BAER RINGS AND BAER *-RINGS

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FOREWORD

The theory of Baer *-rings was set down in definitive form by Irving Kaplansky (its creator) in 1968, but the subject refuses to stop evolving; these notes are an attempt to record the present state of its evolution, taking into account especially the simplifications due to S. Maeda, S.S. Holland and D. Handelman.

Also noted are the connections (first explored by J.-E. Roos and G. Renault) with the theory of regular self-injective rings exposed in K. R. Goodearl's book [*Von Neumann regular rings*, 1979] and with the theory of continuous geometries, maximal rings of quotients, and von Neumann algebras.

Kaplansky's axiomatic approach for studying simultaneously the classical equivalence relations on projection lattices is developed in detail, culminating in the construction of a dimension function in that context.

The foregoing makes plain that this is less a new venture than it is a consolidation of old debts. I am especially grateful to Professors Maeda, Holland and Handelman for explaining to me several key points in their work; their generous help made it possible for me to comprehend and incorporate into these notes substantial portions of their work.

Each time I survey this theory I learn something new. The most important lessons I learned this time are the following:

1. The incisive results of Maeda and Holland on the interrelations of the various axioms greatly simplify and generalize many earlier results.

2. The connection with regular, right self-injective rings puts the theory in a satisfying general-algebraic framework (the ghost of operator theory surviving only in the involution).

3. There is a wealth of new ideas in Handelman's 1976 *Transactions* paper; it will be awhile before I can absorb it all.

4. Somehow, the touchstone to the whole theory is Kaplansky's proof of direct finiteness for a complete *-regular ring (*Annals of Math.*, 1955). That it is an exhiliarating technical tour de force does not diminish one's yearning for a simpler proof. The lack of one suggests to me that this profound result needs to be better digested by ring theory.

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Poitiers, Spring 1982

Preface to the English version: The original French version (*Anneaux et *-anneaux de Baer*) was informally available at the University of Poitiers in the Spring of 1982. Apart from the correction of errors (and, no doubt, the introduction of new ones), the present version differs from the French only in the addition of a number of footnotes (some of them clarifications, others citing more recent literature).

S.K.B.

Austin, Texas, January, 1988

Preface to the second English version: The first English version (with a circulation of approximately 40) was produced on a 9-pin dot-matrix printer. The present version was produced from a TEX file created using LEO (ABK Software). I am most grateful to Margaret Combs, the University of Texas Mathematics Department's virtuoso TEXperson, for gently guiding me through (and around) the fine points of TEX without insisting that I learn it.

S.K.B.

Austin, Texas, March, 1991

Preface to the third English version: The diagrams on pages 57, 75 and 102, and the block matrices on pages 119–120, have been re-coded so as to avoid hand-drawn elements.

S.K.B.

Austin, Texas, April, 2003

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1. RICKART RINGS, BAER RINGS

By **ring** we mean ring with unity; a **subring** B of a ring A is assumed to contain the unity element of A.

1.1. DEFINITION. [23, p. 510] A **Rickart ring** is a ring such that the right annihilator (resp. left annihilator) of each element is the principal right ideal (resp. left ideal) generated by an idempotent.

1.2. Let A be a Rickart ring, $x \in \mathrm{A}$. Say

$$\{x\}^l = \mathrm{A}(1-e)\,, \quad \{x\}^r = (1-f)\mathrm{A}\,,$$

e and f idempotents.[1] Then

$$\begin{aligned} &yx = 0 \Leftrightarrow y(1-e) = y \Leftrightarrow ye = 0\,, \\ &xz = 0 \Leftrightarrow (1-f)z = z \Leftrightarrow fz = 0\,; \end{aligned}$$

whence $x^2 = 0 \Leftrightarrow fe = 0$. And $(1-e)x = 0 = x(1-f)$, so $ex = x = xf$.

1.3. If A is a *-ring (that is, a ring with involution) then "right" is enough in Definition 1.1: one has $\{x\}^l = (\{x^*\}^r)^*$, so if $\{x^*\}^r = g\mathrm{A}$ with g idempotent, then $\{x\}^l = \mathrm{A}g^*$.

1.4. DEFINITION. [23, p. 522] A **Rickart *-ring** is a *-ring such that the right annihilator of each element is the principal right ideal generated by a projection (a self-adjoint idempotent).

1.5. In a Rickart *-ring, all's well with "left": if $\{x^*\}^r = g\mathrm{A}$, where $g^* = g = g^2$, then $\{x\}^l = (g\mathrm{A})^* = \mathrm{A}g$.

1.6. The projection in 1.4 is unique. {Proof: If $e\mathrm{A} = f\mathrm{A}$ with e and f projections, then $fe = e$ and $ef = f$, so $e = e^* = (fe)^* = e^*f^* = ef = f$.}

1.7. DEFINITION. Let A be a Rickart *-ring, $x \in \mathrm{A}$, and write

$$\{x\}^l = \mathrm{A}(1-e)\,, \quad \{x\}^r = (1-f)\mathrm{A}$$

[1] For any subset S of a ring A, $\mathrm{S}^l = \{x \in \mathrm{A} : \ xs = 0 \ \ (\forall\ s \in \mathrm{S})\}$, $\mathrm{S}^r = \{x \in \mathrm{A} : \ sx = 0 \ \ (\forall\ s \in \mathrm{S})\}$.

with e and f projections. Then e and f are unique (1.6), and (cf. 1.2)

$$yx = 0 \Leftrightarrow ye = 0\,,\ xz = 0 \Leftrightarrow fz = 0\,,\ ex = x = xf\,.$$

One writes $e = \mathrm{LP}(x)$, $f = \mathrm{RP}(x)$, called the **left projection** and the **right projection** of x. Thus

$$\{x\}^l = \mathrm{A}(1 - \mathrm{LP}(x))\,,\quad \{x\}^r = (1 - \mathrm{RP}(x))\mathrm{A}\,.$$

By 1.5, one has $\mathrm{LP}(x^*) = \mathrm{RP}(x)$, $\mathrm{RP}(x^*) = \mathrm{LP}(x)$.

1.8. For idempotents e, f in a ring A, one writes $e \le f$ in case $e \in f\mathrm{A}f$, that is, $ef = fe = e$. For projections e, f in a *-ring A, the following conditions are equivalent: $e \le f$, $e = ef$, $e = fe$, $e\mathrm{A} \subset f\mathrm{A}$, $\mathrm{A}e \subset \mathrm{A}f$.

1.9. In a Rickart *-ring A, for an element $x \in \mathrm{A}$ and a projection $g \in \mathrm{A}$, one has

$$gx = x \Leftrightarrow g \ge \mathrm{LP}(x)\,.$$

{Proof: If $e = \mathrm{LP}(x)$, then $\{x\}^l = \mathrm{A}(1-e)$, so $gx = x \Leftrightarrow (1-g)x = 0 \Leftrightarrow 1 - g \in \mathrm{A}(1-e) \Leftrightarrow 1 - g \le 1 - e \Leftrightarrow e \le g$.}

1.10. In a Rickart *-ring, the involution is **proper**, that is, $x^*x = 0 \Rightarrow x = 0$. {Proof: If $x^*x = 0$ then (cf. 1.7) $0 = \mathrm{RP}(x^*)\mathrm{LP}(x) = \mathrm{LP}(x)\mathrm{LP}(x) = \mathrm{LP}(x)$, hence $x = \mathrm{LP}(x)\cdot x = 0$.}

1.11. PROPOSITION. [14] *Let* A *be a *-ring. The following conditions are equivalent*:

(a) A *is a Rickart *-ring*;

(b) A *is a Rickart ring and* $\mathrm{A}e = \mathrm{A}e^*e$ *for every idempotent* e.

Proof. (a) ⇒ (b): Let $e \in \mathrm{A}$ be idempotent. Then $\mathrm{A}e = \{1-e\}^l = \mathrm{A}g$ with g a projection. One has $eg = e$ and $ge = g$, whence $ege = e$ and $g = g^* = (ge)^* = e^*g^* = e^*g$; therefore $e = ege = e(e^*g)e$, whence $e\mathrm{A} \subset ee^*\mathrm{A}$. But $ee^*\mathrm{A} \subset e\mathrm{A}$ trivially, so $e\mathrm{A} = ee^*\mathrm{A}$. Taking adjoints, $\mathrm{A}e^* = \mathrm{A}ee^*$; applying this to e^* in place of e, one obtains $\mathrm{A}e = \mathrm{A}e^*e$.

(b) ⇒ (a): Let $e \in \mathrm{A}$ be idempotent. By (b), $e = ae^*e$ for suitable a. Set $f = ae^*$. Then

$$ff^* = (ae^*)(ea^*) = (ae^*e)a^* = ea^* = f^*\,,$$

whence f is self-adjoint and $f^2 = f$. From $f = ae^*$ one has $\mathrm{A}f \subset \mathrm{A}e^*$; from $e = (ae^*)e = fe$ one has $e^* = e^*f$, so $\mathrm{A}e^* \subset \mathrm{A}f$; thus $\mathrm{A}e^* = \mathrm{A}f$.

Now let $x \in \mathrm{A}$ be arbitrary. Write $\{x\}^l = \mathrm{A}g$, g idempotent. Set $e = g^*$; by the preceding, there exists a projection f such that $\mathrm{A}f = \mathrm{A}e^* = \mathrm{A}g = \{x\}^l$, thus A is a Rickart *-ring. ◇

1.12. EXAMPLE. Every (von Neumann) regular ring [7, p. 1] is a Rickart ring.

{Proof: Let A be a regular ring, $x \in \mathrm{A}$. Choose $y \in \mathrm{A}$ with $x = xyx$ and let $e = xy$; then e is idempotent, $x\mathrm{A} = e\mathrm{A}$, and $\{x\}^l = (x\mathrm{A})^l = (e\mathrm{A})^l =$

$A(1-e)$. Similarly, $f = yx$ is idempotent, $Ax = Af$, and $\{x\}^r = (1-f)A$. Incidentally,

$$\{x\}^{lr} = (A(1-e))^r = \{1-e\}^r = eA = xA\,,$$

similarly $\{x\}^{rl} = Ax$; thus $(xA)^{lr} = xA$ and $(Ax)^{rl} = Ax$.}

1.13. PROPOSITION. [27, p. 114, Th. 4.5] *The following conditions on a *-ring* A *are equivalent*:

(a) A *is a regular Rickart *-ring*;

(b) A *is regular and the involution is proper*;

(c) *for every* $x \in A$ *there exists a projection* e *such that* $xA = eA$.

Proof. (a) ⇒ (b): Every Rickart *-ring has proper involution (1.10).

(b) ⇒ (a): By 1.12, A is a Rickart ring. Let $x \in A$. Since the involution is proper, $xy = 0 \Leftrightarrow x^*xy = 0$; for, if $x^*xy = 0$ then $(xy)^*(xy) = y^*(x^*xy) = 0$. Thus $\{x\}^r = \{x^*x\}^r$, whence $\{x\}^{rl} = \{x^*x\}^{rl}$, in other words (proof of 1.12) $Ax = Ax^*x$. In particular, $Ae = Ae^*e$ for all idempotents e, so A is a Rickart *-ring (1.11).

(a) ⇒ (c): Let $x \in A$. Write $\{x\}^l = Af$, f a projection. Then $\{x\}^{lr} = (1-f)A$; but $\{x\}^{lr} = xA$ (proof of 1.12) so $xA = (1-f)A$ and $e = 1-f$ fills the bill.

(c) ⇒ (a): Let $x \in A$, and write $xA = eA$ with e a projection. Then $ex = x$, and $e = xy$ for suitable y, whence $x = (xy)x$; thus A is regular. And $\{x\}^l = (xA)^l = (eA)^l = A(1-e)$, so A is a Rickart *-ring. {Note, incidentally, that $e = \mathrm{LP}(x)$ by 1.7. Similarly, if $f = \mathrm{RP}(x)$ then $Ax = Af$.} ♢

1.14. DEFINITION. A *-ring satisfying the conditions of 1.13 is called a *-**regular** ring.

1.15. PROPOSITION. [16, Lemma 5.3] *The projections of a Rickart *-ring form a lattice, with*

$$e \cup f = f + \mathrm{RP}[e(1-f)]\,, \quad e \cap f = e - \mathrm{LP}[e(1-f)]\,.$$

Proof. Write $x = e(1-f)$ and let $f' = \mathrm{RP}(x)$. Obviously $f' \le 1-f$, so $f + f'$ is a projection; we are to show that $f + f'$ serves as $\sup\{e, f\}$.

From $x = xf' = e(1-f)f' = ef'$ we have $e - ef = ef'$, $e = ef + ef' = e(f+f')$, thus $e \le f + f'$; so $f + f'$ majorizes both e and f. Suppose also $e \le g$ and $f \le g$ (g a projection); then $f = fg = gf$, so $xg = e(1-f)g = eg(1-f) = e(1-f) = x$, whence $f' \le g$ and therefore also $f + f' \le g$. Thus $e \cup f$ exists and is equal to $f + f'$. This establishes the first formula, and the second follows from it by duality: $e \cap f$ exists and

$$\begin{aligned} e \cap f &= 1 - [(1-f) \cup (1-e)] \\ &= 1 - \{(1-e) + \mathrm{RP}[(1-f)e]\} \\ &= e - \mathrm{RP}[(1-f)e] \\ &= e - \mathrm{LP}[((1-f)e)^*] = e - \mathrm{LP}[e(1-f)]\,. \ ♢ \end{aligned}$$

1.16. COROLLARY. *Let* A *be a* *-*regular ring. Then*:

(1) *The set of principal right ideals of* A *is a modular lattice with a canonical complementation* $\mathrm{I} \mapsto (\mathrm{I}^*)^r$.

(2) *If* e, f *are projections in* A, *then*

$$e\mathrm{A} \cap f\mathrm{A} = (e \cap f)\mathrm{A}\,, \quad e\mathrm{A} + f\mathrm{A} = (e \cup f)\mathrm{A}\,.$$

Proof. As for any regular ring, the set $\mathcal{R}$ of principal right ideals of A is a (complemented) modular lattice for the order relation $\subset$, with

$$\sup\{\mathrm{I}, \mathrm{J}\} = \mathrm{I} + \mathrm{J}\,, \quad \inf\{\mathrm{I}, \mathrm{J}\} = \mathrm{I} \cap \mathrm{J}$$

[7, p. 15, Th. 2.3]. Since A is a Rickart *-ring (1.14) its projections form a lattice P (1.15); and since every $\mathrm{I} \in \mathcal{R}$ has the form $\mathrm{I} = e\mathrm{A}$ with e a unique projection (1.13), the mapping $e \mapsto e\mathrm{A}$ is a bijection $\mathrm{P} \to \mathcal{R}$, clearly an order isomorphism. Therefore

$$(e \cup f)\mathrm{A} = e\mathrm{A} + f\mathrm{A} \ \text{ and } \ (e \cap f)\mathrm{A} = e\mathrm{A} \cap f\mathrm{A}$$

for all e, f in P. Moreover, since $e \mapsto 1 - e$ is an order anti-automorphism (of P), so is

$$e\mathrm{A} \mapsto (1 - e)\mathrm{A} = (\mathrm{A}e)^r = (e\mathrm{A})^{*r}$$

(of $\mathcal{R}$), and $e\mathrm{A}$, $(1-e)\mathrm{A}$ are obviously complementary in $\mathcal{R}$: $e\mathrm{A} \cap (1-e)\mathrm{A} = 0$, $e\mathrm{A} + (1 - e)\mathrm{A} = \mathrm{A}$. ♢

1.17. PROPOSITION. *Let* A *be a Rickart ring*, $\mathcal{R}$ (resp. $\mathcal{L}$) *the set of all idempotent-generated principal right* (resp. *left*) *ideals of* A, *ordered by inclusion. Then*:

(i) $\mathrm{J} \mapsto \mathrm{J}^l$ *is an order anti-isomorphism* $\mathcal{R} \to \mathcal{L}$, *with inverse mapping* $\mathrm{I} \mapsto \mathrm{I}^r$.

(ii) *If* (J_α) *is a family in* $\mathcal{R}$ *that possesses an infimum* J *in* $\mathcal{R}$, *then* $\mathrm{J} = \bigcap \mathrm{J}_\alpha$.

(iii) *If* (J_α) *is a family in* $\mathcal{R}$ *that possesses a supremum* K *in* $\mathcal{R}$, *then* $\mathrm{K} = (\bigcup \mathrm{J}_\alpha)^{lr} = (\sum \mathrm{J}_\alpha)^{lr}$.

Proof. (i) If $e \in \mathrm{A}$ is idempotent, then $(e\mathrm{A})^l = \{e\}^l = \mathrm{A}(1 - e) \in \mathcal{L}$ and $(\mathrm{A}e)^r = \{e\}^r = (1 - e)\mathrm{A} \in \mathcal{R}$, whence $(e\mathrm{A})^{lr} = e\mathrm{A}$ and $(\mathrm{A}e)^{rl} = \mathrm{A}e$.

(ii) Suppose there exists $\mathrm{J} = \inf(\mathrm{J}_\alpha)$ in $\mathcal{R}$. Then $\mathrm{J} \subset \mathrm{J}_\alpha$ for all α, so $\mathrm{J} \subset \bigcap \mathrm{J}_\alpha$. Conversely, let $x \in \bigcap \mathrm{J}_\alpha$. Since A is a Rickart ring, $\{x\}^l = \mathrm{A}e$ for an idempotent e, whence $\{x\}^{lr} = (1 - e)\mathrm{A} \in \mathcal{R}$. For all α, $\{x\} \subset \mathrm{J}_\alpha$, so $\{x\}^{lr} \subset \mathrm{J}_\alpha^{lr} = \mathrm{J}_\alpha$; therefore $\{x\}^{lr} \subset \mathrm{J}$ and in particular $x \in \mathrm{J}$.

(iii) Suppose there exists $\mathrm{K} = \sup(\mathrm{J}_\alpha)$ in $\mathcal{R}$. In view of (i), there exists $\inf(\mathrm{J}_\alpha^l)$ in $\mathcal{L}$ and $\mathrm{K} = [\inf(\mathrm{J}_\alpha^l)]^r$. By the dual of (ii) (apply (ii) in the opposite ring A°) one has $\inf(\mathrm{J}_\alpha^l) = \bigcap(\mathrm{J}_\alpha^l) = \left(\bigcup \mathrm{J}_\alpha\right)^l$, whence $\mathrm{K} = \left(\bigcup \mathrm{J}_\alpha\right)^{lr}$. ♢

1.18. COROLLARY. [11, Prop. 2.1]. *Let* A *be a Rickart* *-*ring*, P *its projection lattice* (1.15), (e_i) *a family of projections in* A.

(i) *If* $\inf e_i$ *exists in* P, *then* $(\inf e_i)\mathrm{A} = \bigcap(e_i\mathrm{A})$.

(ii) *If* $\sup e_i$ *exists in* P, *then* $(\sup e_i)\mathrm{A} = \big(\bigcup e_i\mathrm{A}\big)^{lr} = \big(\sum e_i\mathrm{A}\big)^{lr}$.

In particular, for any finite set of projections $e_1, \ldots, e_n$ *in* A, *one has*

$$(e_1 \cap \ldots \cap e_n)\mathrm{A} = e_1\mathrm{A} \cap \ldots \cap e_n\mathrm{A}$$

and $(e_1 \cup \ldots \cup e_n)\mathrm{A} = (e_1\mathrm{A} \cup \ldots \cup e_n\mathrm{A})^{lr} = (e_1\mathrm{A} + \ldots + e_n\mathrm{A})^{lr}$.

Proof. As in 1.17, let $\mathcal{R}$ be the set of idempotent-generated principal right ideals of A. If $u \in \mathrm{A}$ is idempotent, then $u\mathrm{A} = \{1-u\}^r = e\mathrm{A}$ for some projection e (clearly $e = \mathrm{LP}(u)$); it follows that $e \mapsto e\mathrm{A}$ is an order-isomorphism $\mathrm{P} \to \mathcal{R}$ (in particular, $\mathcal{R}$ is a lattice), and the corollary is immediate from 1.17. ♢

1.19. DEFINITION. [18, p. 3] A **Baer ring** is a ring A such that, for every subset S of A, the right annihilator of S is the principal right ideal generated by an idempotent.

1.20. In a Baer ring, left annihilators are also idempotent-generated (hence Baer ⇒ Rickart). {Proof: One has $\mathrm{S}^l = \mathrm{S}^{lrl}$; if $\mathrm{S}^{lr} = e\mathrm{A}$, e idempotent, then $\mathrm{S}^l = (e\mathrm{A})^l = \mathrm{A}(1-e)$.}

1.21. PROPOSITION. [24, Lemma 1.3] *Let* A *be a ring*, $\mathcal{R}$ *the set of idempotent-generated principal right ideals*; *order* $\mathcal{R}$ *by inclusion. The following conditions are equivalent*:

(a) A *is a Baer ring*;

(b) A *is a Rickart ring and* $\mathcal{R}$ *is a complete lattice.*

In this case, *if* (J_α) *is any family in* $\mathcal{R}$ *then*

$$\inf(\mathrm{J}_\alpha) = \bigcap \mathrm{J}_\alpha, \quad \sup(\mathrm{J}_\alpha) = \big(\bigcup \mathrm{J}_\alpha\big)^{lr}.$$

Proof. In any case, note that for every $\mathrm{J} \in \mathcal{R}$ one has $\mathrm{J} = \mathrm{J}^{lr}$.

(a) ⇒ (b): By hypothesis, $\mathcal{R}$ is the set of all right ideals S^r, where $\mathrm{S} \subset \mathrm{A}$. As noted in 1.20, A is a Rickart ring. Let (J_α) be any family in $\mathcal{R}$. Then

$$\bigcap \mathrm{J}_\alpha = \bigcap(\mathrm{J}_\alpha^{lr}) = \big(\bigcup \mathrm{J}_\alpha^{l}\big)^r \in \mathcal{R},$$

whence $\bigcap \mathrm{J}_\alpha$ obviously serves as $\inf(\mathrm{J}_\alpha)$ in $\mathcal{R}$. Let $\mathrm{J} = \big(\bigcup \mathrm{J}_\alpha\big)^{lr} \in \mathcal{R}$. Obviously $\mathrm{J} \supset \mathrm{J}_\alpha$ for all α. And if $\mathrm{J}_\alpha \subset \mathrm{K} \in \mathcal{R}$ for all α, then $\bigcup \mathrm{J}_\alpha \subset \mathrm{K}$ so

$$\mathrm{J} = \big(\bigcup \mathrm{J}_\alpha\big)^{lr} \subset \mathrm{K}^{lr} = \mathrm{K}.$$

Thus J serves as $\sup(\mathrm{J}_\alpha)$ in $\mathcal{R}$.

(b) ⇒ (a): Let $\mathrm{S} = \{x_\alpha : \alpha \in \Omega\}$ be any subset of A; we are to show that $\mathrm{S}^r \in \mathcal{R}$. Now,

$$\mathrm{S}^r = \bigcap_\alpha \{x_\alpha\}^r.$$

Since A is a Rickart ring, $\{x_\alpha\}^r \in \mathcal{R}$. Write $\mathrm{J}_\alpha = \{x_\alpha\}^r$ and let $\mathrm{J} = \inf(\mathrm{J}_\alpha)$ in $\mathcal{R}$, which exists by hypothesis. By (ii) of 1.17, $\mathrm{J} = \bigcap \mathrm{J}_\alpha$. But

$$\bigcap \mathrm{J}_\alpha = \bigcap \{x_\alpha\}^r = \mathrm{S}^r\,,$$

so $\mathrm{S}^r = \mathrm{J} \in \mathcal{R}$. {Note: It suffices to assume that A is a Rickart ring and that every family in $\mathcal{R}$ has an infimum in $\mathcal{R}$.} ♢

1.22. COROLLARY. *Let* A *be a regular ring*, $\mathcal{R}$ *its lattice of principal right ideals. The following conditions are equivalent*:

(a) A *is a Baer ring*;

(b) $\mathcal{R}$ *is a complete lattice.*

Proof. As noted in the proof of 1.16, $\mathcal{R}$ *is* a lattice and is the set of idempotent-generated right ideals; since A is a Rickart ring (1.12), the corollary is immediate from 1.21. {In particular, the lattice operations in $\mathcal{R}$ are given by the formulas in 1.21.} ♢

1.23. DEFINITION. [18, p. 27] A **Baer *-ring** is a *-ring A such that, for every subset S of A, the right annihilator of S is the principal right ideal generated by a projection. {In view of the formula $\mathrm{S}^l = ((\mathrm{S}^*)^r)^*$, left annihilators are also generated by projections (cf. 1.20).}

1.24. PROPOSITION. *Let* A *be a *-ring. The following conditions are equivalent*:

(a) A *is a Baer *-ring*;

(b) A *is a Rickart *-ring whose projection lattice* (1.15) *is complete*;

(c) A *is a Rickart *-ring and a Baer ring.*

In such a ring, if $\mathrm{S} \subset \mathrm{A}$ *then* $\mathrm{S}^r = (1-e)\mathrm{A}$, *where*

$$e = \sup\{\mathrm{RP}(s) :\ s \in \mathrm{S}\}\,;$$

and if (e_α) *is any family of projections in* A *then*

$$(\inf e_\alpha)\mathrm{A} = \bigcap e_\alpha \mathrm{A}\,, \quad (\sup e_\alpha)\mathrm{A} = \left(\bigcup e_\alpha \mathrm{A}\right)^{lr}.$$

Proof. (a) ⇒ (c): Obvious.

