau)\, dv\, dy$$

and the mass in the left and right compartments, separately,

$$\mathcal{M}_L(\tau) = \int_0^{Y(\tau)} \int \pi(y, v, \tau)\, dv\, dy$$

$$\mathcal{M}_R(\tau) = \int_{Y(\tau)}^1 \int \pi(y, v, \tau)\, dv\, dy$$

and the total kinetic energy

$$2\mathcal{E}_{\rm tot}(\tau) = \int_0^1 \int v^2 \pi(y, v, \tau)\, dv\, dy$$

The following lemma is left as a (simple) exercise:

Lemma 2.3 *The quantities $\mathcal{M}_{\rm tot}$, $\mathcal{M}_L$, $\mathcal{M}_R$, and $\mathcal{E}_{\rm tot}$ remain constant in τ.*

The main equation (2.7) also preserves the total momentum $\int\int v\pi(y,v,\tau)\,dv\,dy$, but this quantity changes due to collisions with the walls.

Remark. Previously, Lebowitz, Piasecki and Sinai [LPS] studied the piston dynamics under essentially the same initial conditions as our (P1)–(P5). They argued heuristically that the piston dynamics could be approximated by certain deterministic equations in the original (microscopic) variables x and t. In fact, the present work grew as a continuation of [LPS]. The deterministic equations found in [LPS] correspond to our (2.2)–(2.6) with obvious transformation back to the variables x, t, but our main equation (2.7) has a different counterpart in the context of [LPS], which reads

$$\frac{d}{dt}V(t) = a\left[\int_V^\infty (v - V(t))^2\pi(Y-0,v,t)\,dv - \int_{-\infty}^V (v-V(t))^2\pi(Y+0,v,t)\,dv\right] \quad (2.9)$$

Here $X = X(t)$ and $V = V(t) = \dot{X}(t)$ denote the deterministic position and velocity of the piston and $\pi(x,v,t)$ the density of the gas (the constant a appeared in (1.4)). We refer to [LPS] for more details and a heuristic derivation of (2.9). Since (2.9), unlike our (2.7), is a differential equation, the initial velocity $V(0)$ has to be specified separately, and it is customary to set $V(0) = 0$. Equation (2.9) can be reduced to (2.7) in the limit $L \to \infty$ as follows. One can show (we omit details) that (2.9) is a dissipative equation whose solution with any (small enough) initial condition $V(0)$ converges to the solution of (2.7) during a t-time interval of length $\sim \ln L$. That interval has length $\sim L^{-1}\ln L$ on the τ axis, and so it vanishes as $L \to \infty$, this is why we replace (2.9) with (2.7) and ignore the initial condition $V(0)$ when working with the thermodynamic variables τ and y. For the same reasons, it will be convenient to reset the initial value of the piston velocity in the mechanical model of Section 1 to from $V(0) = 0$ to $V(0) = W(0)$, see Theorem 3.5 below. The equation (2.9) will not be used anymore in this paper.

We now describe the solution of the hydrodynamical equations (H1)–(H4) in more detail. Assume that for some $\tau > 0$ the gas density $\pi(y,v,\tau)$ satisfies the following requirements, similar to (P1)–(P5) imposed on the initial function $\pi_0(y,v)$ in Section 1:

(P1') *Smoothness.* $\pi(y,v,\tau)$ is a piecewise C^1 function with uniformly bounded partial derivatives, i.e. $|\partial\pi/\partial y| \le D_1'$ and $|\partial\pi/\partial v| \le D_1'$ for some $D_1' > 0$.

(P2') *Discontinuity lines.* $\pi(y,v,\tau)$ has a finite number ($\le K_1'$) of discontinuity lines in the (y,v)-plane with strictly positive slopes (each line is given by an equation $v = f(y)$ where $f(y)$ is C^1 and $0 < c_1' < f'(y) < c_2' < \infty$).

(P3') *Density bounds.* Let

$$\pi(y,v,\tau) > \pi'_{\min} > 0 \qquad \text{for} \ \ v_1' < |v| < v_2' \quad (2.10)$$

for some $0 < v_1' < v_2' < \infty$, and

$$\sup_{y,v} \pi(y, v, \tau) = \pi'_{\max} < \infty \tag{2.11}$$

(P4') *Velocity "cutoff"*. Let

$$\pi(y, v, \tau) = 0, \quad \text{if} \quad |v| \le v'_{\min} \quad \text{or} \quad |v| \ge v'_{\max} \tag{2.12}$$

with some $0 < v'_{\min} < v'_{\max} < \infty$.

Lastly, we want to assume, similarly to (P5), that $\pi(y, v, \tau)$ is nearly symmetric about the piston, but this assumption requires a little extra work, since the piston does not have to stay at the middle point $Y(0) = 0.5$. For every $Y \in (0, 1)$ denote by h_Y the unique homeomorphism of $[0, 1]$ such that $h_Y(0) = 1$, $h_Y(1) = 0$, $h_Y(Y) = Y$ and h_Y is linear on the subsegments $[0, Y]$ and $[Y, 1]$. Next, we consider $[0, Y] \times \mathbb{R}$ as a manifold in which points $(0, v)$ and $(0, -v)$ are identified for all $v > 0$, and so are the points (Y, v) and $(Y, -v)$ for $v > 0$. Similarly, let $[Y, 1] \times \mathbb{R}$ be a manifold in which one identifies $(1, v)$ with $(1, -v)$ and (Y, v) and $(Y, -v)$ for all $v > 0$. We denote by d_Y the distance on each of these two manifolds induced by the Euclidean metric $(dy^2 + dv^2)^{1/2}$. The reason why we need this special distance will be clear later, in the proof of Proposition 2.10.

(P5') *Approximate pressure balance.* We require that

$$|Y(\tau) - 0.5| < \varepsilon_0' \tag{2.13}$$

and for any point (y, v) with $0 \le y \le 1$ and $v'_{\min} \le |v| \le v'_{\max}$ there is another point (y_*, v_*) "across the piston", i.e. such that $(y - Y)(y_* - Y) < 0$, where $Y = Y(\tau)$, satisfying

$$d_Y((y_*, v_*), (h_Y(y), -v)) < \varepsilon_0' \tag{2.14}$$

and

$$|\pi(y, v, \tau) - \pi(y_*, v_*, \tau)| < \varepsilon_0' \tag{2.15}$$

for some sufficiently small $\varepsilon_0' > 0$. In addition, we require that

$$\varepsilon_0' < C_0' \varepsilon_0 \tag{2.16}$$

with some constant $C_0' > 0$.

Actually, the map $(y, v) \mapsto (y_*, v_*)$ involved in (P5'), which we will denote by R_τ, is one-to-one and will be explicitly constructed below, in the proof of Proposition 2.10.

Again, we think of D_1', K_1', c_1', c_2', v_1', v_2', $v'_{\min}$, $v'_{\max}$, $\pi'_{\min}$, $\pi'_{\max}$, and now also C_0', as global constants. They must be bounded on the time interval on which we consider the dynamics (and $v'_{\min}$, $\pi'_{\min}$ must be bounded away from zero), hence we may treat all these constants as independent of τ. By (2.16), ε_0' is, just like ε_0 in (P5), a small adjustable parameter.

Now we derive rather elementary but important consequences of the above assumptions. Since the density $\pi(y, v, \tau)$ vanishes for $|v| < v'_{\min}$, so does the function $q(v, \tau; Y, W)$ defined by (2.8). Moreover, for all $|W| < v'_{\min}$, the function $q(v, \tau; Y, W)$ will be independent of W and can be redefined by

$$q(v, \tau; Y) = \begin{cases} \pi(Y + 0, v, \tau) & \text{if} \quad v < 0 \\ \pi(Y - 0, v, \tau) & \text{if} \quad v > 0 \end{cases} \tag{2.17}$$

Also, the equation (2.7) can be simplified: the factor $\operatorname{sgn}(v - W)$ can be replaced by $\operatorname{sgn} v$. Then, expanding the squares in (2.7) reduces it to a simple quadratic equation for W:

$$\mathcal{Q}_0 W^2 - 2\mathcal{Q}_1 W + \mathcal{Q}_2 = 0 \tag{2.18}$$

where

$$\mathcal{Q}_0 = \int \operatorname{sgn} v \cdot q(v, \tau; Y)\, dv \tag{2.19}$$

$$\mathcal{Q}_1 = \int v \operatorname{sgn} v \cdot q(v, \tau; Y)\, dv \tag{2.20}$$

$$\mathcal{Q}_2 = \int v^2 \operatorname{sgn} v \cdot q(v, \tau; Y)\, dv \tag{2.21}$$

with $Y = Y(\tau)$. The integrals $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ have the following physical meaning:

$$m\mathcal{Q}_0 = m_L - m_R$$

$$m\mathcal{Q}_1 = p_L - p_R$$

$$m\mathcal{Q}_2 = 2(e_L - e_R)$$

where m_L, p_L, e_L represent the total mass, momentum and energy of the incoming gas particles (per unit length) on the left hand side of the piston, and m_R, p_R, e_R – those on the right hand side of it. The value $\mathcal{Q}_2$ also represents the net pressure exerted on the piston by the gas if the piston did not move. Of course, if $\mathcal{Q}_2(\tau) = 0$, then we must have $W(\tau) = 0$, which agrees with (2.18). The following lemma easily follows from (P1')–(P5'). It means that the function $q(v, \tau; Y(\tau))$ is nearly symmetric in v about $v = 0$.

Lemma 2.4 *For any smooth function $f(v)$ defined for $v > 0$ we have*

$$\left| \int_0^\infty f(v)\, q(v, \tau; Y(\tau))\, dv \; - \; \int_{-\infty}^0 f(-v)\, q(v, \tau; Y(\tau))\, dv \right| \le C_f\, \varepsilon_0$$

where the factor $C_f > 0$ depends on f but not on ε_0.

Convention. We call constants that do not depend on our small adjustable parameter ε_0 involved in (P5) and (P5') *global constants* (such as C_f in the above lemma). All the constants in the requirements (P1)–(P5) and (P1')–(P5') are global, except ε_0 itself and the related ε_0'. In many cases, we will denote various global constants by C_i, $i \geq 0$, or just by C.

Lemma 2.4 implies that $\mathcal{Q}_0$ and $\mathcal{Q}_2$ are small, more precisely

$$\max\{|\mathcal{Q}_0|, |\mathcal{Q}_2|\} \leq C\varepsilon_0 \tag{2.22}$$

where $C > 0$ is a global constant. At the same time, the assumption (P3') guarantees that

$$\mathcal{Q}_1 \geq \mathcal{Q}_{1,\min} > 0 \tag{2.23}$$

where $\mathcal{Q}_{1,\min}$ is another global constant.

If ε_0 is small enough, there is a unique root of the quadratic polynomial (2.18) on the interval $(-v'_{\min}, v'_{\min})$, which corresponds to the only solution of (2.7). Since this root is smaller, in absolute value, than the other root of (2.18), it can expressed by

$$W = \frac{\mathcal{Q}_1 - \sqrt{\mathcal{Q}_1^2 - \mathcal{Q}_0\mathcal{Q}_2}}{\mathcal{Q}_0} \tag{2.24}$$

where the sign before the radical is "−", not "+". Of course, (2.24) applies whenever $\mathcal{Q}_0 \neq 0$, while for $\mathcal{Q}_0 = 0$ we simply have

$$W = \frac{\mathcal{Q}_2}{2\mathcal{Q}_1} \tag{2.25}$$

Corollary 2.5 *If ε_0 is small enough, then*

$$|W(\tau)| \leq \mathcal{B}\varepsilon_0 < v'_{\min}/3 \tag{2.26}$$

with some global constant $\mathcal{B} > 0$.

Proof. This immediately follows from equations (2.22)–(2.25). □

We now make an important remark.

Remark (Extension). Consider the density of the incoming gas particles on the left hand side of the piston, i.e. $\pi(y, v, \tau)$ for $y = Y(\tau) - 0$ and $v > v'_{\min}$. This function "terminates" on the piston, i.e. has a discontinuity in y at $y = Y(\tau)$. But it can be naturally extended smoothly "across the piston", i.e. for $y > Y(\tau)$ if one ignores the interaction of the gas coming from the left compartment with the piston at times $s \in (\tau - \delta, \tau)$ and applies the rule (H1) instead, as if the gas "passed through the piston". This defines a smooth extension of $\pi(y, v, \tau)$ from the region $y \leq Y(\tau)$ to the region $Y(\tau) < y < Y(\tau) + O(\delta)$ for all $v \geq v'_{\min}$. This extension allows us to differentiate $q(v, \tau; Y)$ defined by (2.17) with respect to Y for any $v \geq v'_{\min}$. A similar extension can be made for the density $\pi(y,v,\tau)$ from the region $y \geq Y(\tau)$ to the region $Y(\tau) - O(\delta) < y < Y(\tau)$ for all $v \leq -v'_{\min}$, hence $q(v,\tau;Y)$ becomes differentiable with respect to Y for $v \leq -v'_{\min}$. We note that our extension can be unambiguously defined because we only need it for $|v| \geq v'_{\min}$ while the piston's velocity remains smaller than $v'_{\min}$.

Now the quantities $\mathcal{Q}_0$, $\mathcal{Q}_1$, and $\mathcal{Q}_2$ defined by (2.19)–(2.21) become differentiable in Y for each fixed τ, and the assumptions (P1')–(P4') easily imply that

$$|d\mathcal{Q}_i/dY| \leq C_1, \qquad i = 0,1,2 \tag{2.27}$$

where $C_1 > 0$ is a global constant.

Corollary 2.6 *The piston acceleration $A(\tau) = dW(\tau)/d\tau$ satisfies*

$$|A(\tau)| \leq C\varepsilon_0 \tag{2.28}$$

with a global constant $C > 0$.

Proof. We differentiate the quadratic equation (2.18) with respect to τ and get

$$A(\tau) = \frac{(d\mathcal{Q}_0/d\tau)W^2 - 2(d\mathcal{Q}_1/d\tau)W + (d\mathcal{Q}_2/d\tau)}{2(\mathcal{Q}_1 - \mathcal{Q}_0 W)}$$

Clearly, the denominator is bounded away from zero, and the numerator has an upper bound of order ε_0, because $|d\mathcal{Q}_i/d\tau| = |(d\mathcal{Q}_i/dY)W| \leq \text{const}\cdot\varepsilon_0$ by (2.27) and (2.26). □

More importantly, we can now derive the existence and uniqueness of the solution of the hydrodynamical equations (H1)–(H4) as long as the conditions (P1')–(P5') continue holding:

Lemma 2.7 *If the hydrodynamical equations* (H1)–(H4) *have a solution on an interval $0 \leq \tau \leq T$ and the conditions* (P1')–(P5') *hold on this interval, then the solution is unique at $\tau = T$ and can be extended immediately beyond the point $\tau = T$.*

Proof. The only differential equation in our system (H1)–(H4) is (2.6), in which $W(\tau)$ is the root of the quadratic equation (2.18) given by (2.24)–(2.25). Due to the above Extension Remark we can think of W as an implicit function of Y, i.e. effectively $W = F(Y,\tau)$. Then the differential equation (2.6) takes a canonical form

$$\frac{d}{d\tau}Y(\tau) = F(Y(\tau),\tau) \tag{2.29}$$

For this equation to have a unique solution, it suffices that $F(Y,\tau)$ has a bounded partial derivative with respect to Y.

Since W is a root of the quadratic equation (2.18), we can differentiate (2.18) with respect to Y and get

$$\frac{\partial F(Y,\tau)}{\partial Y} = \frac{(d\mathcal{Q}_0/dY)W^2 - 2(d\mathcal{Q}_1/dY)W + (d\mathcal{Q}_2/dY)}{2(\mathcal{Q}_1 - \mathcal{Q}_0 W)}$$

We already know that the denominator is bounded away from zero. It follows from (2.27) that the numerator stays bounded above, hence

$$\left|\frac{\partial F(Y,\tau)}{\partial Y}\right| \le \kappa \tag{2.30}$$

with a global constant $\kappa > 0$. □

Next we consider the evolution of a point (y, v) in the domain

$$\mathcal{G} := \{(y,v):\ 0 \le y \le 1\}$$

under the rules (H1)–(H3), i.e. as it moves freely with constant velocity and collides elastically with the walls and the piston. Denote by (y_τ, v_τ) its position and velocity at time $\tau \ge 0$. Then (H1) translates into $\dot{y}_\tau = v_\tau$ and $\dot{v}_\tau = 0$ whenever $y_\tau \notin \{0, 1, Y(\tau)\}$, (H2) becomes $(y_{\tau+0}, v_{\tau+0}) = (y_{\tau-0}, -v_{\tau-0})$ whenever $y_{\tau-0} \in \{0,1\}$, and (H3) gives

$$(y_{\tau+0}, v_{\tau+0}) = (y_{\tau-0}, 2W(\tau) - v_{\tau-0}) \tag{2.31}$$

whenever $y_{\tau-0} = Y(\tau)$. Note that (2.31) corresponds to a special case of the mechanical collision rules (1.2)–(1.3) with $\varepsilon = 0$ (equivalently, $m = 0$). Hence the point (y, v) moves in $\mathcal{G}$ as if it was a gas particle with zero mass.

The motion of points in (y, v) is described by a one-parameter family of transformations $\mathcal{F}^\tau : \mathcal{G} \to \mathcal{G}$ defined by $\mathcal{F}^\tau(y_0, v_0) = (y_\tau, v_\tau)$ for $\tau > 0$. We will also write $\mathcal{F}^{-\tau}(y_\tau, v_\tau) = (y_0, v_0)$. According to (H1)–(H3), the density $\pi(y, v, \tau)$ satisfies a simple equation

$$\pi(y_\tau, v_\tau, \tau) = \pi(\mathcal{F}^{-\tau}(y_\tau, v_\tau), 0) = \pi_0(y_0, v_0) \tag{2.32}$$

for all $\tau \ge 0$. Also, it is easy to see that for each $\tau > 0$ the map $\mathcal{F}^\tau$ is one-to-one and preserves area, i.e. $\det |D\mathcal{F}^\tau(y, v)| = 1$.

Now, because of (P4), the initial density $\pi_0(y, v)$ can only be positive in the region

$$\mathcal{G}^+ := \{(y,v):\ 0 \le y \le 1,\ v_{\min} \le |v| \le v_{\max}\}$$

hence we will restrict ourselves to points $(y, v) \in \mathcal{G}^+$ only. At any time $\tau > 0$, the images of those points will be confined to the region $\mathcal{G}^+(\tau) := \mathcal{F}^\tau(\mathcal{G}^+)$. In particular, $\pi(y, v, \tau) = 0$ for $(y, v) \notin \mathcal{G}^+(\tau)$.

We now make an important observation. If a point (y_τ, v_τ) collides with a piston whose velocity is slow, $|W(\tau)| \ll |v_\tau|$, they cannot recollide too soon: the point must travel to a wall, bounce off it, and then travel back to the piston before it hits it again. This is quantified in the following lemma:

Lemma 2.8 *Let a point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ collide with the piston, i.e. $y_\tau = Y(\tau)$. Then during the interval $(\tau, \tau + \Delta)$ with*

$$\Delta = \frac{1 - 2\mathcal{B}\varepsilon_0\tau}{v'_{\max} + 3\mathcal{B}\varepsilon_0}$$

it cannot recollide with the piston, i.e. $y_s \ne Y(s)$ for $s \in (\tau, \tau + \Delta)$, provided (P1')–(P5') *continue holding during this interval.*

Proof. The point's speed after the collision is at least $v'_{\min} - 2\mathcal{B}\varepsilon_0$ and at most $v'_{\max} + 2\mathcal{B}\varepsilon_0$. The piston cannot "catch up" with it, since $|W(\tau)| < v'_{\min} - 2\mathcal{B}\varepsilon_0$ by (2.26). So, the point travels to the wall, bounces off it, and travels back to the piston, and all that will take time

$$\Delta \geq 2D/(v'_{\max} + 3\mathcal{B}\varepsilon_0)$$

where $D = \min\{Y(\tau), 1 - Y(\tau)\} \geq 0.5 - \mathcal{B}\varepsilon_0\tau$. □

Therefore, as long as (P1')–(P4') hold, the collisions of each moving point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ with the piston occur at well separated time moments, which allows us to effectively count them. For $(x, v) \in \mathcal{G}^+$

$$N(y, v, \tau) = \#\{s \in (0, \tau) : \; y_s = Y(s), \; v_s \neq W(s)\}$$

is the number of collisions of the point (y, v) with the piston during the interval $(0, \tau)$. For each $\tau > 0$, we partition the region $\mathcal{G}^+(\tau)$ into subregions

$$\mathcal{G}_n^+(\tau) := \{\mathcal{F}^\tau(y, v) : \; (y, v) \in \mathcal{G}^+ \;\; \& \;\; N(y, v, \tau) = n\}$$

so $\mathcal{G}_n^+(\tau)$ is occupied by the points that at time τ have experienced exactly n collisions with the piston during the interval $(0, \tau)$.

Now, for each $n \geq 1$ we define $\tau_n > 0$ to be the first time when a point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ experiences its $(n+1)$-st collision with the piston, i.e.

$$\tau_n = \sup\{\tau > 0 : \; \mathcal{G}_{n+1}^+(\tau) = \emptyset\}$$

In particular, $\tau_1 > 0$ is the earliest time when a point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ experiences its first recollision with the piston. Hence, no recollisions occur on the interval $[0, \tau_1)$, and we call it the *zero-recollision interval*. Similarly, on the interval (τ_1, τ_2) no more than one recollision with the piston is possible for any point, and we call it the *one-recollision interval*.

The time moment τ_* mentioned in Theorem 1.2 is the earliest time when a point $(y_\tau, v_\tau) \in \mathcal{G}^+(\tau)$ either experiences its third collision with the piston or has its second collision with the piston given that the first one occurred after τ_1. Hence, $\tau_* \leq \tau_2$, and actually τ_* is very close to τ_2, see the next lemma.

Lemma 2.9 *Let* (P1')–(P5') *hold on the interval* $(0, n/v_{\max} + \delta)$ *for some* $n \geq 1$ *and* $\delta > 0$. *Then, for all sufficiently small* ε_0

$$|\tau_k - k/v_{\max}| \leq C\varepsilon_0$$

for all $1 \leq k \leq n$, *where* $C > 0$ *is a global constant that may depend on* n. *Also,*

$$|\tau_* - 2/v_{\max}| \leq C\varepsilon_0$$

Proof. The necessary lower bounds on τ_k follow from Lemma 2.8. The necessary upper bounds are just as easy to obtain, we omit details. □

It is clear at this point that the hydrodynamical equations (H1)–(H4) will have a unique and "well behaved" solution as long as the conditions (P1')–(P5') continue holding with some small ε_0. Our next goal is to show that this is indeed the case.

Proposition 2.10 *Let $T > 0$. If the initial density $\pi_0(y, v)$ satisfies* (P1)–(P5) *and ε_0 in* (P5) *is small enough (for the given T), then the conditions* (P1')–(P5') *will hold on the interval* $0 < \tau < T$.

Note: The corresponding global constants in (P1')–(P5') will depend on T as specified below.

Proof. The main idea is to show that the restrictions on $\pi(y, v, \tau)$ imposed by (P1)–(P5) at $\tau = 0$ "deteriorate" very slowly, as time goes on, so that (P1')–(P5') will continue holding ("propagate") with the respective global constants slowly changing in time.

We first note that as long as (P1')–(P5') hold, the number of collisions grows at most linearly in τ, i.e. on any interval $(0, \tau)$ on which (P1')–(P5') hold, every moving point $(y_s, v_s) \in \mathcal{G}^+(s)$ experiences at most $\tau v'_{\max} + 1$ collisions with the piston, if ε_0 is small enough, see Lemmas 2.8–2.9. Next, we examine the conditions (P1')–(P5') individually and show that each of them should hold up to time T, provided that the others do.

We start with (P1). Due to (H1) we have

$$\frac{\partial \pi(y, v, \tau + s)}{\partial y} = \frac{\partial \pi(y - sv, v, \tau)}{\partial y}$$

and

$$\frac{\partial \pi(y, v, \tau + s)}{\partial v} = \frac{\partial \pi(y - sv, v, t)}{\partial v} - s \frac{\partial \pi(y - sv, v, \tau)}{\partial y}$$

for all $s > 0$ such that the moving point located at (y, v) at time $\tau + s$ did not experience collisions with the piston during the interval $(\tau, \tau + s)$. Thus, between collisions with the walls and the piston, the partial derivatives of $\pi(y, v, \tau)$ can grow at most linearly with τ. Collisions with the walls could only change the sign of the derivatives of p, but not their absolute values.

Now consider the effect of interactions with the piston. We evaluate the partial derivatives of $\pi(y, v, \tau)$ at a point (y, v) after a collision with the piston at some earlier time $s \in (0, \tau)$. For simplicity, assume that there are no other collisions of the moving point (y, v) with the piston or the walls on the interval (s, τ). Then s satisfies the equation

$$Y(s) = y - (\tau - s)v \tag{2.33}$$

Due to (H3) and (H1) we have

$$\begin{aligned}
\pi(y, v, \tau) &= \pi(y - (\tau - s)v, v, s + 0) \\
&= \pi(y - (\tau - s)v, 2W - v, s - 0) \\
&= \pi(y - (\tau - s)v - (s - s_0)(2W - v), 2W - v, s_0)
\end{aligned}$$

where $s_0 < s$ is any earlier time (that we consider fixed) and $W = W(s)$ is the piston velocity at the time of collision. Let $y_0 = y-(\tau-s)v-(s-s_0)(2W-v)$ and $v_0 = 2W-v$. Then

$$\begin{aligned}\frac{\partial\pi(y,v,\tau)}{\partial y} &= \frac{\partial\pi(y_0,v_0,s_0)}{\partial y}\left[1+v\frac{ds}{dy}-2(s-s_0)\frac{dW}{dy}-(2W-v)\frac{ds}{dy}\right]\\ &\quad+\frac{\partial\pi(y_0,v_0,s_0)}{\partial v}\cdot 2\,\frac{dW}{dy}\end{aligned}$$

Differentiating (2.33) with respect to y gives

$$\frac{dY}{ds}\cdot\frac{ds}{dy} = 1+v\frac{ds}{dy}$$

hence

$$\frac{ds}{dy} = \frac{1}{W-v}$$

Also,

$$\frac{dW}{dy} = \frac{dW}{ds}\cdot\frac{ds}{dy} = \frac{A}{W-v}$$

where $A = A(s)$ is the piston acceleration at the time of collision. Now, as long as (P1')–(P5') hold, we have $W = O(\varepsilon_0)$ and $A = O(\varepsilon_0)$, hence $ds/dy = -v^{-1}+O(\varepsilon_0)$ and so

$$\frac{\partial\pi(y,v,\tau)}{\partial y} = -\frac{\partial\pi(y_0,v_0,s_0)}{\partial y}+O(\varepsilon_0)$$

In other words, the piston (due to its low speed and acceleration) acts almost as a wall, which only changes the sign of $\partial\pi/\partial y$. A similar calculation (we omit it) holds for the partial derivative with respect to v.

Thus, as long as (P1')–(P5') hold, the density $\pi(y,v,\tau)$ remains piecewise C^1 and its partial derivatives can grow at most linearly with τ.

Next, we check the condition (P2'). We begin with three special discontinuity lines that do not explicitly appear in (P2). They are created immediately by the reflections at the walls and the piston at time $\tau = 0$, since the initial density $\pi(y,v,0)$ does not have to satisfy (H2)–(H3). Those discontinuity lines are $y = 0$, $y = 0.5$ and $y = 1$ at $\tau = 0$, and their images at $\tau > 0$ will be slanted lines

$$y = v\tau, \qquad y = 0.5+v\tau, \qquad y = 1+v\tau \tag{2.34}$$

respectively, see Fig. 2. So, their slope at any time τ is positive and constant: $dy/dv = \tau$. It is not bounded away from zero as $\tau\to 0$, so we have a technical violation of (P2') for small τ, but it will be clear immediately why this does not bother us.

The singularity lines (2.34) only exist in the region $v_{\min} \le |v| \le v_{\max}$ (elsewhere $p \equiv 0$), hence they cannot intersect the piston $y = Y(\tau)$ for small τ. It will take some time, at least

$$\tau^* = \frac{0.5}{v_{\max}+\mathcal{B}\varepsilon_0} > 0$$

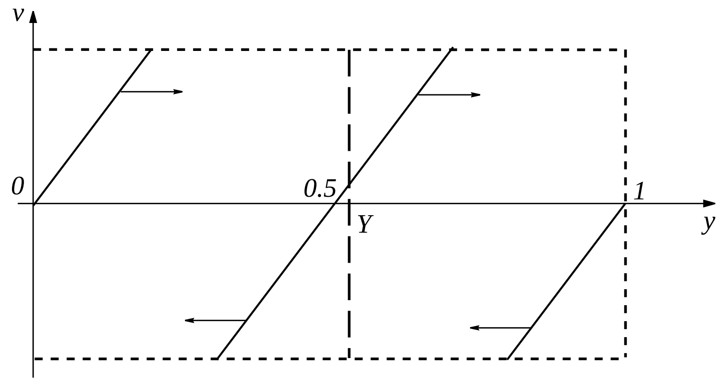

Figure 2: Slanted discontinuity lines.
The dashed vertical line shows the piston position.

before any of these singularity lines "reaches" the piston and its effect has to be reckoned with. At that time the slopes of those lines will be bounded away from zero: $dy/dv \geq \tau^* > 0$, hence (P2') will hold.

We now consider the evolution of all discontinuity curves of the function $\pi(y, v, \tau)$ as τ increases. Let a discontinuity curve of the function $\pi(y, v, s)$ at time s be given by equation $y = g_s(v)$, and its slope is then $h_s(v) = dg_s(v)/dv$. Since the curve and its slope change in time, the function g_s and its derivative h_s depend on s. According to (2.2), we have $g_{s+r}(v) = g_s(v) + vr$ between collisions with the piston and the walls, hence

$$\frac{dg_s(v)}{ds} = v \qquad \text{and} \qquad \frac{dh_s(v)}{ds} = 1 \tag{2.35}$$

Hence, between collisions with the piston, the slope of discontinuity curves grows linearly with τ (note that, in particular, it remains positive).

Now, let the curve $y = g_s(v)$ cross the piston at some point

$$g_s(v) = Y(s) \tag{2.36}$$

(this equation makes v a function of s). After the collision with the piston, this point transforms to $(Y, 2W - v)$, according to the rule (H3), here $W = W(s)$ is the piston velocity. If $\tau > s$ is some fixed time, then the image of our point at time τ is $(Y + (\tau - s)(2W - v), 2W - v)$. Such points make a curve on the y, v plane, parameterized by s (the collision time). This will be the discontinuity curve for the density $\pi(y, v, \tau)$ at time τ. Let $y_s = Y + (\tau - s)(2W - v)$ and $v_s = 2W - v$ be the coordinates of a point on that curve. To compute the slope dy_s/dv_s of that curve, we first differentiate y_s and v_s with respect to the parameter s:

$$\frac{dy_s}{ds} \;=\; W + (\tau - s)\left[2\frac{dW}{ds} - \frac{dv}{ds}\right] - (2W - v)$$

$$= \quad v - W + (\tau - s)\left[2A - \frac{dv}{ds}\right]$$

and

$$\frac{dv_s}{ds} = 2\frac{dW}{ds} - \frac{dv}{ds} = 2A - \frac{dv}{ds}$$

where $A = A(s)$ is the piston acceleration (at the collision time s). Also, differentiating (2.36) with respect to s and using (2.35) gives

$$\frac{dg_s(v)}{dv} \cdot \frac{dv}{ds} + v = W$$

hence

$$\frac{dv}{ds} = \frac{W - v}{h_s(v)}$$

Therefore, the slope of our singularity curve at time τ is

$$\frac{dy}{dv}(\tau) = \frac{(v - W)\,[h_s(v) + \tau - s] + 2Ah_s(v)(\tau - s)}{v - W + 2Ah_s(v)} \tag{2.37}$$

As long as (P1')–(P5') hold, we have $W = O(\varepsilon_0)$ and $A = O(\varepsilon_0)$, hence

$$\frac{dy}{dv}(\tau) = h_s(v) + \tau - s + O(\varepsilon_0) \tag{2.38}$$

Hence, every collision with the piston only adds a $O(\varepsilon_0)$ correction to the linear growth of the slopes of discontinuity curves.

Next we check the conditions (P3')–(P5') based on the following lemma:

Lemma 2.11 *Let* (P1')–(P5') *hold on an interval* $(0, \tau)$. *Then for every point* $(y, v) \in \mathcal{G}^+(\tau)$ *there is another point* $(y_0, v_0) \in \mathcal{G}^+$ *such that* $\pi(y, v, \tau) = \pi(y_0, v_0, 0)$ *and*

$$|\,|v| - |v_0|\,| = 2(v'_{\max}\tau + 1)\mathcal{B}\varepsilon_0$$

Proof. We set $(y_0, v_0) = \mathcal{F}^{-\tau}(y, v)$ and use (2.32). At each collision of the point (y_0, v_0) with the piston, its speed $|v|$ changes by $2|W| \le 2\mathcal{B}\varepsilon_0$ according to (2.31) and (2.26), and the number of collisions is bounded by $v'_{\max}\tau + 1$. □

Lemma 2.11 immediately implies that (P3') and (P4') continue holding with global constants v'_1, v'_2, $v'_{\min}$ and $v'_{\max}$ slowly changing with time – they change at most by $CT\varepsilon_0$ on the interval $(0, T)$, with a global constant $C > 0$. In particular, v'_1 and $v'_{\min}$ remain positive, provided ε_0 is small enough. The constants $\pi'_{\min}$ and $\pi'_{\max}$ do not change at all.

To check (P5'), we explicitly construct the map $R_\tau : (y, v) \mapsto (y_*, v_*)$ involved in (2.14) and (2.15), it is defined here by $R_\tau = \mathcal{F}^\tau \circ R_0 \circ \mathcal{F}^{-\tau}$, where $R_0(y, v) = (1 - y, -v)$

is a simple reflection "across the piston" at time $\tau = 0$. Now, (2.13) follows from (2.26), and (2.15) follows from (P5).

Lastly, we derive (2.14) from the Lemma 2.11. Let (y, v) be a moving point at time τ and $(y_0, v_0) = \mathcal{F}^{-\tau}(y, v) \in \mathcal{G}^+$ its preimage to time zero. Compare the evolution of the point (y_0, v_0) and its mirror image $R_0(y_0, v_0) = (1 - y_0, -v_0) \in \mathcal{G}^+$ "across the piston" during the interval $(0, \tau)$. Due to (2.13) and (2.26), these two points will experience collisions with the walls and the piston at time moments that differ at most by $O(\varepsilon_0)$. And their velocities will also differ at most by $O(\varepsilon_0)$, hence their positions at time T will be almost symmetric about the piston, up to $O(\varepsilon_0)$. This implies (2.14).

Note that by the given time T the above two moving points may have experienced a different number of collisions, as one point may have just collided with the piston or a wall, while the other may be about to collide with it. To take care of this case, we introduced the special distance d_Y in (P5'). □

We summarize our main results in the following theorem:

Theorem 2.12 *Let $T > 0$ be given. If the initial density $\pi_0(y, v)$ satisfies* (P1)–(P5) *with a sufficiently small ε_0, then*

(a) *the solution of our hydrodynamical equations* (H1)–(H4) *exists and is unique on the interval* $(0, T)$*;*

(b) *the density $\pi(y, v, \tau)$ satisfies* (P1')–(P5') *for all* $0 < \tau < T$*;*

(c) *The piston velocity and acceleration remain small, $|W(\tau)| = O(\varepsilon_0)$ and $|A(\tau)| = O(\varepsilon_0)$, and its position remains close to the midpoint* 0.5 *in the sense* $|Y(\tau) - 0.5| = O(\varepsilon_0)$*, for all* $0 < \tau < T$*;*

(d) *we have $|\tau_k - k/v_{\max}| = O(\varepsilon_0)$ for all $1 \le k < Tv_{\max}$, and if $Tv_{\max} > 2$, then also $|\tau_* - \tau_2| = O(\varepsilon_0)$.*

Corollary 2.13 *If $\varepsilon_0 = 0$, so that the initial density $\pi_0(y, v)$ is completely symmetric about the piston, the solution is trivial: $Y(\tau) \equiv 0.5$ and $W(\tau) \equiv 0$ for all $\tau > 0$.*

Lastly, we demonstrate the reason for our assumption that all the discontinuity curves of the initial density $\pi(y, v)$ must have positive slopes. It would be quite tempting to let $\pi(y, v)$ have more general discontinuity lines, e.g. allow it be smooth for $v_{\min} < |v| < v_{\max}$ and abruptly drop to 0 at $v = v_{\min}$ and $v = v_{\max}$. The following example shows why this is not acceptable.

Example. Suppose the initial density $\pi_0(y, v)$ has a horizontal discontinuity line $v = v_0$ (say, $v_0 = v_{\min}$ or $v_0 = v_{\max}$). After one interaction with the piston the image of this discontinuity line can oscillate up and down, due to the fluctuations of the piston acceleration (Fig. 3). As time goes on, this oscillating curve will "travel" to the wall and come back to the piston, experiencing some distortions on its way, caused by the differences

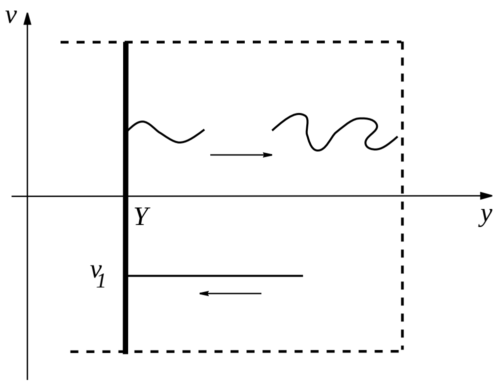

Figure 3: A horizontal discontinuity line (bottom) comes off the piston as an oscillating curve (top).

in velocities of its points (Fig. 3). When this curve comes back to the piston again, it may well have "turning points" where its tangent line is vertical, or even contain vertical segments of positive length. This produces unwanted singularities or even discontinuities of the piston velocity and acceleration. The same phenomena can also occur when a discontinuity line of the initial density $\pi_0(y, v)$ has a negative slope.

3 Dynamics before the first recollision

In this section we begin to study the mechanical model of the piston in the ideal gas described in Section 1. We will show that the random trajectory of the piston described by the functions $Y_L(\tau, \omega) = X_L(\tau L, \omega)/L$ and $W_L(\tau, \omega) = V_L(\tau L, \omega)$, cf. (1.6), converges in probability, as $L \to \infty$, to the solution of the hydrodynamical equations $Y(\tau)$ and $W(\tau)$ found in the previous section, on the zero-recollision interval $(0, \tau_1)$.

Convention. For brevity of notation, we will suppress the dependence of L and ω in our expressions, when it does not cause confusion. For example, we will write $X(t)$ and $V(t)$ instead of $X_L(t, \omega)$ and $V_L(t, \omega)$, respectively, etc.

We will work here with the microscopic time t. First, we define the "microscopic" gas density, which we will denote by $p(x, v, t)$, for all $t \geq 0$. For $t = 0$ it is initialized by $p(x, v, 0) = \pi_0(x/L, v)$, see (1.7). For $t > 0$, its evolution is defined by the rules similar to (H1)–(H3): the free motion between collisions

$$p(x, v, t) = p(x - vs, v, t - s) \tag{3.1}$$

for $s > 0$ such that $x - vr \notin \{0, X(t - r), L\}$ for all $r \in (0, s)$; reflections at the walls

$$p(0, v, t) = p(0, -v, t) \qquad \text{and} \qquad p(L, v, t) = p(L, -v, t) \tag{3.2}$$

and elastic collisions with the piston

$$p(X(t) \pm 0, v, t) \quad = \quad p(X(t) \pm 0, 2V(t) - v, t) \tag{3.3}$$

Since the last equation involves the random functions $X(t)$ and $V(t)$, the density $p(x, v, t)$ will depend on ω, i.e. it is now a random function.

The evolution of the density $p(x, v, t)$ can be conveniently described with the help of a one-parameter family of transformations F^t similar to $\mathcal{F}^\tau$ defined in Section 2. Let (x, v) be a point in the domain

$$G := \{(x, v) : \ 0 \le x \le L\}$$

Its trajectory (x_t, v_t) for $t > 0$ is defined by the free motion inside the container, $\dot{x}_t = v_t$ and $\dot{v}_t = 0$ whenever $x_t \notin \{0, X(t), L\}$, reflections at the walls $(x_{t+0}, v_{t+0}) = (x_{t-0}, -v_{t-0})$ whenever $x_{t-0} \in \{0, L\}$, and collisions with the piston

$$(x_{t+0}, v_{t+0}) = (x_{t-0}, 2V(t) - v_{t-0}) \tag{3.4}$$

whenever $x_{t-0} = X(t)$. Now the family of transformations F^t is defined by $F^t(x_0, v_0) = (x_t, v_t)$ for $t > 0$. We will also write $F^{-t}(x_t, v_t) = (x_0, v_0)$. Now we simply have

$$p(x_t, v_t, t) = p(F^{-t}(x_t, v_t), 0) = p(x_0, v_0, 0) \tag{3.5}$$

Note that for each $t > 0$ the map F^t is a bijection of G and preserves area, i.e.

$$\det |DF^t(x, v)| = 1 \tag{3.6}$$

We emphasize that the transformation F^t, just as the density $p(x, v, t)$, is random, i.e. depends on ω.

Remark. The piston velocity $V(t)$ is a piecewise constant function updated at the moments of collision with gas atoms by the rules (1.2)–(1.3). If t is such a collision moment, then $V(t)$ in equation (3.3) must be replaced by the average of its one-sided limit values $(V(t-0) + V(t+0))/2$. This modification is important, since it makes the rule (3.3) equivalent to (1.2)–(1.3) when (x_t, v_t) represents an actual gas particle of mass m. Otherwise, it will correspond to the motion of a particle of zero mass, and we may call it a *virtual particle*.

Because of (P4), the initial density $p(x, v, 0)$ can only be positive in the region

$$G^+ := \{(x, v) : \ 0 \le x \le L, \ v_{\min} \le |v| \le v_{\max}\}$$

which therefore contains all the gas particles at time $t = 0$. For any $t > 0$, the region $G^+(t) := F^t(G^+)$ contains all the actual gas particles at time t, and $p(x, v, t) = 0$ for all $(x, v) \notin G^+(t)$.

For each point $(x, v) \in G$ and $t > 0$ we define the number of collisions with the piston during the interval $(0, t)$

$$N(x, v, t) = \#\{s \in (0, t) : \ x_s = X(s), \ v_s \neq V(s)\}$$

Then we partition the region G into subregions

$$G_n(t) := \{F^t(x, v) : \ (x, v) \in G \ \ \& \ \ N(x, v, t) = n\}$$

and put $G_n^+(t) := G^+(t) \cap G_n(t)$. The region $G_n^+(t)$ is occupied by the points that have experienced exactly n collisions with the piston during the time interval $(0, t)$.

We emphasize that our transformations F^t and the regions $G_n(t)$ and $G_n^+(t)$ depend on ω, i.e. are random. We note, however, that they are completely determined by the trajectory of the piston, i.e. by the function $X(s)$, $0 < s < t$.

The family of transformations $\mathcal{F}^\tau : \mathcal{G} \to \mathcal{G}$ introduced in Section 2 induces a (deterministic) family $\tilde{F}^t : G \to G$ defined as follows: if $\mathcal{F}^\tau(y, v) = (y_\tau, v_\tau)$, then we put $\tilde{F}^{\tau L}(yL, v) := (y_\tau L, v)$. The transformations $\tilde{F}^t$ define a deterministic density function on G by $\tilde{p}(x, v, t) := p(\tilde{F}^t(x, v), 0)$, which is related to the density $\pi(y, v, \tau)$ studied in Section 2 by $\tilde{p}(x, v, t) = \pi(x/L, v, t/L)$. We also put $\tilde{G}^+(t) := \mathcal{F}^t(G^+)$.

The gas particles in $G_0(t)$ make a Poisson process, as the following lemma shows. Let $\omega \in \Omega$ and $t > 0$. Fix the trajectory of the piston $X(s)$, $0 < s < t$. That completely specifies the region $G_0(t)$ and the density $p(x, v, t)$.

Lemma 3.1 *The conditional distribution of the gas particles in $G_0(t)$ (given the trajectory $X(s)$, $0 < s < t$, of the piston) is Poisson with density function $L^2 p(x, v, t)$.*

Proof. Let $D \subset G_0(t)$ be any domain. Then its preimage $F^{-(t-s)}(D)$ stays positive distance away from the piston $X(s)$ for all $s \in (0, t)$. Hence, the particles starting out in the region $F^{-t}(D)$ and ending up in the region D could not affect the piston during the time interval $(0, t)$. Therefore, the number of particles in D at time t, being equal to the number of particles in $F^{-t}(D)$ at time 0, is independent of the piston trajectory, so it is a Poisson random variable with parameter

$$\lambda_D(t) = L^2 \iint_{F^{-t}(D)} p(x, v, 0)\, dx\, dv = L^2 \iint_D p(x, v, t)\, dx\, dv$$

The identity of the above integrals follows from (3.5) and (3.6). $\square$

Remark. For any domain $D \subset G_0(t)$ its preimage $F^{-t}(D)$ is actually independent of ω. Indeed, let F_0^t be another family of transformations on G defined by the free motion on the entire interval $0 < x < L$ and elastic reflections at the walls $x = 0$ and $x = L$ only (as if the piston did not exist). Then we have $F^{-t}(D) = F_0^{-t}(D)$ for any domain $D \subset G_0(t)$.

For $n \geq 1$, we define T_n to be the earliest time the piston interacts with points from $G_n^+(t)$ (thus creating the region $G_{n+1}^+(t)$), or, equivalently,

$$T_n = \sup_{t>0}\{G_{n+1}^+(t) = \emptyset\} \tag{3.7}$$

The time moments $T_n = T_n(\omega)$ are random analogues of τ_n introduced in Section 2. In particular, T_1 is the time of the first recollision in the system (by an actual or a virtual particle).

Lemma 3.2 *For all* $\omega \in \Omega$

$$T_1 \leq T_{1,\max} := L/v_{\max} \tag{3.8}$$

Proof. The fastest particles $(x, v) \in G^+$ that collide with the piston at time 0 will move with the speed $v_{\max}$ and recollide with the piston at time t that satisfies $v_{\max}t + |X(t) - L/2| = L$. For all such t we have

$$T_1 \leq t = \frac{L - |X(t) - L/2|}{v_{\max}}$$

This proves the lemma. □

During the time interval $(0, T_1)$ the piston interacts with particles in $G_0^+(t)$. Denote by

$$\begin{aligned}\mathcal{X}_0(t) = \{(x,v) :\, x = X(t) + 0,\, -v_{\max} < v < -v_{\min}\} \\ \cup\{(x,v) :\, x = X(t) - 0,\, v_{\min} < v < v_{\max}\}\end{aligned}$$

two immediate one-sided vicinities of the piston which contain all "incoming" particles, which are about to collide with the piston.

In order to study the piston dynamics on the zero-recollision interval, it is convenient to assume that the piston is slow enough and only interacts with the original particles that started in G^+ at time 0. A subinterval $(0, S_1) \subset (0, T_1)$ where the piston satisfies these requirements will be called a "slow" interval:

Definition We define $(0, S_1) \subset (0, T_1)$ to be the maximal time interval on which
(a) $|V(t)| < v_{\min}$;
(b) $\mathcal{X}_0(t) \subset G_0(t)$.

Note that by the condition (b) all the incoming particles that are about to hit the piston at time t have started out in G^+ at time zero and have never interacted with the piston during the time interval $(0, t)$.

Almost all of our considerations in this section are restricted to the "slow" interval $(0, S_1)$. But in the end of the section we will see that for typical $\omega \in \Omega$ the "slow" interval coincides with the entire interval $(0, T_1)$, i.e. $S_1 = T_1$.

Now, for every $t \in (0, S_1)$ we define the density of colliding particles "on the piston" $q(v,t)$ by

$$q(v,t) = \begin{cases} p(X(t)-0, v, t) & \text{if} \quad v > 0 \\ p(X(t)+0, v, t) & \text{if} \quad v < 0 \end{cases} \tag{3.9}$$

cf. (2.17). Next, we define

$$Q_0(t) = \int (\operatorname{sgn} v)\, q(v,t)\, dv \tag{3.10}$$

$$Q_1(t) = \int v\, (\operatorname{sgn} v)\, q(v,t)\, dv \tag{3.11}$$

$$Q_2(t) = \int v^2\, (\operatorname{sgn} v)\, q(v,t)\, dv \tag{3.12}$$

in a way similar to (2.19)–(2.21) in Section 2.

Since $p(x,v,t)$, restricted to the domain $G_0(t)$, coincides with $\tilde{p}(x,v,t) = \pi(x/L, v, t/L)$, the conditions (P1')–(P2') imply

Lemma 3.3 *The density $p(x,v,t)$ restricted to the region $G_0(t)$ is piecewise C^1 smooth on the x,v plane. We also have $|\partial p(x,v,t)/\partial x| \le D_1'/L$. The discontinuity lines of $p(x,v,t)$ within the region $G_0(t)$ have slope of order $O(1/L)$ (for large L, they are almost parallel to the x axis).*

Due to the above lemma the quantities Q_i, $i = 0,1,2$, are, as functions of the piston position X, smooth and have derivatives

$$\left| \frac{\partial Q_i}{\partial X} \right| \le \frac{\text{const}}{L} \tag{3.13}$$

where const is a global constant.

The following theorem gives a key technical estimate of this section.

Theorem 3.4 *For sufficiently large L there is a set $\Omega_0^* \subset \Omega$ of initial configurations of gas particles such that*
(i) *for some constant $c > 0$*

$$P(\Omega_0^*) > 1 - L^{-c \ln \ln L}$$

(ii) *for each configuration $\omega \in \Omega_0^*$, for each time interval*

$$(t, t + \Delta t) \subset (0, S_1)$$

such that

$$\frac{1}{L^2} < \Delta t < \frac{1}{L^{2/3} \ln L} \tag{3.14}$$

the change of the velocity of the piston during $(t, t+\Delta t)$ satisfies

$$V(t+\Delta t) - V(t) = \mathcal{D}(t)\, \Delta t + \chi \tag{3.15}$$

where

$$\mathcal{D}(t)/a = Q_0(t)V^2(t) - 2Q_1(t)V(t) + Q_2(t) \tag{3.16}$$

and

$$|\chi| \leq C\,\frac{\ln L\,\sqrt{\Delta t}}{L} \tag{3.17}$$

with some constant $C > 0$.

Remark. The function $\mathcal{D}(t)/a$ in (3.16) is the random analogue of the quadratic polynomial (2.18). The term $\mathcal{D}(t)\,\Delta t$ in (3.15) is the main ("deterministic") component of the dynamics of the piston velocity. The term χ represents random fluctuations.

Proof. The set Ω_0^* will consist of all configurations that satisfy certain requirements. We start with *preliminary requirements*.

Consider a discrete set of time moments $t_i = i/L^2$, where $i = 0, 1, \dots, I$ and $I = [T_{1,\max}L^2]$. Partition the domain G into the strips $S_j := \{(x, v) : \; j/L^2 \leq x < (j+1)/L^2\}$, where $j = 0, 1, \dots, L^3 - 1$. For each i and j denote by $N_{i,j}$ the number of gas particles in the region $S_j \cap G_0(t_i)$ at time t_i. Our preliminary requirements are

$$N_{i,j} \leq \ln L \tag{3.18}$$

for all $0 \leq i \leq I$ and $0 \leq j < L^3$. We observe that $N_{i,j}$ equals the number of gas particles in the region $F^{-t_i}(S_j \cap G_0(t_i))$ at time 0, and

$$F^{-t_i}(S_j \cap G_0(t_i)) \subset F_0^{-t_i}(S_j)$$

So, $N_{i,j}$ does not exceed the number of gas particles in $F_0^{-t_i}(S_j)$ at time 0, denote the latter by $\tilde{N}_{i,j}$. Now, $\tilde{N}_{i,j}$ is a Poisson random variable whose parameter $\lambda_{i,j}$ is bounded by

$$\lambda_{i,j} \leq L^2\pi_{\max}|S_j| \leq 2\pi_{\max}(v_{\max} - v_{\min})$$

According to Corollary A.4, for each i, j our requirement (3.18) will fail with probability $< L^{-d\ln\ln L}$ with some $d > 0$. The total number of pairs i, j equals $L^3 I = L^5 T_{1,\max} \leq L^6/2v_{\max}$. Hence, all our preliminary requirements hold with probability $> 1 - L^{-c'\ln\ln L}$ with some global constant $c' > 0$.