(c) ⇒ (b): Let P be the projection lattice of A, $\mathcal{R}$ the set of idempotent-generated principal right ideals of A; as noted in the proof of 1.18, $e \mapsto e\mathrm{A}$ is an order-isomorphism $\mathrm{P} \to \mathcal{R}$. Since $\mathcal{R}$ is complete (1.21), so is P.

(b) ⇒ (a): Let $\mathrm{S} \subset \mathrm{A}$ and let

$$e = \sup\{\mathrm{RP}(s) :\ s \in \mathrm{S}\}$$

(assumed to exist by (b)). Then (cf. 1.7)

$$\begin{aligned}
\mathrm{S}^r &= \{x \in \mathrm{A} :\ sx = 0\ (\forall s \in \mathrm{S})\}\\
&= \{x :\ \mathrm{RP}(s)\mathrm{LP}(x) = 0\ (\forall s \in \mathrm{S})\}\\
&= \{x :\ \mathrm{RP}(s) \le 1 - \mathrm{LP}(x)\ (\forall s \in \mathrm{S})\}\\
&= \{x :\ e \le 1 - \mathrm{LP}(x)\}\\
&= \{x :\ e\mathrm{LP}(x) = 0\} = \{x :\ ex = 0\}\\
&= \{e\}^r = (1-e)\mathrm{A}\,,
\end{aligned}$$

thus S is a Baer *-ring. The asserted formulas are immediate from those in 1.21 and the fact that $e \mapsto e\mathrm{A}$ is an order-isomorphism $\mathrm{P} \to \mathcal{R}$. ♢

1.25. COROLLARY. *Let* A *be a *-ring. The following conditions are equivalent*:

(a) A *is a regular Baer *-ring*;

(b) A *is a *-regular ring whose projection lattice* (cf. 1.16) *is complete*;

(c) A *is a *-regular Baer ring.*

Proof. (a) ⇒ (b): The involution of A is proper (1.10), so A is *-regular (1.13, 1.14), and its projection lattice is complete by (b) of 1.24.

(b) ⇔ (c): Under either hypothesis, A is a Rickart *-ring, so (b) ⇔ (c) by 1.24.

(c) ⇒ (a): Since A is a Rickart *-ring (1.14, 1.13) it is a Baer *-ring by criterion (c) of 1.24. ♢

Because of criterion (b), such rings are also called "complete *-regular rings".

1.26. EXAMPLE. The endomorphism ring of a vector space is a regular Baer ring.

{Proof: Let V be a vector space (left or right) over a division ring D, and let $\mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V})$ be the ring of all D-linear mappings $u : \mathrm{V} \to \mathrm{V}$.

Regularity: Let $u \in \mathrm{A}$; we seek $v \in \mathrm{A}$ with $u = uvu$, that is, $u(x) = u(v(u(x)))$ for all $x \in \mathrm{V}$. Let W be any supplement of $\mathrm{Ker}\, u$ in V: $\mathrm{V} = \mathrm{W} \oplus \mathrm{Ker}\, u$. Since $\mathrm{W} \cap \mathrm{Ker}\, u = 0$, the restriction of u to W is injective, whence an isomorphism

$$u_0 : \mathrm{W} \to u(\mathrm{W}),$$

where u_0 has the graph of $u|\mathrm{W}$. Consider

$$u_0^{-1} : u(\mathrm{W}) \to \mathrm{W} \subset \mathrm{V}$$

and let $v \in \mathrm{A}$ extend u_0^{-1} (for example, take v to be 0 on some supplement of $u(\mathrm{W})$). Then

$$(\forall\, y \in \mathrm{W}) \quad v(u(y)) = u_0^{-1}(u_0(y)) = y,$$

therefore

$$(\forall\, y \in \mathrm{W}) \quad u(v(u(y))) = u(y).$$

Thus $uvu = u$ on W; also $uvu = 0 = u$ on $\mathrm{Ker}\, u$, so $uvu = u$ on V.

The Baer property: Let $\mathrm{S} \subset \mathrm{A}$, say $\mathrm{S} = \{u_i : i \in \mathrm{I}\}$. Then

$$\begin{aligned} u \in \mathrm{S}^r &\Leftrightarrow u_i u = 0 \ (\forall i) \\ &\Leftrightarrow u(\mathrm{V}) \subset \mathrm{Ker}\, u_i \quad (\forall i) \\ &\Leftrightarrow u(\mathrm{V}) \subset \bigcap \mathrm{Ker}\, u_i\,. \end{aligned}$$

Let $e \in \mathrm{A}$ be an idempotent whose range is $\bigcap \mathrm{Ker}\, u_i$; then

$$\begin{aligned} u \in \mathrm{S}^r &\Leftrightarrow u(\mathrm{V}) \subset e(\mathrm{V}) \\ &\Leftrightarrow e(u(x)) = u(x) \quad (\forall\, x \in \mathrm{V}) \\ &\Leftrightarrow u = eu \\ &\Leftrightarrow u \in e\mathrm{A}, \end{aligned}$$

thus $S^r = eA$.}

*[1] **1.27.** EXAMPLE. The algebra of bounded operators on a Hilbert space is a Baer *-ring.

{Proof: Let $A = L(H)$ be the algebra of all continuous linear mappings $u : H \to H$, where H is a Hilbert space; with $u^* =$ adjoint of u, A is a *-ring. Let $S \subset A$, say $S = \{u_i : i \in I\}$. Then $N = \bigcap \operatorname{Ker} u_i$ is a closed linear subspace of H. Let $e \in A$ be the projection with range N (and kernel $N^\perp$). As in 1.26, $S^r = \{u \in A : u(H) \subset e(H)\} = eA$.}

1.28. Let A be a ring. A right ideal I of A is said to be **essential** in A if $I \cap J \neq 0$ whenever J is a nonzero right ideal. One says that A is **right nonsingular** if the only $x \in A$ for which $\{x\}^r$ is an essential right ideal is $x = 0$ (in other words, if $x \neq 0$ then there exists a nonzero right ideal J such that $\{x\}^r \cap J = 0$). Dually for "essential left ideals" and "left nonsingularity".

1.29. Every Rickart ring (in particular, every regular ring and every Rickart *-ring) is both right and left nonsingular.

{Proof: Let $x \in A$, $x \neq 0$. Then $\{x\}^r = eA$ with e idempotent, $e \neq 1$, and $J = (1-e)A$ is a nonzero right ideal with $\{x\}^r \cap J = 0$.}

1.30. EXAMPLE. If A is a right self-injective ring and A is right nonsingular, then A is a Baer ring. (In 1.32 we shall see that A is regular.)

{Proof: The assumptions are that (i) A_A is an injective (right) A-module, and (ii) if $x \in A$ is such that the right ideal $\{x\}^r$ is essential, then $x = 0$. Given $S \subset A$, let $I = S^r$; we seek an idempotent $e \in A$ such that $I = eA$.

Let $\hat{I}$ be an injective envelope of I in $\operatorname{Mod} A$ (the category of right A-modules); that is, $\hat{I} \in \operatorname{Mod} A$, $\hat{I}$ is injective, and I is an essential submodule of $\hat{I}$ [19, p. 92, Prop. 10]. Since A_A is injective, the identity mapping $I \to A_A$ extends to a monomorphism $\hat{I} \to A_A$ [19, p. 91, Lemma 4]; thus one can view $\hat{I}$ as a submodule of A_A, that is, as a right ideal of A. Since $\hat{I}$ is injective, it is a direct summand of A_A, say $A_A = \hat{I} \oplus K$, K a suitable right ideal of A; if e is the component of 1 in $\hat{I}$ for this decomposition, one sees easily that e is idempotent and $\hat{I} = eA$, thus it will suffice to show that $I = \hat{I}$. Of course $I \subset \hat{I}$. Conversely, let $u \in \hat{I}$; we are to show that $u \in I = S^r$. Let $s \in S$; to show that $su = 0$ it will suffice, by the hypothesis (ii), to show that $\{su\}^r$ is an essential right ideal of A. Let

$$(I : u) = \{a \in A : ua \in I\},$$

clearly a right ideal of A; since $u \in \hat{I}$ and I is essential in $\hat{I}$, it follows that $(I : u)$ is an essential right ideal of A. So it will suffice to show that $\{su\}^r \supset (I : u)$. If $a \in (I : u)$ then $ua \in I = S^r$, so $0 = s(ua) = (su)a$; thus $a \in \{su\}^r$ and we have shown that $(I : u) \subset \{su\}^r$.}

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*The asterisk signals material mainly functional-analytic in nature. Some of it is needed for §19 (dimension function); until then, it serves only to provide examples and applications of the algebraic development. The reader planning to omit dimension can omit the functional analysis altogether.

Example 1.26 is a special case of 1.30 [7, p. 11, Cor. 1.23].

1.31. EXAMPLE. If A is a right nonsingular ring, then the maximal ring of right quotients Q of A is a regular, right self-injective ring (hence is a Baer ring).

{Proof: Q is right self-injective [19, p. 107, Cor. of Prop. 2] and regular [19, p. 106, Prop. 2] hence right nonsingular (1.29), therefore Q is a Baer ring (1.30).}

1.32. EXAMPLE. If A is a right self-injective ring and A is right nonsingular, then A is a regular Baer ring. (Cf. 1.41.)

{Proof: With Q as in 1.31, it will suffice to show that $A = Q$. Now, $A \subset Q$ and Q is a ring of right quotients of A, hence A_A is essential in Q_A [19, p. 99, proof of Prop. 8]. Since A_A is injective, it is a summand of Q_A, say $Q_A = A_A \oplus J$. Then $A_A \cap J = 0$ yields $J = 0$ (A_A is essential), so $Q = A$.}

1.33. A *-ring A is said to be **symmetric** if, for every $x \in A$, $1 + x^*x$ is invertible in A. { *EXAMPLE: Any C*-algebra with unity.}

1.34. LEMMA. [18, p. 34, Th. 26] *If A is a symmetric *-ring, then for every idempotent $e \in A$ there exists a projection $f \in A$ such that $eA = fA$.*

Proof. Let $z = 1 + (e^* - e)^*(e^* - e) = 1 - e - e^* + ee^* + e^*e$ and let $t = z^{-1}$; since z is self-adjoint, so is t. One has $ez = ee^*e = ze$, therefore $et = te$ and $te^* = e^*t$. Set $f = ee^*t = tee^*$; then $f^* = f$ and $f^2 = ee^*t \cdot ee^*t = (ee^*e)te^*t = (ez)te^*t = e(zt)e^*t = ee^*t = f$, thus f is a projection. From $ef = f$ one has $fA \subset eA$, and from $fe = ee^*t \cdot e = ee^*et = ezt = e$ one has $eA \subset fA$. {Note that the conclusion of the lemma also holds in any Rickart *-ring A (symmetric or not), since $eA = \{1 - e\}^r$.} ♢

1.35. PROPOSITION. [18, p. 34, Cor.] *Let A be a symmetric *-ring. If A is a Baer ring* (resp. *Rickart ring*) *then it is a Baer *-ring* (resp. *Rickart *-ring*).

Proof. Suppose, for example, that A is a Baer ring, and let $S \subset A$. Write $S^r = eA$, e idempotent; by 1.34, $S^r = fA$ with f a projection. ♢

1.36. EXERCISE. Let A be a *-regular ring, n a positive integer, $M_n(A)$ the *-ring of $n \times n$ matrices over A, with *-transpose as the involution. The following conditions are equivalent: (a) $M_n(A)$ is *-regular; (b) the involution of $M_n(A)$ is proper; (c) the involution of A is n-**proper**, that is,

$$\sum_{i=1}^{n} x_i^* x_i = 0 \text{ implies } x_1 = \ldots = x_n = 0 .$$

{Hint: By a theorem of von Neumann, every full matrix ring over a regular ring is regular [7, p. 4, Th. 1.7]; cf. 1.13.}

1.37. EXERCISE. If A is a *-regular ring whose involution is 2-proper ($x^*x + y^*y = 0 \Rightarrow x = y = 0$) then A is symmetric.

{Hint: In a Rickart *-ring with n-proper involution, one has $\mathrm{RP}(x_1^*x_1 + \ldots + x_n^*x_n) = \mathrm{RP}(x_1) \cup \ldots \cup \mathrm{RP}(x_n)$ [2, p. 225].}

***1.38.** An AW*-**algebra** is a C*-algebra A (with unity) that is also a Baer *-ring. {Since every C*-algebra with unity is symmetric, it is the same to say that A is a C*-algebra and a Baer ring (1.35).} Example: $\mathrm{A} = \mathrm{L(H)}$, H a Hilbert space (1.27). A **Rickart** C*-**algebra** is a C*-algebra that is also a Rickart *-ring (equivalently, a C*-algebra that is a Rickart ring). For an example of a Rickart C*-algebra that is not an AW*-algebra, see [2, p. 15, Example 2].

***1.39.** The commutative AW*-algebras are the algebras $\mathcal{C}(\mathrm{T})$, where T is a *Stonian space* (a compact space in which the closure of every open set is open) [2, p. 40, Th. 1]. The commutative Rickart C*-algebras are the algebras $\mathcal{C}(\mathrm{T})$, where T is compact, the closed-open sets of T are basic for its topology, and the closure of the union of any sequence of closed-open sets is open [2, p. 44, Th. 1].

***1.40.** A commutative C*-algebra A with unity is an AW*-algebra if and only if (i) A is the closed linear span of its projections, and (ii) every orthogonal family of projections has a supremum [2, p. 43, Exer. 1]. A commutative C*-algebra A with unity is a Rickart C*-algebra if and only if (i) A is the closed linear span of its projections, and (ii) every orthogonal sequence of projections in A has a supremum [2, p. 46, Prop. 3].

1.41. Every right continuous regular ring is a Baer ring.

{Proof: Let A be a regular ring and $\mathcal{R}$ its lattice of principal right ideals. The hypothesis is that $\mathcal{R}$ is upper continuous [7, pp. 160-161]; in particular, $\mathcal{R}$ is complete, so A is a Baer ring by 1.22. Incidentally, since every regular, right self-injective ring is right continuous [7, p. 162, Cor. 13.5], 1.30 and 1.31 are special cases of 1.41.}

1.42. (i) Every ring without divisors of 0 is a Baer ring whose only idempotents are 0 and 1. (ii) Conversely, if A is a Rickart ring whose only idempotents are 0 and 1, then A has no divisors of 0 (hence is a Baer ring).

{Proof: (i) is obvious. (ii) Let $x \in \mathrm{A}$, $x \neq 0$. Write $\{x\}^r = e\mathrm{A}$, e idempotent. By hypothesis, $e = 0$ or $e = 1$; since $x \neq 0$, necessarily $e = 0$, thus $\{x\}^r = \{0\}$.}

1.43. (i) Every *-ring without divisors of 0 is a Baer *-ring whose only projections are 0 and 1. (ii) Conversely, if A is a Rickart *-ring whose only projections are 0 and 1, then A has no divisors of 0 (hence is a Baer *-ring).

{Proof: (i) is obvious. (ii) Same proof as in 1.42, with e taken to be a projection.}

1.44. (i) Every division ring is a regular Baer ring whose only idempotents are 0 and 1. (ii) Conversely, if A is a regular ring whose only idempotents are 0 and 1, then A is a division ring.

{Proof: (ii) If $x \in \mathrm{A}$, $x \neq 0$, then $x\mathrm{A} = 1\mathrm{A}$ by the hypothesis, thus x is right-invertible.}

1.45. (i) Every involutive division ring is a regular Baer *-ring whose only projections are 0 and 1. (ii) Conversely, if A is a *-regular ring whose only projections are 0 and 1, then A is an involutive division ring. {Proof: Cf. 1.43 and 1.44.}

2. CORNERS

2.1. If A is a ring and $e \in A$ is idempotent, the ring eAe (with unity element e) is called a **corner** of A. {Reason: If $A = M_n(B)$ and $e = \mathrm{diag}(1, 0, \dots, 0)$, then $eAe \cong B$ is the 'northwest corner' of the matrix ring A.} For example, the following proposition says that every corner of a Baer ring is itself a Baer ring:

2.2. PROPOSITION. [18, p. 6, Th. 4] *If* A *is a Baer ring and* $e \in A$ *is idempotent, then* eAe *is a Baer ring.*

Proof. Let $S \subset eAe$. The right annihilator of S in eAe is $(eAe) \cap S^r$. Write $S^r = fA$, f idempotent. Since $S \subset eAe$ one has $1 - e \in S^r$, hence $1 - e = f(1-e) = f - fe$; left-multiplying by e, one has $0 = ef - efe$, thus $ef = efe \in eAe$. Write $g = ef$, which is an idempotent of eAe: $g^2 = (ef)(ef) = (efe)f = (ef)f = ef = g$. It will suffice to show that $(eAe) \cap S^r = g \cdot eAe$. From $Sg = S(ef) = (Se)f = Sf = 0$ one has $g \cdot eAe \subset (eAe) \cap S^r$. Conversely, if $x \in (eAe) \cap S^r$ then $x = fx = f(ex) = (fe)x = gx \in g \cdot eAe$, thus $(eAe) \cap S^r \subset g \cdot eAe$. ♢

2.3. Every corner of a Rickart ring is a Rickart ring.

{Proof: If $e \in A$ is idempotent and $x \in eAe$, apply the proof of 2.2 with $S = \{x\}$.}

2.4. Every corner of a regular ring is regular.

{Proof: Let A be regular, $e \in A$ idempotent, $x \in eAe$. Choose $y \in A$ with $x = xyx$. Then $xyx = (xe)y(ex) = x(eye)x$, so replacing y by eye one can suppose that $y \in eAe$.}

2.5. If A is a Rickart *-ring and $e \in A$ is a projection, then eAe is a Rickart *-ring.

{Proof: Since $(eae)^* = ea^*e$, eAe is a *-ring. Let $x \in eAe$ and let $S = \{x\}$. Write $S^r = fA$, f a projection; then the idempotent $g = ef = efe$ (see the proof of 2.2) is self-adjoint, hence is a projection, and the proof continues as in 2.2. Note, incidentally, that $\mathrm{RP}(x) = 1 - f$, whereas the right projection of x calculated in eAe is $e - g$ (1.7); but $1 - e \in S^r = fA$, so $1 - e \le f$, $1-e = (1-e)f = f - ef = f - g$, whence $1 - f = e - g$, thus the right projection of x is the same whether calculated in A or in eAe. Similarly for the left projection. Briefly, LP's and RP's in eAe are unambiguous.}

2.6. PROPOSITION. [18, p. 30] *If* A *is a Baer *-ring and* $e \in A$ *is a projection, then* eAe *is a Baer *-ring.*

Moreover, if (e_i) *is any family of projections in* eAe, *then* $\sup e_i$ *is the same whether computed in* A *or in* eAe, *and the same is true of* $\inf e_i$ (*briefly*, sups *and* infs *in* eAe *are unambiguous*).

Proof #1: eAe is a Baer ring (2.2) and a Rickart *-ring (2.5) hence a Baer *-ring (1.24).

Proof #2: Let $S \subset eAe$; in the notation of the proof of 2.2, one can take f to be a projection, and then $g = ef = efe$ is a projection.

Finally, let (e_i) be a family of projections in eAe, $h = \sup e_i$ in A. Since $e_i \le e$ for all i, one has $h \le e$; thus $h \in eAe$ clearly serves as supremum of the e_i in eAe. Similarly for $\inf e_i$. {Alternatively, $\inf e_i = e - \sup(e - e_i)$.} ♢

2.7. Let A be a *-regular ring, $x \in A$, $e = LP(x)$, $f = RP(x)$.

(i) There exists a unique $y \in fA$ such that $xy = e$. (One calls y the **relative inverse** of x.)

(ii) Moreover, $y \in fAe$, $yx = f$, $xyx = x$, $yxy = y$.

(iii) The relative inverse of y is x.

{Proof: (i), (ii) One has $eA = xA$ (cf. 1.13), say $e = xy$. Right-multiplying by e, one can suppose $y \in Ae$; and $e = xy = (xf)y = x(fy)$, so replacing y by fy one can suppose $y \in fAe$. If also $y' \in fA$ with $xy' = e$, then $x(y'-y) = e-e = 0$, so $f(y'-y) = 0$, $fy' = fy$, $y' = y$. Moreover, $x(yx-f) = xyx - xf = ex - xf = x - x = 0$, so $f(yx-f) = 0$, whence $yx = f$. Finally, $xyx = ex = x$ and $yxy = fy = y$.

(iii) From $y \in fAe$ one has $yA \subset fA$, and from $yx = f$ one has $fA \subset yA$, thus $yA = fA$ and so $f = LP(y)$. Similarly $e = RP(y)$, and then (iii) is immediate from (i).}

2.8. PROPOSITION. *If* A *is a *-regular ring and* $e \in A$ *is a projection, then* eAe *is *-regular. More precisely, if* $x \in eAe$ *and* y *is the relative inverse of* x *in* A, *then* $y \in eAe$.

Proof. eAe is regular (2.4) and its involution (induced by that of A) is proper, hence it is *-regular (1.13). Let $x \in eAe$ and write $f = LP(x)$, $g = RP(x)$ for the left and right projections of x as calculated in A. As noted in 2.5, $f, g \in eAe$, thus if y is the relative inverse of x in A then $y \in gAf \subset eAe$. ♢

2.9. If A is a regular Baer *-ring and $e \in A$ is a projection, then eAe is a regular Baer *-ring. {Proof: 2.4 and 2.6.}

2.10. Every corner of a regular, right self-injective ring also has these properties [7, p. 98, Prop. 9.8].

2.11. Every corner of a right continuous regular ring also has these properties [7, p. 162, Prop. 13.7].

***2.12.** Every self-adjoint corner of a von Neumann algebra is a von Neumann algebra (see 4.14 below).

3. CENTER

If A is a ring, we systematically write Z for the center of A.

3.1. The idempotents of Z form a Boolean algebra (that is, a complemented, distributive lattice) with $u \cap v = uv$, $u \cup v = u + v - uv$, $u' = 1 - u$ ($u \le v$ being defined by $u = uv$).

3.2. PROPOSITION. [18, p. 8, Th. 7] *The center of a Baer ring is a Baer ring.*

Proof. Let A be a Baer ring with center Z and let $\mathrm{S} \subset \mathrm{Z}$. Write $\mathrm{S}^r = v\mathrm{A}$, $\mathrm{S}^l = \mathrm{A}w$, v and w idempotents. Since $\mathrm{S} \subset \mathrm{Z}$ one has $\mathrm{S}^r = \mathrm{S}^l$, thus $v\mathrm{A} = \mathrm{A}w$. Then $v \in \mathrm{A}w$ and $w \in v\mathrm{A}$, so $v = vw = w$. Therefore $v\mathrm{A}(1-v) = \mathrm{A}v(1-v) = 0$, similarly $(1-v)\mathrm{A}v = 0$; thus for all $a \in \mathrm{A}$, $va(1-v) = (1-v)av = 0$, $va = vav = av$, whence $v \in \mathrm{Z}$. It follows that $\mathrm{Z} \cap \mathrm{S}^r = v\mathrm{Z}$, thus Z is a Baer ring. ♢

3.3. The central idempotents of a Baer ring form a complete Boolean algebra.

{Proof: Let $(u_i)_{i\in\mathrm{I}}$ be any family of idempotents in Z. Applying the proof of 3.2 to $\mathrm{S} = \{u_i :\ i \in \mathrm{I}\}$, one has $\mathrm{S}^r = v\mathrm{A}$ with v a central idempotent. Write $u = 1 - v$. For $x \in \mathrm{A}$ one has $u_i x = 0$ for all $i \Leftrightarrow x \in v\mathrm{A} \Leftrightarrow vx = x \Leftrightarrow ux = 0$. In particular, if $e \in \mathrm{A}$ is idempotent then $u_i \le e$ for all $i \Leftrightarrow u_i(1-e) = 0$ for all $i \Leftrightarrow u(1-e) = 0 \Leftrightarrow u \le e$, thus u serves as a supremum for the u_i in the set of all idempotents of A (for the order $e \le f$ defined by $e \in f\mathrm{A}f$); *a fortiori*, $u = \sup u_i$ in the set of idempotents of Z. It then follows that $1 - \sup(1-u_i)$ serves as $\inf u_i$.}

The proof shows:

3.4. If A is a Baer ring, (u_i) a family of central idempotents of A, and $u = \sup u_i$, then for $x \in \mathrm{A}$ one has $xu = 0 \Leftrightarrow xu_i = 0$ for all i.