We now turn to the proof of (3.14)–(3.16). Let $i = [L^2 t]$ and $t_1 = (i+1)/L^2$. Note that $t_1 - t \leq L^{-2}$. One can easily derive from our preliminary requirements that the number of gas particles colliding with the piston on the time interval (t, t_1) is less than const$\cdot \ln L$. Hence, the piston velocity V does not change by more than const$\cdot \ln L/L^2$ during this interval. This amount is less than the bound on χ in (3.17) for all Δt satisfying (3.14). Therefore, we can ignore the interval (t, t_1) and assume that $t = t_1$. Note that the quantities Q_0, Q_1, Q_2 in (3.16) will change, as the result of the substitution $t = t_1$, also, but only by the amount $<$ const$/L^3$ due to Lemma 3.3. This change can be obviously ignored, too. So, we suppose that $t = i/L^2$ for some $i = 0, 1, \dots, I$

Now we state our *main requirements*. We again partition the x axis into intervals $S_j := \{j/L^2 \leq x < (j+1)/L^2\}$, where $j = 0, 1, \dots, L^3 - 1$. For each j we put $x_j = j/L^2$.

For each integer p, $|p| \le v_{\min} L^2$, we put $v_p = p/L^2$ and for each integer $1 \le q \le L^2$ we put $d_q = q/L^2$. For each triple (j, p, q) we now define two trapezoid-like domains on the x, v plane (see Fig. 4 below):

$$D^-_{j,p,q} := \left\{ (x, v) : \ \frac{v - v_p}{x - x_j} < -\frac{1}{d_q}, \quad v_{\min} < v < v_{\max} \right\} \tag{3.19}$$

and

$$D^+_{j,p,q} := \left\{ (x, v) : \ \frac{v - v_p}{x - x_j} < -\frac{1}{d_q}, \quad -v_{\max} < v < -v_{\min} \right\} \tag{3.20}$$

(here $-1/d_q$ is the slope of the oblique side of these trapezoids), and two strips

$$U^-_{j,p,q} := \{(x, v) : \ |x - x_j + d_q(v - v_p)| < 10 v_{\max}/L^2, \quad v_{\min} < v < v_{\max}\}$$

and

$$U^+_{j,p,q} := \{(x, v) : \ |x - x_j + d_q(v - v_p)| < 10 v_{\max}/L^2, \quad -v_{\max} < v < -v_{\min}\}$$

Note that $U^\pm_{j,p,q}$ are the neighborhoods of the oblique sides of the trapezoids $D^\pm_{j,p,q}$.

Consider all time moments $t_i = i/L^2$ for $i = 0, 1, \dots, I$. Denote by $N^\pm_{i,j,p,q}$ the number of particles in the region $F_0^{-t_i}(D^\pm_{j,p,q})$ at time 0. And denote by $M^\pm_{i,j,p,q}$ the number of particles in the region $F_0^{-t_i}(U^\pm_{j,p,q})$ at time 0. These are Poisson random variables. The parameter of the variable $N^\pm_{i,j,p,q}$ is

$$\lambda^\pm_{i,j,p,q} = E(N^\pm_{i,j,p,q}) = L^2 \int_{F_0^{-t_i}(D^\pm_{j,p,q})} p(x, v, 0)\, dx\, dv$$

One can verify directly that $\lambda^\pm_{i,j,p,q}$ are uniformly bounded below by a positive constant (even for the smallest d_p, i.e. for $d_q = 1/L^2$), due to the assumption (1.8) on the initial density. Also note that the parameters of $M^\pm_{i,j,p,q}$ are uniformly bounded above, by

$$E(M^\pm_{i,j,p,q}) < \pi_{\max} |U^\pm_{j,p,q}| < 20 v_{\max}(v_{\max} - v_{\min}) \pi_{\max}$$

Our main requirements are

$$|N^\pm_{i,j,p,q} - \lambda^\pm_{i,j,p,q}| \le \ln L \sqrt{\lambda^\pm_{i,j,p,q}} \tag{3.21}$$

and

$$M^\pm_{i,j,p,q} \le \ln L \tag{3.22}$$

By Lemma A.3 and Corollary A.4 the probability that any of these requirements fails will be less than $L^{-d \ln \ln L}$ with some constant $d > 0$. The total number of quadruples (i, j, p, q) does not exceed $L^9 T_{1,\max} \le L^{A''}$ with some fixed $A'' > 0$. Therefore, all our main requirements hold with probability $> 1 - L^{-c'' \ln \ln L}$ with some constant $c'' > 0$.

In addition, let

$$Z^{\pm}_{i,j,p,q} = \sum_{(x,v)\in F_0^{-t_i}(D^{\pm}_{j,p,q})} v$$

taken at time 0. This is an "integrated" Poisson random variable, as defined in Appendix. (Technically, we require there that the domain must be on one side of the x axis, and now it may happen here that the region $F_0^{-t_i}(D^{\pm}_{j,p,q})$ crosses the wall $x = 0$ or $x = L$ and then lies on both sides on the x-axis; in that case we need to replace t_i by a nearby time moment $t_{i'} < t_i$ so that $F_0^{-t_{i'}}(D^{\pm}_{j,p,q})$ lies entirely on one side of the x axis and define $Z^{\pm}_{i,j,p,q}$ at time $t_i - t_{i'}$ rather than 0; Some obvious modifications need to be made then, we omit details.) The estimates obtained in Appendix yield

$$E(Z^{\pm}_{i,j,p,q}) = L^2 \int_{F_0^{-t_i}(D^{\pm}_{j,p,q})} v\, p(x,v,0)\, dx\, dv \tag{3.23}$$

$$\mathrm{Var}(Z^{\pm}_{i,j,p,q}) = L^2 \int_{F_0^{-t_i}(D^{\pm}_{j,p,q})} v^2 p(x,v,0)\, dx\, dv \tag{3.24}$$

Our last main requirement is

$$|Z^{\pm}_{i,j,p,q} - E(Z^{\pm}_{i,j,p,q})| \le \ln L \sqrt{\mathrm{Var}(Z^{\pm}_{i,j,p,q})} \tag{3.25}$$

for all i, j, p, q. The probability of failure for these requirements is estimated exactly as above, by using Lemma A.6.

We now turn to the estimation of the piston velocity on the time interval $(t, t + \Delta t)$. Recall that $t = t_i$ for some $i = 0, 1, \ldots, I$.

Velocity decomposition scheme. Here we obtain a general formula for the piston velocity, which we will use in the proof of several theorems. The laws of elastic collisions imply [LPS]

$$V(t + \Delta t) = (1 - \varepsilon)^k V(t) + \varepsilon \sum_{j=1}^{k} (1 - \varepsilon)^{k-j} \cdot v_j \tag{3.26}$$

Here k is the number of particles colliding with the piston during the time interval $(t, t+\Delta t)$, and v_j are their velocities numbered in the order in which the particles collide. Equation (3.26) can be easily verified by induction on k.

We modify the formula (3.26) as follows:

$$V(t + \Delta t) = (1 - \varepsilon k)V(t) + \varepsilon \sum_{j=1}^{k} v_j + \chi^{(1)} + \chi^{(2)} \tag{3.27}$$

where

$$\chi^{(1)} = V(t)[(1 - \varepsilon)^k - 1 + \varepsilon k]$$

and

$$\chi^{(2)} = \varepsilon \sum_{j=1}^{k} v_j [(1 - \varepsilon)^{k-j} - 1]$$

Let us assume that the fluctuations of the velocity $V(s)$ on the interval $(t, t+\Delta t)$, are bounded:

$$\sup_{s\in(t,t+\Delta t)} |V(s) - V(t)| \le \delta V \tag{3.28}$$

Consider two regions on the x, v plane:

$$D_1 = \left\{(x,v):\ \frac{v - V(t) - (\operatorname{sgn} v)\,\delta V}{x - X(t)} < -\frac{1}{\Delta t},\quad v_{\min} < |v| < v_{\max}\right\} \tag{3.29}$$

and

$$D_2 = \left\{(x,v):\ \frac{v - V(t) + (\operatorname{sgn} v)\,\delta V}{x - X(t)} < -\frac{1}{\Delta t},\quad v_{\min} < |v| < v_{\max}\right\} \tag{3.30}$$

Each of them is a union of two trapezoids $D_i = D_i^+ \cup D_i^-$, $i = 1, 2$, where D_i^- denotes the upper and D_i^+ the lower trapezoid, see Fig. 4.

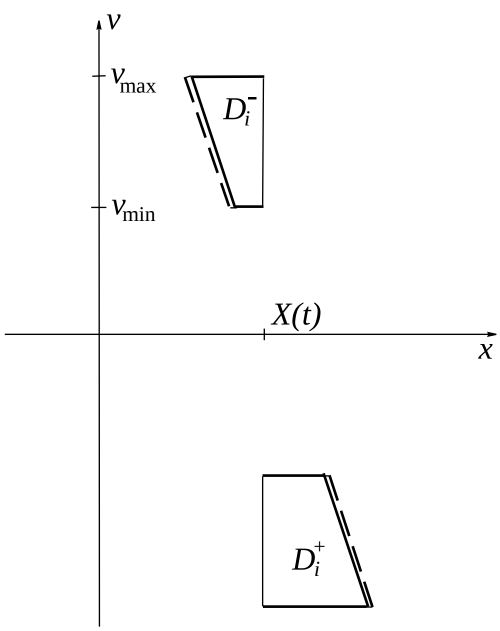

Figure 4: Region D_1 is bounded by solid lines. Region D_2 is bounded by a dashed line.

Note that $D_1 \subset D_2$. The bound (3.28) implies that all the particles in the region D_1 necessarily collide with the piston during the time interval $(t, t+\Delta t)$. Moreover, the trajectory of every point $(x, v) \in D_1$ hits the piston within time Δt, hence $D_1 \subset G_0(t)$ by the condition (b) in the definition of the "slow" interval $(0, S_1)$. The bound (3.28) also implies that all the particles actually colliding with the piston during the interval $(t, t+\Delta t)$ are contained in D_2 (though it is not necessarily true that $D_2 \subset G_0(t)$).

We now obtain an upper bound on k. Since the velocities of the gas particles and the piston are bounded by $v_{\max}$ all the particles colliding with the piston during the interval $(t, t+\Delta t)$ are contained in the region $S' \cap G_0(t)$, where

$$S' = \{(x,y) : \; |x - X(t)| < 2v_{\max}\Delta t\}$$

Hence, k does not exceed the number of particles in the region $F_0^{-t}(S')$ at time zero. Our preliminary requirements imply

$$k \le 2v_{\max}\Delta t \, L^2 \ln L \tag{3.31}$$

This allows us to bound the quantities $\chi^{(1)}$ and $\chi^{(2)}$, for large L, by

$$|\chi^{(u)}| \le v_{\max}\varepsilon^2 k^2 \le 4a^2 v_{\max}^3 (\ln L)^2 \, (\Delta t)^2 \tag{3.32}$$

for $u = 1, 2$. It also follows from (3.27) that

$$\delta V \le 5av_{\max}\Delta t \, \ln L \tag{3.33}$$

We denote by $k_r^{\pm}$ the number of particles in the regions $D_r^{\pm} \cap G_0(t)$ for $r = 1, 2$ at time $t = t_i$. We also denote by k^- the number of particles colliding with the piston "on the left", and by k^+ that number "on the right" (of course, $k^- + k^+ = k$). Clearly,

$$k_1^{\pm} \le k^{\pm} \le k_2^{\pm} \tag{3.34}$$

Since $D_1 \subset G_0(t)$, then $k_1^{\pm}$ equals the number of particles in the region $F_0^{-t}(D_1^{\pm})$ at time 0. Similarly, $k_2^{\pm}$ equals the number of particles in the region $F_0^{-t}(D_2^{\pm} \cap G_0(t))$ at time 0.

The trapezoids $D_r^{\pm}$, $r = 1, 2$, can be well approximated by some trapezoids $D_{j,p,q}^{\pm}$ defined earlier in our main requirements. In fact, the horizontal sides $v = \pm v_{\min}$ and $v = \pm v_{\max}$ are common for all trapezoids, only the vertical side $x = X(t)$ and the oblique side need approximation. The symmetric difference between $D_r^{\pm}$ and the approximating $D_{j,p,q}^{\pm}$ will lie inside some strips S_j and $U_{j,p,q}^{\pm}$ also defined above. Then our main requirements will guarantee that

$$k_1^{\pm} \ge \lambda_1^{\pm} - \ln L \sqrt{\lambda_1^{\pm}}$$

and

$$k_2^{\pm} \le \lambda_2^{\pm} + \ln L \sqrt{\lambda_2^{\pm}}$$

where

$$\lambda_1^{\pm} = L^2 \int_{F_0^{-t}(D_1^{\pm})} p(x,v,0)\, dx\, dv = L^2 \int_{D_1^{\pm}} p(x,v,t)\, dx\, dv$$

and

$$\begin{aligned} \lambda_2^{\pm} &= L^2 \int_{F_0^{-t}(D_2^{\pm} \cap G_0(t))} p(x,v,0)\, dx\, dv \\ &\le \lambda_1^{\pm} + L^2 \pi_{\max} |D_2^{\pm} \setminus D_1^{\pm}| \end{aligned}$$

where we used the boundedness of the density (1.9). The area of the domain $D_2^\pm \setminus D_1^\pm$ is bounded by

$$|D_2^\pm \setminus D_1^\pm| < 4v_{\max}\,\delta V\,\Delta t$$

Also, (1.9) implies that

$$\lambda_1^\pm \le L^2\pi_{\max}|D_1^\pm| \le 2\pi_{\max}v_{\max}^2 L^2\,\Delta t$$

Combining the above estimates gives

$$|k^\pm - \lambda_1^\pm| \le \text{const}\cdot(L\ln L\sqrt{\Delta t} + L^2\,\delta V\,\Delta t) \tag{3.35}$$

We now turn to the quantity

$$Z = \sum_{j=1}^{k} v_j$$

also involved in the main equation (3.27). Again we can decompose $Z = Z^- + Z^+$, where Z^- and Z^+ denote the sum of velocities of the particles colliding with the piston "on the left" and "on the right", respectively. We put

$$Z_1^\pm = \sum_{(x,v)\in D_1^\pm} v$$

(taken at time t). Analysis similar to the previous one and the requirement (3.25) with formulas (3.23)–(3.24) give

$$|Z^\pm - E(Z_1^\pm)| \le \text{const}\cdot(L\ln L\sqrt{\Delta t} + L^2\,\delta V\,\Delta t) \tag{3.36}$$

We now combine (3.27) with all the subsequent estimates and obtain

$$V(t+\Delta t) - V(t) = -\varepsilon\lambda_1 V(t) + \varepsilon E(Z_1) + \chi^{(3)} \tag{3.37}$$

where

$$\lambda_1 = \lambda_1^+ + \lambda_1^- = L^2\int_{D_1} p(x,v,t)\,dx\,dv$$

$$E(Z_1) = E(Z_1^+) + E(Z_1^-) = L^2\int_{D_1} v\,p(x,v,t)\,dx\,dv$$

and

$$|\chi^{(3)}| \le \text{const}\cdot\left[L^{-1}\ln L\sqrt{\Delta t} + \delta V\,\Delta t + (\ln L)^2\,(\Delta t)^2\right]$$

Using Lemma 3.3 (as we already did in deriving (3.13)) to bound possible fluctuations of the density $p(x,v,t)$ within D_1 gives

$$\int_{D_1} p(x,v,t)\,dx\,dv = (Q_1(t) - Q_0(t))\,V(t)\,\Delta t + \chi^{(4)} \tag{3.38}$$

and

$$\int_{D_1} v\, p(x,v,t)\, dx\, dv = (Q_2(t) - Q_1(t))\, V(t)\, \Delta t + \chi^{(5)} \tag{3.39}$$

with

$$|\chi^{(u)}| \le \text{const} \cdot \left[(\Delta t)^2/L + \delta V\, \Delta t\right] \tag{3.40}$$

for $u = 4, 5$.

Therefore, we get

$$V(t+\Delta t) - V(t) = \mathcal{D}(t)\, \Delta t + \chi \tag{3.41}$$

where

$$|\chi| \le \text{const} \cdot \left[L^{-1} \ln L \sqrt{\Delta t} + \delta V\, \Delta t + (\ln L)^2\, (\Delta t)^2\right] \tag{3.42}$$

By using (3.14) and (3.33) it is easy to incorporate the second and the third terms in (3.42) into the first one. Theorem 3.4 is proved. □

For the next theorem, we rewrite (P5) in terms of the microscopic coordinates:

$$|p(x,v,0) - p(L-x,-v,0)| < \varepsilon_0 \tag{3.43}$$

We also assume also that the initial velocity of the piston is set to $V(0) = W(0)$ rather than to 0, see a remark Section 2 and another remark below.

Theorem 3.5 *Assume that $\varepsilon_0 > 0$ is small enough. For all sufficiently large L, for each configuration $\omega \in \Omega_0^*$ and for all $t \in (0, S_1)$ we have*
(i) *there is a constant $B > 0$ such that*

$$|V(t)| < B\varepsilon_0 \tag{3.44}$$

(ii) *there is a constant $C_0 > 0$ such that*

$$|V(t) - V_0(t)| < \frac{C_0 (\ln L)^{3/2}}{L^{2/3}} \tag{3.45}$$

where $V_0(t)$ is defined by

$$V_0(t) = \frac{Q_1(t) - \sqrt{Q_1^2(t) - Q_0(t) Q_2(t)}}{Q_0(t)} \tag{3.46}$$

whenever $Q_0(t) \ne 0$ and by

$$V_0(t) = \frac{Q_2(t)}{2Q_1(t)} \tag{3.47}$$

otherwise.

Proof. We will start proving (i) and obtain (ii) as a "side result".

Assume that (i) is false and let $t_* < S_1$ be the first time (3.44) fails, i.e. let

$$|V(t_* + 0)| \geq B\varepsilon_0 \tag{3.48}$$

Since (3.44) holds for $t < t_*$, the piston's position satisfies

$$|X(t) - L/2| \leq B\varepsilon_0 t \tag{3.49}$$

for all $t < t_*$. Assume for the moment that the piston did not move at all, i.e. $X(s) = L/2$ for all $0 < s < t$. Then, by (3.43), the density $q(v,t)$ of the gas "on the piston" would be almost symmetric, i.e.

$$|q(v,t) - q(-v,t)| < \varepsilon_0$$

Hence, we would have

$$|Q_i(t)| < C_1\varepsilon_0, \qquad i = 0, 2 \tag{3.50}$$

with $C_1 = v_{\max}^3$. We emphasize that C_1 does not depend on the choice of B in (3.44). Below we will introduce some more constants C_i, $i \geq 2$, so that none of them will depend on B.

When the piston actually moves and covers the distance $X(t) - L/2$, then (3.13) and (3.50) imply

$$|Q_i(t)| < C_1\varepsilon_0 + C_2 L^{-1}|X(t) - L/2|, \qquad i = 0, 2 \tag{3.51}$$

with some constant $C_2 > 0$. At the same time, $Q_1(t)$ stays bounded above and below by positive constants for all $t < t_*$:

$$0 < Q_{1,\min} \leq Q_1(t) \leq Q_{1,\max} < \infty \tag{3.52}$$

where $Q_{1,\min}$ and $Q_{1,\max}$ are constants determined by $\pi_{\min}$ and $\pi_{\max}$ in (P3).

By (3.49), (3.51) and (3.8) we have

$$|Q_i(t)| < (C_1 + Bv_{\max}^{-1})\,\varepsilon_0, \qquad i = 0, 2 \tag{3.53}$$

for all $t < t_*$. This and (3.52) imply that the quadratic polynomial (3.16) has two real roots, and the smaller one (the one closer to zero) is given by (3.46)–(3.47), cf. (2.24) and (2.25) in Section 2. Due to (2.22) and (3.52) we have

$$|V_0(t)| < C_3\varepsilon_0 + C_4 L^{-1}|X(t) - L/2| \tag{3.54}$$

for all $t < t_*$ and some constants $C_3, C_4 > 0$ (independent of the choice of B).

We note that (3.54) and (3.8) imply the boundedness of $V_0(t)$

$$|V_0(t)| \leq \text{ const } = C_3\varepsilon_0 + C_4 v_{\max}^{-1} \tag{3.55}$$

for all $t < t_*$.

Next, we need to estimate the derivative $dV_0(t)/dt$. (Note: the function $V_0(t)$ defined by (3.46)–(3.47) is continuous and piecewise differentiable, it should not be confused with the piston velocity $V(t)$, which is piecewise constant and hence not even continuous). Due to (3.13)

$$\left|\frac{dQ_i(t)}{dt}\right| = \left|\frac{dQ_i(t)}{dX}\cdot V(t)\right| \leq \frac{\text{const}\cdot\varepsilon_0}{L} \tag{3.56}$$

for $i = 0, 1, 2$. Differentiating the quadratic equation

$$Q_0(t)V_0^2(t) - 2Q_1(t)V_0(t) + Q_2(t) = 0$$

with respect to t gives

$$\frac{dV_0(t)}{dt} = \frac{(dQ_0/dt)V_0^2 - 2(dQ_1/dt)V_0 + (dQ_2/dt)}{2(Q_1 - Q_0V_0)} \tag{3.57}$$

Due to (2.22), (3.52), (3.55) and (3.56) we have

$$\left|\frac{dV_0(t)}{dt}\right| \leq \frac{E_0\,\varepsilon_0}{L} \tag{3.58}$$

where $E_0 > 0$ is a constant.

Now consider the quantity $\mathcal{D}$ defined by (3.16) as a function of V (with fixed Q_i, $i = 0, 1, 2$). Its derivative is

$$\frac{\partial\mathcal{D}}{\partial V} = 2a[Q_0V - Q_1]$$

Due to (3.52) and (3.53) there are positive constants $0 < E_1 < E_2$ such that for all $t < t_*$ we have

$$-E_2 < \frac{\partial\mathcal{D}}{\partial V} < -E_1$$

therefore, by the mean value theorem, for all $t < t_*$

$$-E_2 < \frac{\mathcal{D}(t)}{V(t) - V_0(t)} < -E_1 \tag{3.59}$$

We now prove (3.45) for all $t < t_*$ with some constant $C_0 > 0$ independent of the choice of B in (3.44). Recall that we have set the initial velocity of the piston to $V(0) = W(0)$ and that $W(0) = V_0(0)$, see Section 2. Now, by way of contradiction, let $t \in (0, t_*)$ be the first time (3.45) fails. Denote by

$$\Delta_0 = \frac{1}{L^{2/3}\ln L}$$

the maximal allowed time increment in Theorem 3.4. Let $s = t - \Delta_0$. Due to Theorem 3.4

$$V(t) = V(s) + \mathcal{D}(s)\Delta_0 + \chi \tag{3.60}$$

with

$$|\chi| \le \frac{C\sqrt{\Delta_0}\ln L}{L} = \frac{C\sqrt{\ln L}}{L^{4/3}}$$

Due to (3.58) we have

$$V_0(t) = V_0(s) + \chi_0 \tag{3.61}$$

with

$$|\chi_0| \le \frac{E_0\,\varepsilon_0\,\Delta_0}{L} = \frac{E_0\,\varepsilon_0}{L^{5/3}\ln L}$$

For brevity, put $U(s) = V(s) - V_0(s)$ for all s. Subtracting (3.61) from (3.60) then gives

$$U(t) = U(s) + \mathcal{D}(s)\Delta_0 + \chi' \tag{3.62}$$

with $\chi' = \chi - \chi_0$, so that for large L

$$|\chi'| \le \frac{2C\sqrt{\ln L}}{L^{4/3}} \tag{3.63}$$

Now assume, without loss of generality, that $U(t) > 0$. Since (3.45) fails at time t, we have

$$U(t) \ge \frac{C_0(\ln L)^{3/2}}{L^{2/3}} \tag{3.64}$$

Now consider two cases. If $U(s) \le 0$, then by (3.59)

$$U(t) \le |\mathcal{D}(s)|\Delta_0 + |\chi'| \le E_2\,|U(s)|\Delta_0 + |\chi'| \ll \frac{(\ln L)^{3/2}}{L^{2/3}}$$

for large L, which contradicts to (3.64). If $U(s) > 0$, then, again due to (3.62) and (3.59),

$$U(t) < U(s)[1 - E_1\Delta_0] + \chi',$$

hence

$$\begin{aligned} U(s) &> \frac{U(t) - \chi'}{1 - E_1\Delta_0} > (U(t) - \chi')(1 + E_1\Delta_0) \\ &> U(t) + U(t)E_1\Delta_0 - 2\chi' \end{aligned} \tag{3.65}$$

Now, if C_0 in (3.45) is large enough, say $C_0 = 5C/E_1$, then $U(t)E_1\Delta_0 > 2\chi'$ by (3.64) and (3.63). This fact and (3.65) imply $U(s) > U(t)$, so (3.45) fails at an earlier time $s < t$, a contradiction. Hence, (3.45) is proved for all $t < t_*$ and $C_0 = 5C/E_1$.