3.5. The center of a Rickart ring is a Rickart ring.

{Proof: In the proof of 3.2 let $\mathrm{S} = \{z\}$, where $z \in \mathrm{Z}$.}

3.6. The center of a regular ring is regular.

{Proof: Let A be regular with center Z and let $z \in \mathrm{Z}$. Write $z\mathrm{A} = v\mathrm{A}$, $\mathrm{A}z = \mathrm{A}w$ with v and w idempotents. Then $z\mathrm{A} = \mathrm{A}z$ yields $v = w \in \mathrm{Z}$ as in the proof of 3.2. Thus $z\mathrm{A} = v\mathrm{A}$ with v a central idempotent. Write $v = zz'$, $z' \in \mathrm{A}$; replacing z' by vz', one can suppose $z' \in v\mathrm{A}$. Since $zz'z = vz = z$, it will suffice to show that $z' \in \mathrm{Z}$. Indeed, for all $x \in \mathrm{A}$ one has

$$(xz' - z'x)z = xz'z - z'zx = xv - vx = 0\,,$$

whence $0 = (xz' - z'x)zz' = (xz' - z'x)v = xz' - z'x$.}

3.7. The center of a regular Baer ring is a regular Baer ring. {Proof: 3.2 and 3.6.}

3.8. In a Rickart *-ring, every central idempotent is a projection.

{Proof: Let A be a Rickart *-ring with center Z, $u \in \mathrm{Z}$ idempotent; we are to show that $u^* = u$. As noted in the proof of 1.18, $u\mathrm{A} = f\mathrm{A}$ with f a projection, whence $u = fu = uf = f$.}

3.9. PROPOSITION. (S. Maeda [23, Lemma 2.1]) *Let* A *be a Rickart *-ring,* Z *its center,* $e \in \mathrm{A}$ *a projection. The following conditions are equivalent:*

(a) $e \in \mathrm{Z}$;

(b) $ef = fe$ *for all projections* f *of* A;

(c) e *has a unique complement in the projection lattice of* A (*namely* $1-e$).

Proof. (a) ⇒ (b): Trivial.

(b) ⇒ (c): Let f be a complement of e in the projection lattice of A, that is, $e \cup f = 1$ and $e \cap f = 0$; we are to show that $f = 1 - e$. By hypothesis, $ef = fe$, therefore $e \cap f = ef$ and $e \cup f = e + f - ef$; thus $ef = 0$ and $1 = e + f - 0$, so $f = 1 - e$.

(c) ⇒ (a): Let $x \in \mathrm{A}$; we are to show that $xe = ex$. Set $a = e + ex - exe$. Then $ea = a$ and $ae = e$; it follows that $e\mathrm{A} = a\mathrm{A}$, whence $e = \mathrm{LP}(a)$. And $a^2 = a(ea) = (ae)a = ea = a$, so a is idempotent. Write $f = \mathrm{LP}(1-a)$; as noted in the proof of 1.18, $(1-a)\mathrm{A} = f\mathrm{A}$. From $a(1-a) = 0$ we infer that $af = 0$. By the formulas of 1.18,

$$(e \cap f)\mathrm{A} = e\mathrm{A} \cap f\mathrm{A} = a\mathrm{A} \cap (1-a)\mathrm{A} = 0\,,$$
$$(e \cup f)\mathrm{A} = (e\mathrm{A} + f\mathrm{A})^{lr} = [a\mathrm{A} + (1-a)\mathrm{A}]^{lr} = \mathrm{A}^{lr} = \mathrm{A}\,,$$

thus $e \cap f = 0$ and $e \cup f = 1$, that is, f is a complement of e. By the hypothesis (c), $f = 1 - e$, so

$$0 = af = a(1-e) = a - ae = a - e\,,$$

thus $a = e$. That is, $e + ex - exe = e$, whence $ex = exe$; similarly $ex^* = ex^*e$, whence $xe = exe = ex$. ♢

This result is of capital importance from §13 onward (cf. the proof of 13.8).

3.10. PROPOSITION. [18, p. 30, Cor.] *The center of a Baer *-ring is a Baer *-ring, with unambiguous sups and infs.*

Proof. In the proof of 3.2 one can take v to be a projection, thus Z is a Baer *-ring.

Now suppose $(u_i)_{i \in \mathrm{I}}$ is a family of projections in Z and let $u = \sup u_i$ as calculated in A (1.24). Writing $\mathrm{S} = \{u_i :\ i \in \mathrm{I}\}$, we know that $\mathrm{S}^r = (1-u)\mathrm{A}$

(by 1.24). But, as noted in the proof of 3.2, we have $1-u \in \mathrm{Z}$, that is, $u \in \mathrm{Z}$, and $\mathrm{Z} \cap \mathrm{S}^r = (1-u)\mathrm{Z}$; therefore u is the supremum of the u_i as calculated in Z (1.24 applied to Z). Thus sups in Z are unambiguous; by duality, so are infs. $\diamondsuit$

3.11. The center of a *-regular ring is *-regular.

{Proof: If A is a *-regular ring with center Z, then Z is a *-ring with proper involution and is regular (3.6), so it is *-regular (1.14).}

3.12. The center of a regular Baer *-ring is a regular Baer *-ring. {Proof: 3.6 and 3.10.}

3.13. If A is a right continuous regular ring, then its center Z is regular and continuous (both right and left).

{Proof: A is a regular Baer ring (1.41) so Z is a regular Baer ring (3.7), and the idempotents of Z form a complete Boolean algebra B (3.3). Since Z is regular, its lattice of principal ideals is isomorphic to B, hence is continuous; thus Z is continuous [cf. 7, pp. 160-161].}

3.14. If A is a regular, right self-injective ring, then its center Z is regular and self-injective (right and left).

{Proof [1, p. 418]: By 1.29 and 1.32, A is a regular Baer ring. The strategy of the proof is to verify Baer's criterion for Z. Suppose I is an ideal of Z and $f : \mathrm{I} \to \mathrm{Z}$ is Z-linear; we seek to extend f to a Z-linear map $\mathrm{Z} \to \mathrm{Z}$ (equivalently, we seek $z \in \mathrm{Z}$ such that $f(y) = zy$ for all $y \in \mathrm{I}$). One can suppose that I is essential in Z; for, there exists an ideal K of Z with $\mathrm{I} \cap \mathrm{K} = 0$ and $\mathrm{I} + \mathrm{K}$ essential in Z [19, p. 60, Lemma 1], and one can extend f to $\mathrm{I} \oplus \mathrm{K}$, for example by annihilating K.

For use later in the proof, we now show that the right annihilator I^r of I (in A) is 0. Write $\mathrm{I}^r = u\mathrm{A}$, $u \in \mathrm{A}$ idempotent. Since $\mathrm{I} \subset \mathrm{Z}$ one knows that $u \in \mathrm{Z}$ (proof of 3.2) and so $\mathrm{Z} \cap \mathrm{I}^r = u\mathrm{Z}$. Writing $^\circ$ for annihilator in Z, we have

$$\mathrm{I}^\circ = \mathrm{Z} \cap \mathrm{I}^r = u\mathrm{Z},$$

hence $\mathrm{I}^{\circ\circ} = (1-u)\mathrm{Z}$. Then $\mathrm{I} \cap \mathrm{I}^\circ \subset \mathrm{I}^{\circ\circ} \cap \mathrm{I}^\circ = 0$; since I is essential in Z, we conclude that $\mathrm{I}^\circ = 0$, thus $u = 0$, whence $\mathrm{I}^r = 0$.

Let $\mathrm{J} = \mathrm{AI} = \mathrm{IA}$ be the ideal of A generated by I. We propose to define a right A-linear extension $f^* : \mathrm{J} \to \mathrm{A}$ of f by the formula

$$f^*\left(\sum_{i=1}^n y_i a_i\right) = \sum_{i=1}^n f(y_i) a_i \qquad (y_i \in \mathrm{I},\ a_i \in \mathrm{A}).$$

To see that this is well-defined, suppose $\sum_{i=1}^n y_i a_i = 0$. Since Z is regular (3.6), one can write

$$y_1\mathrm{Z} + \ldots + y_n\mathrm{Z} = v\mathrm{Z}$$

with $v \in \mathrm{Z}$ idempotent [7, p. 1, Th. 1.1]. Then $vy_i = y_i$ for all i, so

$$\sum f(y_i)a_i = \sum f(vy_i)a_i = \sum f(v)y_i a_i = f(v)\sum y_i a_i = 0,$$

thus f^* is well-defined (and clearly right A-linear). Since A is right self-injective, there exists $t \in \mathrm{A}$ with

$$f^*(x) = tx \quad \text{for all} \quad x \in \mathrm{J}\,.$$

In particular,

$$(\forall\, y \in \mathrm{I}) \quad f(y) = f^*(y) = ty\,,$$

so it will suffice to show that $t \in \mathrm{Z}$. If $a \in \mathrm{A}$ we wish to show that $ta = at$; since $\mathrm{I}^r = 0$ it will suffice to show that $\mathrm{I}(ta - at) = 0$. Indeed, for all $y \in \mathrm{I}$ one has $yt = ty = f(y) \in \mathrm{Z}$, so $y(ta - at) = yta - yat = (yt)a - a(yt) = 0$.}

3.15. DEFINITION. Let A be a Baer ring, $x \in \mathrm{A}$. The **central cover** of x is the central idempotent $\mathrm{C}(x)$ defined by the formula

$$\mathrm{C}(x) = \inf\{u :\ ux = x\,,\quad u \text{ a central idempotent of } \mathrm{A}\}$$

(cf. 3.3). An element $x \in \mathrm{A}$ is said to be **faithful** if $\mathrm{C}(x) = 1$ (in other words, $u = 0$ is the only central idempotent such that $ux = 0$).[1]

3.16. $\mathrm{C}(x)$ is the smallest central idempotent u such that $ux = x$.

{Proof: Let S be the set of all central idempotents u such that $ux = x$; it clearly suffices to show that $\mathrm{C}(x) \in \mathrm{S}$. For all $u \in \mathrm{S}$ one has $x(1-u) = 0$, so by 3.4 we have

$$\begin{aligned} 0 &= x \sup\{1 - u :\ u \in \mathrm{S}\} \\ &= x[1 - \inf\{u :\ u \in \mathrm{S}\}] = x[1 - \mathrm{C}(x)]\,, \end{aligned}$$

whence $x\mathrm{C}(x) = x$ as desired.}

3.17. PROPOSITION. *Let* A *be a Baer ring*, $x \in \mathrm{A}$, u *a central idempotent of* A. *Then*:

(i) $ux = 0 \Leftrightarrow u\mathrm{C}(x) = 0$.

(ii) $\mathrm{C}(ux) = u\mathrm{C}(x)$.

Proof. (i) $ux = 0 \Leftrightarrow (1-u)x = x \Leftrightarrow \mathrm{C}(x) \le 1 - u \Leftrightarrow u\mathrm{C}(x) = 0$.

(ii) Let $v = \mathrm{C}(ux)$. From $u(ux) = ux$ and $\mathrm{C}(x) \cdot ux = u\mathrm{C}(x)x = ux$ one has $v \le u$ and $v \le \mathrm{C}(x)$. But $ux = v(ux) = (vu)x = vx$, $(u - v)x = 0$, so $(u - v)\mathrm{C}(x) = 0$ by (i), thus $u\mathrm{C}(x) = v\mathrm{C}(x) = v$. ♢

3.18. DEFINITION. [19, pp. 54-56] A ring A is said to be **semiprime** if $x\mathrm{A}x = 0 \Rightarrow x = 0$. An equivalent condition is that 0 is the only nilpotent ideal of A (it does not matter if one means left, right or bilateral ideal). It is the same to suppose that $\mathrm{I} = 0$ is the only ideal with $\mathrm{I}^2 = 0$ (left, right or bilateral–it does not matter). A ring A is **prime** if $x\mathrm{A}y = 0 \Rightarrow x = 0$ or $y = 0$ (an equivalent condition is that there are no ideal divisors of 0).

[1] Note that if x is any element of A and if $u = \mathrm{C}(x)$, then x is faithful in $u\mathrm{A}$. For, the central idempotents of $u\mathrm{A}$ are the central idempotents of A that are $\le u$. A central idempotent v of $u\mathrm{A}$ such that $vx = x$ must therefore be $\ge \mathrm{C}(x) = u$, hence equal to u.

3.19. Every regular ring is semiprime.

{Proof: Every nonzero left or right ideal contains a nonzero idempotent.}

3.20. Every Baer *-ring is semiprime [18, p. 31].

{Proof: Suppose $x\mathrm{A}x = 0$, that is, $x \in (x\mathrm{A})^r$. Since $\mathrm{I} = x\mathrm{A}$ is a right ideal, I^r is a bilateral ideal. Write $\mathrm{I}^r = u\mathrm{A}$, u a projection. Since I^r is bilateral, $\mathrm{A}(u\mathrm{A}) \subset u\mathrm{A}$, so $(1-u)\mathrm{A}u\mathrm{A} = 0$, $(1-u)\mathrm{A}u = 0$, whence (on taking adjoints) $u\mathrm{A}(1-u) = 0$; therefore $au = uau = ua$ for all $a \in \mathrm{A}$, so u is central. By hypothesis, $x \in \mathrm{I}^r$, so $ux = x$; but $\mathrm{I}u = 0$, in particular $xu = 0$, whence $x = ux = xu = 0$.}

3.21. PROPOSITION. [18, p. 15, Th. 13] *Let* A *be a semiprime Baer ring.*

(i) *If* I *is a right ideal of* A, *then* I^r *is a direct summand of* A.

(ii) *For* x, y *in* A, $x\mathrm{A}y = 0 \Leftrightarrow \mathrm{C}(x)\mathrm{C}(y) = 0$.

(iii) *If* $e \in \mathrm{A}$ *is idempotent, then the central idempotents of the ring* $e\mathrm{A}e$ *are the* eu *with* u *a central idempotent of* A (*thus* $e\mathrm{Z}$ *and the center of* $e\mathrm{A}e$ *contain the same idempotents*); *more precisely, if* f *is a central idempotent of* $e\mathrm{A}e$, *then* $f = e\mathrm{C}(f)$.

(iv) *If* $e \in \mathrm{A}$ *is an idempotent and* $x \in e\mathrm{A}e$, *then* $\mathrm{C}_e(x) = e\mathrm{C}(x)$, *where* C_e *denotes central cover relative to the Baer ring* $e\mathrm{A}e$.

Proof. (i) Write $\mathrm{I}^r = u\mathrm{A}$, u idempotent. As argued in the proof of 3.20, one has $(1-u)\mathrm{A}u = 0$. Setting $\mathrm{J} = u\mathrm{A}(1-u) = \mathrm{I}^r(1-u)$, J is a left ideal of A (because $u\mathrm{A} = \mathrm{I}^r$ is) such that $\mathrm{J}^2 = u\mathrm{A}(1-u)\cdot u\mathrm{A}(1-u) = 0$; since A is semiprime, $\mathrm{J} = 0$, thus $u\mathrm{A}(1-u) = 0$. As argued in 3.20, u is central.

(ii) Suppose $x\mathrm{A}y = 0$. Let $\mathrm{I} = x\mathrm{A}$; by (i), $\mathrm{I}^r = u\mathrm{A}$ with u a central idempotent. By hypothesis $y \in \mathrm{I}^r$, thus $uy = y$, so $\mathrm{C}(y) \le u$. But $\mathrm{I}u = 0$, so $xu = 0$, hence $\mathrm{C}(x) \le 1-u$; therefore $\mathrm{C}(x)\mathrm{C}(y) = 0$. The reverse implication results from the formula $x\mathrm{A}y = x\mathrm{C}(x)\mathrm{A}y\mathrm{C}(y) = x\mathrm{A}y\mathrm{C}(x)\mathrm{C}(y)$.

(iii) Let f be a central idempotent of $e\mathrm{A}e$. Then $(e-f)\mathrm{A}f = (e-f)e\mathrm{A}(ef) = (e-f)(e\mathrm{A}e)f = (e-f)f(e\mathrm{A}e) = 0$, so $\mathrm{C}(e-f)\mathrm{C}(f) = 0$ by (ii); then $(e-f)\mathrm{C}(f) = 0$, so $e\mathrm{C}(f) = f\mathrm{C}(f) = f$.

(iv) Let $x \in e\mathrm{A}e$. Since $x \cdot e\mathrm{C}(x) = (xe)\mathrm{C}(x) = x\mathrm{C}(x) = x$, one has $\mathrm{C}_e(x) \le e\mathrm{C}(x)$. By (iii) one can write $\mathrm{C}_e(x) = eu$ with u a central idempotent of A; then $x = x\mathrm{C}_e(x) = xeu = xu$, so $\mathrm{C}(x) \le u$, whence $e\mathrm{C}(x) \le eu = \mathrm{C}_e(x)$. $\diamondsuit$

3.22. If A is a Baer *-ring and (e_i) is any family of projections in A, then $\mathrm{C}(\sup e_i) = \sup \mathrm{C}(e_i)$.

{Proof: Write $e = \sup e_i$, $u_i = \mathrm{C}(e_i)$; we are to show that $\mathrm{C}(e) = \sup u_i$. One has $\mathrm{C}(e) \ge e \ge e_i$, so $\mathrm{C}(e) \ge \mathrm{C}(e_i) = u_i$ for all i. If v is a central projection with $u_i \le v$ for all i, then $e_i \le u_i \le v$ for all i, whence $e \le v$, so $\mathrm{C}(e) \le v$.}

3.23. If, in a *-ring, (e_i) is a family of projections possessing a supremum e (cf. 1.18, 1.24), then for every central projection u the family (ue_i) has ue as supremum; briefly, $u(\sup e_i) = \sup(ue_i)$. Similarly, if (e_i) has an infimum f, then $uf = \inf(ue_i)$ for every central projection u.

{Proof: Suppose $e = \sup e_i$ and u is a central projection. From $e_i = e_i e$ one infers $ue_i = (ue_i)(ue)$, thus $ue_i \le ue$ for all i. Suppose g is a projection with $ue_i \le g$ for all i. Then $0 = ue_i(1-g) = e_i u(1-g)$, so $e_i \le 1 - u(1-g)$ for all i; therefore $e \le 1 - u(1-g)$, whence $eu(1-g) = 0$, thus $ue \le g$. This proves that ue serves as supremum for the family (ue_i). The second assertion follows from the first by virtue of the order anti-isomorphism $g \mapsto 1-g$.}

***3.24.** (U. Sasaki) If A is an AW*-algebra (1.38) with center Z, and if $e \in \mathrm{A}$ is a projection, then the center of $e\mathrm{A}e$ is $e\mathrm{Z}$.

{Proof [2, p. 37, Cor. 2 of Prop. 4]: From 2.6 one sees that $e\mathrm{A}e$ is an AW*-algebra; let Z_e be its center (also an AW*-algebra, by 3.10). By 3.21, (iii), Z_e and $e\mathrm{Z}$ contain the same projections; since each is the closed linear span of its projections (1.40), $\mathrm{Z}_e = e\mathrm{Z}$.}

See also 3.34.

3.25. If A is a ring with center Z, and if u is a central idempotent of A, then $u\mathrm{A}$ has center $u\mathrm{Z}$. {Proof: $\mathrm{A} = u\mathrm{A} \times (1-u)\mathrm{A}$.}

3.26. (L. Jérémy [15, Lemma 0.2]) If A is a regular, right self-injective ring with center Z, and if $e \in \mathrm{A}$ is idempotent, then the center of $e\mathrm{A}e$ is $e\mathrm{Z}$.

{Proof [4]: Let $u = \mathrm{C}(e)$, the central cover of e. Then $e\mathrm{A}e = (eu)\mathrm{A}e = e(u\mathrm{A})e$ and $e\mathrm{Z} = e(u\mathrm{Z})$, where $u\mathrm{Z}$ is the center of $u\mathrm{A}$ (3.25); dropping down to $u\mathrm{A}$, we can suppose that $\mathrm{C}(e) = 1$. Then $(\mathrm{A}e)^l = 0$ by 3.19 and 3.21, (ii), so there exists an isomorphism of rings $\varphi : \mathrm{A} \to \mathrm{End}_{e\mathrm{A}e}(\mathrm{A}e)$, where $\mathrm{A}e$ is regarded as a right $e\mathrm{A}e$-module in the natural way and, for $a \in \mathrm{A}$, $\varphi(a)$ is left-multiplication by a [7, p. 98, Prop. 9.8].

Let Z_e be the center of $e\mathrm{A}e$. It is obvious that $e\mathrm{Z} \subset \mathrm{Z}_e$. Conversely, let $t \in \mathrm{Z}_e$. Define $\alpha : \mathrm{A}e \to \mathrm{A}e$ by $\alpha(xe) = (xe)t$; since t is central in $e\mathrm{A}e$, α is right $e\mathrm{A}e$-linear, so $\alpha \in \mathrm{End}_{e\mathrm{A}e}(\mathrm{A}e)$. Therefore $\alpha = \varphi(a)$ for suitable $a \in \mathrm{A}$. Then for all $x \in \mathrm{A}$ one has $\alpha(xe) = (\varphi(a))(xe)$, $xet = axe$, $xt = axe$; for $x = 1$ this yields $t = ae$, so it will suffice to show that $a \in \mathrm{Z}$. Since α is a right-multiplication on $\mathrm{A}e$, it commutes with every left-multiplication on $\mathrm{A}e$, hence $\alpha\varphi(b) = \varphi(b)\alpha$ for all $b \in \mathrm{A}$, that is, α is in the center of $\mathrm{End}_{e\mathrm{A}e}(\mathrm{A}e)$; since $\alpha = \varphi(a)$ and φ is a ring isomorphism, a is in the center of A.}

3.27. If A is a right continuous regular ring with center Z, and if $e \in \mathrm{A}$ is idempotent, then $e\mathrm{A}e$ has center $e\mathrm{Z}$.

{Proof: One has $\mathrm{A} = \mathrm{B} \times \mathrm{C}$ with B 'abelian' (all idempotents central) and C right self-injective [7, p. 169, Th. 13.17]; so we may consider these cases separately. In view of 3.26, we need only consider the case that A is abelian; but then every idempotent $e \in \mathrm{A}$ is central and our assertion follows trivially from 3.25.}

3.28. If A is a regular Baer *-ring with center Z, and if $e \in \mathrm{A}$ is idempotent, then $e\mathrm{A}e$ has center $e\mathrm{Z}$.

{Proof: By a theorem of Kaplansky ([17, Th. 3] or [18, p. 117, Th. 69]; cf. 20.10 below), A satisfies the hypotheses of 3.27.}

3.29. DEFINITION. [4] We say that a ring A, with center Z, is **compressible** if, for every idempotent $e \in A$, the center of eAe is eZ. {Examples: Every regular, right (or left) self-injective ring (3.26); every right (or left) continuous regular ring (3.27); every regular Baer *-ring (3.28).}

3.30. If A is a compressible ring, then every corner of A is compressible.

{Proof: Let Z be the center of A. Let $f \in A$ be idempotent, and write Z_f for the center of fAf; by the hypothesis, $Z_f = fZ$. Assuming $e \in fAf$ idempotent, we are to show that $e{\cdot}fAf{\cdot}e$ has center eZ_f; indeed, $e{\cdot}fAf{\cdot}e = eAe$ has (by the compressibility of A) center $eZ = efZ = eZ_f$.}

3.31. If A is an 'abelian' ring (all idempotents central), then A is compressible. {Proof: Obvious from 3.25.}

3.32. Every ring isomorphic to a compressible ring is compressible. {Obvious.}

3.33. If A is a Rickart *-ring (with center Z) such that, for every projection f, fAf has center fZ, then A is compressible.

{Proof: Let $e \in A$ be idempotent. Since A is a Rickart *-ring, one has $eA = \{1-e\}^r = fA$ for a suitable projection f. From $eA = fA$ and the idempotence of e and f, one infers that e and f are similar (5.5 below), say $f = tet^{-1}$. Let $\varphi : A \to A$ be the inner automorphism of A induced by t: $\varphi(a) = t^{-1}at$. By hypothesis, fAf has center fZ, hence $\varphi(fAf)$ has center $\varphi(fZ)$; but $\varphi(fAf) = \varphi(f)\varphi(A)\varphi(f) = eAe$ and $\varphi(fZ) = \varphi(f)\varphi(Z) = eZ$, thus eAe has center eZ as desired.}

***3.34.** Every AW*-algebra is compressible. {Proof: 3.24 and 3.33.}

3.35. EXAMPLE. (E. P. Armendariz) Not every Baer ring is compressible.

{Proof [cf. 4, Example 10]: Let $\mathbb{H}$ be the division ring of real quaternions, and let

$$B = \begin{pmatrix} \mathbb{H} & \mathbb{H} \\ 0 & \mathbb{H} \end{pmatrix}$$

be the ring of all upper triangular 2×2 matrices

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

with $x, y, z \in \mathbb{H}$. Since $\mathbb{H}$ is a division ring, B is a Baer ring [18, p. 16, Exer. 2]. One readily calculates that the idempotents of B are $0, 1$ and the matrices

$$\begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$$

with $y \in \mathbb{H}$. Regard $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ in the usual way. Let A be the subring of B defined by

$$A = \begin{pmatrix} \mathbb{C} & \mathbb{H} \\ 0 & \mathbb{H} \end{pmatrix},$$

that is, A is the set of all matrices of the form

$$\begin{pmatrix} c & x \\ 0 & y \end{pmatrix}$$

where $c \in \mathbb{C}$ and $x, y \in \mathbb{H}$. Since A contains every idempotent of the Baer ring B, A is itself a Baer ring; for, if $\mathrm{S} \subset \mathrm{A}$ and S^r is the right annihilator of S in B, say $\mathrm{S}^r = e\mathrm{B}$ with e idempotent, then $e \in \mathrm{A}$ and so $\mathrm{A} \cap \mathrm{S}^r = e\mathrm{A}$. The center Z of A is the set of all matrices

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

with $r \in \mathbb{R}$, and in particular Z is one-dimensional over $\mathbb{R}$. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{A}\,;$$

then

$$e\mathrm{A}e = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix}$$

is commutative (hence is its own center) and is 2-dimensional over $\mathbb{R}$, whereas

$$e\mathrm{Z} = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{pmatrix}$$

is 1-dimensional over $\mathbb{R}$. The evident relation $e\mathrm{A}e \neq e\mathrm{Z}$ shows that A is not compressible. (In fact, $e\mathrm{A}e$ is not even isomorphic to $e\mathrm{Z}$.)}

3.36. If A is a ring such that, for some positive integer n, $\mathrm{M}_n(\mathrm{A})$ is compressible, then A is compressible.