Now, combining (3.54) and (3.45) gives, for large L and all $t < t_*$

$$|dX(t)/dt| < 2C_3\varepsilon_0 + C_4L^{-1}|X(t) - L/2|$$

Using the standard Gronwall inequality in differential equations, see, e.g., Lemma 2.1 in [TVS], gives

$$|X(t) - L/2| < 2\varepsilon_0 C_3 C_4^{-1} L(e^{C_4L^{-1}t} - 1)$$

and

$$|V(t)| < 2\varepsilon_0 C_3 e^{C_4 L^{-1} t}$$

for all $t < t_*$. By (3.8) we have

$$|V(t)| < 2\varepsilon_0 C_3 e^{C_4 v_{\max}^{-1}} \tag{3.66}$$

Now we choose $B = 3C_3 e^{C_4 v_{\max}^{-1}}$. Clearly, (3.66) then contradicts (3.48). This completes the proof of (3.44) for all $t < S_1$. Theorem 3.5 is now proved. □

Corollary 3.6 *Assume that* $\varepsilon_0 > 0$ *in (1.11) is small enough. Then, for all large* L *and all* $\omega \in \Omega_0^*$*, we have* $S_1 = T_1$*, i.e. the previous theorems hold on the entire zero-recollision interval* $(0, T_1)$*.*

Proof. Recall that the "slow" interval $(0, S_1) \subset (0, T_1)$ is defined by two conditions, (a) and (b). If $S_1 < T_1$, then either (a) or (b) fails at S_1. Clearly, (a) cannot fail "abruptly" since (3.44) holds for all $t < S_1$.

Suppose (b) fails at some $s < T_1$, while (a) still holds. The failure of (b) means that at time s the piston "collides" with a point (x, v) such that $v_{\min} < |v| < v_{\max}$ and $(x, v) \notin G_0(s)$. Therefore, the backward trajectory $F^{-(s-t)}(x, v)$, $0 < t < s$, of the point (x, y) hits the piston at some time $t > 0$. Now, during the time interval $(0, t)$ the piston covers the distance $|X(t) - L/2| \leq B\varepsilon_0 t$, and during the time interval (t, s) the trajectory of our point covers the distance $|v|(s-t) < v_{\max}(s-t)$. Hence we have

$$\begin{aligned} L &\leq |X(t) - L/2| + |v|(s-t) + |X(s) - L/2| \\ &\leq |X(s) - L/2| + v_{\max} s - (v_{\max} - B\varepsilon_0) t \end{aligned}$$

On the other hand, since $s < T_1$, we have

$$L > |X(s) - L/2| + v_{\max} s$$

see the proof of Lemma 3.2. This contradiction shows that (b) cannot fail either. The proof of Corollary 3.6 is completed. □

Remark. We have reset the initial velocity of the piston to $V(0) = W(0)$ here, while in Section 1 it was set to zero. If $V(0) = 0$, then Theorem 3.4 would imply that $V(t)$ converges to $V_0(t)$ exponentially fast in t, until it gets δ-close to $V_0(t)$ with $\delta = C_0(\ln L)^{3/2} L^{-2/3}$. After that all our results will apply without change. The initial interval on which the convergence takes place will be of order $\ln L$, and in the hydrodynamical time it is $L^{-1} \ln L$, which vanishes as $L \to \infty$. This is why we simply opted for the most convenient setting $V(0) = W(0)$ here.

The following theorem improves the results of Theorems 3.4 and 3.5.

Theorem 3.7 *Assume that $\varepsilon_0 > 0$ in (1.11) is small enough. Then there is a constant $C > 0$ such that for all large L, all $\omega \in \Omega_0^*$ and all $t < T_1$*

$$|V(t) - V_0(t)| < \frac{C \ln L}{L} \tag{3.67}$$

and for any time interval $(t, t + \Delta t) \subset (0, T_1)$ such that $L^{-2} < \Delta t \leq 1$ we have

$$|V(t + \Delta t) - V(t)| < C\, \frac{\ln L \sqrt{\Delta t}}{L} \tag{3.68}$$

Proof. Due to our choice of the initial velocity, $V(0) = V_0(0)$, hence (3.67) holds for at least small t. Assume that it fails at some $t_* < T_1$, and t_* is the earliest time of failure. Without loss of generality, assume

$$V(t_*) - V_0(t_*) \geq \frac{C \ln L}{L} \tag{3.69}$$

Let $0 < t_0 < t_*$ be the latest time when

$$V(t_0) - V_0(t_0) \leq \frac{C \ln L}{2L} \tag{3.70}$$

Then we have

$$V(t_0) - V_0(t_0) \leq V(t) - V_0(t) \leq V(t_*) - V_0(t_*) \tag{3.71}$$

for all $t \in (t_0, t_*)$. Let

$$\Delta t = \min\{1, t_* - t_0\} \tag{3.72}$$

We will analyze the dynamics of the piston during the time interval $(t_0, t_0 + \Delta t)$. Due to (3.58) we have

$$|V_0(t) - V_0(t_0)| < \delta V := E_0\, \varepsilon_0 L^{-1} \Delta t \tag{3.73}$$

for all $t \in (t_0, t_0 + \Delta t)$, hence (3.71) implies

$$V(t) > V(t_0) - \delta V \tag{3.74}$$

We note that Δt is not too small, it is at least $\Delta t > (L^{2/3} \ln L)^{-1}$. Indeed, otherwise we would have $t_* = t_0 + \Delta t$ and then (3.69), (3.70) and (3.73) would imply

$$V(t_*) - V(t_0) \geq 2^{-1} C L^{-1} \ln L - E_0\, \varepsilon_0 L^{-5/3}$$

which would contradict Theorem 3.4, since $\mathcal{D}(t_0) < 0$ (because $V(t_0) > V_0(t_0)$).

Next, we develop a generalized version of the velocity decomposition (3.27) in the proof of Theorem 3.4. We partition the interval $(t_0, t_0 + \Delta t)$ into subintervals of length

δ (to be chosen shortly) with endpoints $t_i = t_0 + i\delta$, $i = 0, 1, \ldots, I$, where $I = \Delta t/\delta$. We select δ so that

$$\frac{0.5}{L \ln L} < \delta < \frac{1}{L \ln L} \tag{3.75}$$

and $\Delta t/\delta$ is an integer (for convenience). Preliminary requirements in the proof of Theorem 3.4 allow us to adjust time so that $L^2 t_0$ and $L^2 \delta$ are integers, hence $L^2 t_i$ will be an integer for every i. The velocity decomposition in the proof of Theorem 3.4 now applies to each subinterval (t_i, t_{i+1}) of length δ. In particular, (3.27) implies

$$V(t_{i+1}) - V(t_i) = -\varepsilon k_i V(t_i) + \varepsilon \sum_{j=1}^{k_i} v_j + \chi_i^{(1)} \tag{3.76}$$

where k_i is the number of particles colliding with the piston during the time interval (t_i, t_{i+1}) and v_j, $1 \leq j \leq k_i$, are their velocities. The fluctuation term $\chi_i^{(1)}$ can be bounded by (3.32):

$$|\chi_i^{(1)}| \leq 8a^2 v_{\max}^3 (\ln L)^2 \delta^2 \tag{3.77}$$

Due to (3.74), the expansion (3.76) can be rewritten as

$$V(t_{i+1}) - V(t_i) \leq -\varepsilon k_i V(t_0) + \varepsilon \sum_{j=1}^{k_i} v_j + \chi_i^{(1)} + \chi_i^{(2)} \tag{3.78}$$

with

$$|\chi_i^{(2)}| \leq \varepsilon k_i\, \delta V \leq 2a v_{\max} E_0\, \varepsilon_0 L^{-1} \ln L\, \Delta t\, \delta$$

where in the last step we used (3.31) and (3.73). Summing (3.78) up over i yields

$$V(t_0 + \Delta t) - V(t_0) \leq -\varepsilon k V(t_0) + \varepsilon \sum_{j=1}^{k} v_j + \chi^{(3)} \tag{3.79}$$

where k is the number of particles colliding with the piston during the time interval $(t_0, t_0 + \Delta t)$ and v_j, $1 \leq j \leq k$, are their velocities, and we have

$$\begin{aligned} |\chi^{(3)}| &\leq 8a^2 v_{\max}^3 (\ln L)^2 \delta \Delta t + 2a v_{\max} E_0\, \varepsilon_0 L^{-1} \ln L\, (\Delta t)^2 \\ &\leq (8a^2 v_{\max}^3 + 2a v_{\max} E_0\, \varepsilon_0)\, L^{-1} \ln L\, \Delta t \end{aligned} \tag{3.80}$$

(where we used (3.73) and the assumption $\Delta t \leq 1$).

The expansion (3.79) can be analyzed similarly to (3.27) in the proof of Theorem 3.4. Define a region on the x, v plane:

$$D_1 = \left\{ (x, v) : \ \frac{v - V(t_0) + \delta V}{x - X(t_0)} < -\frac{1}{\Delta t}, \quad v_{\min} < |v| < v_{\max} \right\} \tag{3.81}$$

It is the union of two trapezoids $D_1 = D_1^+ \cup D_1^-$, where D_1^- denotes the upper and D_1^+ the lower one, see Fig. 4. The bound (3.74) implies that all the particles in the region

D_1^+ necessarily collide with the piston during the time interval $(t_0, t_0 + \Delta t)$ and all the particles actually colliding with the piston on its left hand side during this interval of time are contained in D_1^-.

Since $v > V(t_0)$ for all particles $(x, v) \in D_1^-$ and $v < V(t_0)$ for all $(x, v) \in D_1^+$, we can remove from (3.79) the particles that do not belong in D_1^+ and simultaneously add to (3.79) the particles that belong in D_1^- but do not collide with the piston. This modification only makes the right hand side of (3.79) larger, hence

$$V(t_0 + \Delta t) - V(t_0) \leq -\varepsilon k_1 V(t_0) + \varepsilon \sum_{j=1}^{k_1} v_j + \chi^{(3)} \tag{3.82}$$

where k_1 is the number of particles in D_1 at time t_0, and the summation runs over all those particles. Let $Z_1 = \sum_{(x,v)\in D_1} v$. Just like in the proof of Theorem 3.4, our main requirements stated there guarantee that

$$|k_1 - E(k_1)| \leq c_3 L \ln L \sqrt{\Delta t}$$

and

$$|Z_1 - E(Z_1)| \leq c_4 L \ln L \sqrt{\Delta t}$$

where the constant $c_3, c_4 > 0$ do not depend on the choice of C in (3.67), which we have not made yet. Now, computing the mean values of k_1 and Z_1 as in the proof of Theorem 3.4 we arrive at

$$V(t_0 + \Delta t) - V(t_0) \leq \mathcal{D}(t_0)\,\Delta t + \chi^{(3)} + \chi^{(4)} \tag{3.83}$$

with

$$|\chi^{(4)}| \leq a(c_3 + c_4)L^{-1} \ln L \sqrt{\Delta t} + c_5\,\delta V\,\Delta t \tag{3.84}$$

where $c_5 > 0$ is a constant independent of the choice of C in (3.67). The last term in (3.84) comes from the adjustment δV to the velocity $V(t_0)$ in (3.81). This last term is bounded by $c_5 E_0\,\varepsilon_0 L^{-1}(\Delta t)^2$, and since $\Delta t \leq 1$, it can be incorporated into the first term in (3.84). Recall that $V(t_0) - V_0(t_0) \approx 2^{-1} C L^{-1} \ln L > 0$ (here we have an approximation up to a quantity of order $1/L^2$, since the piston velocity changes by $O(1/L^2)$ at each collision). Then due to (3.59) we have

$$\mathcal{D}(t_0) \leq -E_1(V(t_0) - V_0(t_0)) \approx -2^{-1} C E_1 L^{-1} \ln L$$

Therefore, combining the above estimates gives

$$V(t_0 + \Delta t) - V(t_0) \leq -2^{-1} C E_1 L^{-1} \ln L\,\Delta t + \chi^{(5)} \tag{3.85}$$

with $\chi^{(5)} = \chi^{(3)} + \chi^{(4)}$ bounded by (3.80) and (3.84):

$$|\chi^{(5)}| \leq c_6(L^{-1} \ln L\,\Delta t + L^{-1} \ln L \sqrt{\Delta t})$$

where $c_6 > 0$ is a constant independent of the choice of C in (3.67). Now we chose the constant C there as

$$C = \max\{c_6, 6E_1^{-1}c_6\} \tag{3.86}$$

Then (3.85) implies

$$V(t_0 + \Delta t) - V(t_0) \le c_6 L^{-1} \ln L\,(-2\Delta t + \sqrt{\Delta t})$$

and hence, due to (3.73),

$$\begin{aligned} V(t_0 + \Delta t) - V_0(t_0 + \Delta t) \ \ \le\ \ & V(t_0) - V_0(t_0) \\ & + c_6 L^{-1} \ln L\,(-2\Delta t + \sqrt{\Delta t}) + E_0\,\varepsilon_0 L^{-1} \Delta t \end{aligned} \tag{3.87}$$

We now have two cases. First, let $t_* - t_0 \le 1$, hence $\Delta t \le 1$. The expression $-2\Delta t + \sqrt{\Delta t}$ has a maximum, equal to 1/8, at the point $\Delta t = 1/16$. Therefore, (3.87) implies

$$V(t_*) - V_0(t_*) < V(t_0) - V_0(t_0) + 8^{-1}c_6 L^{-1} \ln L + E_0\,\varepsilon_0 L^{-1}$$

This contradicts (3.69) and (3.70) when L is large enough, recall our choice of C in (3.86). Consider the second case: $t_* - t_0 > 1$. Then $\Delta t = 1$ and (3.87) implies, for large L,

$$V(t_0 + 1) - V_0(t_0 + 1) < V(t_0) - V_0(t_0) - 2^{-1}c_6 L^{-1} \ln L$$

which contradicts (3.71). This completes the proof of (3.67).

We now prove (3.68). If $\Delta t < (L^{2/3} \ln L)^{-1}$, then we can use Theorem 3.4:

$$|V(t + \Delta t) - V(t)| \le |\mathcal{D}(t)|\,\Delta t + C L^{-1} \ln L \sqrt{\Delta t}$$

The early estimates (3.59) and (3.67) imply

$$|\mathcal{D}(t)| \le E_2 |V(t) - V_0(t)| \le C E_2 L^{-1} \ln L \tag{3.88}$$

so that (3.68) follows (with some larger value of C than above).

Now let $(L^{2/3} \ln L)^{-1} \le \Delta t \le 1$. Without loss of generality, assume that $V(t + \Delta t) > V(t)$. Moreover, we can assume that

$$V(s) > V(t) \qquad \text{for all } s \in (t, t + \Delta t) \tag{3.89}$$

Indeed, if this is not the case, we can replace t by $t' = \max\{s < t + \Delta t : V(s) \le V(t)\}$ and prove (3.68) for the smaller interval $(t', t + \Delta t)$.

Next, our plan is to apply some estimates from the proof of (3.67) and then argue along the lines of the proof of Theorem 3.4. Denote $t_0 = t$ and partition the interval $(t_0, t_0 + \Delta t)$ into subintervals of length δ satisfying (3.75). Then we again have decomposition (3.76)–(3.77). Due to (3.89) we have $V(t_i) > V(t_0)$ for all i, hence (3.76) implies

$$V(t_{i+1}) - V(t_i) < -\varepsilon k_i V(t_0) + \varepsilon \sum_{j=1}^{k_i} v_j + \chi_i^{(1)}$$

Summing this up over i gives (3.79) with the bound (3.80), in which the second term can be simply removed, since we do not have $\chi_i^{(2)}$ anymore. Next, possible fluctuations of the piston velocity $V(s)$ during the time interval $(t_0, t_0 + \Delta t)$ can be estimated with the help of (3.67) and (3.73):

$$|V(s) - V(t_0)| \le \delta V := 2CL^{-1} \ln L + E_0\, \varepsilon_0 L^{-1} \Delta t$$

for all $s \in (t_0, t_0 + \Delta t)$. Then we estimate the random variables k and $Z = \sum_j v_j$ along the lines of the proof of Theorem 3.4, starting with construction of two domains D_1 and D_2 by (3.29)–(3.30), etc. Repeating the argument almost word by word we arrive at an analogue of (3.41):

$$V(t + \Delta t) - V(t) < \mathcal{D}(t)\, \Delta t + \chi'$$

where

$$|\chi'| \le \text{const} \cdot \left[L^{-1} \ln L \sqrt{\Delta t} + \delta V\, \Delta t + L^{-1} \ln L\, \Delta t\right]$$

where the last term comes from (3.80), which we have now, instead of (3.32) (note: (3.32) would not be nearly enough anymore, since Δt is large; this is why we needed to partition the interval $(t, t + \Delta t)$ into smaller subintervals). We now combine the above estimates with (3.88) and complete the proof of (3.68) and Theorem 3.7. □

We finally prove the convergence, as $L \to \infty$, of the random trajectory of the piston to the solution $Y(\tau), W(\tau)$ of the hydrodynamical equations found in Section 2.

Theorem 3.8 *Assume that $\varepsilon_0 > 0$ in (1.11) is small enough. Then, for all large L and all $\omega \in \Omega_0^*$, there is a constant $C > 0$ such that*

$$|Y_L(\tau, \omega) - Y(\tau)| \le \frac{C \ln L}{L} \tag{3.90}$$

and

$$|W_L(\tau, \omega) - W(\tau)| \le \frac{C \ln L}{L} \tag{3.91}$$

for all $0 < \tau < \min\{\tau_1, T_1/L\}$ and

$$|\tau_1 - T_1/L| \le \frac{C \ln L}{L} \tag{3.92}$$

Proof. In Section 2 we defined the function $F(Y, \tau)$ so that the hydrodynamical solution $Y(\tau)$ satisfies

$$dY(\tau)/d\tau = F(Y, \tau), \qquad Y(0) = 1/2 \tag{3.93}$$

see (2.29). Now Theorem 3.7 implies that for all $\omega \in \Omega_0^*$ the random trajectory satisfies

$$\partial Y_L(\tau, \omega)/\partial \tau = F(Y, \tau) + \chi(\tau, \omega), \qquad Y_L(0, \omega) = 1/2 \tag{3.94}$$

with some

$$|\chi(\tau,\omega)| \le \frac{C\,\ln L}{L}$$

Recall that $|\partial F(Y,\tau)/\partial Y| \le \kappa$, see (2.30). Therefore, the difference $Z_L(\tau,\omega) := Y_L(\tau,\omega) - Y(\tau)$ satisfies

$$|Z_L'(\tau,\omega)| \le \kappa |Z_L(\tau,\omega)| + \frac{C\,\ln L}{L}$$

and $Z_L(0,\omega) = 0$. By the standard Gronwall inequality in differential equations, see, e.g., Lemma 2.1 in [TVS], we have

$$|Z_L(\tau,\omega)| \le \frac{C\,\ln L}{\kappa L}\left(e^{\kappa\tau} - 1\right)$$

and

$$|Z_L'(\tau,\omega)| \le \frac{C\,\ln L}{L}\,e^{\kappa\tau}$$

for all $\tau < \min\{\tau_1, T_1/L\}$, which imply (3.90) and (3.91).

Lastly, we verify (3.92). By (3.91), random fluctuations of the piston velocity are bounded by $CL^{-1}\ln L$. Hence, random fluctuations of the velocities of particles that have had one collision with the piston are bounded by $2CL^{-1}\ln L$. The random fluctuations of the positions of both the piston and particles at every moment of time $t < \min\{\tau_1 L, T_1\}$ are bounded by the same quantities (with, possibly, a different value of C) in the coordinate $y = x/L$. On the other hand, the relative velocity of the piston and the particles stays bounded away from zero (by, say, $v_{\min} - 4B\varepsilon_0 > 0$). Hence the time of the first recollision T_1/L can differ from τ_1 by at most const$\cdot L^{-1}\ln L$. Theorem 3.8 is proved. □

4 Dynamics between the first and second recollisions

In this section we study the one-recollision interval (τ_1, τ_2), on which gas particles experience the second collision (i.e., the first *re*collision) with the piston.

The particles that have collided with the piston no longer make a Poisson process, hence their distribution is much harder to control. This is our main trouble. On the other hand, we will be satisfied with much weaker estimates than those in the previous section. Also, many arguments and constructions in this section are similar to those in Section 3, and we omit some details. We will focus on new ideas.

Here our analysis is always restricted to the configurations $\omega \in \Omega_0^*$. Later on we will put additional requirements on ω.

Recall that for $\omega \in \Omega_0^*$ the piston velocity is small, $|V(t)| < B\varepsilon_0$, see (3.44), on the zero recollision interval $(0, T_1)$. Hence, the velocities of gas particles that experience one collision with the piston on the interval $(0, T_1)$ are bounded

$$v_{1,\min} < |v| < v_{1,\max} \tag{4.1}$$

with

$$v_{1,\min} := v_{\min} - 2B\varepsilon_0 \qquad \text{and} \qquad v_{1,\max} := v_{\max} + 2B\varepsilon_0 \tag{4.2}$$

The first time of the second recollision $T_2 = T_2(\omega)$ is defined by $T_2 = \sup_{t>0}\{G_3^+ = \emptyset\}$, see (3.7). Due to (3.44) and (4.1) the following bound can be easily obtained as in the proof of Lemma 3.2:

$$T_2 \le \frac{L}{v_{\max}} + \frac{L}{v_{\max} - 2B\varepsilon_0} \le \frac{3L}{v_{\max}} \tag{4.3}$$

Now let $(x,v) \in G^+$ and $(x_t, v_t) = F^t(x,v)$ for $t > 0$. Denote by

$$s_1(x,v) = \min\{t : \, x_t = X(t)\}$$

the time of the first collision with the piston. The region

$$G_*^+(t) := \{(x_t, y_t) \in G_1^+(t) : \; s_1(x,v) < T_1\}$$

is occupied by points that by the time t have experienced one collision with the piston, which occurred before time T_1. By removing the superscript $+$ in the above formula we define $G_*(t)$. Let $T_* \le T_2$ be the earliest time the piston interacts with the particles

$$(x,v) \in [G_1^+(t) \setminus G_*^+(t)] \cup G_2^+(t)$$

The time T_* is a random analogue of τ_* introduced in Section 2. During the interval (T_1, T_*) the piston only interacts with the particles from $G_0^+(t) \cup G_*^+(t)$, hence their velocities must be bounded by (4.1). Denote by

$$\begin{aligned} \mathcal{X}_1(t) = \{(x,v) : \, x = X(t) + 0, \, -v_{1,\max} < v < -v_{1,\min}\} \\ \cup\{(x,v) : \, x = X(t) - 0, \, v_{1,\min} < v < v_{1,\max}\} \end{aligned} \tag{4.4}$$

two immediate one-sided vicinities of the piston which contain all "incoming" particles for every $t \in (T_1, T_*)$.

Again, as in the previous section, we define a subinterval $(T_1, S_2) \subset (T_1, T_*)$ on which the piston is slow enough:

Definition Let $(T_1, S_2) \subset (T_1, T_*)$ be the maximal time interval during which
(a) $|V(t)| < v_{1,\min}$;
(b) $\mathcal{X}_1(t) \subset G_0(t) \cup G_*(t)$.

The condition (b) means that the particles with velocities $v_{1,\min} < |v| < v_{1,\max}$ that are about to interact with the piston at time t have interacted with the piston during the interval $(0,t)$ at most once, and if they did, the interaction occurred before T_1.

Next we estimate how large the interval (T_1, S_2) is. Suppose

$$|V(t) - W(t/L)| \le \Delta \tag{4.5}$$

for all $t < S_2$ and some small Δ (we will later estimate Δ and show that $\Delta \to 0$ as $L \to \infty$). This immediately implies $|V(t)| \leq \Delta + \mathcal{B}\varepsilon_0$, according to (2.26). Integrating (4.5) with respect to t gives

$$|X(t) - LY(t/L)| \leq t\Delta$$

and hence

$$|X(t) - L/2| \leq (\Delta + \mathcal{B}\varepsilon_0)t \tag{4.6}$$

Proposition 4.1 *If (4.5) holds for $t < S_2$ with some small $\Delta > 0$, then $T_2 - S_2 \leq CL(\Delta + \varepsilon_0)$, where $C > 0$ is a constant.*

Proof. If $S_2 = T_2$, then the statement is trivial. If $S_2 < T_2$, then either $S_2 < T_*$, and so the condition (a) or (b) in the previous definition fails at time S_2, or $S_2 = T_* < T_2$. Note that the condition (a) cannot fail abruptly, since we assume $|V(t)| \leq \Delta + \mathcal{B}\varepsilon_0$, on $(0, S_2)$, i.e. $V(t)$ remains small. If (b) fails, then at time S_2 the piston collides with a point (x, v) such that $v_{1,\min} \leq |v| \leq v_{1,\max}$ and the past trajectory $(x_t, v_t) := F^{t-S_2}(x, v)$ of that point for $t \in (0, S_2)$ hits the piston at some time $t_1 \geq T_1$. If $S_2 = T_* < T_2$, then at time S_2 the piston recollides with a gas particle (x, v) whose past trajectory $(x_t, v_t) := F^{t-S_2}(x, v)$ experiences the first collision with the piston at some time $t_1 \geq T_1$. In the last case, by (4.5)

$$|v| \leq v_{\max} + 2|V(t_1)| \leq v_{\max} + 2(\Delta + \mathcal{B}\varepsilon_0)$$

In either of the above two cases, the trajectory (x_t, v_t) collides with the piston twice - once at time $t_1 \geq T_1$ and the second time at $t_2 = S_2$. Denote by X_1, X_2 the positions of the piston and by V_1, V_2 its velocities at times t_1, t_2, respectively. Without loss of generality, assume that our trajectory (x_t, v_t) lies to the right of the piston. Note that the speed $|v| = |v_t|$ for $t_1 < t < t_2$ satisfies

$$\begin{aligned} |v| &\leq \max\{v_{1,\max}, v_{\max} + 2(\Delta + \mathcal{B}\varepsilon_0)\} \\ &\leq v_{\max} + 2(\Delta + B_1\varepsilon_0) \end{aligned} \tag{4.7}$$

with $B_1 = \max\{B, \mathcal{B}\}$. Then we write an obvious identity

$$|v|(t_2 - t_1) = (L - X_1) + (L - X_2)$$

hence, by (4.6) and (4.7)

$$t_2 - t_1 \geq \frac{L - (\Delta + \mathcal{B}\varepsilon_0)(t_1 + t_2)}{v_{\max} + 2(\Delta + B_1\varepsilon_0)}$$

On the other hand, consider the particle that experiences the *very first* recollision with the piston (this happens at time T_1). After the collision, that particle acquires velocity

$|v(T_1+0)| \geq v_{\max} - 2|V(T_1)|$. The next collision of this particle with the piston occurs *after* T_2. Therefore,

$$\begin{aligned} T_2 - T_1 &\leq \frac{L/2 + |X(T_1) - L/2| + L/2 + |X(T_2) - L/2|}{v_{\max} - 2|V(T_1)|} \\ &\leq \frac{L + (\Delta + \mathcal{B}\varepsilon_0)(T_1 + T_2)}{v_{\max} - 2(\Delta + \mathcal{B}\varepsilon_0)} \end{aligned}$$

Combining the above estimates gives

$$T_2 - S_2 \leq (T_2 - T_1) - (t_2 - t_1) \leq CL(\Delta + \varepsilon_0)$$

with some $C > 0$ determined by B and $\mathcal{B}$. □

Next, we study the dynamics during the time interval (T_1, S_2). We again define the density of colliding particles "on the piston" $q(v,t)$ by the equation (3.9) and the functions $Q_i(t)$, $i = 0, 1, 2$, by (3.10)–(3.12). We emphasize that now, unlike what we had in the previous section, the density $p(x,v,t)$ *essentially* depends on ω (at least for $(x,v) \in G_*^+(t)$), hence $q(v,t)$ and $Q_i(t)$ will depend on ω not only through the piston position $X(t)$ but also through the surrounding density $p(x,v,t)$.