{Proof: Since A is isomorphic to a corner of $\mathrm{M}_n(\mathrm{A})$, this is immediate from 3.30 and 3.32.}

3.37. PROBLEM. If A is compressible, is $\mathrm{M}_2(\mathrm{A})$ compressible?[2]

3.38. PROBLEM. Is every regular Baer ring compressible?[3]

3.39. PROBLEM. Is every Baer *-ring compressible?

For several other fragmentary results on compressibility, see [4].

[2] Answered in the negative by G. M. Bergman [*Comm. Algebra* 12 (1984), 1-8].

[3] My guess is that the answer is no: D. Castella [*Comm. Algebra* 15 (1987), 1621-1635] has proved that if A is a regular Baer ring without abelian summand (cf. 8.27), then A is compressible if and only if the center of A coincides with the center of its maximal ring of right quotients. So the task is to construct a regular Baer ring without abelian summand, whose center fails to satisfy the indicated condition. (For complements to Castella's paper, see E. P. Armendariz and S. K. Berberian [*Comm. Algebra* 17 (1989), 1739-1758].)

4. COMMUTANTS

4.1. If S is a subset of a ring A, the **commutant** of S in A is the set

$$\mathrm{S}' = \{x \in \mathrm{A} :\ xs = sx \ \text{ for all } \ s \in \mathrm{S}\},$$

which is a subring of A. One also writes $\mathrm{S}'' = (\mathrm{S}')'$ (the **bicommutant** of S) and $\mathrm{S}''' = (\mathrm{S}'')' = (\mathrm{S}')''$. One has (i) $\mathrm{S} \subset \mathrm{S}''$, and (ii) $\mathrm{S} \subset \mathrm{T} \Rightarrow \mathrm{S}' \supset \mathrm{T}'$. It follows that $\mathrm{S}' = \mathrm{S}'''$. {For, $\mathrm{S} \subset \mathrm{S}''$ yields $\mathrm{S}' \supset \mathrm{S}'''$, whereas $\mathrm{S}' \subset \mathrm{S}'''$ by (i).} For a subring B of A, one has $\mathrm{B} = \mathrm{S}'$ for some subset S of A if and only $\mathrm{B} = \mathrm{B}''$.

If A is a $*$-ring and S is a $*$-subset of A ($s \in \mathrm{S} \Rightarrow s^* \in \mathrm{S}$), then S' is a $*$-subring of A; the $*$-subrings B of A satisfying $\mathrm{B} = \mathrm{B}''$ are the subrings S' with S a $*$-subset of A.

If S is a commutative subset of a ring A (that is, $\mathrm{S} \subset \mathrm{S}'$), then $\mathrm{B} = \mathrm{S}''$ is a commutative subring of A such that $\mathrm{S} \subset \mathrm{B}$ and $\mathrm{B} = \mathrm{B}''$.

{Commutativity of B: From $\mathrm{S} \subset \mathrm{S}'$ one infers that $\mathrm{S}' \supset \mathrm{S}'' = \mathrm{B}$, thus $\mathrm{B} \subset \mathrm{S}' = \mathrm{S}''' = \mathrm{B}'$.}

4.2. If S is a subset of a ring A and if $\mathrm{B} = \mathrm{S}'$, then B contains inverses ($b \in \mathrm{B}$ invertible $\Rightarrow$ $b^{-1} \in \mathrm{B}$).

{Proof: Let $b \in \mathrm{B}$ be invertible. For all $s \in \mathrm{S}$ one has $sb = bs$, $b^{-1}(sb)b^{-1} = b^{-1}(bs)b^{-1}$, $b^{-1}s = sb^{-1}$.}

4.3. Let A be a Rickart ring, S a subset of A, $x \in \mathrm{S}'$. Write $\{x\}^r = e\mathrm{A}$, e idempotent. Then for all $s \in \mathrm{S}$, (i) $se = ese$, and (ii) $(1-e)s = (1-e)s(1-e)$.

{Proof: For all $s \in \mathrm{S}$ one has $xse = sxe = s \cdot 0 = 0$, thus $se \in \{x\}^r = e\mathrm{A}$, whence $e(se) = se$. Condition (ii) is equivalent to condition (i).}

4.4. PROPOSITION. *Let* A *be a Rickart *-ring*, S *a* $*$*-subset of* A. *If* $x \in \mathrm{S}'$ *then* $\mathrm{LP}(x) \in \mathrm{S}'$ *and* $\mathrm{RP}(x) \in \mathrm{S}'$, *thus* S' *is a Rickart* $*$*-ring with unambiguous* LP'*s and* RP'*s.*

Proof. Let $f = \mathrm{RP}(x)$; thus $\{x\}^r = (1-f)\mathrm{A}$. By (ii) of 4.3, $fs = fsf$ for all $s \in \mathrm{S}$; also $fs^* = fs^*f$ for all $s \in \mathrm{S}$ (because $\mathrm{S}^* \subset \mathrm{S}$), whence $sf = fsf$. Thus $fs = fsf = sf$ for all $s \in \mathrm{S}$, so $f \in \mathrm{S}'$. Writing $\mathrm{B} = \mathrm{S}'$, we thus have $\mathrm{B} \cap \{x\}^r = (1-f)\mathrm{B}$, whence the proposition. ♢

4.5. PROPOSITION. [18, p. 30, Th. 20] *If* A *is a Baer* $*$*-ring and* S *is a* $*$*-subset of* A, *then* S' *is a Baer* $*$*-ring with unambiguous sups*, *infs*, LP'*s and* RP'*s* .

Proof. Write $\mathrm{B} = \mathrm{S}'$. By 4.4, $x \in \mathrm{B} \Rightarrow \mathrm{LP}(x), \mathrm{RP}(x) \in \mathrm{B}$. Let (e_i) be a family of projections in B, $e = \sup e_i$ in A; we assert that $e \in \mathrm{B}$. If $s \in \mathrm{S}$ then for all i one has $(es - se)e_i = ese_i - se_i = ee_is - se_i = e_is - se_i = 0$, whence $(es - se)e = 0$ (cf. 1.24), thus $se = ese$; therefore also $s^*e = es^*e$ (because $\mathrm{S}^* \subset \mathrm{S}$), whence $es = ese = se$. Thus $e \in \mathrm{S}' = \mathrm{B}$. By 4.4, B is a Rickart $*$-ring, and its projection lattice is complete by the preceding, therefore B is a Baer $*$-ring (1.24). ◊

4.6. There exists a regular Baer $*$-ring A with a subset S such that S' is not a Rickart ring (hence is not regular). (For the good news, see 4.7.)

{Proof: Let F be a field, $\mathrm{A} = \mathrm{M}_2(\mathrm{F})$ the ring of 2×2 matrices over F, and let $\mathrm{S} = \{s\}$, where s is the matrix

$$s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let us calculate S':

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix},$$

which says that $c = 0$ and $d = a$. Thus the ring $\mathrm{B} = \mathrm{S}'$ consists of all matrices

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

with $a, b \in \mathrm{F}$. The only idempotents of B are $0, 1$, and B contains nilpotent elements, namely the matrices

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix};$$

it follows that B is not a Rickart ring (1.42). However, A is a regular Baer ring (1.26), indeed, a regular self-injective ring (right and left) [7, p. 11, Cor. 1.23]. (So far, F need not be commutative–it can be any division ring.)

If, moreover, F has no element a such that $a^2+1 = 0$ (equivalently, $a^2+b^2 = 0 \Rightarrow a = b = 0$) then A is $*$-regular with transpose as involution (1.36), hence is a regular Baer $*$-ring (1.25).}

4.7. PROPOSITION. *If* A *is a $*$-regular ring and* S *is a $*$-subset of* A, *then* S' *is a $*$-regular ring. More precisely, if* $x \in \mathrm{S}'$ *and* y *is the relative inverse of* x (2.7), *then* $y \in \mathrm{S}'$.

Proof. Write $\mathrm{B} = \mathrm{S}'$; since A is a Rickart $*$-ring (1.14) so is B (4.4). Let $x \in \mathrm{B}$ and write $e = \mathrm{LP}(x)$, $f = \mathrm{RP}(x)$; by 4.4, $e, f \in \mathrm{B}$. Let y be the relative inverse of x in A, thus $y \in f\mathrm{A}e$, $xy = e$, $yx = f$. Given $s \in \mathrm{S}$, we are to show that $ys = sy$. One has

$$(ys - sy)x = ysx - syx = yxs - syx = fs - sf = 0$$

because $f \in \mathrm{B}$, therefore

$$0 = (ys - sy)e = yse - sye = yes - sye = ys - sy$$

(recall that $e \in \mathrm{B}$, so $se = es$). ♢

For an application, see 14.26.

4.8. COROLLARY. *If* A *is a regular Baer* $*$*-ring and* S *is a* $*$*-subset of* A, *then* S' *is a regular Baer* $*$*-ring* (*with unambiguous sups*, *infs*, LP'*s*, RP'*s and relative inverses*).

Proof. 4.5 and 4.7. ♢

4.9. The ring $\mathrm{A} = \mathrm{M}_2(\mathrm{F})$ of 4.6 is self-injective; thus 4.6 runs somewhat counter to 3.14, which says that $\mathrm{A}' = \mathrm{Z}$ ($\cong \mathrm{F}$) is regular and self-injective.

***4.10.** If A is an AW*-algebra (1.38) and S is a $*$-subset of A, then S' is an AW*-algebra (with unambiguous sups, infs, LP's and RP's).

{Proof: $\mathrm{B} = \mathrm{S}'$ is clearly closed in A for the norm topology, thus is a C*-algebra; quote 4.5. (It is essential that S be a $*$-subset: let $\mathrm{F} = \mathbb{C}$ in 4.6, with conjugate-transpose for the involution of $\mathrm{A} = \mathrm{M}_2(\mathbb{C})$.}

***4.11.** DEFINITION. [6, p. 2, Def. 1] If H is a Hilbert space and $\mathrm{L(H)}$ is the $*$-algebra of all bounded operators on H (1.27), then a $*$-subalgebra A of $\mathrm{L(H)}$ such that $\mathrm{A} = \mathrm{A}''$ is called a **von Neumann algebra** on H; thus (4.1) the von Neumann algebras on H are the algebras S', where S is a $*$-subset of $\mathrm{L(H)}$.

***4.12.** Every von Neumann algebra is an AW*-algebra (1.38 and 4.10). But not conversely:

***4.13.** Let A be a commutative C*-algebra with unity and write $\mathrm{A} = \mathcal{C}(\mathrm{T})$, T compact. {Thus A is an AW*-algebra if and only if T is Stonian (1.39).} In order that A be ($*$-isomorphic to) a von Neumann algebra on a suitable Hilbert space, it is necessary and sufficient that T be *hyperstonian* (= Stonian with sufficiently many 'normal' measures [5]).

***4.14.** If A is a von Neumann algebra on a Hilbert space H, and if $e \in \mathrm{A}$ is a projection, then $e\mathrm{A}e$ may be identified (via restriction) with a von Neumann algebra on the Hilbert space $e(\mathrm{H})$ [6, p. 16, Prop. 1].

***4.15.** If A is a von Neumann algebra on a Hilbert space H, and if S is a $*$-subset of A, then the commutant B of S in A is a von Neumann algebra on H. {Proof: Writing $'$ for commutant in $\mathrm{L(H)}$, one has

$$\mathrm{B} = \mathrm{A} \cap \mathrm{S}' = (\mathrm{A}')' \cap \mathrm{S}' = (\mathrm{A}' \cup \mathrm{S})',$$

where $\mathrm{A}' \cup \mathrm{S}$ is a $*$-subset of $\mathrm{L(H)}$.} In particular, the center $\mathrm{Z} = \mathrm{A} \cap \mathrm{A}'$ of A is a von Neumann algebra on H. (Note, incidentally, that $\mathrm{A} \cap \mathrm{A}'$ is the center of both A and A'.)

****4.16.** If A is a von Neumann algebra on the Hilbert space H, and A′ is the commutant of A in L(H), then A′ is a von Neumann algebra on H. {Proof: Immediate from 4.1: $(\mathrm{A}')'' = \mathrm{A}''' = \mathrm{A}'$.}

***4.17.** If A is a von Neumann algebra on a Hilbert space H, and if n is a positive integer, then $\mathrm{M}_n(\mathrm{A})$ may be identified with a von Neumann algebra on the Hilbert space sum $n\mathrm{H}$ of n copies of H [6, p. 23, Lemma 2].

5. EQUIVALENCE OF IDEMPOTENTS

5.1. If A is a ring, viewed as a right A-module A_A, then the direct summands of A_A are the principal right ideals eA with e idempotent.

5.2. PROPOSITION. *For idempotents* e, f *of a ring* A, *the following conditions are equivalent*:

(a) $eA \cong fA$ *as right* A*-modules*;

(b) $Ae \cong Af$ *as left* A*-modules*;

(c) *there exist* x, y *in* A *with* $xy = e$ *and* $yx = f$;

(d) *there exist* $x \in eAf$, $y \in fAe$ *with* $xy = e$ *and* $yx = f$.

Proof. (a) ⇒ (d): Let $\varphi : eA \to fA$ be an isomorphism of right A-modules. Set $y = \varphi(e) \in fA$ and $x = \varphi^{-1}(f) \in eA$; then $ye = \varphi(e)e = \varphi(ee) = \varphi(e) = y$, so $y \in fAe$, and similarly $x \in eAf$. For all $s \in eA$,

$$\varphi(s) = \varphi(es) = \varphi(e)s = ys\,,$$

similarly $\varphi^{-1}(t) = xt$ for all $t \in fA$. Then $xy = \varphi^{-1}(y) = \varphi^{-1}(\varphi(e)) = e$, similarly $yx = f$.

(d) ⇒ (c): Trivial.

(c) ⇒ (d): If $xy = e$, $yx = f$, set $x' = exf \in eAf$ and $y' = fye \in fAe$. Then $x'y' = exfye = ex(yx)ye = e(xy)(xy)e = e^4 = e$, similarly $y'x' = f$.

(d) ⇒ (a): Let x, y be as in (d). For $s \in eA$ one has $ys \in (fAe)s \subset fA$, so a map $\varphi : eA \to fA$ is defined by $\varphi(s) = ys$; φ is right A-linear. Similarly, a right A-linear mapping $\psi : fA \to eA$ is defined by $\psi(t) = xt$. For all $s \in eA$, $\psi(\varphi(s)) = x(ys) = es = s$, thus $\psi \circ \varphi = 1_{eA}$; similarly $\varphi \circ \psi = 1_{fA}$.

Thus (a) ⇔ (c) ⇔ (d). Since (c) holds for A if and only if it holds for the opposite ring A°, we conclude that (c) ⇔ (b). ♢

5.3. DEFINITION. [18, p. 22] Idempotents e, f of a ring A are said to be **equivalent** (or 'algebraically equivalent') in A, written $e \overset{a}{\sim} f$, if they satisfy the conditions of 5.2. (One also refers to $\overset{a}{\sim}$ as 'ordinary equivalence', as contrasted with the '$*$-equivalence', defined in the next section, for projections of a $*$-ring.) From condition (a) of 5.2, it is obvious that $\overset{a}{\sim}$ is an equivalence relation in the set of idempotents of A.

5.4. PROPOSITION. [18, p. 23, Th. 15] *With notations as in 5.2, (d), the mapping* $s \mapsto ysx$ *is an isomorphism of rings* $\theta : eAe \to fAf$, *with inverse mapping* $t \mapsto xty$. *For every idempotent* $g \le e$ *one has* $g \overset{a}{\sim} \theta(g)$.

Proof. For $s \in eAe$ write $\theta(s) = ysx$; since $y \in fA$ and $x \in Af$, one has $\theta(s) \in fAf$. The mapping $\theta : eAe \to fAf$ is additive, $\theta(e) = f$, and for s, s' in eAe one has

$$\theta(ss') = yss'x = yses'x = ysxys'x = \theta(s)\theta(s'),$$

thus θ is a homomorphism of rings. If $\theta(s) = 0$, then $0 = ysx$, $0 = x(ysx)y = ese = s$, thus θ is injective; and if $t \in fAf$ then $t = ftf = yxtyx = \theta(s)$, where $s = xty \in eAe$, so θ is surjective. Finally, if $g \le e$ (cf. 1.8) let $x' = gx$, $y' = yg$; then $x'y' = gxyg = geg = g$ and $y'x' = ygx = \theta(g)$, thus $g \overset{a}{\sim} \theta(g)$. ♢

5.5. PROPOSITION. [18, p. 24] *The equivalence of idempotents in a ring* A *has the following properties*:

(1) $e \overset{a}{\sim} 0 \Rightarrow e = 0$.

(2) $e \overset{a}{\sim} f \Rightarrow ue \overset{a}{\sim} uf$ *for all central idempotents* u.

(3) *If* $e \overset{a}{\sim} f$ *and if* $e_1, \ldots, e_n$ *are pairwise orthogonal idempotents with* $e = e_1 + \ldots + e_n$, *then there exist pairwise orthogonal idempotents* $f_1, \ldots, f_n$ *such that* $f = f_1 + \ldots + f_n$ *and* $e_i \overset{a}{\sim} f_i$ *for all* i.

(4) *If* $e_1, \ldots, e_n$ *are orthogonal idempotents with sum* e, *if* $f_1, \ldots, f_n$ *are orthogonal idempotents with sum* f, *and if* $e_i \overset{a}{\sim} f_i$ *for all* i, *then* $e \overset{a}{\sim} f$.

(5) e, f *are similar if and only if both* $e \overset{a}{\sim} f$ *and* $1 - e \overset{a}{\sim} 1 - f$.

(6) *If* $eA = fA$ *then* e *and* f *are similar*.

Proof. {Recall that idempotents e, f are said to be **orthogonal** if $ef = fe = 0$ (in which case $e + f$ is an idempotent with $e \le e + f$ and $f \le e + f$.}

(1) $eA0 = 0Ae = 0$, so in the notation of 5.2, (d), $x = y = 0$, whence $e = xy = 0$.

(2) is obvious.

(3) With $\theta : eAe \to fAf$ as in 5.4, the idempotents $f_i = \theta(e_i)$ clearly fill the bill.

(4) If $x_i \in e_iAf_i$, $y_i \in f_iAe_i$ with $x_iy_i = e_i$ and $y_ix_i = f_i$, then the elements

$$x = \sum_{i=1}^{n} x_i \in eAf \quad \text{and} \quad y = \sum_{i=1}^{n} y_i \in fAe$$

effect an equivalence $e \overset{a}{\sim} f$.

(5) Suppose $x \in eAf$, $y \in fAe$ with $xy = e$, $yx = f$, and that $x' \in (1-e)A(1-f)$, $y' \in (1-f)A(1-e)$ with $x'y' = 1 - e$, $y'x' = 1 - f$; writing $s = x + x'$, $t = y + y'$, we have $st = ts = 1$ and $f = tes = s^{-1}es$.

(6) If $eA = fA$ then also $A(1-e) = (eA)^l = (fA)^l = A(1-f)$. From 5.2 it is obvious that $e \overset{a}{\sim} f$ and $1 - e \overset{a}{\sim} 1 - f$, so e and f are similar by (5). ♢

5.6. In a Rickart $*$-ring, every idempotent is similar to a projection.

{Proof: If $e \in A$ is idempotent, then $eA = \{1-e\}^r = fA$ for a suitable projection f, so e and f are similar by (6) of 5.5. Incidentally, $A(1-f) = \{f\}^l = \{e\}^l = A[1 - \mathrm{LP}(e)]$ by 1.7, so $f = \mathrm{LP}(e)$. Similarly $Ae = Ag$ with $g = \mathrm{RP}(e)$.}

5.7. If A is a regular ring, $x \in \mathrm{A}$, and e, f are idempotents with $x\mathrm{A} = e\mathrm{A}$ and $\mathrm{A}x = \mathrm{A}f$, then $e \overset{a}{\sim} f$.

{Proof: Let $y \in \mathrm{A}$ with $x = xyx$ and set $e' = xy$, $f' = yx$; e', f' are idempotents such that $e' \overset{a}{\sim} f'$. As in 1.12, $e\mathrm{A} = x\mathrm{A} = e'\mathrm{A}$, thus e, e' are similar (5.5); likewise $\mathrm{A}f = \mathrm{A}x = \mathrm{A}f'$, so f, f' are similar. Thus $e \overset{a}{\sim} e' \overset{a}{\sim} f' \overset{a}{\sim} f$, so $e \overset{a}{\sim} f$.}

5.8. If A is a $*$-regular ring, then $\mathrm{LP}(x) \overset{a}{\sim} \mathrm{RP}(x)$ for all $x \in \mathrm{A}$. {Proof: Immediate from 1.13 and 5.7.}

5.9. In a $*$-regular ring,

$$e \cup f - f \overset{a}{\sim} e - e \cap f$$

for all projections e and f. {Proof: Apply 5.8 to $x = e(1-f)$ and cite 1.15.}

5.10. In a $*$-regular ring A, if e and f are projections such that $e\mathrm{A}f \neq 0$, then there exist nonzero projections $e_0 \leq e$ and $f_0 \leq f$ with $e_0 \overset{a}{\sim} f_0$.

{Proof: Choose $x \in e\mathrm{A}f$ with $x \neq 0$, let $e_0 = \mathrm{LP}(x)$, $f_0 = \mathrm{RP}(x)$, and cite 5.8.}[1]

5.11. DEFINITION. For idempotents e, f in a ring, one writes $e \precsim_a f$ if $e \overset{a}{\sim} f'$ for some idempotent $f' \leq f$ (cf. 1.8). {We then say that e is **dominated** by f.}

5.12. If $e \precsim_a f$ and $f \precsim_a g$ then $e \precsim_a g$.

{Proof: Say $e \overset{a}{\sim} f' \leq f$ and $f \overset{a}{\sim} g' \leq g$. If $\theta : f\mathrm{A}f \to g'\mathrm{A}g'$ is the isomorphism given by 5.4, then

$$e \overset{a}{\sim} f' \overset{a}{\sim} \theta(f') \leq g' \leq g\,,$$

thus $e \overset{a}{\sim} \theta(f') \leq g$.}

5.13. $e \precsim_a 0 \;\Rightarrow\; e = 0$. {Clear from 5.5, (1).}

5.14. $e \precsim_a f \;\Rightarrow\; ue \precsim_a uf$ for all central idempotents u. {Clear from 5.5, (2).}

5.15. If u is a central idempotent and $e \precsim_a u$, then $e \leq u$.

{Proof: By 5.14, $(1-u)e \precsim_a (1-u)u = 0$, so $(1-u)e = 0$ by 5.13.}

5.16. In a Baer ring, $e \precsim_a f \;\Rightarrow\; \mathrm{C}(e) \leq \mathrm{C}(f)$; $e \overset{a}{\sim} f \;\Rightarrow\; \mathrm{C}(e) = \mathrm{C}(f)$.

{Proof: Immediate from 5.15 (here C denotes central cover, defined in 3.15).}

5.17. LEMMA. [18, p. 28, Th. 18] *Let* A *be a Rickart $*$-ring*, e *and* f *idempotents of* A *such that* $f \leq e$ (cf. 1.8). *Then* $e - f \overset{a}{\sim} \mathrm{RP}(e) - \mathrm{RP}(f)$.

---

[1] The analogous remark holds for idempotents in a regular ring, with the roles of $\mathrm{LP}(x)$ and $\mathrm{RP}(x)$ played by idempotent generators of $x\mathrm{A}$ and $\mathrm{A}x$ (cf. 5.7). A little fussing is needed to assure that $e_0 \in e\mathrm{A}e$ and $f_0 \in f\mathrm{A}f$. {Cf. E. P. Armendariz and S. K. Berberian [*Comm. Algebra* 17 (1989), 1739-1758], p. 1752, 7.5.}

Proof. Let $g = \mathrm{RP}(e)$, $h = \mathrm{RP}(f)$. One knows (cf. 5.6) that $\mathrm{A}h = \mathrm{A}f \subset \mathrm{A}e = \mathrm{A}g$; since h and g are self-adjoint, this implies that $h \le g$. Let $x = e(1-h)$, $y = g(1-f)$; it will suffice to show that $xy = e - f$ and $yx = g - h$. Indeed,

$$\begin{aligned} xy &= e(1-h)g(1-f) = (eg - ehg)(1-f) \\ &= (e - eh)(1-f) = e(1-h)(1-f) \\ &= e(1-h-f+hf) = e(1-h-f+h) \\ &= e(1-f) = e - ef = e - f\,, \end{aligned}$$

whereas

$$\begin{aligned} yx &= g(1-f)e(1-h) = (ge - gfe)(1-h) \\ &= (g - gf)(1-h) = g(1-f)(1-h) \\ &= g(1-f-h+fh) = g(1-f-h+f) \\ &= g(1-h) = g - gh = g - h\,. \diamondsuit \end{aligned}$$

5.18. THEOREM. [18, p. 42, Th. 28] *Let* A *be a Baer $*$-ring,* e *and* f *projections in* A *such that* $e \overset{a}{\sim} f$, *and* $(e_i)_{i\in\mathrm{I}}$ *an orthogonal family of projections in* A *with* $e = \sup e_i$. *Then there exists an orthogonal family of projections* $(f_i)_{i\in\mathrm{I}}$ *with* $f = \sup f_i$, *such that* $e_i \overset{a}{\sim} f_i$ *for all* i.