In the previous sections we also introduced the deterministic density $\tilde{p}(x,v,t)$. So, now we can define the corresponding deterministic density "on the piston"

$$\tilde{q}(v,t) = \begin{cases} \tilde{p}(X(t)+0, v, t) & \text{if} \quad v < 0 \\ \tilde{p}(X(t)-0, v, t) & \text{if} \quad v > 0 \end{cases} \tag{4.8}$$

cf. (2.17) and the deterministic functions $\tilde{Q}_i(t)$, $i = 0, 1, 2$ by the equations similar to (3.10)–(3.12) but with tildes over the corresponding functions. We use tildes to distinguish these deterministic functions from the random ones.

We now compare the random functions p, q, Q_i with their deterministic counterparts on the interval (T_1, S_2). Since the transformations $F^t : G^+ \to G^+(t)$ and $\tilde{F}^t : \tilde{G}^+ \to \tilde{G}^+(t)$ (recall that $\tilde{G}^+(t) = \tilde{F}^t(G^+)$) are invertible area-preserving maps, then so is the map

$$\Psi^t = \tilde{F}^t \circ F^{-t}$$

which takes $G^+(t)$ onto $\tilde{G}^+(t)$. The next lemma easily follows from Theorem 3.8:

Lemma 4.2 *Let $t < S_2$ and $\omega \in \Omega_0^*$. For every $(x,v) \in G_*^+(t)$ put $(\tilde{x}, \tilde{v}) := \Psi^t(x,v)$. Then*

$$|x - \tilde{x}| \leq C \ln L \qquad \text{and} \qquad |v - \tilde{v}| \leq CL^{-1} \ln L$$

with a constant $C > 0$. We also have

$$p(x,v,t) = \tilde{p}(\tilde{x}, \tilde{v}, t)$$

Next, the properties of the density $p(x, v, t)$ stated in Lemma 3.3 obviously hold for the deterministic density $\tilde{p}(x, v, t)$ on the entire region $\tilde{G}^+$. That is, the density $\tilde{p}(x, v, t)$ is piecewise C^1 smooth with $|\partial\tilde{p}(x, v, t)/\partial x| \le D'_1/L$ and the discontinuity lines of $\tilde{p}(x, v, t)$ have slope of order $O(1/L)$ (so they are almost parallel to the x axis). Therefore, as in (3.13), we have

$$\left|\frac{\partial\tilde{Q}_i}{\partial X}\right| \le \frac{\text{const}}{L} \tag{4.9}$$

Lemma 4.2 allows us to compare Q_i and $\tilde{Q}_i$ considered as functions of X in the following way:

Lemma 4.3 *Let $t < S_2$ and $\omega \in \Omega_0^*$. Then for $i = 0, 1, 2$*

$$|Q_i(t) - \tilde{Q}_i(t)| \le CL^{-1}\ln L$$

Note that $\tilde{Q}_i(t)$ is defined through $\tilde{q}(v, t)$ which uses the (random) position of the piston $X(t)$ for the given ω, see (4.8). The above lemma shows that the dependence of $Q_i(t)$ on ω through the density $p(x, v, t)$ (which itself depends on ω) is very weak, because $L^{-1}\ln L$ is small. In other words, the density $p(x, v, t)$ of the gas surrounding the piston fluctuates with ω very little.

The following theorem is an analogue of Theorem 3.4.

Theorem 4.4 *For all sufficiently large L there is a set $\Omega_1^* \subset \Omega_0^*$ of initial configurations of particles such that*
(i) *there is a constant $c > 0$ such that*

$$P(\Omega_1^*) > 1 - L^{-c\ln\ln L} \tag{4.10}$$

(ii) *for each configuration $\omega \in \Omega_1^*$, for each time interval*

$$(t, t + \Delta t) \subset (T_1, S_2)$$

such that

$$\frac{(\ln L)^2}{L^{1/3}} \le \Delta t \le \frac{1}{L^{1/7}} \tag{4.11}$$

the change of the velocity of the piston satisfies

$$V(t + \Delta t) - V(t) = \tilde{\mathcal{D}}(t)\,\Delta t + \chi \tag{4.12}$$

where

$$\tilde{\mathcal{D}}(t) = a[\tilde{Q}_0(t)V^2(t) - 2\tilde{Q}_1(t)V(t) + \tilde{Q}_2(t)] \tag{4.13}$$

and

$$|\chi| \le C\,\frac{\ln L\,(\Delta t)^{1/4}}{L^{1/4}} \tag{4.14}$$

with some global constant $C > 0$.

Remark. Note that our bound (4.14) on random fluctuations represented by χ is much weaker than (3.17) in Theorem 3.4. This is due to the lack of a good control over large deviations for the distribution of gas particles, as it will be clear from the proof.

Proof. Our argument basically goes along the lines of the proof of Theorem 3.4. But it involves a good deal of new constructions, which we describe in detail. The first step is the velocity decomposition scheme, see (3.27),

$$V(t+\Delta t) = (1-\varepsilon k)V(t) + \varepsilon \sum_{j=1}^{k} v_j + \chi^{(1)} + \chi^{(2)} \tag{4.15}$$

The error terms $\chi^{(1)}$ and $\chi^{(2)}$ are defined in Section 3 after (3.27), and they are bounded by

$$|\chi^{(u)}| \leq v_{\max}\varepsilon^2 k^2$$

for $u = 1, 2$, see (3.32). We will see later that

$$k \leq \text{ const} \cdot L^2 \, \Delta t \tag{4.16}$$

hence

$$|\chi^{(u)}| \leq \text{ const} \cdot (\Delta t)^2 \tag{4.17}$$

for $u = 1, 2$.

Next, we need a crude upper bound δV on possible fluctuations of the piston velocity $V(s)$ during the interval $(t, t+\Delta t)$, as defined by (3.28). One can be easily derived from (4.15):

$$\delta V \leq 2v_{\max}\varepsilon k \leq \text{ const} \cdot \Delta t \tag{4.18}$$

We now define two regions $D_1 = D_1^+ \cup D_1^-$ and $D_2 = D_2^+ \cup D_2^-$ on the x, v plane by equations (3.29)–(3.30), where $v_{\min}$ and $v_{\max}$ must be replaced by $v_{1,\min}$ and $v_{1,\max}$, respectively. As it is explained in Section 3, all the particles in D_1 (at time t) will necessarily collide with the piston during the interval $(t, t+\Delta t)$. And all the particle that actually collide with the piston during that interval are contained in D_2. Therefore, we have

$$k_1^\pm \leq k^\pm \leq k_2^\pm \tag{4.19}$$

where $k^\pm, k_1^\pm, k_2^\pm$ are defined around (3.34). This gives upper and lower bounds on the number of colliding particles.

Our next step is to estimate the numbers $k_i^\pm$, $i = 1, 2$. Since $k_i^\pm$ no longer have Poisson distribution, we estimate them by using a new approach. Let D be one of the four trapezoids $D_i^\pm$, $i = 1, 2$, and let $k_{D,\omega}$ be the (random) number of particles in D at time t. Obviously, $k_{D,\omega}$ is equal to the number of particle in $F^{-t}(D)$ at time zero.

We note that the trapezoid D has height $v_{1,\max} - v_{1,\min} =$const and width $O(\Delta t)$, hence its area is bounded by

$$d_1 \, \Delta t \leq |D| \leq d_2 \, \Delta t \tag{4.20}$$

with some constants $0 < d_1 < d_2 < \infty$.

Let us examine the region $F^{-t}(D)$ more closely. Since $(t, t+\Delta t) \subset (T_1, S_2)$, it follows from the condition (b) in the definition of S_2 that $D \subset G_0(t) \cup G_*(t)$. Put $D_0 = D \cap G_0(t)$ and $D_1 = D \cap G_*(t)$, then $F^{-t}(D) = F^{-t}(D_0) \cup F^{-t}(D_1)$. To emphasize the dependence of the flow F^t on ω we will write F^t_ω for F^t. Now the part $F^{-t}_\omega(D_0)$ will be obtained by a simple linear transformation of D_0 without collisions with the piston, hence it will be actually independent of ω. The part $F^{-t}_\omega(D_1)$ is also obtained by pulling the domain D_1 back in time, but one collision with the piston will occur along the way. Since the position and velocity of the piston are random, then the domain $F^{-t}_\omega(D_1)$ will depend on ω. Hence, the domain $D_\omega := F^{-t}_\omega(D)$ will depend on ω. The initial number of particles, $k_{D,\omega}$, in a randomly selected domain D_ω certainly need not be a Poisson random variable.

To estimate $k_{D,\omega}$ we fix some Δt satisfying (4.11) and construct finitely many domains $D_n \subset G$, $1 \leq n \leq N_*$, which have the following property. For every $\omega \in \Omega_0^*$, every $t \in (T_1, S_2 - \Delta t)$, and every trapezoid D defined above, there are two domains $D_{n'}, D_{n''}$ such that

$$D_{n'} \subset F^{-t}_\omega(D) \subset D_{n''} \tag{4.21}$$

and the area of the difference is relatively small:

$$|D_{n''} \setminus D_{n'}| \leq \chi^{(3)} |D_\omega| \tag{4.22}$$

with some $\chi^{(3)} \to 0$ as $L \to \infty$. We say that $D_{n'}$ and $D_{n''}$ approximate D_ω "from inside" and "from outside", respectively. We denote the collection of the domains D_n, for the given Δt, by

$$\mathcal{C} = \mathcal{C}_{\Delta t} = \{D_n\}_{n=1}^{N_*}$$

We postpone the construction of D_n's for the moment and derive immediate benefits of the above approximation. The inclusion (4.21) implies

$$k_{n',\omega} \leq k_{D,\omega} \leq k_{n'',\omega} \tag{4.23}$$

where $k_{n,\omega}$ is the (random) number of particles in the domain D_n at time zero. Since D_n, for each n, is fixed (independent of ω), the random variable $k_{n,\omega}$ does have a Poisson distribution with mean value

$$\lambda_n := L^2 \int_{D_n} p(x, v, 0)\, dx\, dv$$

According to Lemma A.2, we have

$$P\left(\omega : |k_{n,\omega} - \lambda_n| > B\sqrt{\lambda_n}\right) \leq 2e^{-cB^2} \tag{4.24}$$

for any $B < b\sqrt{\lambda_n}$, where $b > 0$ is a constant and $c = c(b) > 0$ another constant. We will specify the value of $B = B_{\Delta t}$ (one for all D_n's in $\mathcal{C}_{\Delta t}$) later. Due to (4.20), it is enough to require

$$B_{\Delta t} < L\sqrt{\Delta t} \tag{4.25}$$

Then we define $\Omega_1^*(\Delta t)$ as the set of configurations $\omega \in \Omega_0^*$ satisfying

$$|k_{n,\omega} - \lambda_n| \leq B_{\Delta t}\sqrt{\lambda_n} \qquad \text{for } \ 1 \leq n \leq N_* \tag{4.26}$$

Then by (4.24) we have

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \leq 2N_* e^{-cB_{\Delta t}^2} \tag{4.27}$$

Next, for all $\omega \in \Omega_1^*(\Delta t)$ the bounds (4.23) and (4.26) imply

$$\lambda_{n'} - B_{\Delta t}\sqrt{\lambda_{n'}} \leq k_{D,\omega} \leq \lambda_{n''} + B_{\Delta t}\sqrt{\lambda_{n''}} \tag{4.28}$$

Furthermore, consider the quantity

$$\lambda_{D,\omega} = L^2 \int_{F_\omega^{-t}(D)} p(x,v,0)\, dx\, dv = L^2 \int_D p(x,v,t)\, dx\, dv$$

Due to the inclusion (4.21) we have

$$\lambda_{n'} \leq \lambda_{D,\omega} \leq \lambda_{n''}$$

It follows from (4.22) that

$$(1 - c\chi^{(3)})\lambda_{D,\omega} \leq \lambda_{n'} \leq \lambda_{n''} \leq (1 + c\chi^{(3)})\lambda_{D,\omega} \tag{4.29}$$

for some constant $c > 0$ (determined by $\pi_{\max}$, $\pi_{\min}$, v_1, and v_2 in (P3)).

Consider the deterministic quantity

$$\tilde{\lambda}_D = L^2 \int_D \tilde{p}(x,v,t)\, dx\, dv$$

It easily follows from Lemma 4.2 and the properties of the function $\tilde{p}(x,v,t)$ that

$$|\lambda_{D,\omega} - \tilde{\lambda}_D| \leq CL^{-1}\ln L\, \tilde{\lambda}_D \tag{4.30}$$

with some constant $C > 0$. Combining (4.29) and (4.30) we arrive at

$$(1 - \chi^{(4)})\tilde{\lambda}_D \leq \lambda_{n'} \leq \lambda_{n''} \leq (1 + \chi^{(4)})\tilde{\lambda}_D \tag{4.31}$$

with

$$\chi^{(4)} = \text{ const} \cdot (\chi^{(3)} + L^{-1}\ln L) \tag{4.32}$$

The bounds (4.28) and (4.31) will give the desired estimate on the number $k_{D,\omega}$.

We now construct the domains D_n that approximate the domains $D_\omega = F_\omega^{-t}(D)$ for all $\omega \in \Omega_0^*$ and all trapezoids D defined above. We first fix $\omega \in \Omega_0^*$ and a trapezoid D and will construct two special domains D', D'' that approximate $D_\omega = F_\omega^{-t}(D)$ from inside and from outside, i.e. such that $D' \subset D_\omega \subset D''$.

The domain D_ω is obtained by pulling D back in time. We consider its "trajectory" $D_s^- = F_\omega^{s-t}(D)$ for $0 < s < t$, so that $D_0^- = D_\omega$ and $D_t^- = D$. We examine the shape of the domain D_s^- and how it changes as s runs from t down to 0. Recall that the trapezoid D is adjacent to the piston at time t (it is about to run into the piston at that time). As s goes from t downward, the domain D_s^- comes off the piston and travels to a wall (as in a movie running backward). During that period, the map $F_\omega^{-(t-s)}$ restricted to D is linear, hence D_s^- is still a trapezoid. But since the velocities of points $(x, v) \in D$ vary (from $v_{1,\min}$ to $v_{1,\max}$), the trapezoid D_s^- will be "skewed" – its "outer" edge $|v| = v_{1,\max}$ will move toward the wall faster than the other edge $|v| = v_{1,\min}$. By the time it reaches the wall, D_s^- will be a long slanted trapezoid stretched the distance $O(L)$ along the x axis. Every vertical line (parallel to the v axis) will intersect D_s^- in a segment of length $O(\Delta t/L)$. As s runs farther down, a collision with the wall occurs, and a new part of D_s^- appears directly across the x axis, moving now toward the piston. Its shape will be also that of a long narrow trapezoid, whose vertical "thickness" is $O(\Delta t/L)$. Eventually it will move all the way (the distance $O(L)$) from the wall to the piston and contact the piston at some time $s_* = s_*(D)$, see Fig. 5. Note that so far D_s^- is completely independent of ω.

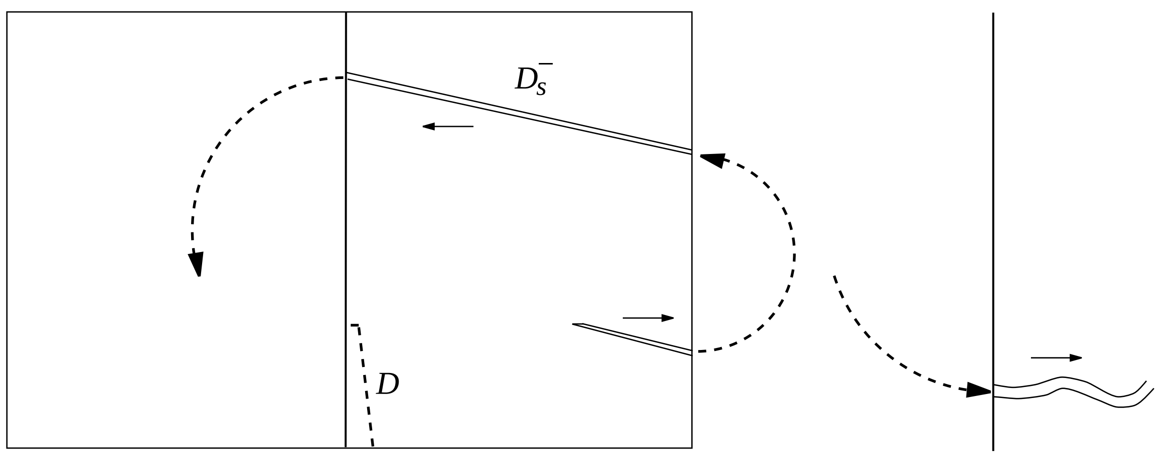

Figure 5: The domain D_s^-. The arrows show its motion as s decreases.
The "outgoing" part of D_s^-, as it comes off the piston, is shown on the right.

After the collision with the piston, a new part of D_s^- appears across the x axis, coming off the piston and moving back to the wall. This is the most interesting part, it will be determined by the piston position and velocity, hence it will actually depend on ω. Recall that the part of D_s^- running into the piston before the collision (the "incoming" part) is a narrow trapezoid bounded by two almost horizontal lines with slope $O(1/L)$ lying a distance $O(\Delta t/L)$ apart. We call them the upper and lower boundaries of D_s^-. The part of D_s^- coming off the piston after the collision (the "outgoing" part) also has upper and lower boundaries (due to the reversal of velocities at collision, though, the upper boundary after the collision is the image of the lower boundary before the collision, and vice versa), see Fig. 5. The boundaries of the outgoing part of D_s^- are quite irregular and depend on ω. It is them who need careful approximation.

The shape of the outgoing boundary of D_s^- will be determined by the piston velocity $V(u)$ during the interval $s < u < s_*$. Since the collision between D_s^- and the piston occurs during the zero-recollision interval ($s_* < T_1$), we can apply the results of Section 3. By Theorem 3.7 the piston velocity behaves as a Hölder continuous[3] function:

$$|V(u+h) - V(u)| \leq CL^{-1} \ln L \sqrt{h} =: d_h \tag{4.33}$$

for all $h \in (L^{-2}, 1)$. We will only consider h satisfying $L^{-a_1} < h < L^{-a_2}$ with some $0 < a_2 < a_1 < 2$. Note that in this case

$$Ld_h/h \to \infty \qquad \text{and} \qquad d_h/h \to 0 \tag{4.34}$$

as $L \to \infty$.

Let us consider the upper boundary of the outgoing domain D_s^- after the collision. It is a continuous curve that can be parameterized by the collision time (or contact time between D_s^- and the piston), call it u, and then it becomes a parametric curve $(x_s(u), v_s(u))$ with parameter $u \in (s, s_*)$. This means, in particular, that

$$F^w(x_s(u), v_s(u)) = (x_{s+w}(u), v_{s+w}(u))$$

for $s < u < s_*$ and $w \leq u - s$, and the point

$$F^{u-s}(x_s(u), v_s(u)) = (x_u(u), v_u(u))$$

is the endpoint of the upper boundary of D_u^-, i.e. $x_u(u) = X(u)$ is the piston coordinate at time u.

Claim 1. For each $u \in (0, s_* - h)$ and all sufficiently large L we have

$$|x_{u+h}(u+h) - x_u(u)| \leq B\varepsilon_0 h, \tag{4.35}$$

and

$$|v_{u+h}(u+h) - v_u(u)| \leq 3d_h \tag{4.36}$$

Proof. The first bound means that $|X(u+h) - X(u)| \leq B\varepsilon_0 h$ and follows from (3.44). To prove the second, we use the collision rule

$$v_u(u) = -v_u^-(u) + 2V(u)$$

where $v_u^-(u)$ is the v coordinate of the lower boundary of the "incoming" part of D_u where it contacts the piston (at time u). Similarly,

$$v_{u+h}(u+h) = -v_{u+h}^-(u+h) + 2V(u+h)$$

[3] For any $d > 0$, d-dense sets in the space of Hölder continuous functions were constructed by Kolmogorov and Tihomirov [KT], our constructions here are in the same spirit.

Since the lower boundary of the "incoming" part of D_s^- is a straight line with slope $O(1/L)$, and the piston velocity is bounded, then

$$|v_{u+h}^-(u+h) - v_u^-(u)| \le \text{const} \cdot h/L < d_h$$

where the last bound follows from(4.34). This and (4.33) imply (4.36). □

We now fix some h satisfying

$$(\Delta t/L)^{1/2} \le h \le 2(\Delta t/L)^{1/2} \tag{4.37}$$

With our restrictions (4.11) on Δt, this implies $L^{-2/3} \le h \le L^{-4/7}$, hence (4.34) will hold. Also, we note that

$$d_h/(\Delta t/L) \to 0 \quad \text{ as } \ L \to \infty \tag{4.38}$$

hence d_h will be much smaller than the "thickness" of the domain D_s^-.

Next we put a lattice on the x, v plane with the x-spacing rh and the v-spacing rd_h, where $r > 0$ is a sufficiently small constant (for example, $r = B\varepsilon_0/10$). The lattice sites are

$$(x_i, v_i) = (rhi, rd_hj), \qquad i, j \in \mathbb{Z} \tag{4.39}$$

Now we are ready to define the boundary of $D' \subset D_0^- = F_\omega^{-t}(D)$ approximating D_0^- from inside. We start with the upper boundary of the "outgoing" part of D_0^-. It is parameterized as $(x_0(u), v_0(u))$, see above, with $0 < u < s_*$. We consider discrete parameter values $u_k = kh$, $k = 0, 1, \ldots, [s_*/h]$. For each such k, consider the point

$$F^{kh}(x_0(kh), v_0(kh)) = (x_{kh}(kh), v_{kh}(kh))$$

This is the endpoint of the upper boundary of D_{kh}^-, so that $x_{kh}(kh) = X(kh)$ is the piston coordinate at time kh. Now we pick a site of the lattice (4.39) closest to the point $(x_{kh}(kh), v_{kh}(kh))$ and lying on the same side of the piston as the domain D_{kh}^-. Call this site (x_k^s, y_k^s). Next, we adjust this site by moving it down (along the v axis) the fixed distance $10d_h$ and obtain the adjusted point (x_k^a, v_k^a):

$$x_k^a = x_k^s, \qquad v_k^a = v_k^s - 10d_h \tag{4.40}$$

We note that Claim 1 (along with the smallness of r) implies

$$|x_k^a - x_{k+1}^a| = |x_k^s - x_{k+1}^s| \le 2B\varepsilon_0 h \tag{4.41}$$

and

$$|v_k^a - v_{k+1}^a| = |v_k^s - v_{k+1}^s| \le 4d_h \tag{4.42}$$

Next, let

$$(x_k^0, v_k^0) := F^{-kh}(x_k^a, v_k^a)$$

Now join the points (x_k^0, v_k^0) and (x_{k+1}^0, v_{k+1}^0) with a straight line segment, call it L_k. Then the upper boundary of the domain D' is made by the segments L_k, i.e. it is

$$\cup_{k=0}^{[s_*/h]-1} L_k$$

Claim 2. The upper boundary of D' lies completely inside the domain $D_0^- = F_\omega^{-t}(D)$.

Proof. It is enough to show that each link L_k lies inside D_0^-. It is easy to see that the map F^{kh} restricted to the link L_k is a linear map, and the image $L_k^a := F^{kh}(L_k)$ is a straight line segment joining the points (x_k^a, v_k^a) and

$$F^{-h}(x_{k+1}^a, v_{k+1}^a) = (x_{k+1}^a - v_{k+1}^a h, v_{k+1}^a)$$

We denote by

$$|L_k^a|_x = |x_{k+1}^a - x_k^a - v_{k+1}^a h|, \qquad |L_k^a|_v = |v_{k+1}^a - v_k^a|$$

the lengths of the projections of L_k^a onto the x and v axes, respectively. Note that $v_{1,\min} - B\varepsilon_0 \le |v_{k+1}^a| \le v_{1,\max} + B\varepsilon_0$. Hence (4.35) implies that

$$(v_{1,\min} - 2B\varepsilon_0)h \le |L_k^a|_x \le (v_{1,\max} + 2B\varepsilon_0)h$$

i.e. $|L_k^a|_x$ is of order h. By (4.36), $|L_k^a|_v < 3d_h$. Hence, the slope of L_k^a is $O(d_h/h) \to 0$ by (4.34). Therefore, L_k^a is almost a horizontal segment. The inequality (4.36) shows that the upper boundary $(x_{kh}(u), v_{kh}(u))$ of D_{kh}^- for $kh \le u \le kh+h$, which lies directly above L_k^a, oscillates in the v direction by less than $3d_h$. The adjustment (4.40) then ensures that the segment L_k^a lies entirely below the upper boundary of the domain D_{kh}^-. Also, (4.38) implies that the domain D_{kh}^- is much "thicker" than the distance between its upper boundary and L_k^a, hence L_k^a lies entirely inside that domain. Hence, $L_k = F^{-kh}(L_k^a)$ lies entirely inside $D_0^- = F^{-kh}(D_{kh}^-)$, proving our claim. □

It is also clear from the above argument that the segment L_k^a lies the distance $< 20d_h$ below the actual upper boundary of the domain D_{kh}^- so that the area between them is bounded by const$\cdot hd_h$. Hence, the total area between the upper boundaries of D_0^- and D' is bounded by

$$\text{const} \cdot hd_h \cdot (s_*/h) \le \text{ const} \cdot Ld_h$$

In the same way we construct the lower boundary of the new domain D'. The only difference is that we adjust the selected sites of the lattice (4.39) by moving them up, so that the joining segments will be again inside D_0^-. The upper and lower parts of the boundary will give us a new domain approximating the "outgoing" part of D_0^- after the collision with the piston, i.e. the part $F^{-t}(D \cap G_*(t))$. Note that this part may itself experience one collision with the wall (during the interval $(0, s_*)$), then it will consist of two connected components lying across the x axis.

It remains to approximate the "good" part of $F_\omega^{-t}(D) = D_0^-$, which has not interacted with the piston, i.e. the part $F_\omega^{-t}(D \cap G_0(t))$. That one, as described above, consists of one or two trapezoids bounded by two almost horizontal straight lines a distance $O(\Delta t/L)$

apart. This is an easy task. We simply replace the upper (lower) boundary of the domain $F^{-t}(D \cap G_0(t))$ with a polygonal line joining some sites of the lattice (4.39), picked one on each vertical line $x = x_i$ crossing the domain D_s^- and lying inside (outside) of this domain and no farther than d_h from the original boundary (recall that r in (4.39) is very small, so the above choice is possible). That gives the boundary of D'. Note that if we number the selected sites consecutively (say, from left to right), and call them (x_k^s, y_k^s), then the neighboring sites will again satisfy (4.41)–(4.42).

The complicated construction described above produces a domain $D' \subset D_\omega = F_\omega^{-t}(D)$ such that

$$|D_\omega \setminus D'| \leq \text{ const} \cdot d_h L = \text{ const} \cdot \ln L \sqrt{h} \tag{4.43}$$

In a completely similar way we construct another domain D'' that approximates D_ω from outside and satisfies

$$|D'' \setminus D_\omega| \leq \text{ const} \cdot d_h L = \text{ const} \cdot \ln L \sqrt{h} \tag{4.44}$$

(we just need to adjust the sites along the upper boundary by moving them up and the sites along the lower boundary by moving them down, so that the new boundaries will be completely outside of D_ω). Since $|D| = O(\Delta t)$, the inequalities (4.43)–(4.44) imply

$$|D'' \setminus D'| \leq \text{ const} \cdot \ln L \sqrt{h} \, (\Delta t)^{-1} \, |D|$$

hence we get (4.22) with

$$\chi^{(3)} = \text{const} \cdot \ln L \sqrt{h} \, (\Delta t)^{-1} \tag{4.45}$$

Note that $\chi^{(3)} \to 0$ as $L \to \infty$ due to our choice of h in (4.37) and restrictions on Δt in (4.11).

Our domains D', D'' are constructed around D_ω, hence they depend on ω. However, the boundaries of D', D'' are polygonal lines whose vertices are derived from the sites of a fixed lattice (4.39). Therefore, the total number of *distinct* domains D', D'' is finite. Now we estimate their number N_*.

First, the total number of the sites of the lattice (4.39) in the relevant area $0 < x < L$, $|v| < v_{1,\max}$ is

$$K_1 = \text{ const} \cdot (L/h) \cdot (2v_{\max}/d_h) \leq \text{ const} \cdot L \, (hd_h)^{-1} \tag{4.46}$$

Recall that the domain $D_0 = F^{-t}(D)$ consists of at most four connected components. The part $F^{-t}(D \cap G_*(t))$, or the "outgoing" part of D_0 created after the collision with the piston, consists of at most two components. And the part $F^{-t}(D \cap G_0(t))$ consists of at most two components, each is a regular, trapezoidal region. The upper (lower) part of the boundary of D' (D'') will then consists of at most four disjoint polygonal lines constructed from the lattice sites (4.39). In each polygonal line, the first point can be constructed from any of the K_1 sites of the lattice, see (4.46). But other points can be constructed consecutively, and each point must be constructed from a site selected from

$$K_0 := (4B\varepsilon_0/r) \cdot (8/r) = \text{ const} \tag{4.47}$$

nearest neighbors of the previously selected site, according to (4.41)–(4.42). It is also clear that each polygonal line has at most

$$M := \text{const} \cdot L/h$$

links (vertices). Therefore, the total number of ways to construct one polygonal line does not exceed $K_1 K_0^M$. The total number of ways to construct the entire boundary of D' (D'') is then less than $K_1^8 K_0^{8M}$. This gives an upper bound on the number of distinct domains D' (D''):

$$N_* \le K_1^8 K_0^{8M} \le \text{const} \cdot L^8 (h d_h)^{-8} \cdot e^{\text{const} \cdot L/h} \tag{4.48}$$

Of course, the exponential factor is dominant here and can absorb all the others.

We now set the value of $B_{\Delta t}$ in (4.25)–(4.28):

$$B_{\Delta t} = c_7 \sqrt{L/h} \tag{4.49}$$

with a sufficiently small constant $c_7 > 0$. Then (4.27) and (4.48) imply

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \le \text{const} \cdot e^{-c_8 L/h} \tag{4.50}$$

with some constant $c_8 > 0$. Recall that h satisfies (4.37), hence

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \le \text{const} \cdot e^{-\text{const} \cdot L^{3/2} (\Delta t)^{-1/2}}$$

Next, since Δt satisfies (4.11), we have

$$P(\Omega_0^* \setminus \Omega_1^*(\Delta t)) \le \text{const} \cdot e^{-\text{const} \cdot L^{11/7}} \tag{4.51}$$

Also, by (4.49) and (4.37)

$$B_{\Delta t} \le c_7 L^{3/4} (\Delta t)^{-1/4}$$

which implies (4.25) since $\Delta t \gg L^{-1/3}$.

Next, for each $\omega \in \Omega_1^*(\Delta t)$, each $t \in (T_1, S_2 - \Delta t)$, and any trapezoid D defined above, the number of particles $k_{D,\omega}$ in D satisfies (4.28)–(4.32) with $\chi^{(3)}$ given by (4.45) and h given by (4.37), hence

$$k_{D,\omega} = \tilde{\lambda}_D + \chi_{D,\omega}$$

with

$$\begin{aligned} |\chi_{D,\omega}| &\le \text{const} \cdot \left(B_{\Delta t} \, L \sqrt{\Delta t} + L^2 \ln L \sqrt{h} + L \ln L \right) \\ &\le \text{const} \cdot L^{7/4} \ln L \, (\Delta t)^{1/4} \end{aligned}$$

We note that $\tilde{\lambda}_D = O(L^2 \, \Delta t)$, hence $|\chi_{D,\omega}| \ll \tilde{\lambda}_D$ for all Δt satisfying (4.11). By the way, this easily implies a rough bound $k_{D,\omega} \le 2\tilde{\lambda}_D \le \text{const} \cdot L^2 \, \Delta t$, which justifies (4.16) and hence (4.18).

Next, using (4.19) and applying the above estimates to each of $k_i^{\pm}$, $i = 1, 2$, gives

$$k = L^2 \int_{D_1} \tilde{p}(x, v, t)\, dx\, dv + \chi'$$

with

$$|\chi'| \leq \text{ const} \cdot [L^{7/4} \ln L\, (\Delta t)^{1/4} + L^2\, \delta V\, \Delta t]$$

In a similar way we can estimate the other random factor in the main decomposition formula (4.15), which is

$$Z = \sum_{j=1}^{k} v_j$$

and get

$$Z = L^2 \int_{D_1} v\, \tilde{p}(x, v, t)\, dx\, dv + \chi''$$

with the same upper bound on χ'' as that on χ' above.

Using the smoothness properties of the function $\tilde{p}(x, v, t)$ described around (4.9) gives, cf. (3.38)–(3.40),

$$\int_{D_1} \tilde{p}(x, v, t)\, dx\, dv = (\tilde{Q}_1(t) - \tilde{Q}_0(t))\, \Delta t + \chi^{(4)}$$

and

$$\int_{D_1} v\, \tilde{p}(x, v, t)\, dx\, dv = (\tilde{Q}_2(t) - \tilde{Q}_1(t))\, \Delta t + \chi^{(5)}$$

with

$$|\chi^{(u)}| \leq \text{const} \cdot \Big[(\Delta t)^2/L + \delta V\, \Delta t\Big]$$

for $u = 4, 5$.

Therefore, we get

$$V(t + \Delta t) - V(t) = \tilde{\mathcal{D}}(t)\, \Delta t + \chi \tag{4.52}$$

where

$$|\chi| \leq \text{const} \cdot \Big[L^{-1/4} \ln L\, (\Delta t)^{1/4} + \delta V\, \Delta t + (\Delta t)^2\Big] \tag{4.53}$$

Recall that $\delta V = O(\Delta t)$ by (4.18). It is now easy to check that the second and the third terms are much smaller than the first one. This proves (4.12)–(4.14). It does not prove Theorem 4.4 yet, because we have fixed one (arbitrary) value of Δt, and our set Ω_1^* depended on Δt.

Actually, for our main purpose it is enough to prove Theorem 4.4 for just one value of Δt, namely for $\Delta t = 1/L^{1/7}$, as we will see later. But at a little extra effort we can prove our theorem for all Δt satisfying (4.11). We do that next. Divide the interval (4.11) into subintervals of length e^{-L}. That is, fix a finite collection of points $(\Delta t)_n = ne^{-L}$ for $n = n_1, \dots, n_2$ with $n_1 = L^{-1/3}(\ln L)^2 e^L$ and $n_2 = L^{-1/7} e^L$. Then we define

$$\Omega_1^* = \cap_{n=n_1}^{n_2} \Omega_1^*((\Delta t)_n)$$

The bound (4.51) then implies

$$P(\Omega_0^* \setminus \Omega_1^*) \leq \text{const} \cdot e^{L - \text{const} \cdot L^{11/7}}$$

which is obviously sufficient to maintain the bound (4.10).

Now for any Δt satisfying (4.11) we find $(\Delta t)_n$ such that $|\Delta t - (\Delta t)_n| \leq e^{-L}$. For any $\omega \in \Omega_1^* \subset \Omega_1^*((\Delta t)_n)$ we have all the above estimates with Δt replaced by $(\Delta t)_n$. This replacement only causes an exponentially small error, e^{-L}, in our estimates. It will not spoil our bounds, which are all polynomial in L. This completes the proof of Theorem 4.4. □

The next theorem is an analogue of Theorem 3.5.

Theorem 4.5 *Assume that $\varepsilon_0 > 0$ is small enough. For all sufficiently large L, for each configuration $\omega \in \Omega_1^*$ and for all $t \in (T_1, S_2)$ we have*
(i) *there is a constant $B > 0$ such that*

$$|V(t)| < B\varepsilon_0 \tag{4.54}$$

(ii) *there is a constant $C_0 > 0$ such that*

$$|V(t) - V_0(t)| < C_0\, L^{-1/7} \ln L \tag{4.55}$$

where $V_0(t)$ is defined by

$$V_0(t) = \frac{\tilde{Q}_1(t) - \sqrt{\tilde{Q}_1^2(t) - \tilde{Q}_0(t)\tilde{Q}_2(t)}}{\tilde{Q}_0(t)} \tag{4.56}$$

whenever $\tilde{Q}_0(t) \neq 0$ and by

$$V_0(t) = \frac{\tilde{Q}_2(t)}{2\tilde{Q}_1(t)} \tag{4.57}$$

otherwise.

Proof of this theorem very much repeats that of Theorem 3.5. The first half of it, up to the formula (3.59) can be copied almost verbatim, with only replacement of S_1 by S_2, Q_i by $\tilde{Q}_i$, and $\mathcal{D}$ by $\tilde{\mathcal{D}}$. We omit that part. The rest of the proof requires more substantial modifications, and we give it in detail.

First, recall that Theorem 3.5 deals with $t \in (0, S_1)$. On the interval $(0, S_1)$ the functions $Q_i(t)$, $i = 0, 1, 2$, are independent of ω, they are defined by equations (3.9)–(3.12) where $p(x, v, t)$ was in fact the deterministic density now denoted by $\tilde{p}(x, v, t)$. Therefore, our functions $\tilde{Q}_i(t)$ for $t > T_1$, are natural continuations of $Q_i(t)$ beyond the interval $(0, S_1) = (0, T_1)$ and they have the same properties, cf. (3.13) and (4.9). Hence, the function $V_0(t)$ defined by (4.56)–(4.57) for $t > T_1$ is the continuation of $V_0(t)$ defined by (3.46)–(3.47) on the interval $(0, T_1)$.

Next, in the first half of the proof (which we omitted since it almost coincided with that of Theorem 3.5), we must introduce $t_* < S_2$ as the first time when (4.54) fails. Now we prove (4.55) for all $t < t_*$ with some constant $C_0 > 0$ (independent of the choice of B in (4.54), which is to be made yet).

The bound (3.45) proved for all $t \le T_1$ implies that (4.55) holds for at least some $t > T_1$. Next, if (4.55) fails for any $t < t_*$, then let $t \in (T_1, t_*)$ be the first time (4.55) fails. Denote by

$$\Delta_0 = L^{-1/7}$$

the maximal allowed time increment in Theorem 4.4. Let $s = t - \Delta_0$. Due to Theorem 4.4

$$V(t) = V(s) + \tilde{\mathcal{D}}(s)\Delta_0 + \chi \tag{4.58}$$

with

$$|\chi| \le CL^{-1/4} \ln L\, (\Delta_0)^{1/4} = CL^{-2/7} \ln L$$

Due to the analogue of (3.58) obtained in the first half of the proof of the theorem,

$$V_0(t) = V_0(s) + \chi_0 \tag{4.59}$$

with

$$|\chi_0| \le \frac{E_0\, \varepsilon_0\, \Delta_0}{L} = \frac{E_0\, \varepsilon_0}{L^{8/7}}$$

For brevity, put $U(s) = V(s) - V_0(s)$ for all s. Subtracting (4.59) from (4.58) then gives

$$U(t) = U(s) + \tilde{\mathcal{D}}(s)\Delta_0 + \chi' \tag{4.60}$$

with $\chi' = \chi - \chi_0$, so that for large L

$$|\chi'| \le 2CL^{-2/7} \ln L \tag{4.61}$$

Now assume, without loss of generality, that $U(t) > 0$. Since (4.55) fails at time t, we have

$$U(t) \ge C_0 L^{-1/7} \ln L \tag{4.62}$$

Now consider two cases. If $U(s) \le 0$, then by the analogue of (3.59)

$$U(t) \le |\tilde{\mathcal{D}}(s)|\Delta_0 + |\chi'| \le E_2\, |U(s)|\Delta_0 + |\chi'| \ll L^{-1/7} \ln L$$

for large L, which contradicts to (4.62). If $U(s) > 0$, then, again due to (4.60) and the analogue of (3.59),

$$U(t) < U(s)[1 - E_1\Delta_0] + \chi',$$

hence

$$\begin{aligned} U(s) \quad &> \quad \frac{U(t) - \chi'}{1 - E_1\Delta_0} > (U(t) - \chi')(1 + E_1\Delta_0) \\ &> \quad U(t) + U(t)E_1\Delta_0 - 2\chi' \end{aligned} \tag{4.63}$$

Now, if C_0 in (4.55) is large enough, say $C_0 = 3C/E_1$, then $U(t)E_1\Delta_0 > 2\chi'$ by (4.62) and (4.61). This fact and (4.63) imply $U(s) > U(t)$, so (4.55) fails at an earlier time $s < t$, a contradiction. Hence, (4.55) is proved for all $t < t_*$ and $C_0 = 3C/E_1$.

Lastly, the remaining part of the proof of Theorem 3.5 can be repeated verbatim, concluding the proof of Theorem 4.5. □

We finally prove the convergence, as $L \to \infty$, of the random trajectory of the piston to the solution $Y(t/L)$, $W(t/L)$ of the hydrodynamical equations described in Section 2 for all $t \in (T_1, S_2)$.

Theorem 4.6 *Assume that* $\varepsilon_0 > 0$ *in* (P5) *is small enough. Then, for all large* L *and all* $\omega \in \Omega_1^*$, *there is a constant* $C > 0$ *such that*

$$|Y_L(\tau,\omega) - Y(\tau)| \le \frac{C\,\ln L}{L^{1/7}} \tag{4.64}$$

and

$$|W_L(\tau,\omega) - W(\tau)| \le \frac{C\,\ln L}{L^{1/7}} \tag{4.65}$$

for all $\min\{\tau_1, T_1/L\} < \tau < \min\{\tau_2, S_2/L\}$. *We also have*

$$T_2 - S_2 \le C\varepsilon_0 L \tag{4.66}$$

and

$$\tau_2 - S_2/L \le C\varepsilon_0 \tag{4.67}$$

The fluctuations of the function $S_2 = S_2(\omega)$ *for* $\omega \in \Omega_1^*$ *are bounded by*

$$\sup_{\omega,\omega' \in \Omega_1^*} |S_2(\omega) - S_2(\omega')| \le \frac{C\,\ln L}{L^{1/7}} \tag{4.68}$$

Proof. According to (2.29), the deterministic function $Y(\tau)$ satisfies

$$dY(\tau)/d\tau = F(Y,\tau), \qquad Y(0) = 1/2 \tag{4.69}$$

Now Theorems 3.7 and 4.5 imply that for all $\omega \in \Omega_1^*$ the random trajectory satisfies

$$\partial Y_L(\tau,\omega)/\partial\tau = F(Y,\tau) + \chi(\tau,\omega), \qquad Y_L(0,\omega) = 1/2 \tag{4.70}$$

with some

$$|\chi(\tau,\omega)| \le \frac{C\,\ln L}{L^{1/7}}$$

Recall that $|\partial F(Y,\tau)/\partial Y| \le \kappa$, see (2.30). Therefore, the difference $Z_L(\tau,\omega) := Y_L(\tau,\omega) - Y(\tau)$ satisfies

$$|Z_L'(\tau,\omega)| \le \kappa |Z_L(\tau,\omega)| + \frac{C\,\ln L}{L^{1/7}}$$

and $Z_L(0,\omega) = 0$. Using the standard Gronwall inequality in differential equations, see, e.g., Lemma 2.1 in [TVS], gives

$$|Z_L(\tau,\omega)| \le \frac{C \ln L}{\kappa L^{1/7}} \left(e^{\kappa\tau} - 1\right)$$

and

$$|Z_L'(\tau,\omega)| \le \frac{C \ln L}{L^{1/7}} e^{\kappa\tau}$$

for all $\tau < S_2/L$, which imply (4.64) and (4.65).

Next, (4.65) enables us to apply Proposition 4.1 with $\Delta = CL^{-1/7} \ln L$ and thus prove (4.66). Now we employ the same argument as in the proof of Theorem 3.8. By (4.65), random fluctuations of the piston velocity are bounded by $CL^{-1/7} \ln L$. Hence, random fluctuations of the velocities of particles that have had one or two collisions with the piston are bounded by $4CL^{-1/7} \ln L$. The random fluctuations of the positions of both piston and particles at every moment of time $t < \min\{\tau_2 L, S_2\}$ are bounded by the same quantities (with, possibly, a different value of C) in the coordinate $y = x/L$. This implies (4.68) in the same way, as (3.92) in Theorem 3.8. Now we can apply Proposition 4.1 to the deterministic dynamics constructed in Section 2 with the same result, and thus prove (4.67). □

The bound (4.68) shows that the limit

$$\tau_{**} := \lim_{L\to\infty} S_2(\omega)/L$$

does not depend on $\omega \in \Omega_1^*$. We have $|\tau_{**} - 2/v_{\max}| \le \text{const} \cdot \varepsilon_0$, due to (4.67) and the estimates in Lemma 2.9.

The convergence claimed in Theorem 1.2 is now proved on the interval $(0, \tau_{**})$, but, generally, $\tau_{**} < \tau_*$ with $\tau_* - \tau_{**} = O(\varepsilon_0)$, so we may still be $O(\varepsilon_0)$ short of our target value τ_*. To extend our results all the way to τ_* we need to redefine S_2 and the neighborhood $\mathcal{X}_1$ introduced by (4.4) more accurately. We need to set

$$\begin{aligned} \mathcal{X}_1(t) = \{(x,v) \in G^+(t) :\, x = X(t) + 0,\ v < 0\} \\ \cup\{(x,v) \in G^+(t) :\, x = X(t) - 0,\ v > 0\} \end{aligned} \tag{4.71}$$

The domain $G^+(t) = F^t(G^+)$ is random (it depends of ω), i.e. we should write $G^+(t) = G^+(t,\omega)$, and this is why we could not adopt the above definition of $\mathcal{X}_1(t)$ earlier and opted for a cruder one (4.4). But now, as we have just shown in the proof of Theorem 4.6, random fluctuations of the velocities and positions (in the y coordinate) of the particles and the piston are bounded by $\text{const} \cdot L^{-1/7} \ln L$. Hence, at every time moment $t < \min\{\tau_2 L, S_2\}$ all the domains $G^+(t,\omega)$, are close to each other – the distance between $G^+(t,\omega)$ and $G^+(t,\omega')$ for $\omega, \omega' \in \Omega_1^*$ in the Hausdorff metric on the coordinate plane y, v is bounded by $\text{const} \cdot L^{-1/7} \ln L$. By the same reason, every domain $G^+(t,\omega)$ will be $O(L^{-1/7} \ln L)$-close in the Hausdorff metric to the deterministic domain $\mathcal{G}^+(t/L)$ defined in Section 2. Hence we can easily make $G^+(t)$ in (4.71) independent of ω by, say, taking the union

$$G^+(t) = \cup_{\omega \in \Omega_1^*} G^+(t, \omega) \tag{4.72}$$

It is important to note that all the gas particles colliding with the piston at time t for every $\omega \in \Omega_1^*$ will be in $\mathcal{X}_1$ defined by (4.71)–(4.72). With this new definition of $\mathcal{X}_1$ replacing (4.4) and with S_2 changing accordingly, we have

$$\lim_{L \to \infty} |S_2/L - \tau_*| = 0$$

which follows from the $O(L^{-1/7} \ln L)$-closeness of the random dynamics to the deterministic dynamics on the y, v plane. Thus we extend all our results to the interval $(0, \tau_*)$.

5 Beyond the second recollision

The main goal of our analysis in Sections 3 and 4 is to prove that under suitable initial conditions random fluctuations in the motion of a massive piston in a closed container filled with an ideal gas are small and vanish in the thermodynamic limit. We are, however, able to control those fluctuations effectively only as long as the surrounding gas of particles can be described by a Poisson process, i.e. during the zero-recollision interval $0 < \tau < \tau_1$. In that case the random fluctuations are bounded by const$\cdot L^{-1} \ln L$, see Theorem 3.8. Up to the logarithmic factor, this bound is optimal, according to Theorem 1.1 by Holley.

During the one-recollision interval $\tau_1 < \tau < \tau_*$, the situation is different. The probability distribution of gas particles that have experienced one collision with the piston is no longer a Poisson process, it has intricate correlations. We are only able to show that random fluctuations remain bounded by $L^{-1/7}$, see Theorem 4.6. Perhaps, our bound is far from optimal, but our numerical experiments reported below demonstrate that random fluctuations indeed grow during the one-recollision interval and beyond.

At present, we do not know if our methods or results can be extended beyond the critical time τ_*, this remains an open question. We emphasize, however, that our Theorem 4.6 is the first rigorous treatment of the evolution of a piston in an ideal gas where most or all of the particles experience more than one collision with the piston.

In order to understand what is going on beyond the critical time τ_*, and in particular whether random fluctuations grow or remain small, we undertook experimental and heuristic studies of the piston dynamics on a large time scale. Below we describe our findings and discuss further research in this direction. A detailed account of our work can be found in [CL].

We set the initial density of the gas to

$$\pi_0(y, v) = \pi_0(|v|) = \begin{cases} 1 & \text{if } \ 0.5 \le |v| \le 1 \\ 0 & \text{elsewhere} \end{cases} \tag{5.1}$$

It satisfies our requirements (P1)–(P5), in particular $v_{\min} = 0.5$ and $v_{\max} = 1$, and most importantly $\varepsilon_0 = 0$. Therefore, by Corollary 2.13, the solution of the hydrodynamical equations is trivial: $Y(\tau) \equiv 0.5$, $W(\tau) \equiv 0$, and $\pi(y, v, \tau) \equiv \pi_0(y, v)$ for all $\tau > 0$.

To generate an initial configuration of particles, we used a random number generator described in [MN]. For our density (5.1), the x and v coordinates of all the particles are independent random variables uniformly distributed in their ranges $0 < x < L$ and $v_{\min} \le |v| \le v_{\max}$. Our computer program first selects the number of particles N according to the Poisson law with mean L^3, and then generates all (x_i, v_i), $1 \le i \le N$, independently according to their uniform distributions. The parameter L changed in our simulations from $L = 30$ to $L = 300$. For $L = 300$ the system contains $\approx L^3 = 27,000,000$ particles.

Once the initial data is generated randomly, the program computes the dynamics by using the elastic collision rules (1.2)–(1.3). All calculations were performed in double precision, with coordinates and velocities of all particles stored and computed individually.

Figure 6 presents a typical trajectory of the piston. Here $L = 100$. The position and time are measured in hydrodynamic variables $Y = X/L$, $0 < Y < 1$, and $\tau = t/L$.