Proof. Say $x \in e\mathrm{A}f$, $y \in f\mathrm{A}e$ with $xy = e$, $yx = f$, and let $\theta : e\mathrm{A}e \to f\mathrm{A}f$ be the ring isomorphism defined by $\theta(s) = ysx$ (5.4). Let us assume that I is infinite (for I finite, the argument is an evident simplification of what follows). We can suppose that I is well-ordered, with $\mathrm{I} = \{\alpha :\ \alpha < \Omega\}$, Ω a limit ordinal (let Ω be the first ordinal whose cardinality is that of I). For every $\alpha \in \Omega$ write

$$g_\alpha = \sup\{e_\beta :\ \beta < \alpha\}\,;$$

evidently (g_α) is an increasing family of projections with supremum e. And, for each $\alpha < \Omega$, one has $g_{\alpha+1} = g_\alpha + e_\alpha$, thus $e_\alpha = g_{\alpha+1} - g_\alpha$. The idempotents $u_\alpha = \theta(g_\alpha)$ of $f\mathrm{A}f$ clearly satisfy $u_\alpha \le u_\beta \le f$ for $\alpha \le \beta$. Let $h_\alpha = \mathrm{RP}(u_\alpha)$, thus $\mathrm{A}u_\alpha = \mathrm{A}h_\alpha$ (cf. 5.6); by 5.17, if $\alpha \le \beta$ then $h_\alpha \le h_\beta$ and $u_\beta - u_\alpha \overset{a}{\sim} h_\beta - h_\alpha$. Set

$$f_\alpha = h_{\alpha+1} - h_\alpha \quad (\alpha < \Omega)\,;$$

clearly (f_α) is an orthogonal family of projections $\le f$, and

$$\begin{aligned} f_\alpha \overset{a}{\sim} u_{\alpha+1} - u_\alpha &= \theta(g_{\alpha+1}) - \theta(g_\alpha) \\ &= \theta(g_{\alpha+1} - g_\alpha) = \theta(e_\alpha) \overset{a}{\sim} e_\alpha \end{aligned}$$

(the last equivalence by 5.4), thus $f_\alpha \overset{a}{\sim} e_\alpha$. It remains only to show that $\sup f_\alpha = f$. It is the same to show that $\sup h_\alpha = f$. Let $h = \sup h_\alpha$; then $h \le f$ and we are to show that $f - h = 0$. Since $f - h$ is an idempotent of $f\mathrm{A}f$, the element $u = \theta^{-1}(f-h)$ is an idempotent of $e\mathrm{A}e$, thus $u \le e$. For all α, we have

$$h_\alpha(f-h) = h_\alpha f - h_\alpha h = h_\alpha - h_\alpha = 0\,;$$

since $h_\alpha = \mathrm{RP}(u_\alpha)$, we thus have

$$0 = u_\alpha(f-h) = \theta(g_\alpha)\theta(u) = \theta(g_\alpha u)\,,$$

so $g_\alpha u = 0$; since $\sup g_\alpha = e$ it follows that $eu = 0$. But $u \in e\mathrm{A}e$, so $u = eu = 0$, that is, $\theta^{-1}(f-h) = 0$, whence $f - h = 0$. $\diamondsuit$

5.19. If e, f are projections in a $*$-regular ring and if e, f are *perspective* (that is, have a common complement), then $e \overset{a}{\sim} f$.

{Proof: By hypothesis, there exists a projection g such that $e \cup g = f \cup g = 1$ and $e \cap g = f \cap g = 0$. Citing 5.9, one has

$$e = e - e \cap g \overset{a}{\sim} e \cup g - g = 1 - g$$

and similarly $f \overset{a}{\sim} 1 - g$, whence $e \overset{a}{\sim} f$.}

5.20. [18, p. 48, Exer. 4] If e, f are orthogonal projections in a Rickart $*$-ring A such that $e \overset{a}{\sim} f$, then e, f are perspective.

{Proof: Dropping down to $(e+f)\mathrm{A}(e+f)$, which is permissible, one can suppose that $e + f = 1$. Let $x \in e\mathrm{A}f$, $y \in f\mathrm{A}e$ with $xy = e$, $yx = f = 1 - e$. Then

$$\begin{aligned}(e+x)(f+y) &= ef + ey + xf + xy \\ &= 0 + 0 + x + e = e + x\,,\end{aligned}$$

thus $\mathrm{A}(e+x) \subset \mathrm{A}(f+y)$; similarly $\mathrm{A}(f+y) \subset \mathrm{A}(e+x)$, so $\mathrm{A}(e+x) = \mathrm{A}(f+y)$. One sees easily that $e+x$ and $f+y$ are idempotents. Therefore (cf. 5.6) writing $g = \mathrm{RP}(e+x)$, we have $\mathrm{A}(e+x) = \mathrm{A}g$; similarly,

$$\mathrm{A} \cdot \mathrm{RP}(f+y) = \mathrm{A}(f+y) = \mathrm{A}(e+x) = \mathrm{A}g\,,$$

so also $\mathrm{RP}(f+y) = g$.

We assert that $e \cup g = 1$. Writing $h = e \cup g$, we have $(f+y)h = f+y$ because $h \geq g = \mathrm{RP}(f+y)$, and $yh = y$ because $h \geq e = \mathrm{RP}(y)$; therefore

$$f = (f+y) - y \in \mathrm{A}h + \mathrm{A}h = \mathrm{A}h\,,$$

so $f \leq h = e \cup g$. Therefore $e \cup f \leq e \cup g$; but $e \cup f = e + f = 1$, so $e \cup g = 1$.

Similarly $f \cup g = 1$.

We assert that $e \cap g = 0$. For, let $t \in \mathrm{A}(e \cap g)$. Since (1.18)

$$\mathrm{A}(e \cap g) = \mathrm{A}e \cap \mathrm{A}g = \mathrm{A}e \cap \mathrm{A}(f+y)\,,$$

one can write $t = re = s(f+y)$ for suitable r, s in A. Then

$$sf = re - sy \in \mathrm{A}e + \mathrm{A}e = \mathrm{A}e\,,$$

thus $sf \in \mathrm{A}f \cap \mathrm{A}e = \mathrm{A}(1-e) \cap \mathrm{A}e = 0$, so $sf = 0$. Then $se = s(1-f) = s - sf = s$, whence $sy = (se)y = s(ey) = s \cdot 0 = 0$. Then $t = sf + sy = 0 + 0$, thus $t = 0$ and the assertion is proved.

Similarly $f \cap g = 0$, thus g is a common complement of e and f.}

6. $*$-EQUIVALENCE OF PROJECTIONS

6.1. DEFINITION. Projections e, f in a $*$-ring A are said to be $*$-**equivalent** in A, written $e \overset{*}{\sim} f$, if there exists $x \in \mathrm{A}$ such that $xx^* = e$ and $x^*x = f$. {Then $e \overset{a}{\sim} f$, and one can suppose $x \in e\mathrm{A}f$ (5.2); cf. 6.2.}

6.2. [18, p. 32, Th. 23] Let A be a $*$-ring with proper involution ($aa^* = 0 \Rightarrow a = 0$) and let $x \in \mathrm{A}$ with $xx^* = e$, e a projection (such an element x is called a **partial isometry**). Then x^*x is a projection f, and $x \in e\mathrm{A}f$.

{Proof: One has

$$\begin{aligned}(x - ex)(x - ex)^* &= (x - ex)(x^* - x^*e) \\ &= xx^* - xx^*e - exx^* + exx^*e \\ &= e - e^2 - e^2 + e^3 = 0\,;\end{aligned}$$

since the involution is proper, we conclude that $x = ex$. The element $f = x^*x$ is self-adjoint and $f^2 = x^*(xx^*)x = x^*ex = x^*x = f$, thus f is a projection, and $x^* = fx^*$ by the preceding argument; thus $x \in e\mathrm{A}f$.}

6.3. In any $*$-ring A, $*$-equivalence is an equivalence relation.

{Proof: In question is transitivity. Suppose e, f, g are projections with $e \overset{*}{\sim} f$ and $f \overset{*}{\sim} g$. Say $xx^* = e$, $x^*x = f$ and $yy^* = f$, $y^*y = g$. We can suppose $x \in e\mathrm{A}f$, $y \in f\mathrm{A}g$ (6.1). Then

$$(xy)(xy)^* = xyy^*x^* = xfx^* = xx^* = e$$

and similarly $(xy)^*(xy) = g$, thus $e \overset{*}{\sim} g$.}

6.4. PROPOSITION. [18, p. 33, Th. 25] *Let* A *be a $*$-ring, e and f projections in* A *such that $e \overset{*}{\sim} f$, and let $x \in e\mathrm{A}f$ with $xx^* = e$, $x^*x = f$. Then the mapping $s \mapsto x^*sx$ is an isomorphism of $*$-rings $\theta : e\mathrm{A}e \to f\mathrm{A}f$, with inverse mapping $t \mapsto xtx^*$. For every projection $g \le e$, one has $g \overset{*}{\sim} \theta(g)$.*

Proof. To the proof of 5.4 we need only add the following computations: $\theta(s^*) = x^*s^*x = (x^*sx)^* = (\theta(s))^*$; and $(gx)(gx)^* = gxx^*g = geg = g$, $(gx)^*(gx) = x^*gx = \theta(g)$. $\diamondsuit$

6.5. PROPOSITION. *The $*$-equivalence of projections in a $*$-ring* A *has the following properties*:

(1) $e \overset{*}{\sim} 0 \ \Rightarrow \ e = 0$.

(2) $e \overset{*}{\sim} f \ \Rightarrow \ ue \overset{*}{\sim} uf$ *for all central projections* u (cf. 3.8).

(3) *If* $e \overset{*}{\sim} f$ *and if* $e_1, \ldots, e_n$ *are pairwise orthogonal projections with* $e = e_1 + \ldots + e_n$, *then there exist pairwise orthogonal projections* $f_1, \ldots, f_n$ *such that* $f = f_1 + \ldots + f_n$ *and* $e_i \overset{*}{\sim} f_i$ *for all* i.

(4) *If* $e_1, \ldots, e_n$ *are orthogonal projections with sum* e, *if* $f_1, \ldots, f_n$ *are orthogonal projections with sum* f, *and if* $e_i \overset{*}{\sim} f_i$ *for all* i, *then* $e \overset{*}{\sim} f$.

(5) e, f *are unitarily equivalent if and only if both* $e \overset{*}{\sim} f$ *and* $1-e \overset{*}{\sim} 1-f$.

Proof. (1)-(4): See the proof of 5.5.

(5) If $t^*et = f$ with t unitary ($t^*t = tt^* = 1$) then $(tf)(tf)^* = tft^* = e$ and $(tf)^*(tf) = ft^*tf = f$, thus $e \overset{*}{\sim} f$; also, $t^*(1-e)t = t^*t - t^*et = 1 - f$, so $1 - e \overset{*}{\sim} 1 - f$. Suppose, conversely, that $x \in e\mathrm{A}f$, $y \in (1-e)\mathrm{A}(1-f)$ with $xx^* = e$, $x^*x = f$ and $yy^* = 1 - e$, $y^*y = 1 - f$; then $t = x + y$ is unitary and $t^*et = f$. ♢

6.6. If A is a Baer $*$-ring, e and f are projections in A such that $e \overset{*}{\sim} f$, and $(e_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections such that $e = \sup e_i$, then there exists an orthogonal family of projections $(f_i)_{i\in\mathrm{I}}$ such that $f = \sup f_i$ and $e_i \overset{*}{\sim} f_i$ for all i.

{Proof: Immediate from 6.4.}

6.7. LEMMA. [18, p. 44, Th. 29] *Let* A *be a Rickart $*$-ring*, e *and* f *projections in* A *with* $ef = 0$. *Then* $e \overset{*}{\sim} f$ *if and only if there exists a projection* g *such that* $e = 2ege$, $f = 2fgf$, $g = 2geg = 2gfg$.

Proof. [cf. 2, p. 99, Prop. 2] If such a projection g exists, then the element $x = 2egf$ satisfies $xx^* = e$ and $x^*x = f$, thus $e \overset{*}{\sim} f$. (One does not need $ef = 0$ here.)

Conversely, suppose $x \in e\mathrm{A}f$ with $xx^* = e$ and $x^*x = f$. Set $g = \mathrm{RP}(e + x)$. Since $e + x \in (e + f)\mathrm{A}(e + f)$, one has

$$g \le e + f. \tag{i}$$

By the definition of g,

$$e + x = (e + x)g = eg + xg. \tag{ii}$$

Also

$$\begin{aligned}(e + x)(e - x^*) &= e - ex^* + xe - xx* \\ &= e - (xe)^* + xe - xx^* \\ &= e - 0 + 0 - e = 0\end{aligned}$$

whence $g(e - x^*) = 0$, $gx^* = ge$, thus

$$xg = eg. \tag{iii}$$

Substituting (iii) into (ii),

$$e + x = 2eg. \tag{iv}$$

Since $xe = xfe = 0$, right-multiplication of (iv) by e yields

$$e = 2ege\,. \tag{v}$$

Since $xx = (xf)(ex) = 0$, one has

$$(e + x)(x - f) = ex - ef + xx - xf = x - 0 + 0 - x = 0\,,$$

whence $g(x - f) = 0$, thus

$$gx = gf\,. \tag{vi}$$

Left-multiplying (iv) by g and citing (vi), one has

$$2geg = ge + gx = ge + gf = g(e + f) = g$$

(the last equality by (i)), thus

$$g = 2geg\,. \tag{vii}$$

From (ii) and (iii), one has $e+x = 2xg$; left multiplying by x^*, $x^*e+x^*x = 2x^*xg$, thus

$$x^* + f = 2fg\,. \tag{viii}$$

Right-multiplying (viii) by f yields $0 + f = 2fgf$, thus

$$f = 2fgf\,. \tag{ix}$$

Left-multiplying (viii) by g and recalling that $gx^* = ge$ (see (iii)), one has

$$2gfg = gx^* + gf = ge + gf = g(e + f) = g\,,$$

thus

$$g = 2gfg\,. \tag{x}$$

The equations (v), (vii), (ix), (x) establish the lemma. ♢

6.8. PROPOSITION. [18, p. 46, Th. 30] *Let* A *be a Baer* $*$*-ring,* $(e_i)_{i\in \mathrm{I}}$ *an orthogonal family of projections in* A *with* $\sup e_i = e$, $(f_i)_{i\in\mathrm{I}}$ *an orthogonal family of projections in* A *with* $\sup f_i = f$. *If* $e_i \overset{*}{\sim} f_i$ *for all* i *and if* $ef = 0$, *then* $e \overset{*}{\sim} f$.

Proof. Dropping down to $(e + f)\mathrm{A}(e + f)$, we can suppose that $e + f = 1$. Let $u_i = e_i + f_i$, let $\mathrm{S} = \{u_i : i \in \mathrm{I}\}$, and let $\mathrm{T} = \mathrm{S}'$ be the commutant of S in A (4.1); then T is a Baer $*$-ring "with unambiguous sups and infs" (4.5). Since S is a commutative set, that is, $\mathrm{S} \subset \mathrm{S}' = \mathrm{T}$, one has $\mathrm{T} = \mathrm{S}' \supset \mathrm{S}'' = \mathrm{T}'$, thus the center of T is $\mathrm{T} \cap \mathrm{T}' = \mathrm{T}' = \mathrm{S}''$; in particular, the u_i are orthogonal projections in the center of T, with $\sup u_i = e + f = 1$. Clearly T contains the e_i, the f_i, and the given partial isometries implementing the $*$-equivalences $e_i \overset{*}{\sim} f_i$. Dropping down further to T, we can suppose that the $u_i = e_i + f_i$ are (orthogonal) central projections in A with $\sup u_i = 1$. By the lemma, there exists for each $i \in \mathrm{I}$ a projection g_i in A with $e_i = 2e_ig_ie_i$, etc., and the proof shows that $g_i \in u_i\mathrm{A}$. Then $g_i \le u_i$, so $(g_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections; moreover, setting $g = \sup g_i$, it is clear that $u_ig = g_i$ for all i. {For fixed i, argue that $g_j(u_ig - g_i) = 0$ for all j.} Since the u_i are central, evidently

$$u_i(e - 2ege) = e_i - 2e_ig_ie_i = 0$$

for all i, whence $e - 2ege = 0$. Similarly $f = 2fgf$, etc., therefore $e \overset{*}{\sim} f$ by the lemma. ♢

Under suitable hypotheses on A, the conclusion of 6.8 holds without the restriction $ef = 0$ (cf. 18.14).

6.9. Under the hypotheses of 6.8: If, moreover, $x_i \in \mathrm{A}$ with $x_ix_i^* = e_i$ and $x_i^*x_i = f_i$, one can show that there exists a partial isometry $x \in \mathrm{A}$ such that $xx^* = e$, $x^*x = f$ and $e_ix = x_i = xf_i$ for all i [2, p. 56, Lemma 3]. {For other results in this vein, see Section 14.}

6.10. PROPOSITION. [18, p. 35, Th. 27] *Let* A *be a* $*$*-ring satisfying the following condition*: *for every* $x \in \mathrm{A}$ *there exists* $r \in \{x^*x\}''$ (*the bicommutant of* x^*x) *such that* $x^*x = r^*r$. *If* e, f *are projections in* A *such that* $e \overset{a}{\sim} f$, *then* $e \overset{*}{\sim} f$.

Proof: {In [2, p. 66, Lemma] the condition on A is called the "weak square root axiom" (WSR).}

Let $x \in e\mathrm{A}f$, $y \in f\mathrm{A}e$ with $xy = e$, $yx = f$. Choose $r \in \{yy^*\}''$ with $yy^* = rr^*$ ($= r^*r$, since $\{yy^*\}''$ is commutative) and set $w = xr$; then

$$ww^* = xrr^*x^* = xyy^*x^* = (xy)(xy)^* = ee^* = e\,,$$

so it will suffice to show that $w^*w = f$. Now, $w^*w = r^*x^*xr$. We assert that r commutes with x^*x; indeed,

$$x^*x \cdot yy^* = x^*ey^* = x^*y^* = (yx)^* = f^* = f$$

is self-adjoint, therefore $x^*x \in \{yy^*\}'$, and since $r \in \{yy^*\}''$ we conclude that r commutes with x^*x. Then x^*x also commutes with r^*, so

$$w^*w = r^*x^*xr = x^*xr^*r = x^*xyy^* = f\,,$$

and the proof is complete. ♢

***6.11.** The condition in 6.10 holds for every C*-algebra by spectral theory (one can even take r to be positive, in which case it is unique); in particular, in every Rickart C*-algebra (hence in every AW*-algebra) equivalent projections are $*$-equivalent.

6.12. Let A be a Rickart $*$-ring in which, for projections e and f, $e \overset{a}{\sim} f \Rightarrow e \overset{*}{\sim} f$ (cf. 6.10). Then similar projections are unitarily equivalent.

{Proof: Suppose e, f are similar. Then $e \overset{a}{\sim} f$ and $1-e \overset{a}{\sim} 1-f$ (5.5), therefore, by hypothesis, $e \overset{*}{\sim} f$ and $1-e \overset{*}{\sim} 1-f$, whence e, f are unitarily equivalent (6.5).}

6.13. Let A be as in 6.12. Suppose, moreover, that every 'isometry' in A is unitary (that is, $x^*x = 1 \Rightarrow xx^* = 1$). Then A is *directly finite* ($xy = 1 \Rightarrow yx = 1$).

{Proof: Suppose $xy = 1$. Then $e = yx$ is idempotent, and $e \overset{a}{\sim} 1$. Let $f = \mathrm{LP}(e)$; then $e\mathrm{A} = f\mathrm{A}$ and $f \overset{a}{\sim} e$ (cf. 5.6), so $f \overset{a}{\sim} 1$ by transitivity. By the hypothesis on A, $f \overset{*}{\sim} 1$, say $ww^* = f$ and $w^*w = 1$; this implies, by supposition, that $f = 1$, therefore $e\mathrm{A} = f\mathrm{A} = \mathrm{A}$, whence $e = 1$, that is, $yx = 1$.}

7. DIRECTLY FINITE IDEMPOTENTS IN A BAER RING

7.1. DEFINITION. A ring A is **directly finite** if $yx = 1 \Rightarrow xy = 1$. {If $yx = 1$ then $e = xy$ is idempotent, so direct finiteness means the following condition on the equivalence of idempotents: $e \overset{a}{\sim} 1 \Rightarrow e = 1$.} An idempotent $e \in \mathrm{A}$ is said to be directly finite if the ring $e\mathrm{A}e$ is directly finite (by convention, 0 is directly finite). If A (resp. $e \in \mathrm{A}$) is not directly finite, it is said to be **directly infinite**.

For brevity, in this section we say **finite** and **infinite** for directly finite and directly infinite.

7.2. The following conditions on a ring A are equivalent: (a) A is directly finite; (b) the right A-module A_A is not isomorphic to any proper direct summand of itself; (c) the left A-module ${}_\mathrm{A}\mathrm{A}$ is not isomorphic to any proper direct summand of itself. {Proof: 5.1 and 5.2.}

7.3. If e, f are idempotents of a ring A such that $e \le f$ and f is finite, then e is also finite.

{Proof: If $x, y \in e\mathrm{A}e$ with $yx = e$ then, setting $x' = x + (f - e)$, $y' = y + (f - e)$, one has $x', y' \in f\mathrm{A}f$ with $y'x' = f$, therefore $x'y' = f$; but $x'y' = xy + f - e$, so $xy = e$.}

7.4. If e, f are idempotents of a ring A with $e \precsim_a f$ and f finite, then e is finite.

{Proof: Say $e \overset{a}{\sim} e' \le f$. By 7.3, e' is finite, that is, $e'\mathrm{A}e'$ is finite; since $e\mathrm{A}e \cong e'\mathrm{A}e'$ (5.4) it follows that $e\mathrm{A}e$ is finite, thus e is finite.}

7.5. PROPOSITION. *Let* A *be a Baer ring*, (u_i) *a family of central idempotents in* A, $u = \sup u_i$ (cf. 3.3). *If every* u_i *is finite, then so is* u.

Proof. Suppose $x, y \in u\mathrm{A}$ with $yx = u$. Write $x_i = u_i x$, $y_i = u_i y$. Then $y_i x_i = u_i yx = u_i u = u_i$; since $u_i\mathrm{A}$ is finite, $x_i y_i = u_i$, thus $u_i(xy - u) = 0$ for all i, whence $u(xy - u) = 0$ (3.4), $xy = u$. ♢

7.6. DEFINITION. A Baer ring A is **properly infinite** it it contains no finite central idempotent other than 0. An idempotent $e \in \mathrm{A}$ is said to be properly infinite if the Baer ring $e\mathrm{A}e$ is properly infinite (by convention, 0 is properly infinite) . {Caution: In [18] the term “purely infinite” is used instead; following the usage in [6] we shall employ the latter term for another concept (7.9).}

7.7. COROLLARY. [18, p. 12, Th. 10] *If* A *is any Baer ring, then there exists a unique central idempotent* u *such that* $u\mathrm{A}$ *is finite and* $(1-u)\mathrm{A}$ *is properly infinite.*

Proof. Let T be the set of all finite central idempotents of A (at least $0 \in \mathrm{T}$) and let $u = \sup \mathrm{T}$; by 7.5, $u \in \mathrm{T}$. If $v \le 1-u$ is a finite central idempotent, then $v \in \mathrm{T}$, so $v \le u$, whence $v = 0$; thus $(1-u)\mathrm{A}$ is properly infinite.

If also u' is a central idempotent such that $u'\mathrm{A}$ is finite and $(1-u')\mathrm{A}$ is properly infinite, then $v = u(1-u')$ is finite (because $v \le u$) and infinite (because $v \le 1-u'$) therefore $v = 0$, whence $u \le u'$. Similarly $u' \le u$. ♢

7.8. DEFINITION. A Baer ring A is said to be **semifinite** if it possesses a faithful finite idempotent, that is, there exists a finite idempotent $e \in \mathrm{A}$ such that $\mathrm{C}(e) = 1$ (cf. 3.15). An idempotent $e \in \mathrm{A}$ is said to be semifinite if the Baer ring $e\mathrm{A}e$ is semifinite (by convention, 0 is semifinite).

7.9. DEFINITION. A Baer ring A is said to be **purely infinite** (or of "type III") if 0 is the only finite idempotent of A. An idempotent $e \in \mathrm{A}$ is said to be purely infinite if the Baer ring $e\mathrm{A}e$ is purely infinite (by convention, 0 is purely infinite).

7.10. LEMMA. [18, pp. 12-14] *Let* A *be a Baer ring,* $(u_i)_{i\in \mathrm{I}}$ *a family of pairwise orthogonal central idempotents,* $(e_i)_{i\in\mathrm{I}}$ *a family of idempotents such that* $e_i \le u_i$ *for all* i. *Let* $\mathrm{S} = \{e_i : i \in \mathrm{I}\}$ *and write* $\mathrm{S}^r = (1-e)\mathrm{A}$, e *idempotent. Then* $e_i e = e_i$ *and* $u_i e = e e_i$ *for all* i, *and* $\mathrm{C}(e) = \sup \mathrm{C}(e_i)$.

Proof. {Remark: If the e_i are projections in a Baer $*$-ring and one takes e to be a projection, then $e = \sup e_i$ (1.24).} Since $1-e \in \mathrm{S}^r$, one has $e_i(1-e) = 0$, thus $e_i = e_i e$ for all i.

We assert that $u_i e = e e_i$. For, let $x_i = u_i - e_i$. One has $e_i x_i = e_i u_i - e_i = e_i - e_i = 0$ and, for $j \ne i$, $e_j x_i = (e_j u_j)(u_i x_i) = 0$, thus $x_i \in \mathrm{S}^r = (1-e)\mathrm{A}$, whence $e x_i = 0$, that is, $e u_i = e e_i$.

Let $w_i = \mathrm{C}(e_i)$, $w = \mathrm{C}(e)$, $v = \sup w_i$ (cf. 3.3); we are to show that $v = w$. For all i, $e_i w = (e_i e)w = e_i(ew) = e_i e = e_i$, so $w_i \le w$; therefore $v \le w$. And $e_i \le w_i \le v$, so $e_i(1-v) = 0$ for all i; thus $1-v \in \mathrm{S}^r = (1-e)\mathrm{A}$, so $e(1-v) = 0$, $e = ev$, whence $w \le v$.

{Incidentally, writing $f_i = e e_i = u_i e$, one sees that f_i is an idempotent with $f_i e_i = f_i$ and $e_i f_i = e_i e e_i = e_i e_i = e_i$, so that $\mathrm{A}e_i = \mathrm{A}f_i$. (When the e_i and e are projections in a Baer $*$-ring, $e_i = f_i$.) Note too that (cf. 1.21)

$$(1-e)\mathrm{A} = \mathrm{S}^r = \bigcap \{e_i\}^r = \bigcap (1-e_i)\mathrm{A} = \bigwedge (1-e_i)\mathrm{A}$$

whence $\mathrm{A}e = \mathrm{S}^{rl} = \bigvee \mathrm{A}e_i$, that is, $\mathrm{A}e$ is the supremum of the family $(\mathrm{A}e_i)$ in the lattice $\mathcal{L}$ of idempotent-generated principal left ideals of A.} ♢

7.11. PROPOSITION. [18, p. 12, Th. 12] *If* A *is any Baer ring, there exists a unique central idempotent* u *of* A *such that* $u\mathrm{A}$ *is semifinite and* $(1-u)\mathrm{A}$ *is purely infinite.*

Proof. If A has no finite idempotents other than 0, the proof ends with $u = 0$. Otherwise,[1] let $(u_i)_{i\in I}$ be a maximal family of pairwise orthogonal, nonzero central idempotents such that u_iA is semifinite for all i, and let $u = \sup u_i$. For each $i \in I$ let e_i be a finite idempotent in u_iA such that e_i is faithful in u_iA – which means that $C(e_i) = u_i$ (cf. 3.25). Let e be the idempotent given by 7.10; in particular, $C(e) = u$.

We assert that e is finite. Suppose $x, y \in eAe$ with $yx = e$. Set $x_i = u_ix$, $y_i = u_iy$. Then $y_ix_i = u_i(yx) = u_ie = ee_i$. We have $x_i = u_ix = (xe)u_i = x(eu_i) = x(ee_i)$, whence $x_i \in Ae_i$; similarly, $y_i \in Ae_i$. Left-multiplying $y_ix_i = ee_i$ by e_i, we have $e_iy_ix_i = e_iee_i = e_ie_i = e_i$, so

$$e_i = e_iy_ix_i = e_i(y_ie_i)x_i = (e_iy_i)(e_ix_i)\,,$$

where $e_iy_i, e_ix_i \in e_iAe_i$; since e_iAe_i is finite, it follows that $(e_ix_i)(e_iy_i) = e_i$, thus

$$e_i = e_i(x_ie_i)y_i = e_ix_iy_i = e_iu_i(xy) = e_ixy\,.$$

Thus, $e_i(1-xy) = 0$ for all i, therefore $1-xy \in S^r = (1-e)A$, so $e(1-xy) = 0$, whence $e = xy$. This completes the proof that e is finite. Since $u = C(e)$, we see that uA is semifinite.

Suppose f is a finite idempotent with $f \le 1-u$; we are to show that $f = 0$. Assume to the contrary that $f \ne 0$. Then $C(f) \ne 0$, $C(f) \le 1-u$, and the maximality of the family (u_i) is contradicted.

Suppose also u' has the properties of u. Let e' be a finite idempotent with $C(e') = u'$. Then the idempotent $g = e'(1-u)$ is finite (because $g \le e'$) and $g \le 1-u$; since $(1-u)A$ is purely infinite, necessarily $g = 0$, thus $e' = e'u$, $u' = C(e') \le u$. Similarly, $u \le u'$. ◇

7.12. With notations as in 7.11, if $g \in A$ is any idempotent such that gAg is semifinite, then $g \le u$.

{Proof: Let $h \in gAg$ be a finite idempotent whose central cover in gAg is g. Then $C(h)A$ is semifinite[1], so $C(h) \le u$ by the argument in 7.11. Now, $gC(h)$ is a central idempotent in gAg such that $h \le gC(h)$, therefore $gC(h) = g$ (because h is faithful in gAg). Thus $g \le C(h)$; but $C(h) \le u$, so $g \le u$.}

7.13. The argument in 7.12 shows: If g is any idempotent in a Baer ring A and if $h \in gAg$ is an idempotent that is faithful in gAg, then $g \le C(h)$.[2]

7.14. If A is a Rickart $*$-ring and $e \in A$ is a finite idempotent, then $LP(e)$ and $RP(e)$ are finite projections.

{Proof: Let $f = LP(e)$. We know (5.6) that $eA = fA$, so $e \overset{a}{\sim} f$ (indeed, e and f are similar by 5.5); since e is finite, so is f (7.4).}

7.15. A Baer $*$-ring A is a semifinite Baer ring if and only if it contains a faithful finite projection.

[1] If e is a finite idempotent in A then, since e is faithful in $C(e)A$ (see the footnote for 3.15), $C(e)A$ is semifinite; that gets the Zorn argument started.

[2] It follows that $C(g) \le C(h)$. But $h \le g$, so $C(h) \le C(g)$, therefore $C(h) = C(g)$.

{Proof: Suppose A is a semifinite Baer ring and let $e \in A$ be a faithful finite idempotent. Let $f = \mathrm{LP}(e)$. As noted in 7.14, f is a finite projection and $f \overset{a}{\sim} e$, therefore (5.16) $\mathrm{C}(f) = \mathrm{C}(e) = 1$.}

7.16. A Baer $*$-ring A is a purely infinite Baer ring if and only if 0 is the only finite projection.

{Proof: If 0 is the only finite projection, then by 7.14 it is the only finite idempotent.}

7.17. Suppose A is a Rickart $*$-ring such that for projections f, $f \overset{a}{\sim} 1 \Rightarrow f = 1$. Then A is directly finite, that is, for idempotents e, $e \overset{a}{\sim} 1 \Rightarrow e = 1$.

{Proof: Suppose $e \in A$ is idempotent and $e \overset{a}{\sim} 1$. Let $f = \mathrm{LP}(e)$; then (5.6) $eA = fA$, thus $f \overset{a}{\sim} e \overset{a}{\sim} 1$. Then $f \overset{a}{\sim} 1$, so by hypothesis $f = 1$. Thus $eA = 1A = A$, whence $e = 1$.}

8. ABELIAN IDEMPOTENTS IN A BAER RING; TYPE THEORY

8.1. DEFINITION. [18, p. 10] A ring A is said to be **abelian** if every idempotent of A is central. An idempotent $e \in \mathrm{A}$ is said to be abelian if the ring $e\mathrm{A}e$ is abelian (by convention, 0 is abelian).

8.2. Every division ring is an abelian Baer ring. Every commutative ring is abelian.

8.3. If A is a semiprime Baer ring and $e \in \mathrm{A}$ is idempotent, then e is abelian if and only if $f = e\mathrm{C}(f)$ for every idempotent f of $e\mathrm{A}e$ (3.21).

8.4. Every abelian ring (resp. abelian idempotent) is directly finite.

{Proof: Suppose A is abelian and $x, y \in \mathrm{A}$ with $yx = 1$. Then $e = xy$ is idempotent, by hypothesis central, so $e = 1e = yxe = yex = y(xy)x = (yx)(yx) = 1$.}

8.5. Every subring[1] of an abelian ring is abelian. {Obvious.}

8.6. If e, f are idempotents of a ring A such that $e \precsim_a f$ and f is abelian, then e is abelian.

{Proof: Say $e \overset{a}{\sim} e' \le f$. Then $e'\mathrm{A}e' \subset f\mathrm{A}f$, so $e'\mathrm{A}e'$ is abelian (8.5); and $e\mathrm{A}e \cong e'\mathrm{A}e'$ (5.4), therefore $e\mathrm{A}e$ is abelian.}

8.7. [7, p. 26, Th. 3.2] A regular ring A is abelian if and only if $x\mathrm{A} = \mathrm{A}x$ for all $x \in \mathrm{A}$.[2]

{Proof: Suppose A is abelian and $x \in \mathrm{A}$. Write $x\mathrm{A} = e\mathrm{A}$, e idempotent. By hypothesis, e is central, so $x = ex = xe$, therefore $\mathrm{A}x = \mathrm{A}xe \subset \mathrm{A}e = e\mathrm{A} = x\mathrm{A}$. Similarly $x\mathrm{A} \subset \mathrm{A}x$, so $x\mathrm{A} = \mathrm{A}x$. Conversely, suppose this condition holds and $e \in \mathrm{A}$ is idempotent; then $e\mathrm{A} = \mathrm{A}e$, so $e\mathrm{A}(1-e) = (1-e)\mathrm{A}e = 0$, whence $ex = exe = xe$ for all $x \in \mathrm{A}$.}

8.8. [18, p. 17, Exer. 5] The following conditions on an idempotent e of a ring A are equivalent: (a) e is in the center of A; (b) e commutes with every idempotent of A.

{Proof: (b) $\Rightarrow$ (a): Let $x \in \mathrm{A}$; we are to show that $ex = xe$. The element $f = e + ex(1-e)$ is idempotent. By hypothesis $ef = fe$, thus $f = fe = e$,

[1] Here the subring need not contain the unity element of the ring.

[2] A regular ring is abelian if and only if it is 'reduced' (no nilpotent elements other than 0) [7, p. 26, Th. 3.2].

whence $ex(1-e)=0$, $ex=exe$. Applying this in the opposite ring A° we infer that $xe=exe$, so $ex=exe=xe$.}

8.9. [18, p. 10] A ring is abelian if and only if all of its idempotents commute with each other. {Immediate from 8.8.}

8.10. [18, p. 37, Exer. 2] A Rickart $*$-ring is an abelian ring if and only if all of its projections commute with each other.

{Proof: Let A be a Rickart $*$-ring all of whose projections commute with each other; to show that A is abelian, it will suffice by 8.9 to show that every idempotent of A is a projection. Let $g\in\mathrm{A}$ be idempotent, $e=\mathrm{LP}(g)$, $f=\mathrm{RP}(g)$. As noted in 5.6, $g\mathrm{A}=e\mathrm{A}$ and $\mathrm{A}g=\mathrm{A}f$, thus

$$e=ge\,,\ g=eg\ \text{ and }\ f=fg\,,\ g=gf\,.$$

By hypothesis, $ef=fe$, so

$$ef=e(fg)=f(eg)=fg=f\,,$$

thus $f\le e$; and

$$ef=(ge)f=gfe=ge=e\,,$$

so $e\le f$. Thus $e=f$. Then $g=eg=fg=f$, in particular g is a projection.}

8.11. [18, p. 36] A Rickart $*$-ring is an abelian ring if and only if every projection is central. {Proof: Immediate from 8.10.}

8.12. The following conditions on a Rickart $*$-ring A are equivalent: (a) A is an abelian ring; (b) $\mathrm{LP}(x)=\mathrm{RP}(x)$ for all $x\in\mathrm{A}$.

{Proof: (a) $\Rightarrow$ (b): Every projection e is central, so $ex=x$ if and only if $xe=x$.

(b) $\Rightarrow$ (a): Assuming (b), let $g\in\mathrm{A}$ be idempotent; we are to show that g is central. Let $a\in\mathrm{A}$ and let $x=ga(1-g)$; then $x^2=0$, so $\mathrm{RP}(x)\mathrm{LP}(x)=0$. In view of (b), this means that $x=0$, thus $g\mathrm{A}(1-g)=0$. Similarly $(1-g)\mathrm{A}g=0$, thus g is central.}

8.13. If A is a Rickart $*$-ring and $g\in\mathrm{A}$ is an abelian idempotent, then $\mathrm{LP}(g)$ and $\mathrm{RP}(g)$ are abelian projections.

{Proof: Write $e=\mathrm{LP}(g)$. By 5.6, $e\mathrm{A}=g\mathrm{A}$, so $e\overset{a}{\sim}g$ (indeed, e and g are similar by 5.5); since g is abelian, so is e (8.6).}

8.14. DEFINITION. ([18, p. 11], [6, p. 123, Th. 1]) A Baer ring A is of **type I** (or is "discrete") if it has a faithful abelian idempotent. An idempotent $e\in\mathrm{A}$ is said to be of type I if the Baer ring $e\mathrm{A}e$ is of type I (by convention, 0 is of type I).

8.15. DEFINITION. [6, p. 121, Def. 1 and p. 123, Th. 1] A Baer ring A is **continuous** if 0 is its only abelian idempotent. An idempotent $e\in\mathrm{A}$ is said to be continuous if the Baer ring $e\mathrm{A}e$ is continuous (by convention, 0 is continuous).

8.16. Every Baer ring of type I is semifinite (cf. 7.8, 8.4).

8.17. A Baer $*$-ring A is a Baer ring of type I if and only if it has a faithful abelian projection.

{Proof: Suppose A is a Baer $*$-ring having a faithful abelian idempotent g. Then $e = \mathrm{LP}(g)$ is an abelian projection with $e \overset{a}{\sim} g$ (cf. 8.13); moreover, citing 5.16, $\mathrm{C}(e) = \mathrm{C}(g) = 1$.}

8.18. A Baer $*$-ring is a continuous Baer ring if and only if 0 is the only abelian projection.

{Proof: Suppose 0 is the only abelian projection in the Baer $*$-ring A; then 0 is the only abelian idempotent of A (8.13), so A is continuous.}

8.19. LEMMA. [18, p. 14] *Let* A *be a Baer ring*, (u_i) *a family of pairwise orthogonal central idempotents of* A *such that every* $u_i\mathrm{A}$ *is of type* I, *and let* $u = \sup u_i$. *Then* $u\mathrm{A}$ *is of type* I.

Proof. For each i choose an abelian idempotent $e_i \in u_i\mathrm{A}$ with $\mathrm{C}(e_i) = u_i$. By 7.10 there exists an idempotent e such that $\mathrm{C}(e) = u$, $e_ie = e_i$ and $u_ie = ee_i$ for all i; it will suffice to show that e is abelian. Let $f_i = u_ie = ee_i$; since $e_ie = e_i$ and $ee_i = f_i$, one has $e_i \overset{a}{\sim} f_i$, therefore f_i is abelian (8.6). Given $g \in e\mathrm{A}e$ idempotent, we wish to show that g is central in $e\mathrm{A}e$. Let $x \in e\mathrm{A}e$; we are to show that $gx = xg$, and it will clearly suffice to show that $u_i(gx - xg) = 0$ for all i. Now,

$$u_ix = u_i(ex) = f_ix\,, \quad u_ix = u_i(xe) = x(u_ie) = xf_i\,,$$

thus $u_ix \in f_i\mathrm{A}f_i$. By the same token, $u_ig \in f_i\mathrm{A}f_i$. Since $f_i\mathrm{A}f_i$ is abelian, the idempotent u_ig is central in $f_i\mathrm{A}f_i$, hence commutes with u_ix:

$$0 = (u_ig)(u_ix) - (u_ix)(u_ig) = u_i(gx - xg)\,. \ \diamondsuit$$

8.20. PROPOSITION. [18, pp. 12-14] *If* A *is any Baer ring, there exists a unique central idempotent* u *such that* $u\mathrm{A}$ *is of type* I *and* $(1-u)\mathrm{A}$ *is continuous*.

Proof (cf. 7.11). If A contains no abelian idempotent other than 0, then $u = 0$ fills the bill. Otherwise let (u_i) be a maximal orthogonal family of nonzero central idempotents such that every $u_i\mathrm{A}$ is of type I. Let $u = \sup u_i$ (cf. 3.3). By 8.19, $u\mathrm{A}$ is of type I. Suppose $f \in (1-u)\mathrm{A}$ is an abelian idempotent. Then $\mathrm{C}(f) \le 1-u$ and $\mathrm{C}(f)\mathrm{A}$ is of type I, so $\mathrm{C}(f) = 0$ by maximality, whence $f = 0$. Thus $(1-u)\mathrm{A}$ is continuous.

Suppose also u' is a central idempotent with $u'\mathrm{A}$ of type I and $(1-u')\mathrm{A}$ continuous. Choose abelian idempotents e, e' with $\mathrm{C}(e) = u$, $\mathrm{C}(e') = u'$. Then $(1-u)e'$ is abelian (because it is $\le e'$) and is in $(1-u)\mathrm{A}$, therefore $(1-u)e' = 0$ (because $(1-u)\mathrm{A}$ is continuous); consequently $(1-u)u' = 0$, $u' \le u$. Similarly $u \le u'$, thus $u = u'$. $\diamondsuit$

8.21. With notations as in 8.20, if $g \in \mathrm{A}$ is any idempotent such that $g\mathrm{A}g$ is of type I, then $g \le u$.

{Proof (cf. 7.12): Let $h \in \mathrm{A}$ be an abelian idempotent whose central cover in $g\mathrm{A}g$ is g. Then $\mathrm{C}(h)\mathrm{A}$ is of type I, and $\mathrm{C}(h) \le u$ by the argument in 8.20; but $g \le \mathrm{C}(h)$ by 7.13, whence $g \le u$.}

8.22. DEFINITION. [18, p. 11] A Baer ring A is said to be of **type II** if it is semifinite and continuous (that is, A contains a faithful finite idempotent but no abelian idempotents other than 0). An idempotent $e \in \mathrm{A}$ is said to be of type II if the Baer ring $e\mathrm{A}e$ is of type II (by convention, 0 is of type II).

8.23. A Baer $*$-ring A is a Baer ring of type II if and only if it contains a faithful finite projection but no abelian projections other than 0.

{Proof: If the Baer $*$-ring A is a Baer ring of type II, then A has a faithful finite projection (7.15) but no abelian projections (or even idempotents) other than 0. The converse is immediate from 8.13.}

8.24. THEOREM. [18, p. 12, Th. 11] *Every Baer ring is uniquely the product of Baer rings of types* I, II *and* III.

Proof. If A is any Baer ring, by 8.20 one has $\mathrm{A} = \mathrm{B} \times \mathrm{X}$ with B of type I and X continuous. By 7.11, $\mathrm{X} = \mathrm{C} \times \mathrm{D}$ with C semifinite (and continuous, hence of type II) and D of type III. Then $\mathrm{A} = \mathrm{B} \times \mathrm{C} \times \mathrm{D}$ with the required properties. The proof of uniqueness is routine (cf. 9.21). $\diamondsuit$

8.25. DEFINITION. [18, p. 11] A Baer ring of type I is said to be of **type $\mathbf{I_{fin}}$** if it is directly finite, **type $\mathbf{I_{inf}}$** if it is properly infinite. A Baer ring of type II is said to be of **type $\mathbf{II_{fin}}$** (or type II_1) if it is directly finite, **type $\mathbf{II_{inf}}$** (or type II_∞) if it is properly infinite.

8.26. THEOREM. [18, p. 12, Th. 12] *Every Baer ring is uniquely the product of Baer rings of types* $\mathrm{I_{fin}}$, $\mathrm{I_{inf}}$, $\mathrm{II_{fin}}$, $\mathrm{II_{inf}}$ *and* III.

Proof. Routine (cf. 9.25). $\diamondsuit$

8.27. If A is any Baer ring, there exists a unique central idempotent u of A such that $u\mathrm{A}$ is abelian and $(1-u)\mathrm{A}$ has no abelian central idempotent other than 0 (such rings are called *properly nonabelian*).

{Proof: It is easy to see that if (u_i) is an orthogonal family of abelian central idempotents, then $u = \sup u_i$ is abelian (cf. the proof of 8.19). An obvious exhaustion argument completes the proof.}

8.28. (J.-M. Goursaud and J. Valette [9, p. 95, Th. 1.4]) Let G be a group, K a field such that either K has characteristic $p > 0$, or K has characteristic 0 and contains all roots of unity. Let $\mathrm{A} = \mathrm{KG}$ be the group algebra of G over K, and suppose that A is a regular ring; let Q be the maximal ring of right quotients of A (by 1.31 and 1.32, Q is a regular Baer ring). Then the following conditions are equivalent: (a) Q is of type I; (b) Q is of type $\mathrm{I_{fin}}$ (8.25) and there exists a finite upper bound on the number of pairwise orthogonal, equivalent, nonzero abelian idempotents in Q; (c) G has an abelian subgroup of finite index; (d) A satisfies a polynomial identity.

9. ABSTRACT TYPE DECOMPOSITION OF A BAER *-RING

Throughout this section, A *denotes a Baer* $*$*-ring with an equivalence relation* $\sim$ *defined on the set of projections of* A, *satisfying the following axioms*:

(A) $e \sim 0 \Rightarrow e = 0$.

(B) $e \sim f \Rightarrow ue \sim uf$ *for all central projections* u.

(D) *If* $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_n$) *are pairwise orthogonal projections with sum* e (resp. f) *and if* $e_i \sim f_i$ *for all* i, *then* $e \sim f$.

The labelling of the list of axioms follows that of Kaplansky [18, p. 41]; the list will be rounded out in subsequent sections. Axiom A will not be needed until 9.16.

9.1. EXAMPLES. The most important are $\overset{a}{\sim}$ and $\overset{*}{\sim}$ (5.5 and 6.5, respectively).

9.2. Combining axioms B and D, one sees that if e, f are projections and $u_1, \ldots, u_n$ are orthogonal central projections with sum 1, then $e \sim f$ if and only if $u_i e \sim u_i f$ for all i. So to speak, $\sim$ is compatible with finite direct products.

9.3. If $g \in \mathrm{A}$ is any projection, then the Baer $*$-ring $g\mathrm{A}g$ also satisfies the axioms A, B, D for the restriction of the equivalence relation $\sim$ to its projection lattice.

{Proof: This is obvious for axioms A and D. Suppose e, f are projections in $g\mathrm{A}g$ with $e \sim f$ and suppose v is any central projection of $g\mathrm{A}g$. Since A is semiprime (3.20), $v = ug$ for some central projection u of A — $u = \mathrm{C}(v)$ fills the bill (3.21)—and so $ve = uge = ue \sim uf = ugf = vf$.}

9.4. If $\sim$ is the relation $\overset{a}{\sim}$ (resp. $\overset{*}{\sim}$) on the projection lattice of A, and if $g \in \mathrm{A}$ is a projection, then the relation on the projection lattice of $g\mathrm{A}g$ induced by $\sim$ is the relation $\overset{a}{\sim}$ (resp. $\overset{*}{\sim}$) in the $*$-ring $g\mathrm{A}g$.

{Proof: Suppose e, f are projections in $g\mathrm{A}g$ and $e \overset{a}{\sim} f$ relative to A. One can choose $x \in e\mathrm{A}f$, $y \in f\mathrm{A}e$ with $xy = e$ and $yx = f$ (5.2); then $x, y \in g\mathrm{A}g$, so $e \overset{a}{\sim} f$ relative to $g\mathrm{A}g$. Similarly for $\overset{*}{\sim}$ (cf. 6.1).}

9.5. DEFINITION. We say that A is **finite** relative to the relation $\sim$ in case, for projections $e \in \mathrm{A}$, $e \sim 1 \Rightarrow e = 1$. Finiteness relative to $\overset{a}{\sim}$ is called **direct finiteness**, and means that $yx = 1 \Rightarrow xy = 1$ (7.1, 7.17); finiteness relative to $\overset{*}{\sim}$ will be called $*$-**finiteness**, and means that $x^*x = 1 \Rightarrow xx^* = 1$.[1] {Direct

[1] Every $*$-regular ring is $*$-finite [P. Ara and P. Menal, *Arch. Math.* 42 (1984), 126-130].

finiteness obviously implies $*$-finiteness; sometimes the converse is true (cf. 6.13).} A projection $g \in \mathrm{A}$ is said to be finite relative to $\sim$ if the Baer $*$-ring $g\mathrm{A}g$ is finite relative to the restriction of $\sim$ to its projection lattice (by convention, 0 is finite).

If A is not finite, it is said to be **infinite**; a projection $g \in \mathrm{A}$ is said to be infinite if $g\mathrm{A}g$ is infinite for the restriction of $\sim$ to $g\mathrm{A}g$ (by convention, 0 is infinite).