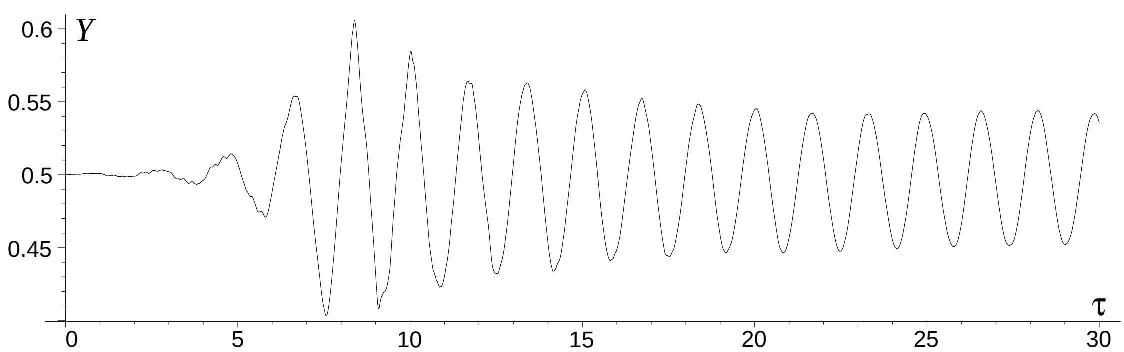

Figure 6: The piston coordinate Y as a function of time τ. Here $L = 100$, $N = 1000229$.

Initially, the piston barely moves about its stationary point $y = 0.5$. Then, at times τ between 3 and 5, the random vibrations of the piston grow and become quite visible on the y-scale, but for a short while they look random, as a trajectory of a Brownian motion. After that the piston starts travelling back and forth along the y axis in a more regular manner, making excursions farther and farther away from the stationary point $y = 0.5$. Very soon, at $\tau = \tau_{\max} \approx 8$, the swinging motion of the piston reaches its maximum, $(\Delta Y)_{\max} = \max |Y_L(\tau) - 0.5| \approx 0.1$. Then the oscillations of the piston dampen in size and seem to stabilize at an amplitude $A \approx 0.04$. At the same time the trajectory of the piston smoothes out and enters an oscillatory mode with a period $\tau_{\text{per}} \approx 1.63$. The velocity of the piston $W(\tau)$ follows similar patterns, see Figure 7.

Both the coordinate and velocity of the piston continue almost perfect harmonic oscillations for a long time with the same period $\tau_{\text{per}} \simeq 1.63$ (this is independent of L) but the amplitudes of both $Y(\tau)$ and $W(\tau)$ are slowly decreasing, see Figure 8.

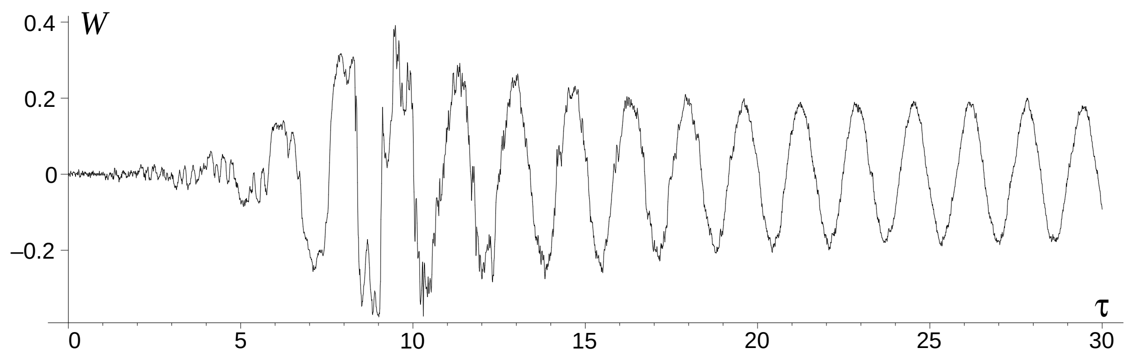

Figure 7: The piston velocity W as a function of time τ. The same run as in Fig. 6.

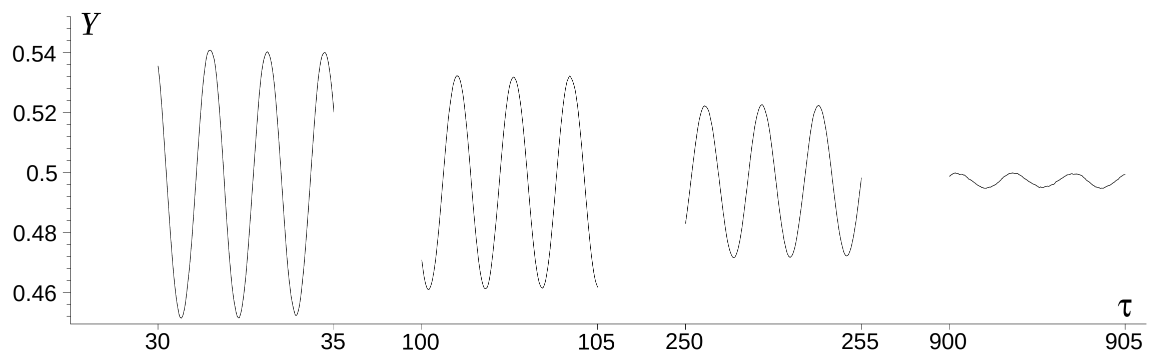

Figure 8: The piston coordinate Y during the intervals $(30, 35)$, $(100, 105)$, $(250, 255)$, and $(900, 905)$. The same run as in Fig. 6 and 7.

The oscillations of the piston with decaying amplitude can be described, in the interval $20 < \tau < 1000$, approximately by

$$Y(\tau) \simeq Ae^{-\lambda(\tau-20)} \sin \omega(\tau - \alpha) \tag{5.2}$$

with $A = 0.046$ and some constant $\lambda > 0$. Correspondingly, $W = dY/d\tau$ in the same interval $20 < \tau < 1000$ is

$$\begin{aligned} W(\tau) &\simeq -\lambda Y(\tau) + Ae^{-\lambda(\tau-20)} \omega \cos \omega(\tau - \alpha) \\ &= Ae^{-\lambda(\tau-20)}[-\lambda \sin \omega(\tau - \alpha) + \omega \cos \omega(\tau - \alpha)] \\ &= A_1 e^{-\lambda(\tau-20)} \sin \omega(\tau - \beta) \end{aligned} \tag{5.3}$$

with $A_1 = A\sqrt{\omega^2 + \lambda^2}$ and some β related to α.

To check how well our prediction (5.2) agrees with the experimental data, we computed the amplitude $A(\tau)$ as a function of time τ, by fitting a sine function $Y(\tau) = A \sin \omega(\tau - \alpha)$ "locally", on the interval $(\tau - 5, \tau + 5)$ for each τ. Fig. 9 shows $A(\tau)$ on the logarithmic scale, which looks almost linear on the interval $30 < \tau < 800$.

We used the least squares fit to estimate $\lambda = 0.00264$ for the run shown on Figs. 6-9. Since λ is small, the oscillations indeed die out very slowly. The "half-life" time (the

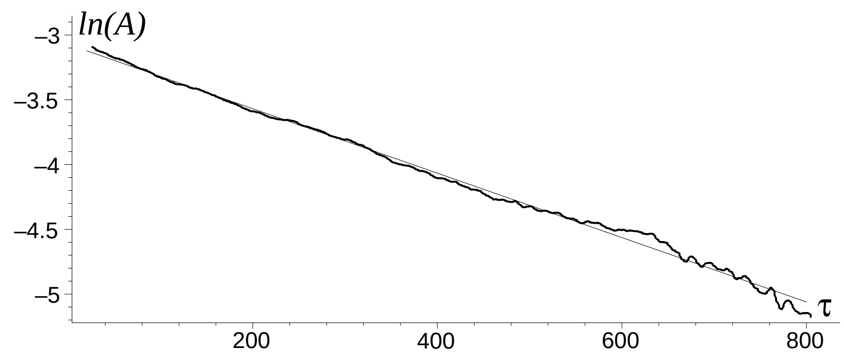

Figure 9: The amplitude $A(\tau)$ on the logarithmic scale: experimental curve (bold) and a linear fit (thin). The same run as the one shown in Fig. 6, 7 and 8.

time it takes to reduce the amplitude by a factor of two) is $\tau_{1/2} = \lambda^{-1} \ln 2 \approx 263$. The parameter λ and hence $\tau_{1/2}$ depend on the system size L. We estimated numerically that $\tau_{1/2} \sim L^{1.3}$, hence $\lambda \sim L^{-1.3}$.

The key characteristics of the piston trajectory described above, in particular, $(\Delta Y)_{\max}$, $W_{\max}$, A, $\tau_{\rm per}$, appear to be independent of L. Even for $L = 300$ (the largest system tested experimentally) the piston experiences large oscillations very similar to the ones shown on Fig. 6 and 7. Some other quantities, such as $\tau_{1/2}$ and the related λ, depend in a systematic way on L.

But most importantly, the time of the largest oscillations $\tau_{\max}$ and the related time of the onset of the instability τ_c, see below, seem to slowly grow with L, very likely as $\log L$. To understand this fact, we looked into the mechanism of the build-up of random fluctuations of the piston position and velocity displayed on Figures 6 and 7. To this end we plotted the histogram of the (empirical) density of gas particles in the y, v plane at various times $0 < \tau < 30$, see samples in Figure 10. The initial density (at time zero) is almost uniform over the domain $0 < x < L$ and $v_{\min} \le |v| \le v_{\max}$ (variations in the initial configuration always exist, because it is generated randomly). Then, for $0 < \tau < 1$, the piston experiences random collisions with particles and acquires a speed of order $M^{-1/2} = O(1/L)$, see Theorem 1.1. These small fluctuations of the piston velocity result in bigger changes of the velocities of the particles which leave the piston after collisions due to the rule (1.3). In particular, the outgoing particles on the right hand side of the piston have velocities in the interval $(v_{\min} + 2W(\tau), v_{\max} + 2W(\tau))$ while those on the left hand side of the piston have velocities in the interval $(-v_{\min} + 2W(\tau), -v_{\max} + 2W(\tau))$. Hence, the region in the y, v plane where the density of the particles is positive is no longer a rectangle with straight sides, now its boundaries are curves whose shape nearly repeats the graph of the randomly evolving piston velocity $W(\tau)$. While the variations of $O(1/L)$ of these boundary curves may seem small, it is crucial that on opposite sides of the piston they go in opposite directions. Indeed, when $W(\tau) > 0$, then the outgoing particles on the right hand side accelerate and those on the left hand side slow down. When $W(\tau) < 0$ the opposite happens.

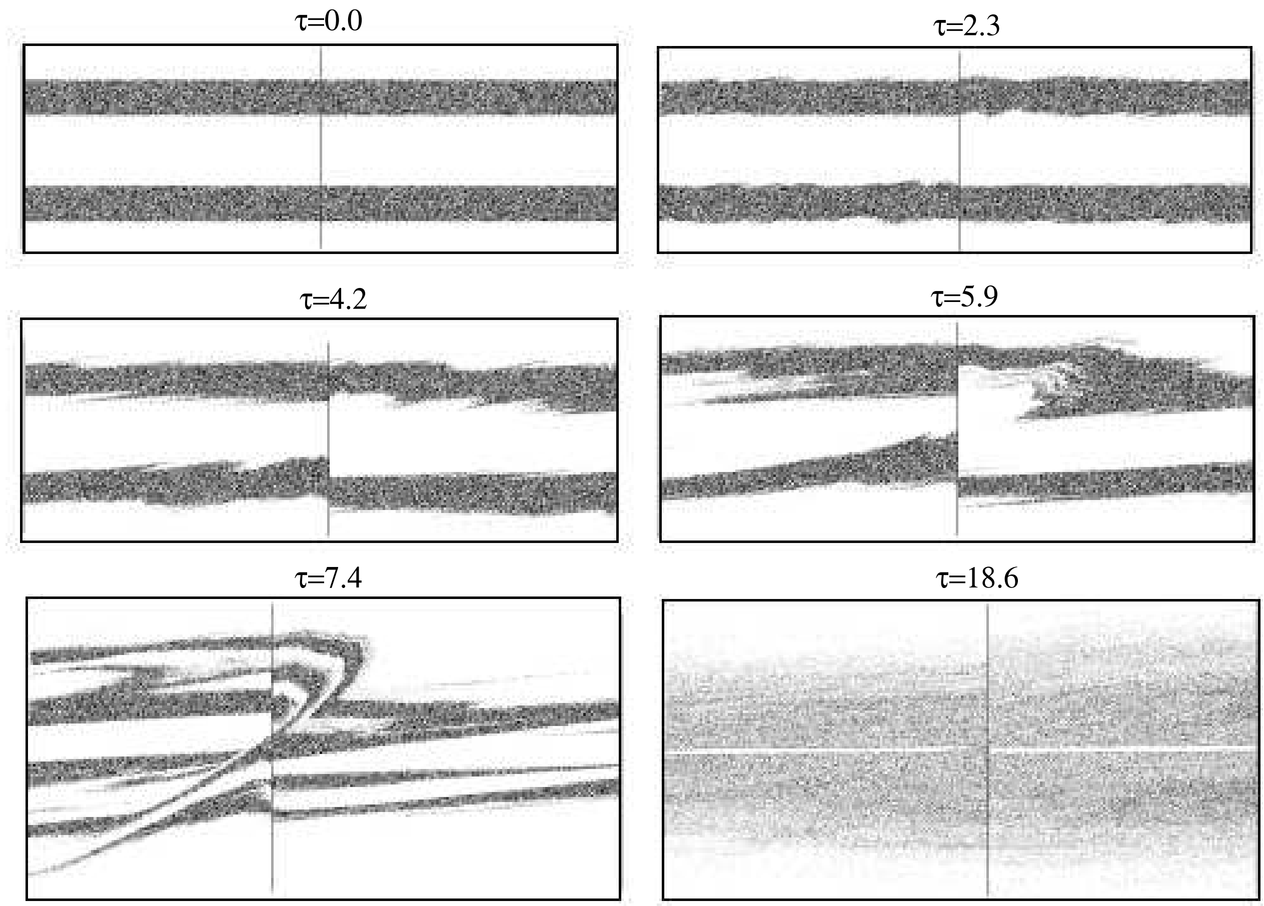

Figure 10: Six snapshots of the empirical gas density (in the x, v plane) at times $\tau = 0$, 2.3, 4.2, 5.9, 7.4 and 18.6.

Next, the particles that have collided with the piston travel to the wall and come back to the piston. Now their densities are less regular than they were initially – the regions in the x, v plane where the density is positive, are curvilinear domains. When they hit the piston, they shake it back and forth more forcefully than before, because the velocities of the incoming particles on the opposite sides of the piston are now negatively correlated. When particles on the right hand side are fast, those on the left hand side are slow, and vice versa. The fluctuations of the gas densities thus "cooperate" to push the piston harder, with a "double" force. This produces a resonance-type effect destabilizing the piston dramatically and the velocity of the piston $W(\tau)$ experiences larger fluctuations than before. The velocities of the newly outgoing particles will again go up and down in opposite direction, on a greater scale than before.

As time goes on, the above phenomenon repeats over and over, with larger and larger fluctuations of the gas and piston velocities, until the distribution of gas particles completely breaks down. For $L = 100$, at times $\tau \sim 10$, two large clusters of particles are formed, one on each side of the piston. When one cluster bombards the piston, the other moves away from it and hits the wall, then they exchange their roles. The clusters have sizes of about 0.3–0.5 in the y direction and the particle velocities range from about 0.2 to just over 1. The average velocity is about 0.5–0.6 and so the clusters hammer the piston periodically with period 1.6–2.0, which is close to the experimentally determined period of piston oscillations, see above.

Fig. 10 shows six snapshots of the empirical density of gas particles taken at different times. At $\tau = 0$ the gas fills (almost uniformly) two rectangles $\{(y, v) :\ 0.5 < |v| < 1,\ 0 < y < 1\}$. At $\tau = 2.3$ one can see some ripples on the boundaries of these rectangles. At time $\tau = 4.2$ the irregularities grow and at $\tau = 5.9$ the rectangular shape is broken down. Two large clusters of particles are formed, both appear in the upper half-plane $v > 0$, i.e. at that time both clusters move to the right (one toward the piston, the other away from it). Later the density undergoes strange formations ($\tau = 7.4$) but eventually smoothes out and enters a slow process of convergence to Maxwellian ($\tau = 18.6$) described below.

The above analysis suggests that the fluctuations of the piston velocity roughly increase by a constant factor during each time interval of length one. Indeed, initial random fluctuations $W_a \sim O(1/L)$ result in additional changes of velocities of outgoing particles by $2W_a$. When those particles come back to the piston (in time $\Delta\tau \approx 1$), they kick its velocity to the level of $2W_a$. Then the newly outgoing particles acquire an additional velocity $4W_a$, etc. Over each time interval of length one the fluctuations double in size. This is an obvious oversimplification of the real dynamics, but it leads to a reasonable conjecture

$$W_a(\tau) \approx \frac{C\, R^{\tau}}{L} \tag{5.4}$$

where $W_a(\tau)$ are typical fluctuations of the piston velocity at time τ and $C, R > 0$ are constants.

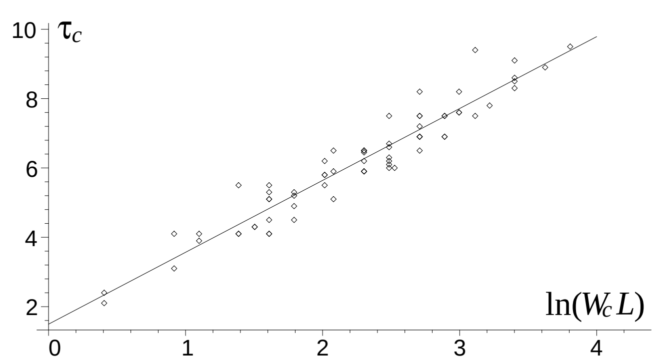

Figure 11: The value τ_c as a function of $\ln(W_c L)$: experimental points and a linear fit.

We tested the above formula numerically as follows. Let $W_c > 0$ be some preset critical value of the piston speed and $\tau_c = \inf\{\tau > 0 :\ |W(\tau)| > W_c\}$ the (random) time when W_c is first exceeded. This time plays the role of the "onset" of large fluctuations of the piston velocity. One would expect, based on (5.4) that

$$\tau_c \approx \ln(W_c L/C)/\ln R \tag{5.5}$$

i.e. τ_c grows as $\ln L$ when L increases.

We found τ_c experimentally for $W_c = 0.1$ and $W_c = 0.15$ and checked that (5.5) agreed well with the data, see Fig. 11. By the least squares fit we estimated $C = 0.45$ and $R = 1.6$.

To summarize our experimental observations, we conclude that the random fluctuations of the density function $p_L(y, v, \tau)$ and the piston coordinate $Y_L(\tau)$ grow exponentially in time τ, and at times $\tau \sim \log L$ they become large even on a macroscopic scale. At that point the evolution of the system deviates far from the solution of the hydrodynamical equations (H1)–(H4), and they become completely separated afterwards.

Interestingly, our observations do not indicate that the convergence (1.12)–(1.13) claimed in Theorem 1.2 fails on any interval of time. In fact, if the random fluctuations behave as CR^τ/L, as predicted by (5.4), then (1.12)–(1.13) should hold as $L \to \infty$ on *every* finite interval $(0, \tau_*)$. However, a rigorous proof of this fact would be a very challenging task.

Next, we also examined numerically and heuristically how the system behaves asymptotically, as $\tau \to \infty$. On physical grounds [B], we expect the system to approach thermal equilibrium, see Section 1, i.e. the velocity distribution of gas particles should converge to a Maxwellian.

We used the Kolmogorov-Smirnov statistical test to verify the convergence of the velocity distribution to a normal law. At any given time $\tau > 0$, let

$$F_\tau(u) = \#\{i : v_i < u\}/N$$

be the empirical (cumulative) distribution function of particle velocities. For the corresponding normal distribution function $\Phi(x)$, we compute

$$D_\tau = \sup_{-\infty < u < \infty} |F_\tau(u) - \Phi(u)|$$

Initially, $D_0 \approx 0.245$ for our choice of $\pi_0(v)$ in (5.1). If the velocities v_i were independent normal random variables, then D_τ would be of order $O(1/\sqrt{N})$ and the product $D_\tau\sqrt{N}$ would have a certain limit distribution, see, e.g. [Lu]. In particular, it is known that the probability $P(D_\tau\sqrt{N} > 1) \approx 0.2$. Based on this, we opted to define the time of convergence to equilibrium by

$$\tau_{\rm eq} = \inf\{\tau > 0 : D_\tau\sqrt{N} < 1\} \tag{5.6}$$

We estimated $\tau_{\rm eq}$ for various L's and found that $\tau_{\rm eq} \approx aL^b$ with some constants $a, b > 0$. By a least squares fit to experimental points we found $a = 0.18$ and $b = 2.47$.

The plot of the product $S = D_\tau\sqrt{N}$ versus τ is given on Fig. 12 (for a particular run with $L = 40$). It shows that, after an initial sharp drop over the period $0 < \tau < 20$, the statistic S decreases exponentially in τ. Another commonly used (and popular among experimentalists) statistic to measure closeness to a normal distribution is

$$S' = 3 - \frac{M_4}{M_2^2}$$

where M_2 and M_4 are the second and the fourth sample moments of the empirical velocity distribution, respectively. Fig. 12 shows that S' converges to zero in a similar manner (for the same run with $L = 40$).

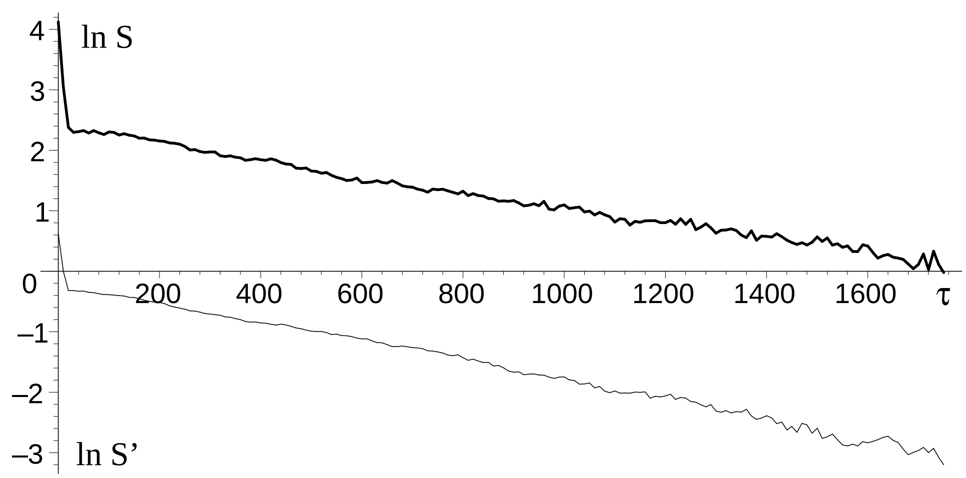

Figure 12: $\ln S$ (thick line) and $\ln S'$ (thin line) as functions of τ.

The convergence to a thermal equilibrium can also be justified mathematically. If one fixes three integrals of motion – the total energy E and the numbers of particles in the left and right compartments N_L and N_R – the dynamics can be reduced to a billiard system in a high-dimensional polyhedron by standard techniques, as we show next.

Let $\{x_i\}$, $i = 1, \dots, N_R$, denote the x-coordinates of the particles to the right of the piston, and $\{x_i\}$, $i = -1, \dots, -N_L$, those to the left of it (ordered arbitrarily). Put $x_0 = X\sqrt{M}$, where X is the coordinate of the piston and M is its mass. Then the configuration space of the system (in the coordinates x_i, $-N_L \le i \le N_R$) is a polyhedron $Q \subset \mathbb{R}^{N+1}$ (recall that $N = N_L + N_R$) defined by inequalities

$$0 \le x_{-N_L}, \dots, x_{-1} \le x_0/\sqrt{M} \le x_1, \dots, x_{N_R} \le L$$

It is known that the dynamics of our mechanical system corresponds to the billiard dynamics in Q, see [CFS]. That is, the configuration point $\mathbf{q} \in Q$ moves freely and experiences specular reflections at the boundary ∂Q. The velocity vector

$$\mathbf{p} = \dot{\mathbf{q}} = \{v_{-N_L}, \dots, v_{-1}, V\sqrt{M}, v_1, \dots, v_{N_R}\}$$

has constant length, since $\|\mathbf{p}\|^2 = 2E = \text{const}$. Therefore, the phase space of the billiard system is $\mathcal{M} = Q \times S_\rho^N$ where S_ρ^N is the N-dimensional sphere of radius $\rho = \sqrt{2E}$.

The billiard system has the Liouville invariant measure μ on $\mathcal{M}$, which is the product of a uniform measure on the polyhedron Q and a uniform (Lebesgue) measure on the sphere S_ρ^N, i.e. $d\mu = dq\, dp$. The properties of billiard dynamics depend heavily on the curvature of the boundary ∂Q. In our case Q is a polyhedron, hence its boundary consists of flat sides with zero curvature. A prototype of such systems is billiard in a polygon. It is well known that (see, e.g., [C])

Fact. For billiards in polygons and polyhedra (and hence, for our mechanical model of a piston in the ideal gas) all Lyapunov exponents vanish, and so does the measure-theoretic (Kolmogorov-Sinai) entropy.

Systems with zero Lyapunov exponents and zero entropy are not regarded as truly chaotic, but they still may be ergodic. In fact, billiards in generic polygons *are* ergodic [KMS]. Moreover, for many nonergodic polygons, the phase space is foliated by invariant subsurfaces on which the dynamics is ergodic.

Even though there are no similar results, to our knowledge, for billiards in high-dimensional polyhedra, one can expect that they, too, have similar properties. That is, they are generically ergodic or become ergodic after trivial reductions. In our case, the billiard in Q is, perhaps, ergodic for typical values of M, or else the phase space is foliated by invariant submanifolds on which the dynamics is ergodic, and that those submanifolds fill $\mathcal{M}$ pretty densely. In the latter case, one would hardly distinguish experimentally between such a nonergodic system and a truly ergodic one.