9.6. EXAMPLE. If e is finite and $e \sim f$, one cannot conclude that f is finite; this will be true under an extra axiom (C′) to be introduced later (15.3). {Counterexample: Let $\mathrm{A} = \mathrm{M}_2(\mathbb{R})$ and declare $e \sim f$ for all nonzero projections e, f. The projection $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is finite (even minimal) and $e \sim 1$, so 1 is not finite.}

9.7. Let e, f be projections, $e \le f$. Then e is a finite projection of $f\mathrm{A}f$ if and only if it is a finite projection of A. {Immediate from $e\mathrm{A}e = e(f\mathrm{A}f)e$.}

9.8. If e, f are projections of A with f finite and $e \le f$, then e is finite.

{Proof: Suppose $g \le e$, $g \sim e$. By axiom D, $g + (f - e) \sim e + (f - e) = f$, so $g + (f - e) = f$ by finiteness of f, whence $g = e$.}

9.9. DEFINITION. We say that A is **properly infinite** (relative to $\sim$) if 0 is the only finite central projection. A projection $e \in \mathrm{A}$ is said to be properly infinite if $e\mathrm{A}e$ is properly infinite (by convention, 0 is properly infinite); this means (cf. 3.20, 3.21) that if u is a central projection of A with ue finite, then $ue = 0$.

9.10. LEMMA. *If* (u_i) *is a family of finite central projections, then* $u = \sup u_i$ *is also finite.*

Proof (cf. 7.5). Suppose $e \in u\mathrm{A}$ is a projection with $e \sim u$. By axiom B, $u_i e \sim u_i u = u_i$ for all i; since $u_i\mathrm{A}$ is finite, necessarily $u_i e = u_i$, thus $u_i(1-e) = 0$ for all i, whence $u(1 - e) = 0$, $u = ue = e$. ♢

9.11. THEOREM. *There exists a unique central projection* u *of* A *such that* $u\mathrm{A}$ *is finite and* $(1 - u)\mathrm{A}$ *is properly infinite.*

Proof. Formally the same as 7.7 (note that 9.8 is needed for the proof of uniqueness). ♢

9.12. DEFINITION. We say that A is **semifinite** (relative to $\sim$) if it contains a faithful finite projection. A projection $e \in \mathrm{A}$ is said to be semifinite if $e\mathrm{A}e$ is semifinite (by convention, 0 is semifinite). We say that A is of **type III**, or "purely infinite" (relative to $\sim$) if 0 is the only finite projection of A. A projection $e \in \mathrm{A}$ is said to be of type III if $e\mathrm{A}e$ is of type III (by convention, 0 is of type III).

9.13. LEMMA. *If* (e_i) *is a family of finite projections whose central covers are pairwise orthogonal, then* $e = \sup e_i$ *is finite.*

Proof. Write $u_i = \mathrm{C}(e_i)$, $u = \sup u_i$, and note that $\mathrm{C}(e) = u$ (3.22). Note that $u_i e = e_i$ (cf. 7.10). {For, $e_j(u_i e - e_i) = e_j u_j(u_i e - e_i) = 0$ for $j \ne i$, and

$e_i(u_ie-e_i)=e_ie-e_i=e_i-e_i=0$, therefore $e(u_ie-e_i)=0$, that is, $u_ie-e_i=0$.} Suppose $f\le e$ and $f\sim e$. Then $u_if\le u_ie$, and $u_if\sim u_ie$ by axiom B; since $u_ie=e_i$ is finite, necessarily $u_if=u_ie$, thus $u_i(e-f)=0$ for all i, therefore $u(e-f)=0$, that is, $e-f=0$. ◊

9.14. THEOREM. *There exists a unique central projection* $u\in\mathrm{A}$ *such that* $u\mathrm{A}$ *is semifinite and* $(1-u)\mathrm{A}$ *is of type* III.

Proof. If A contains no finite projections other than 0, then $u=0$ fills the bill. Otherwise let (u_i) be a maximal orthogonal family of nonzero central projections such that each $u_i\mathrm{A}$ is semifinite, and let $u=\sup u_i$. For each i, let e_i be a finite projection with $\mathrm{C}(e_i)=u_i$ and let $e=\sup e_i$. Then $\mathrm{C}(e)=\sup\mathrm{C}(e_i)=\sup u_i=u$ and e is finite by 9.13, thus $u\mathrm{A}$ is semifinite. If $f\in(1-u)\mathrm{A}$ is a finite projection, then $\mathrm{C}(f)\mathrm{A}$ is semifinite and $\mathrm{C}(f)\le 1-u$, therefore $\mathrm{C}(f)=0$ by maximality, whence $f=0$. Thus $(1-u)\mathrm{A}$ is of type III.

Uniqueness is proved exactly as in 7.11. ◊

9.15. With notations as in 9.14, if $g\in\mathrm{A}$ is any projection such that $g\mathrm{A}g$ is semifinite, then $g\le u$. {Proof: Formally the same as 7.12.}

9.16. If $e\sim f$ then $\mathrm{C}(e)=\mathrm{C}(f)$.

{Proof: By axiom B, $[1-\mathrm{C}(f)]e\sim[1-\mathrm{C}(f)]f=0$, so $[1-\mathrm{C}(f)]e=0$ by axiom A, whence $[1-\mathrm{C}(f)]\mathrm{C}(e)=0$, $\mathrm{C}(e)\le\mathrm{C}(f)$. Similarly $\mathrm{C}(f)\le\mathrm{C}(e)$. Remark: This is the first use of axiom A in this section.}

9.17. Every abelian projection in A is finite.

{Proof: Let $e\in\mathrm{A}$ be abelian and suppose $f\le e$ with $f\sim e$. By 3.20 and 8.3, $f=e\mathrm{C}(f)$; but $\mathrm{C}(f)=\mathrm{C}(e)$ by 9.16, so $f=e\mathrm{C}(e)=e$.} Remark: Only axioms A and B are needed for this.

9.18. (i) If A is of type I then it is semifinite. (ii) If A is of type III then it is continuous.

{Proof: (i) Let $e\in\mathrm{A}$ be a faithful abelian projection. By 9.17, e is finite (relative to $\sim$), so A is semifinite by definition (9.12).

(ii) Suppose A is of type III. If $e\in\mathrm{A}$ is abelian then e is finite (9.17), so $e=0$; thus A is continuous (8.15).}

9.19. DEFINITION. We say that A is of **type II** (relative to $\sim$) if it is semifinite and continuous, that is, if A contains a faithful finite projection but no abelian projections other than 0. A projection $e\in\mathrm{A}$ is said to be of type II if $e\mathrm{A}e$ is of type II (by convention, 0 is of type II).

9.20. For a projection $g\in\mathrm{A}$, each of the following conditions implies that $g=0$. (i) g is both type I and type II. (ii) g is both type I and type III. (iii) g is both type II and type III.

{Proof: Assume to the contrary that $g\ne0$; dropping down to $g\mathrm{A}g$, we can suppose $g=1$. (i) Let $e\in\mathrm{A}$ be a faithful abelian projection. Then $e=0$ (A is continuous), whence $1=\mathrm{C}(e)=0$, a contradiction. (ii) Let $e\in\mathrm{A}$ be a faithful abelian projection. Then e is finite (9.17) hence $e=0$ (A is type III), so $1 = \mathrm{C}(e) = 0$. (iii) Let $e \in \mathrm{A}$ be a faithful finite projection. Then $e = 0$ (A is type III), so $1 = \mathrm{C}(e) = 0$.}

9.21. THEOREM. *The Baer *-ring* A *is uniquely the product of rings of types* I, II *and* III (*relative to the equivalence relation* $\sim$ *satisfying axioms* A, B *and* D).[2]

Proof. By 8.20, there exists a central idempotent u (a projection by 3.8) of the Baer ring A such that $u\mathrm{A}$ is of type I and $(1-u)\mathrm{A}$ is continuous. Apply 9.14 to $(1-u)\mathrm{A}$: let v, w be orthogonal central projections such that $v + w = 1 - u$, $v\mathrm{A}$ is semifinite and $w\mathrm{A}$ is of type III. Since $v\mathrm{A}$ is also continuous (because $v \le 1 - u$) it is of type II. Thus $\mathrm{A} = u\mathrm{A} \times v\mathrm{A} \times w\mathrm{A}$ with the required properties.

Suppose also $1 = u' + v' + w'$ is a central partition of 1 with the indicated properties. Then $uv' = 0$ by (i) of 9.20; similarly $uw' = 0$, so $u = u1 = uu'$, thus $u \le u'$. Similarly $u' \le u$, so $u = u'$. Similarly $v = v'$, $w = w'$. ♢

9.22. With notations as in the proof of 9.21, $u\mathrm{A} \times v\mathrm{A}$ is the semifinite summand of A.

{Proof: By definition, $v\mathrm{A}$ is semifinite; so is $u\mathrm{A}$ (9.18); hence so is $u\mathrm{A} \times v\mathrm{A} = (u + v)\mathrm{A}$.}

9.23. With notations as in the proof of 9.21, $v\mathrm{A} \times w\mathrm{A}$ is the continuous summand of A.

9.24. DEFINITION. If, relative to $\sim$, A is finite and of type I (resp. type II), we say that A is of **type $\mathbf{I_{fin}}$** (resp. **type $\mathbf{II_{fin}}$**, or II_1). If, relative to $\sim$, A is properly infinite and of type I (resp. type II), we say that A is of **type $\mathbf{I_{inf}}$** (resp. **type $\mathbf{II_{inf}}$**).

9.25. THEOREM. *The Baer $*$-ring* A *is uniquely the product of rings of types* $\mathrm{I_{fin}}$, $\mathrm{I_{inf}}$, $\mathrm{II_{fin}}$, $\mathrm{II_{inf}}$ *and* III (*relative to the equivalence relation* $\sim$ *satisfying axioms* A, B *and* D).[2]

Proof. Combine 9.11 and 9.21. ♢

9.26. All of the above carries through for $\sim$ an equivalence relation on the partially ordered set of idempotents of a semiprime Baer ring (cf. 1.8).

{Proof: Routine, notable ingredients being 7.10 and 3.21. Here $e \le f$ means $e \in f\mathrm{A}f$, and e, f are orthogonal if $ef = fe = 0$.}

9.27. If A is finite (relative to $\sim$) and $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ for all $x \in \mathrm{A}$, then A is directly finite.

{Proof: Suppose $yx = 1$. If $xz = 0$ then $yxz = 0$, so $z = 0$; therefore $\mathrm{RP}(x) = 1$. Let $g = xy$, which is idempotent, and let $e = \mathrm{LP}(g)$; then $g\mathrm{A} = e\mathrm{A}$ (5.6). Now, $g\mathrm{A} = xy\mathrm{A} \subset x\mathrm{A}$, whereas $x = x1 = x(yx) = (xy)x = gx$ shows that $x\mathrm{A} \subset g\mathrm{A}$, thus $x\mathrm{A} = g\mathrm{A} = e\mathrm{A}$. It follows that $e = \mathrm{LP}(x)$. By hypothesis, $\mathrm{LP}(x) \sim \mathrm{RP}(x)$, that is, $e \sim 1$; since A is finite relative to $\sim$, $e = 1$, whence $g = 1$, that is, $xy = 1$. (Note that the argument works for a Rickart $*$-ring satisfying the hypotheses.)}

[2] The definitions of 'type I' and 'continuous' are independent of $\sim$; all other 'types' (including 'type $\mathrm{I_{fin}}$' and 'type $\mathrm{I_{inf}}$') depend on the concept of 'finite', therefore on $\sim$.

10. KAPLANSKY'S AXIOMS (A-H, etc.): A SURVEY OF RESULTS

We summarize in this section the principal axioms considered by Kaplansky [18, p. 147] for an equivalence relation $\sim$ on the projection lattice of a Baer $*$-ring A:

(A) $e \sim 0 \;\Rightarrow\; e = 0$. [*Definiteness*]

(B) $e \sim f \;\Rightarrow\; ue \sim uf$ for every central projection u. [*Central compatibility*]

(C) If $(e_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections with $\sup e_i = e$, and if $e \sim f$, then there exists an orthogonal family of projections $(f_i)_{i\in\mathrm{I}}$ such that $f = \sup f_i$ and $e_i \sim f_i$ for all i. [*Induced partitions*]

(C$'$) Same as (C), with the index set I assumed to be finite. [*Induced finite partitions*]

(D) If $e_1, \ldots, e_n$ are orthogonal projections with sum e, and $f_1, \ldots, f_n$ are orthogonal projections with sum f, and if $e_i \sim f_i$ for all i, then $e \sim f$. [*Finite additivity*]

(E) If e, f are projections in A with $e\mathrm{A}f \neq 0$, then there exist nonzero projections $e_0 \le e$, $f_0 \le f$ with $e_0 \sim f_0$. [*Partial comparability*]

(F) If $(e_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections with $\sup e_i = e$, $(f_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections with $\sup f_i = f$, $e_i \sim f_i$ for all $i \in \mathrm{I}$, and $ef = 0$, then $e \sim f$. [*Orthogonal additivity*]

(G) If $(u_i)_{i\in\mathrm{I}}$ is an orthogonal family of central projections with $\sup u_i = 1$, and if e, f are projections such that $u_i e \sim u_i f$ for all i, then $e \sim f$. [*Central additivity*]

(H) $e \cup f - f \sim e - e \cap f$ for every pair of projections e, f. [*Parallelogram law*]

{There are also axioms J and K [18, p. 111], the EP and SR axioms [18, pp. 89-90] (see below), and a "spectral axiom" [18, p. 136]. Properties of this sort have their roots in the Murray–von Neumann theory of operator algebras (1936); for lattice-theoretic antecedents of axioms such as A-H, see Loomis's memoir [20] and S. Maeda's paper [22].}

These axioms are of particular interest for the relations of ordinary equivalence ($\overset{a}{\sim}$) and $*$-equivalence ($\overset{*}{\sim}$), but there are also interesting interactions among the axioms for an abstract relation $\sim$.[1]

Work of Loomis [20, p. 4, axiom (D)] and of S. Maeda and S. S. Holland [26] highlights the following axiom weaker than axiom E:

(E′) If e, f are projections with $ef \neq 0$, then there exist nonzero projections $e_0 \leq e$, $f_0 \leq f$ such that $e_0 \sim f_0$.

We cite the following results of Maeda and Holland:

10.1. In the presence of axioms A and C′, the axioms E and E′ are equivalent (this will be proved in 13.8). Axiom H implies axiom E′ thus axioms A, C′ and H imply E (noted in the proof of 13.9). This is the key to several of the results mentioned below.

Other conditions on $\sim$ contemplated are as follows ($e \precsim f$ means that $e \sim e' \leq f$ for a suitable projection e'):

(GC) For each pair of projections e, f there exists a central projection u such that $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$. [*Generalized comparability*]

Additivity: Same as axiom F, with the condition $ef = 0$ omitted. (This is also called *complete additivity*.)

Continuity of the lattice operations: If (e_i) is an increasingly directed family of projections with supremum e (briefly, $e_i \uparrow e$), then $e_i \cap f \uparrow e \cap f$ for every projection f.

LP $\sim$ RP : For every $x \in \mathrm{A}$, $\mathrm{LP}(x) \sim \mathrm{RP}(x)$.

We now survey some of the principal results to be proved (or, sometimes, merely noted) in the sections that follow.

10.2. For $\overset{a}{\sim}$ in any Baer $*$-ring, axioms A-D hold (see 11.1).

10.3. For $\overset{*}{\sim}$ in any Baer $*$-ring, axioms A-D and F hold (see 11.2).

10.4. Axioms C′, D, F imply the 'Schröder-Bernstein theorem': $e \precsim f$ and $f \precsim e \;\Rightarrow\; e \sim f$ [18, p. 61, Th. 41]. In particular, this holds for $\overset{*}{\sim}$ in any Baer $*$-ring (cf. 6.8). {This would seem to be a fundamental result, hence enormously important. Strangely, it seems to be useless! (At any rate, none of the results in these notes make any use of it.) It has this significance: writing $[e]$ for the equivalence class of e under $\sim$, the relation $[e] \leq [f]$ defined by $e \precsim f$ is a partial ordering, the Schröder-Bernstein theorem providing the 'antisymmetry' property. There is a trivial proof of the Schröder-Bernstein theorem in the finite case (assuming only axioms C′, D and finiteness in the sense of 9.5).}

10.5. Axioms A-G imply complete additivity [18, p. 78, Th. 52] (the proof is sketched in 18.16).

[1] Another concrete example is 'perspectivity' (see 10.7).

10.6. If axioms A-D, F and H hold, then GC (hence E) and complete additivity (hence G) hold [26, Th. 2.1]. {This was proved in [18, p. 82, Th. 54] with axiom E included in the hypothesis.} For the details, see 13.9 and 18.12.

10.7. If, relative to $\sim$, the Baer $*$-ring A is finite (9.5) and satisfies axioms A-D, F and H (cf. 10.6), then the lattice operations are continuous and the relation $\sim$ is the relation of perspectivity (that is, $e \sim f$ if and only if there exists a projection g such that $e \cup g = f \cup g = 1$ and $e \cap g = f \cap g = 0$) [18, Ths. 69, 71]. For the proof, see 20.8.

10.8. If A is a regular Baer $*$-ring then, relative to $\overset{a}{\sim}$, A is finite (i.e., directly finite) and all of the above-mentioned conditions hold [17] (proof sketched in 20.10, leaving out the hardest part).[2]

10.9. Relative to $\overset{*}{\sim}$, GC implies complete additivity [2, p. 129, Th. 1]. This is proved in 18.14.

A 'square-root' condition on A is important in many applications:

(SR) For every $x \in \mathrm{A}$ there exists $r \in \{x^*x\}''$ with $r^* = r$ and $r^2 = x^*x$. [*Square roots*]

If this condition holds, then $\overset{a}{\sim}$ and $\overset{*}{\sim}$ coincide (6.10). The important consequence of (SR) proved by S. Maeda is as follows:

10.10. If (SR) holds in A, then $\overset{*}{\sim}$ satisfies axiom H [25, Th. 2] (proved in 12.13).

The next conditions figure in many applications pertaining to $\overset{*}{\sim}$:

(EP) For every $x \in \mathrm{A}$, $x \neq 0$, there exists $r \in \{x^*x\}''$ such that $r^* = r$ and $(x^*x)r^2$ is a nonzero projection. [*Existence of projections*.]

Addability of partial isometries: If $(e_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections with $\sup e_i = e$, $(f_i)_{i\in\mathrm{I}}$ is an orthogonal family of projections with $\sup f_i = f$, $e_i \overset{*}{\sim} f_i$ for all $i \in \mathrm{I}$, and $(w_i)_{i\in\mathrm{I}}$ is a family of partial isometries such that $w_i^*w_i = e_i$ and $w_iw_i^* = f_i$ for all $i \in \mathrm{I}$, then there exists a partial isometry w such that $w^*w = e$, $ww^* = f$ and $we_i = w_i = f_iw$ for all $i \in \mathrm{I}$. {The term is inelegant, but useful for distinguishing this concept from the closely related concept of 'summability' (see 14.8 below).}

(PD) Every $x \in \mathrm{A}$ can be written $x = wr$ with $r \in \{x^*x\}''$, $r^* = r$, $r^2 = x^*x$ and w a partial isometry such that $ww^* = \mathrm{LP}(x)$, $w^*w = \mathrm{RP}(x)$. [*Polar decomposition*]

Obviously (PD) implies (SR) and $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$. Here are some further results pertaining to the above three conditions:

10.11. If A is a Baer $*$-ring with no abelian summand and if A satisfies GC relative to $\overset{*}{\sim}$ (cf. 10.9), then partial isometries are addable in A [2, p. 129, Th. 1]. In every AW*-algebra, partial isometries are addable (same reference).

[2]See the footnote for 9.5.

10.12. If A is a Baer $*$-ring satisfying (EP), in which partial isometries are addable, then A has (PD) (in particular, A satisfies (SR)) [14, Th. 2.2]. For the details, see 14.23 and 14.29.

10.13. If A is a Baer $*$-ring satisfying (EP), and satisfying GC relative to $\overset{*}{\sim}$, then $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x)$ for all $x \in \mathrm{A}$. (For the proof, see 14.31.)

10.14. If A is a Baer $*$-ring satisfying (EP), and if A is properly infinite relative to $\overset{*}{\sim}$, then A has (PD). (For the proof, see 14.30.)

11. EQUIVALENCE AND $*$-EQUIVALENCE IN BAER $*$-RINGS: FIRST PROPERTIES

11.1. PROPOSITION. [18, p. 47] *For* $\overset{a}{\sim}$, *every Baer* $*$*-ring satisfies the axioms* A–D *of* §10.

Proof. The properties A, B, D are noted in (1), (2), (4) of 5.5, property C in 5.18. ♢

11.2. PROPOSITION. [18, p. 47] *For* $\overset{*}{\sim}$, *every Baer* $*$*-ring satisfies the axioms* A–D *and* F *of* §10.

Proof. The properties A, B, D are noted in (1), (2), (4) of 6.5, property C in 6.6, property F in 6.8. ♢

11.3. PROPOSITION. [18, p. 47] *For* $\overset{a}{\sim}$, *every regular Baer* $*$*-ring satisfies the axioms* A-F *and* H *of* §10.

Proof. Let A be a regular Baer $*$-ring. In view of 11.1, we need only check the properties E, F and H. For E, see 5.10; for H, 5.9.

Property F: Suppose, as in the statement of F, $e = \sup e_i$, $f = \sup f_i$, $e_i \overset{a}{\sim} f_i$ for all i , and $ef = 0$; we are to show that $e \overset{a}{\sim} f$. Let $u_i = e_i + f_i$ and let S be the set of all u_i . Then S is a commutative set, so $\mathrm{S} \subset \mathrm{S}'$, $\mathrm{S}' \supset \mathrm{S}''$. By 4.8, $\mathrm{T} = \mathrm{S}'$ is a regular Baer $*$-ring with unambiguous everything, and its center is $\mathrm{T} \cap \mathrm{T}' = \mathrm{S}' \cap \mathrm{S}'' = \mathrm{S}'' \supset \mathrm{S}$; thus the u_i are central projections in T . If $x_i \in e_i\mathrm{A}f_i$, $y_i \in f_i\mathrm{A}e_i$ with $x_iy_i = e_i$, $y_ix_i = f_i$, then clearly $x_i, y_i \in \mathrm{T}$; we may therefore drop down to T and suppose that the u_i are central in A . Then $u = \sup u_i$ is central in A ; dropping down further to $u\mathrm{A}$, we can suppose that $\sup u_i = 1$. By 5.20, e_i and f_i are perspective in $u_i\mathrm{A}$; let $g_i \in u_i\mathrm{A}$ be a projection with $e_i \cup g_i = f_i \cup g_i = u_i$ and $e_i \cap g_i = f_i \cap g_i = 0$. Let $g = \sup g_i$. Clearly $u_ig = g_i$, $u_ie = e_i$, $u_if = f_i$. By 3.23, one has

$$u_i(e \cup g) = (u_ie) \cup (u_ig) = e_i \cup g_i = u_i \, ,$$

thus $u_i[1 - (e \cup g)] = 0$ for all i , whence $1 - e \cup g = 0$, $e \cup g = 1$. Similarly $f \cup g = 1$, $e \cap g = 0$, $f \cap g = 0$. Thus e, f are perspective in A . Since $\overset{a}{\sim}$ satisfies H, it follows that $e \overset{a}{\sim} f$ (5.19). ♢

{Incidentally, $\mathrm{LP}(x) \overset{a}{\sim} \mathrm{RP}(x)$ for every element x (5.8). These are the properties of $\overset{a}{\sim}$ in a regular Baer $*$-ring that lie nearest the surface, property C being the only one among them that requires a struggle. Deeper results are signaled in §10. The deepest is surely direct finiteness [17, Th. 2]; I am unable to improve on Kaplansky's herculean computation, so can only refer the reader to [17] for the details. (But see 20.4 for a lattice-theoretic approach to this result due to Amemiya and Halperin.) It may be helpful to signal three minor misprints in [17]: on p. 529, in the second sentence after Lemma 19, read $e_{ij}^* \in e_j \mathrm{A} e_i$, and in formula (3) at the bottom of the page read λ_i for λ_1; on p. 530, in the sentence between formulas (5) and (6), the formula should read $e_{i1}^* = \lambda_i^{-1} e_{i1}$; on p. 532, formula (22), the quantifier $r > 0$ refers to the particular r of the induction hypothesis.}

11.4. There exists a Baer ∗-ring in which property E fails for $\overset{*}{\sim}$ [18, p. 43]. {An example is the ∗-ring of 2×2 matrices over the field of three elements (cf. [18, p. 39, Exer. 10], [2, p. 82, Exer. 1]).} Signaled by Kaplansky as open questions are whether axioms E and F hold for $\overset{a}{\sim}$ in every Baer ∗-ring [18, p. 47] .