Hence, we can assume that our system is ergodic or very close to ergodic in the above sense. Then almost every trajectory eventually behaves according to the invariant measure μ, independently of the initial state. In particular, for any initial gas density and velocity distribution (given by the function $\pi_0(y, v)$, see Section 1) the hydrodynamic regime for a finite L is only valid on a finite interval of time – eventually the system will relax to a thermal equilibrium. We expect in fact that in terms of the "macroscopic" variables, say, the one particle distribution function, the system will relax to an effective equilibrium, as defined by (5.6) in terms of τ_{eq}, which is much smaller than the exponentially long time (in L) required for the ergodic theorem. So the real question is how does this time depend on L. According to our earlier discussion $\tau_c \sim \log L$ and $\tau_{\text{eq}} \sim L^{5/2}$.

At equilibrium, the distribution of coordinates x_i and velocities v_i are determined by the Liouville measure μ, which is uniform in the phase space. Physically interesting (and only observable) are its marginal measures, i.e. projections, on lower-dimensional subspaces. The marginal measures of the velocities are approximately normal for large N.

In particular, each individual velocity v_i converges in distribution to a Maxwellian (i.e., normal) random variable with zero mean and variance $2E/N = \text{const}$. The same holds for the "piston" component of the velocity, $\dot{x}_0 = V\sqrt{M}$, hence the piston velocity V will be normally distributed with zero mean and standard deviation $\text{const}/\sqrt{M} =$

const$/L$, as $L \to \infty$. In our case V has standard deviation $\sqrt{7/24}/L \approx 0.5/L$. This conclusion agrees well with Holley's theorem 1.1 and our numerical data.

The equilibrium distribution of the piston coordinate X is also determined by the projection of the uniform measure dq on Q onto the x_0 axis. Before we do that, let us get rid of M in the definition of both Q and x_0. A simple change of variable $X = x_0/\sqrt{M}$ allows us to redefine Q by

$$0 \le x_{-N_L}, \dots, x_{-1} \le X \le x_1, \dots, x_{N_R} \le L$$

Furthermore, rescaling $Y = X/L$ and $y_i = x_i/L$ gives a new, simpler, definition of Q:

$$0 \le y_{-N_L}, \dots, y_{-1} \le Y \le y_1, \dots, y_{N_R} \le 1$$

This is a variation of the so called Brownian bridge. "Integrating away" the variables y_i yields the following equilibrium density for Y:

$$f(Y) = c\, Y^{N_L}(1-Y)^{N_R} \tag{5.7}$$

for $0 < Y < 1$, where c is the normalizing factor that can be computed explicitly:

$$c^{-1} = \int_0^1 Y^{N_L}(1-Y)^{N_R}\, dY = \frac{N_L!N_R!}{(N_L+N_R+1)!} \tag{5.8}$$

Asymptotically, as $L \to \infty$, we have $N_L \sim L^3/2$ and $N_R \sim L^3/2$. Assume, for simplicity, that $N_L = N_R = N/2$ and denote $K = N/2$, then

$$c = \frac{(2K+1)!}{(K!)^2} \simeq \frac{2 \cdot 4^K \sqrt{K}}{\sqrt{\pi}}$$

Put $z = (Y - 0.5)\sqrt{8K}$, then the density of z is given asymptotically by

$$\begin{aligned} f(z) &= \frac{c}{\sqrt{8K}}\left(0.5 + \frac{z}{\sqrt{8K}}\right)^K \left(0.5 - \frac{z}{\sqrt{8K}}\right)^K \\ &= \frac{c}{4^K\sqrt{8K}}\left(1 - \frac{z^2}{2K}\right)^K \\ &\approx \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} \end{aligned}$$

Hence, Y is asymptotically gaussian with mean 0.5 and variance $(4N)^{-1} = (4L^3)^{-1}$. Therefore, in equilibrium

$$|Y - 0.5| \sim \frac{1}{2L\sqrt{L}} \sim \frac{1}{2\sqrt{N}}$$

Note that this estimate is independent of the piston mass.

Next, by using (5.7) one can easily compute the probability that the piston coordinate Y deviates from its mean value 0.5 by a fixed amount $d > 0$, say $d = 0.1$: this probability is $<$ const $\cdot\, e^{-aN}$ with some $a = a(d) > 0$. Therefore, we have observed experimentally a very rare event whose probability was exponentially small in N. If we started with an equilibrium state, such an observation would be practically impossible. But we started with a state described by the density (5.1), which itself has probability less than const $\cdot$ e^{-bN}, $b > 0$, with respect to the equilibrium measure μ. So we only observed how one highly improbable initial state evolved to another highly improbable state along its (very slow) transformation to an equilibrium state. It would be interesting to understand why the system "chooses" such a strange evolution to equilibrium, i.e. why starting with a state described by a "double-peaked" distribution (5.1) the system behaves as a damped harmonic oscillator over an extended time interval, with initial oscillations as large as 1/10 of the system size.

We are currently working on this problem and will report results in a separate paper [CCL]. Below we outline our program and mention some preliminary findings. For simplicity, we assume that $\pi_0(y, v) = \pi_0(|v|)$ and $X(0) = L/2$, $V(0) = 0$. Corollary 2.13 ensures that the hydrodynamical equations have the trivial solution $Y(\tau) \equiv 0.5$, $W(\tau) \equiv 0$, and $\pi(y, v, t) \equiv \pi_0(y, v)$ for all $\tau > 0$. Now, since the initial configuration of particles is generated randomly from a Poisson process with the density $\pi_0(y, v)$, the "actual" (empirical) density of the particles, such as the one shown on Fig. 10 at $\tau = 0$, does not exactly coincide with $\pi_0(y, v)$. Random fluctuations of the empirical density are typically of order $O(1/L)$. Hence, the "actual" initial distribution of particles can be thought of as a small perturbation of the function $\pi_0(y, v) = \pi_0(|v|)$ and can be written as $\pi_0(|v|) + \varepsilon\pi_1(y, v)$ with $\varepsilon = 1/L$ and some (random) function $\pi_1(y, v)$ of order one.

Now, we conjecture that the evolution of the mechanical system closely follows the solutions of the hydrodynamical equations (H1)–(H4) with a perturbed initial density $\pi_0(|v|) + \varepsilon\pi_1(y, v)$, rather than the stationary solution corresponding to the unperturbed density $\pi_0(|v|)$. This accounts for significant differences between the behavior of the mechanical system and the stationary solution, if the latter is unstable. In particular, two trajectories which are initially ε-close (in our case $\varepsilon = 1/L$) can deviate from each other exponentially fast in time τ, and at times $\tau \sim -\log \varepsilon = \log L$ will look completely different. This would be in agreement with our experimental observations and the estimate (5.5) of the time $\tau_c \sim \log L$ of the onset of "instability".

To test our conjecture, we solved the hydrodynamical equations (H1)–(H4) numerically starting with a perturbed initial density obtained by adding to (5.1) a function ε-small in the L^1 metric (with $\varepsilon \simeq 0.01$). We found that the corresponding solution resembled strikingly well the evolution of the mechanical system described above. In particular, the coordinate and velocity of the piston followed large nearly harmonic oscillations during the interval $10 < \tau < 30$. The corresponding plots of the piston position and velocity along the perturbed solutions of (H1)–(H4) were almost indistinguishable from our Figures 6 and 7. Hence, the behavior of the mechanical system can be traced to that of the perturbed solutions of the hydrodynamical equations, and the instability of the latter becomes an important issue.

When we were finishing the present paper, we received a message from E. Caglioti and E. Presutti who (a) proved that the hydrodynamical equations (H1)–(H4) are stable when $\pi_0(|v|)$ is monotonically nonincreasing in $|v|$, i.e. $\pi_0'(|v|) \leq 0$, and (b) suggested that they might be unstable for our class on non-monotone $\pi_0(|v|)$. We checked the suggestion (b) for our particular density (5.1) and found that it was indeed correct; we proved that small perturbations grow exponentially in τ.

Conversely, when we simulated a particle dynamics with a nonincreasing initial density $\pi_0(|v|)$ the oscillations essentially disappeared (to this end we tried a uniform "flat" function given by $\pi_0(|v|) = 1$ for $|v| \leq v_{\max}$ and a triangular one $\pi_0(|v|) = 1 - |v|/v_{\max}$.). On the other hand, the particle velocity distribution still approached a Maxwellian, albeit at a somewhat slower pace.

Finally, we describe some other open problems related to the piston dynamics.

1. It is clear that recollisions of gas particles with the piston have a very "destructive" effect on the dynamics in the system. However, we need to distinguish between two types of recollisions.

We say that a recollision of a gas particle with the piston is *long* if the particle hits a wall $x = 0$ or $x = L$ between the two consecutive collisions with the piston. Otherwise a recollision is said to be *short*. Long recollisions require some time, as the particle has to travel all the way to a wall, bounce off it, and then travel back to the piston before it hits it again. Short recollisions can occur in rapid succession.

We have imposed the velocity cut-off (P4) in order to avoid any recollisions for at least some initial period of time (which we called the zero-recollision interval). More precisely, the upper bound $v_{\max}$ guarantees the absence of long recollisions. Without it, we would have to deal with arbitrarily fast particles that dash between the piston and the walls many times in any interval $(0, \tau)$. On the other hand, the lower bound $v_{\min}$ was assumed to exclude short recollisions.

There are good reasons to believe, though, that short recollisions may not be so destructive for the piston dynamics. Indeed, let a particle experience two or more collisions with the piston in rapid succession (i.e. without hitting a wall in between). This can occur in two cases: (i) the particle's velocity is very close to that of the piston, or (ii) the piston's velocity changes very rapidly. The latter should be very unlikely, since the deterministic acceleration of the piston is very small, cf. Theorem 2.12c. In case (i), the recollisions should have very little effect on the velocity of the piston according to the rule (1.2), so that they may be safely ignored, as it was done already in some earlier studies [H, DGL].

We therefore expect that our results can be extended to velocity distributions without a cut-off from zero, i.e. allowing $v_{\min} = 0$.

2. In our paper, L plays a dual role: it parameterizes the mass of the piston ($M \sim L^2$), and it represents the length of the container ($0 \leq x \leq L$). This duality comes from our assumption that the container is a cube.

However, our model is essentially one-dimensional, and the mass of the piston M and the length of the interval $0 \le x \le L$ can be treated as two independent parameters. In particular, we can assume that the container is infinitely long in the x direction (i.e., *that* L *is infinite*), but the mass of the piston is still finite and given by $M \sim L^2$ (*this* L *is the* size of the container in the y and z directions). In this case there are no recollisions with the piston, as long as its velocity remains small. Hence, our zero-recollision interval is effectively infinite. As a result, Theorem 1.2 can be extended to arbitrarily large times. Precisely, for any $T > 0$ we can prove the convergence in probability:

$$P\left(\sup_{0\le\tau\le T} |Y_L(\tau,\omega) - Y(\tau)| \le C_T \ln L/L\right) \to 1$$

and

$$P\left(\sup_{0\le\tau\le T} |W_L(\tau,\omega) - W(\tau)| \le C_T \ln L/L\right) \to 1$$

as $L \to \infty$, where $C_T > 0$ is a constant and $Y(\tau)$ and $W(\tau) = \dot{Y}(\tau)$ are the solutions of the hydrodynamical equations described in Section 2.

3. Along the same lines as above, we can assume that the container is d-dimensional with $d \ge 4$. Then the mass of the piston and the density of the particles are proportional to L^{d-1} rather than L^2.

When d is large, the gas particles are very dense on the x, v plane. This leads to a much better control over fluctuations of the particle distribution and the piston trajectory. During the zero-recollision interval, for example, the piston trajectory is $L^{-(d-1)/2}$-close to its deterministic trajectory. This is an easy modification of the results of our Section 3. During the one-recollision interval, the piston trajectory is $L^{-(2d-5)/7}$-close to its deterministic trajectory. This can be shown with the methods developed in Section 4 but requires some extra work. Moreover, the methods and results of that section can be extended to the k-recollision interval (τ_k, τ_{k+1}) for any $k \ge 1$. It can be shown that there is a $d_k \ge 3$ such that for all $d \ge d_k$ we have

$$P\left(\sup_{\tau_k<\tau<\tau_{k+1}} |W_L(\tau,\omega) - W(\tau)| \le L^{-b}\right) \to 1$$

as $L \to \infty$, here $b > 0$ depends on k and d. This extension, however, requires quite substantial work, which is beyond the scope of this article. The upshot is that a high dimensional piston is more stable than a lower dimensional one.

It would be interesting to investigate other modifications of our model that lead to more stable regimes. For example, let the initial density $\pi_0(y, v)$ of the gas depend on the factor $a = \varepsilon L^2$ in such a way that $\pi_0(y, v) = a^{-1}\rho(y, v)$, where $\rho(y, v)$ is a fixed function. Then the particle density grows as $a \to 0$. This is another way to increase the density of the particles, but without changing the dimension. One may expect a better control over random fluctuations in this case, too.

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Appendix

In this section, we derive various probabilistic estimates on the distribution of gas particles and their velocities. The number of particles $K = N_D$ in any domain D on the x, v plane at time $t = 0$ is a Poisson random variable, and we need bounds on its large deviations. Perhaps, some of our estimates are known in probability theory, but we include proofs for the sake of completeness.

Lemma A.1 *Let K be a Poisson random variable with parameter $\lambda > 0$. Then for any $A > \lambda$ we have*

$$P(K > A) \leq e^{A-\lambda-A\ln(A/\lambda)}$$

and for any $A < \lambda$

$$P(K < A) \leq e^{A-\lambda-A\ln(A/\lambda)}$$

Proof. The moment generating function of K is

$$\varphi_K(t) = E(e^{tK}) = e^{\lambda(e^t-1)}$$

First, let $A > \lambda$. Then, obviously, for all $t > 0$

$$\varphi_K(t) \geq e^{At} \cdot P(K > A)$$

Hence for all $t > 0$

$$P(K > A) \leq e^{\lambda(e^t-1)-At}$$

The expression on the right hand side takes minimum at

$$t = \ln(A/\lambda) > 0$$

This proves the first part of the lemma.

Now let $A < \lambda$. Then for all $t < 0$

$$\varphi_K(t) \geq e^{At} \cdot P(K < A)$$

Hence for all $t < 0$

$$P(K < A) \le e^{\lambda(e^t - 1) - At}$$

The expression on the right hand side takes minimum at

$$t = \ln(A/\lambda) < 0$$

This proves the second part of the lemma. □

Lemma A.2 *Let K be a Poisson random variable with parameter $\lambda > 0$. For any $b > 0$ there is a $c > 0$ such that for all $0 < B < b\sqrt{\lambda}$ we have*

$$P(|K - \lambda| > B\sqrt{\lambda}) \le 2e^{-cB^2}$$

Proof. A direct application of the previous lemma gives

$$P(|K - \lambda| > B\sqrt{\lambda}) \le 2e^{-B^2 g(q)} \tag{A.1}$$

where

$$g(q) = \frac{(1+q)\ln(1+q) - q}{q^2} = \frac{\int_0^q \ln(1+s)\, ds}{2\int_0^q s\, ds} \tag{A.2}$$

and $q = B/\sqrt{\lambda}$ in (A.1). By direct inspection one can verify that the function $g(q)$ is a positive and strictly monotonically decreasing function on the interval $0 < q < \infty$. We complete the proof by setting $c = g(b)$. □

Lemma A.3 *Let $\lambda_0 > 0$ and $a > 0$. Then for all sufficiently large $L > 0$ and every Poisson random variable with parameters $\lambda \ge \lambda_0$ we have*

$$P(|K - \lambda| > a\sqrt{\lambda}\ln L) \le L^{-d \ln\ln L}$$

where $d = a\sqrt{\lambda_0}/2$.

Proof. Using (A.1) with $B = a\ln L$ gives

$$P(|K - \lambda| > a\sqrt{\lambda}\ln L) \le 2e^{-(a\ln L)^2 g(a\ln L/\sqrt{\lambda_0})}$$

with $g(q)$ defined by (A.2). Observe that, for large L,

$$g\left(\frac{a\ln L}{\sqrt{\lambda_0}}\right) \sim \frac{\sqrt{\lambda_0}\ \ln\ln L}{a\ln L}$$

This complete the proof. □

Remark. In most of our applications, λ is large, of order $\lambda \sim L^b$ with some $b > 0$. Hence, the factor $\ln L$ is small compared to $\sqrt{\lambda}$.

Corollary A.4 *Let $\lambda_0 > 0$ and $a > 0$. Let K be a Poisson random variable with parameter $\lambda \le \lambda_0$. Then for all sufficiently large $L > 0$ we have*

$$P(K > a \ln L) \le L^{-d \ln\ln L}$$

where $d = a/2$.

Proof. The case $\lambda = \lambda_0$ easily follows from the previous lemma. Now, if $\lambda < \lambda_0$, then the event $K > a \ln L$ is even less likely than it is for $\lambda = \lambda_0$. □

Next, we need to study another random variable related to a Poisson process. For any domain D on the x, v plane consider the sum of the velocities

$$Z = Z_D = \sum_{(x,v)\in D} v$$

of the particles in D at time 0. We assume that $D \subset \{v_{\min} < v < v_{\max}\}$ (the case $D \subset \{-v_{\max} < v < -v_{\min}\}$ is completely symmetric and analogous). By projecting the domain D onto the v axis we obtain a Poisson process on the interval

$$I = (v_{\min}, v_{\max})$$

with density

$$\pi(v) = L^2 \int_{D\cap\{u=v\}} p_L(x,u)\, dx$$

Now the random variable Z can be described as follows.

Consider a one-dimensional Poisson process with density $\pi(v)$ on the interval I. This means that for any subinterval $J \subset I$ the number of points in J, call it N_J, is a Poisson random variable with mean

$$E(N_J) = \int_J \pi(v)\, dv$$

Each realization ω of this process is a finite subset of I. Consider a random variable

$$Z(\omega) = \sum_{v\in\omega} v$$

We will call Z an *integrated Poisson random variable.*

If we fix a large $n \ge 1$ and partition I into small intervals

$$\Delta_i = I \cap \left[\frac{i}{n}, \frac{i+1}{n}\right)$$

$i = 0, 1, 2, \ldots$, then we can obviously bound Z by

$$\sum_i \frac{i}{n} N_{\Delta_i} \le Z < \sum_i \frac{i+1}{n} N_{\Delta_i} \tag{A.3}$$

where N_{Δ_i} is the number of points of the process in the interval Δ_i. Note that N_{Δ_i} are independent Poisson random variables with parameters

$$\lambda_i = E(N_{\Delta_i}) = \int_{\Delta_i} \pi(v)\, dv$$

Therefore, the moment generating function $\varphi_Z(t) = E(e^{tZ})$ of Z is bounded by

$$\exp\left[\sum_i \lambda_i (e^{t\frac{i}{n}} - 1)\right] \leq E(e^{tZ}) < \exp\left[\sum_i \lambda_i (e^{t\frac{i+1}{n}} - 1)\right]$$

Taking the limit $n \to \infty$ we obtain

$$\varphi_Z(t) = E(e^{tZ}) = \exp\left[\int_I (e^{tv} - 1)\pi(v)\, dv\right] \tag{A.4}$$

By using (A.3), it is also easy to find the mean value

$$E(Z) = \mu_Z = \int_I v\, \pi(v)\, dv \tag{A.5}$$

and the variance

$$\mathrm{Var}(Z) = \sigma_Z^2 = \int_I v^2 \pi(v)\, dv \tag{A.6}$$

Note that Z is related to a Poisson random variable $K = N_I$ with parameter

$$\lambda_Z = \int_I \pi(v)\, dv$$

In particular, we have

$$v_{\min} K \leq Z \leq v_{\max} K \tag{A.7}$$

hence

$$v_{\min} \lambda_Z \leq \mu_Z \leq v_{\max} \lambda_Z \tag{A.8}$$

We also have

$$v_{\min}^2 \lambda_Z \leq \sigma_Z^2 \leq v_{\max}^2 \lambda_Z \tag{A.9}$$

The random variable Z admits bounds on large deviations similar to the ones we found for Poisson random variables:

Lemma A.5 *For any $b > 0$ there is a $c > 0$ (determined by b, $v_{\min}$ and $v_{\max}$) such that for any integrated Poisson random variable Z and all $0 < B < b\,\sigma_Z$ we have*

$$P(|Z - \mu_Z| > B\sigma_Z) \leq 2e^{-cB^2}$$

Proof. Put, for brevity, $\mu = \mu_Z$ and $\sigma = \sigma_Z$. We will show that

$$P(Z > \mu + B\sigma) \le e^{-cB^2} \tag{A.10}$$

(the same bound for $P(Z < \mu - B\sigma)$ is proved similarly, as we did that in Lemma A.1). For all $t > 0$ we have

$$\varphi_Z(t) \ge e^{(\mu+B\sigma)t} \cdot P(Z > \mu + B\sigma)$$

hence

$$P(Z > \mu + B\sigma) \le \exp\left[\int_I (e^{tv} - 1)\pi(v)\,dv - (\mu + B\sigma)t\right]$$

We substitute $t = Bs/\sigma$ with $s > 0$ to be chosen later and expand e^{tv} into a Taylor series:

$$P(Z > \mu + B\sigma) \le \exp\left[\int_I \left(\sum_{n=1}^{\infty} \frac{(Bsv)^n}{\sigma^n n!}\right)\pi(v)\,dv - \frac{Bs\mu}{\sigma} - B^2 s\right]$$

The first two terms with $n = 1$ and $n = 2$ give

$$\int_I \frac{Bsv}{\sigma}\,\pi(v)\,dv = \frac{Bs\mu}{\sigma}$$

by (A.5) and

$$\int_I \frac{(Bsv)^2}{2\sigma^2}\,\pi(v)\,dv = \frac{B^2 s^2}{2}$$

by (A.6), respectively.

Therefore,

$$\begin{aligned} P(Z > \mu + B\sigma) \;\; &\le \;\; \exp\Big[- B^2 \Big(s - s^2/2 - \\ &\qquad -s^2 \sum_{n=3}^{\infty} \frac{(Bs/\sigma)^{n-2} \int_I v^n\,\pi(v)\,dv}{n!\,\sigma^2} \Big)\Big] \end{aligned}$$

Assuming $B/\sigma < b$ and using (A.6) gives

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{(Bs/\sigma)^{n-2} \int_I v^n\,\pi(v)\,dv}{n!\,\sigma^2} \;\; &< \;\; \sum_{k=1}^{\infty} \frac{(sb)^k\, v_{\max}^{k+2}}{k!\, v_{\min}^2} \\ &< \;\; \frac{v_{\max}^2}{v_{\min}^2}\left(e^{sb\, v_{\max}} - 1\right) \end{aligned}$$

Now if s is small enough, then we have

$$c := s - \frac{s^2}{2} - s^2\,\frac{v_{\max}^2}{v_{\min}^2}\left(e^{sb\, v_{\max}} - 1\right) > 0$$

This proves (A.10), and hence the lemma. □

Lemma A.6 *Let $\lambda_0 > 0$. For all sufficiently large $L > 0$ and any integrated Poisson random variable Z with $\lambda_Z \geq \lambda_0$*

$$P(|Z - \mu_Z| > \sigma_Z \ln L) \leq L^{-d \ln \ln L}$$

where $d > 0$ is a constant determined by $\lambda_0, v_{\min}, v_{\max}$.

Proof. Let

$$b = \frac{2v_{\max}}{v_{\min}^2}$$

First, if $\ln L < b\sigma_Z$, then the result easily follows from the previous lemma.

Now, assume that

$$\ln L \geq b\sigma_Z \tag{A.11}$$

Using the inequalities (A.8)–(A.9) gives

$$\lambda_Z \leq \frac{\sigma_Z^2}{v_{\min}^2} \leq \frac{(\ln L)^2}{b^2 v_{\min}^2}$$

and hence

$$\mu_Z \leq v_{\max}\lambda_Z \leq \frac{v_{\max}(\ln L)^2}{b^2 v_{\min}^2} \tag{A.12}$$

and also

$$\mu_Z \leq v_{\max}\lambda_Z \leq \frac{v_{\max}}{v_{\min}^2}\sigma_Z^2 \tag{A.13}$$

Multiplying (A.12) and (A.13) and taking the square root gives

$$\mu_Z \leq \frac{v_{\max}}{b v_{\min}^2}\sigma_Z \ln L = \frac{1}{2}\sigma_Z \ln L \tag{A.14}$$

Therefore, since Z is a positive random variable, we have

$$\begin{aligned} P(|Z - \mu_Z| > \sigma_Z \ln L) &= P(Z > \mu_Z + \sigma_Z \ln L) \\ &\leq P(Z > \sigma_Z \ln L) \end{aligned}$$

Moreover, combining (A.14) with (A.11) and (A.9) gives

$$\frac{1}{2}\sigma_Z \ln L = \frac{v_{\max}}{b v_{\min}^2}\sigma_Z \ln L \geq \frac{v_{\max}}{v_{\min}^2}\sigma_Z^2 \geq v_{\max}\lambda_Z$$

and also, by (A.9)

$$\frac{1}{2}\sigma_Z \ln L \geq \frac{1}{2} v_{\min}\sqrt{\lambda_Z} \ln L$$

Now, since $Z \leq v_{\max} K$ by (A.7), we have

$$\begin{aligned} P(Z > \sigma_Z \ln L) &\leq P\left(Z > v_{\max}\lambda_Z + \frac{1}{2} v_{\min}\sqrt{\lambda_Z} \ln L\right) \\ &\leq P\left(K > \lambda_Z + \frac{v_{\min}}{2v_{\max}}\sqrt{\lambda_Z} \ln L\right) \end{aligned}$$

Now the result follows from Lemma A.3. □

Corollary A.7 *Let $\lambda_0 > 0$. Let Z be an integrated Poisson random variable with $\lambda_Z \leq \lambda_0$. Then for all sufficiently large $L > 0$ we have*

$$P(Z > \ln L) \leq L^{-d \ln \ln L}$$

where $d > 0$ is determined by $\lambda_0, v_{\min}, v_{\max}$.

This immediately follows from Corollary A.4.

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