***11.5.** Every AW*-algebra satisfies for $\overset{*}{\sim}$ (cf. 6.11) the axiom G of §10 [18, p. 75].

{Proof [2, p. 53, Prop. 2]: Let A be an AW*-algebra, (u_i) an orthogonal family of central projections with $\sup u_i = 1$. We first show that A may be identified with a certain subalgebra B of the product ∗-algebra $\mathrm{C} = \prod u_i \mathrm{A}$. Let B be the set of all families $x = (x_i) \in \mathrm{C}$ with $\|x_i\|$ bounded; it is routine to show that B is a ∗-subalgebra of C, and that B is a C*-algebra for the norm $\|x\| = \sup \|x_i\|$. Since the $u_i\mathrm{A}$ are Baer ∗-rings, so is their product C; and if $(e_i) \in \mathrm{C}$ is a projection, then every e_i is a projection, so $\|e_i\| = \|e_i^* e_i\| = \|e_i\|^2$ shows that $\|e_i\| \le 1$ for all i, whence $(e_i) \in \mathrm{B}$. Thus B contains every projection of C, hence is itself a Baer ∗-ring; so B is an AW*-algebra (1.38), called the C*-**sum** of the family of AW*-algebras $u_i\mathrm{A}$. Define $\varphi : \mathrm{A} \to \mathrm{B}$ by $\varphi(a) = (u_i a)$, clearly a monomorphism of ∗-algebras; it follows that φ is isometric [6, p. 8, Prop. 8], so $\varphi(\mathrm{A})$ is norm-closed in B. If $(e_i) \in \mathrm{B}$ is a projection then, setting $e = \sup e_i$ in A, it is clear that $\varphi(e) = (e_i)$; thus $\varphi(\mathrm{A})$ contains every projection of B. Since B is the closed linear span of its projections (cf. 1.40) and $\varphi(\mathrm{A})$ is closed, it follows that $\varphi(\mathrm{A}) = \mathrm{B}$.

Suppose now that for each i one is given projections e_i, f_i in $u_i\mathrm{A}$ with $e_i \overset{*}{\sim} f_i$. Let $w_i \in e_i \mathrm{A} f_i \subset u_i \mathrm{A}$ with $w_i w_i^* = e_i$, $w_i^* w_i = f_i$. Then $\|w_i\|^2 = \|w_i w_i^*\| = \|e_i\| \le 1$ for all i, so $(w_i) \in \mathrm{B}$. Therefore there exists $w \in \mathrm{A}$ with $\varphi(w) = (w_i)$, that is, $u_i w = w_i$ for all i. It follows easily that $ww^* = e$ and $w^* w = f$, where $e = \sup e_i$, $f = \sup f_i$.}

We shall see in subsequent sections that an AW*-algebra satisfies, for $\overset{*}{\sim}$, all properties mentioned in §10 except continuity of the lattice operations (which it satisfies if and only if the algebra is directly finite; see 20.4 and 20.11).

12. PARALLELOGRAM LAW (AXIOM H)

In this section, A denotes a Rickart $*$-ring and $\sim$ an equivalence relation on the projection lattice of A. We review a definition from §10:

12.1. DEFINITION. The relation $\sim$ is said to satisfy **Axiom H** (or the "parallelogram law") if

$$e \cup f - f \sim e - e \cap f$$

for every pair of projections e, f in A.

12.2. Every $*$-regular ring satisfies axiom H for $\overset{a}{\sim}$ (5.9).

12.3. In the Baer $*$-ring of 11.4, axiom H fails for $\overset{*}{\sim}$ [cf. 2, p. 75, Exer. 1].

12.4. PROPOSITION. *The following conditions are equivalent*:
(a) $\sim$ *satisfies axiom* H;
(b) $e - e \cap (1-f) \sim f - (1-e) \cap f$ *for every pair of projections* e, f;
(c) $e \sim f$ *for every pair of projections* e, f *such that* $e \cap (1-f) = (1-e) \cap f = 0$.

Proof. Let e, f be any pair of projections. By 1.15,

$$\mathrm{LP}(ef) = \mathrm{LP}(e[1-(1-f)] = e - e \cap (1-f),$$

therefore

$$\mathrm{RP}(ef) = \mathrm{LP}((ef)^*) = \mathrm{LP}(fe) = f - f \cap (1-e).$$

Thus (b) says that $\mathrm{LP}(ef) \sim \mathrm{RP}(ef)$ for every pair of projections, whereas (cf. 1.15) axiom H says that $\mathrm{RP}[e(1-f)] \sim \mathrm{LP}[e(1-f)]$ for every pair e, f. Therefore (a) $\Leftrightarrow$ (b).

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (b): Let e, f be any pair of projections and write $e' = \mathrm{LP}(ef)$, $f' = \mathrm{RP}(ef)$. Then $e' \le e$, $f' \le f$, so $ef = e'(ef)f' = (e'e)(ff') = e'f'$; thus

$$e' = \mathrm{LP}(e'f') = e' - e' \cap (1-f'),$$

whence $e' \cap (1-f') = 0$, and

$$f' = \mathrm{RP}(e'f') = f' - (1-e') \cap f',$$

whence $(1-e') \cap f' = 0$. By (c), $e' \sim f'$, that is, $\mathrm{LP}(ef) \sim \mathrm{RP}(ef)$, in other words (b) holds. ♢

12.5. DEFINITION. Projections e, f in a Rickart $*$-ring are said to be in **position** p' if $e \cap (1-f) = (1-e) \cap f = 0$. {In other words, $e \cap (1-f) = 0$ and $e \cup (1-f) = 1$, that is, e and $1-f$ are complementary; equivalently, $1-e$ and f are complementary.}

12.6. Projections e, f are in position p' if and only if $e = \mathrm{LP}(ef)$ and $f = \mathrm{RP}(ef)$. {Clear from the proof of 12.4.}

12.7. Projections in position p' are perspective.

{Proof: Suppose e, f are in position p', that is, e and $1-f$ are complementary; but $f, 1-f$ are also complementary, thus e and f have $1-f$ as a common complement.}

12.8. PROPOSITION. *For every pair of projections* e, f *in a Rickart* $*$*-ring, there exist unique decompositions*

$$e = e' + e'' , \ f = f' + f''$$

such that e', f' *are in position* p' (*hence are perspective by* 12.7) *and* $ef'' = e''f = 0$. *Necessarily* $e' = \mathrm{LP}(ef)$ *and* $f' = \mathrm{RP}(ef)$.

Proof. Suppose $e = e' + e''$, $f = f' + f''$ are decompositions with the indicated properties. Since e', f' are in position p', we know (12.6) that $e' = \mathrm{LP}(e'f')$, $f' = \mathrm{RP}(e'f')$. But $e'f' = (e-e'')(f-f'') = ef$, thus $e' = \mathrm{LP}(ef)$ and $f' = \mathrm{RP}(ef)$, whence uniqueness.

On the other hand, if one defines $e' = \mathrm{LP}(ef)$, $f' = \mathrm{RP}(ef)$ then, as shown in the proof of 12.4, e' and f' are in position p'; also $ef = e'(ef) = e'f$, so $(e-e')f = 0$, and similarly $e(f-f') = 0$, therefore $e'' = e - e'$ and $f'' = f - f'$ fill the bill. ♢

12.9. COROLLARY. *If* $\sim$ *satisfies axiom* H, *then for every pair of projections* e, f *one can write*

$$e = e' + e'' , \ f = f' + f''$$

with $e' \sim f'$ *and* $ef'' = e''f = 0$.

Proof. For the decompositions of 12.8, one has $e' \sim f'$ by condition (c) of 12.4. ♢

12.10. If $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ holds for all $x \in \mathrm{A}$, then axiom H holds. {Immediate from 1.15.} The next results lead up to S. Maeda's theorem that if A satisfies the axiom (SR) of §10, then $\overset{*}{\sim}$ satisfies axiom H.

12.11. DEFINITION. An element s of a $*$-ring is called a **symmetry** if $s^* = s$ and $s^2 = 1$ (that is, s is a self-adjoint unitary). {Note: If e is a projection, then $1 - 2e$ is a symmetry. Conversely, if s is a symmetry and 2 is invertible, then $\frac{1}{2}(1-s)$ is a projection.}

12.12. LEMMA. [25, Th. 1] *The following conditions on a Rickart $*$-ring* A *are equivalent*:

(a) *for every pair of projections* e, f *there exists a symmetry* $s \in \mathrm{A}$ *such that* $s(ef)s = fe$;

(b) *for every pair of projections* e, f *in position* p', *there exists a symmetry* $s \in \mathrm{A}$ *such that* $ses = f$ (hence also $sfs = e$; one says that e and f are **exchanged** by the symmetry s).

These conditions hold if A *satisfies the axiom* (SR) *of* §10.

Proof. (a) $\Rightarrow$ (b): Suppose e, f are in position p'; by 12.6, $e = \mathrm{LP}(ef)$, $f = \mathrm{RP}(ef)$. With s as in (a), the mapping $x \mapsto sxs^{-1} = sxs^* = sxs$ is a $*$-automorphism of A, so $ses = s \cdot \mathrm{LP}(ef) \cdot s = \mathrm{LP}(sefs) = \mathrm{LP}(fe) = \mathrm{RP}((fe)^*) = \mathrm{RP}(ef) = f$.

(b) $\Rightarrow$ (a): Given any pair of projections e, f let $e' = \mathrm{LP}(ef)$, $f' = \mathrm{RP}(ef)$. Then e', f' are in position p' (12.8), so by (b) there exists a symmetry s exchanging e' and f'. Then $s(ef)s = s(e'f')s = (se's)(sf's) = f'e' = (e'f')^* = (ef)^* = fe$.

Now suppose A satisfies (SR). Let $x = e + f - 1$, which is self-adjoint. By hypothesis, there exists $r \in \{x^*x\}'' = \{x^2\}''$ such that $r^* = r$ and $r^2 = x^*x = x^2$. Since $x \in \{x^2\}'$ and $r \in \{x^2\}''$ we have $rx = xr$, therefore $(r+x)(r-x) = r^2 - x^2 = 0$; writing $g = \mathrm{RP}(r + x)$ we have $g(r - x) = 0$, so $gr = gx$. Taking adjoints, $rg = xg$. Then

$$r + x = (r + x)g = rg + xg = 2rg\,. \tag{$*$}$$

Since

$$x^2 = ef + fe - e - f + 1\,,$$

one has $ex^2 = efe = x^2e$, in particular $e \in \{x^2\}'$; since $r \in \{x^2\}''$ it follows that $er = re$. Similarly $f \in \{x^2\}'$ and $fr = rf$. Let $s = 1 - 2g$, which is a symmetry. Then

$$rs = r - 2rg = r - (r + x)$$

by (*), so $rs = -x$; taking adjoints, $sr = -x$, whence $r = -sx$. Finally, since $e = x + 1 - f$ by the definition of x, one has

$$\begin{aligned} sefs &= s(x + 1 - f)fs = sxfs = (-r)fs \\ &= -frs = -f(-x) = fx = f(e + f - 1) = fe\,, \end{aligned}$$

thus (a) holds. ♢

12.13. THEOREM. (S. Maeda [25, Th. 2]) *Consider the following conditions on a Rickart $*$-ring*:

(EU) *For every pair of projections* e, f *there exists a unitary* u *such that* $u(ef)u^* = fe$. [“Existence of unitaries”]

(ES) *For every pair of projections* e, f *there exists a symmetry* s *such that* $s(ef)s = fe$. [“Existence of symmetries”]

Then: (i) (SR) $\Rightarrow$ (ES) $\Rightarrow$ (EU);

(ii) (EU) $\Rightarrow$ *axiom* H *holds for* $\overset{u}{\sim}$ (unitary equivalence);

(iii) (ES) $\Rightarrow$ *axiom* H *holds for* $\overset{s}{\sim}$ (unitary equivalence by a symmetry).

In particular, (SR) $\Rightarrow$ *axiom* H *holds for* $\overset{*}{\sim}$ (in other words, for $\overset{a}{\sim}$ (6.10)).

Proof. (i) The first implication holds by 12.12, the second is trivial.

(ii) Given any pair of projections e, f let u be a unitary such that $ue(1-f)u^* = (1-f)e$. Then (1.15)

$$\begin{aligned} e \cup f - f &= \mathrm{RP}[e(1-f)] = \mathrm{LP}[(1-f)e] \\ &= \mathrm{LP}[ue(1-f)u^*] = u \cdot \mathrm{LP}[e(1-f)] \cdot u^* \\ &= u(e - e \cap f)u^*, \end{aligned}$$

thus $e \cup f - f$ and $e - e \cap f$ are unitarily equivalent.

(iii) Same proof as (ii), with $u^* = u$. $\diamondsuit$

We thus have the diagram (the arrows signify implication):

$$\begin{tikzcd}
 & \mathrm{SR} \arrow[d] & \\
 & \mathrm{ES} \arrow[dl] \arrow[dr] & \\
\mathrm{EU} \arrow[dr] & & \mathrm{H}\ \text{for}\ \overset{s}{\sim} \arrow[dl] \\
 & \mathrm{H}\ \text{for}\ \overset{u}{\sim} \arrow[d] & \\
 & \mathrm{H}\ \text{for}\ \overset{*}{\sim}\ (=\overset{a}{\sim}) &
\end{tikzcd}$$

12.14. If A is a Baer $*$-ring satisfying the axiom (EP) of §10 and if A has GC relative to $\overset{*}{\sim}$, then A satisfies axiom H relative to $\overset{*}{\sim}$, and the relations $\overset{*}{\sim}$, $\overset{a}{\sim}$ coincide.

{Sketch of proof: As is shown in 14.31, $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x)$ for all $x \in \mathrm{A}$, therefore A satisfies axiom H for $\overset{*}{\sim}$ (12.10). If A is abelian, then $e \overset{a}{\sim} f$ implies $e = f$ (9.16), thus both $\overset{a}{\sim}$ and $\overset{*}{\sim}$ coincide with the relation of equality. It now suffices to consider the case that A has no abelian summand (8.27). Then, since A has GC for $\overset{*}{\sim}$, partial isometries are addable in A (10.11); since, moreover, A satisfies (EP), it follows that A has polar decomposition (10.12) and in particular A satisfies (SR). Therefore $\overset{a}{\sim}$ coincides with $\overset{*}{\sim}$ (6.10). (It is the citation of [2] in 10.11 that prevents the foregoing from being a complete proof.)}

12.15. In a Baer $*$-ring satisfying axiom H for $\overset{*}{\sim}$, direct finiteness and $*$-finiteness are equivalent conditions (20.11).

12.16. D. Handelman has constructed a Baer $*$-ring in which axiom H fails for $\overset{a}{\sim}$; his example is, moreover, directly finite—even 'strongly modular' (cf. §21)—and factorial ($0, 1$ the only central projections) [13].

13. GENERALIZED COMPARABILITY)

Throughout this section, A is a Baer $*$-ring, $\sim$ an equivalence relation on its projection lattice (with various axioms added as needed).

13.1. DEFINITION. The Baer $*$-ring A is said to have **generalized comparability** (GC) relative to $\sim$ if, for every pair of projections e, f of A, there exists a central projection u such that

$$ue \precsim uf \quad \text{and} \quad (1-u)f \precsim (1-u)e\,.$$

A weaker condition is **orthogonal** GC: such a u exists whenever $ef = 0$.

13.2. PROPOSITION. *Assume axioms* B *and* D *of* §10 *hold. Then the following conditions on* A *are equivalent*:

(a) A *has* GC;

(b) *for every pair of projections* e, f *of* A *there exist orthogonal decompositions*

$$e = e_1 + e_2\,,\ f = f_1 + f_2$$

with $e_1 \sim f_1$ *and* $\mathrm{C}(e_2)\mathrm{C}(f_2) = 0$ (*where* C *denotes central cover*).

Proof [2, p. 77, Prop. 1]. (a) $\Rightarrow$ (b): Given projections e, f let u be a central projection with $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$, say

$$ue \sim f_1' \le uf \quad \text{and} \quad (1-u)f \sim e_1'' \le (1-u)e\,.$$

Let $e_1' = ue$, $f_1'' = (1-u)f$; thus

$$e_1' \sim f_1'\,,\ e_1'' \sim f_1''\,.$$

Set $e_1 = e_1' + e_1''$, $f_1 = f_1' + f_1''$; by axiom D, $e_1 \sim f_1$. Set $e_2 = e - e_1$, $f_2 = f - f_1$; then $ue_2 = ue - ue_1 = e_1' - e_1' = 0$ and similarly $(1-u)f_2 = 0$. Thus $\mathrm{C}(e_2) \le 1-u$ and $\mathrm{C}(f_2) \le u$, whence $\mathrm{C}(e_2)\mathrm{C}(f_2) = 0$.

(b) $\Rightarrow$ (a): With notations as in (b), let $u = \mathrm{C}(f_2)$. By axiom B, $ue_1 \sim uf_1$ and $(1-u)e_1 \sim (1-u)f_1$. Now, $ue_2 = \mathrm{C}(f_2)e_2 = \mathrm{C}(f_2)\mathrm{C}(e_2)e_2 = 0$ and $(1-u)f_2 = (1-u)\mathrm{C}(f_2)f_2 = (1-u)uf_2 = 0$, so $ue = ue_1 \sim uf_1 \le uf$ and $(1-u)f = (1-u)f_1 \sim (1-u)e_1 \le (1-u)e$. Thus A has GC. ♢

13.3. THEOREM. [18, p. 53, Th. 35] *Assume axioms* B, E *and* F *hold. Then* A *has orthogonal* GC.

Proof. Let e, f be projections in A with $ef = 0$; we seek a central projection u such that $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$. If $e\mathrm{A}f = 0$ then $\mathrm{C}(e)\mathrm{C}(f) = 0$ (by 3.20 and 3.21) and $u = \mathrm{C}(f)$ fills the bill. Assume $e\mathrm{A}f \neq 0$. Let $(e_i), (f_i)$ be a maximal pair of orthogonal families of nonzero projections such that $e_i \le e$, $f_i \le f$ and $e_i \sim f_i$ for all i (Zorn's lemma, using axiom E to get started). Let $g = \sup e_i$, $h = \sup f_i$; by axiom F, $g \sim h$ (recall that $ef = 0$, therefore $gh = 0$). Necessarily $(e-g)\mathrm{A}(f-h) = 0$, since otherwise an application of axiom E would contradict maximality. Therefore $\mathrm{C}(e-g)\mathrm{C}(f-h) = 0$. Set $u = \mathrm{C}(f-h)$. Then $u(e-g) = 0$, so, citing axiom B, one has $ue = ug \sim uh \le uf$; and $(1-u)(f-h) = 0$, so $(1-u)f = (1-u)h \sim (1-u)g \le (1-u)e$. ♢

13.4. COROLLARY. *If* A *is a Baer $*$-ring satisfying axiom* E *for* $\overset{*}{\sim}$, *then* A *has orthogonal* GC *for* $\overset{*}{\sim}$.

Proof. Axioms B and F hold for $\overset{*}{\sim}$ (11.2); quote 13.3. ♢

13.5. COROLLARY. *If* A *is a Baer $*$-ring satisfying the* EP-*axiom*, *then* A *satisfies axiom* E *for* $\overset{*}{\sim}$, *therefore* (13.4) A *has orthogonal* GC *for* $\overset{*}{\sim}$.

Proof. Suppose e, f are projections with $e\mathrm{A}f \neq 0$. Choose $x \in e\mathrm{A}f$ with $x \neq 0$. By the EP-axiom, there exist a nonzero projection g and an element $y \in \{x^*x\}''$ such that $y^* = y$ and $x^*x \cdot y^2 = g$. Then $w = xy$ satisfies $w^*w = yx^*xy = x^*x \cdot y^2 = g$, and $g = y^2x^*x$ shows that $g \le \mathrm{RP}(x) \le f$. Set $h = ww^* = xyw^*$; then $h \le \mathrm{LP}(x) \le e$. Since $g \overset{*}{\sim} h$, axiom E is verified for $\overset{*}{\sim}$. Quote 13.4. ♢

13.6. A regular Baer $*$-ring has orthogonal GC for $\overset{a}{\sim}$ by 11.3 and 13.3; since, moreover, axiom H holds for $\overset{a}{\sim}$ in such a ring (5.9), GC holds by an elementary argument given in the proof of 13.9 below. {Another proof of this is given in 13.11.} The following definitions and lemmas lead up to S. Maeda and S. S. Holland's generalization that, roughly speaking, the parallelogram law implies GC (13.9).

13.7. DEFINITION. Projections e, f in A are **partially comparable** (with respect to $\sim$) if there exist nonzero projections $e_0 \le e$, $f_0 \le f$ such that $e_0 \sim f_0$. {Thus, axiom E is equivalent to the condition: $e\mathrm{A}f \neq 0 \Rightarrow e, f$ are partially comparable.} If e, f are not partially comparable, they are said to be **unrelated** (with respect to $\sim$). {Thus, axiom E is equivalent to the condition: e, f unrelated $\Rightarrow e\mathrm{A}f = 0$. To say that e, f are unrelated means that if $e_0 \le e$, $f_0 \le f$ and $e_0 \sim f_0$, then either $e_0 = 0$ or $f_0 = 0$ (in the presence of axiom A we can say $e_0 = f_0 = 0$).}

13.8. LEMMA. [26, Lemma 2.1] *Assume axioms* A *and* C′ *hold. Then the following conditions are equivalent*:

(a) *axiom* E *holds*;

(b) e, f *unrelated* $\Rightarrow ef = 0$;

(c) *for each projection* e, *there exists a largest projection* e' *unrelated to* e, *and one has* $ee' = 0$.

When these conditions hold, *the projection* e' *of* (c) *is in the center of the ring* A.

Proof. {This is a startling result. Axiom E says: e, f unrelated $\Rightarrow$ $eAf = 0$; this is obviously stronger than condition (b), and one would suppose that it is much stronger. Not so, says the lemma.}

(a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Given a projection $e \in A$, let S be the set of all projections in A that are unrelated to e (at least $0 \in S$), and let $e' = \sup S$. We first show that e, e' are unrelated. Suppose $e_0 \le e$, $e_0' \le e'$, $e_0 \sim e_0'$. Note that if $f \in S$ then e_0, f are unrelated (because $e_0 \le e$ and e, f are unrelated). Since $e_0' \sim e_0$, it follows that e_0', f are unrelated. {For, suppose $e_0'' \le e_0'$, $f'' \le f$, $e_0'' \sim f''$. By axiom C$'$, the equivalence $e_0' \sim e_0$ induces an equivalence $e_0'' \sim g \le e_0$ for some g; thus $f'' \sim e_0'' \sim g$, where $f'' \le f$ and $g \le e_0$. Since f, e_0 are unrelated, it follows that $f'' = 0$ or $g = 0$ hence (axiom A) $f'' = g = 0$ and $e_0'' = 0$.} By (b), $e_0' f = 0$; varying $f \in S$, $e_0' e' = 0$. But $e_0' \le e'$, so $e_0' = e_0' e' = 0$. This shows that e, e' are unrelated. Consequently $e' \in S$, thus S has e' as largest element; moreover, $ee' = 0$ by (b). Thus (c) holds.

(c) $\Rightarrow$ (a): Assuming e, f unrelated, we must show that $eAf = 0$. Choose e' as in (c); thus $f \le e'$. Apply (c) to e': there exists a largest projection e'' unrelated to e', and $e'e'' = 0$. Since e, e' are unrelated, $e \le e''$. It will suffice to show that e' is in the center of A; for then it will follow that $eAf = eAe'f = ee'Af = 0Af = 0$. In view of 3.9, it will further suffice to show that e' has a unique complement (namely $1 - e'$). Assuming g is any complement of e', it will suffice to show that $g = e''$.

We first note that g, e' are unrelated. {For, suppose $g_0 \le g$, $e_0 \le e'$, $g_0 \sim e_0$. Since $e_0 \le e'$ we know that e_0, e are unrelated; since $g_0 \sim e_0$ it follows from axiom C$'$ that g_0, e are unrelated, therefore $g_0 \le e'$. Thus $g_0 \le g \cap e' = 0$ (recall that g is a complement of e'), so $g_0 = 0$.} It follows that $g \le e''$, so we can form the projection $e'' - g$. Since $e'e'' = 0$, one has $e'' \le 1 - e'$, therefore

$$e'' - g \le (1 - e') \cap (1 - g) = 1 - (e' \cup g) = 1 - 1 = 0,$$

thus $g = e''$. ◇

13.9. THEOREM. [cf. 26, Th. 2.1] *Let* A *be a Baer* $*$*-ring*, $\sim$ *an equivalence relation on its projection lattice satisfying the axioms* A, B, C$'$, D, F *and* (*especially*) H *of* §10. *Then* A *has* GC (*relative to* $\sim$).

Proof. Let us first show that axiom E holds. Assuming e, f are unrelated projections, it will suffice by the lemma to show that $ef = 0$. By axiom H and 12.4,

$$e - e \cap (1 - f) \sim f - (1 - e) \cap f;$$

since e, f are unrelated, either $e - e \cap (1 - f) = 0$ or $f - (1 - e) \cap f = 0$ (actually both, by axiom A). Say $f - (1 - e) \cap f = 0$; then $f = (1 - e) \cap f \le 1 - e$, so $ef = 0$.

Since axioms B, E, F hold, A has orthogonal GC (13.3). By axiom H, one can write

$$e = e' + e'', \; f = f' + f''$$

with $e' \sim f'$ and $ef'' = e''f = 0$ (12.9). Then $e''f'' = 0$, so by orthogonal GC and axiom D one can write

$$e'' = e_1 + e_2\,, \ \ f'' = f_1 + f_2$$

with $e_1 \sim f_1$ and $\mathrm{C}(e_2)\mathrm{C}(f_2) = 0$ (see the proof of 13.2). Then

$$e = (e' + e_1) + e_2\,, \quad f = (f' + f_1) + f_2\,,$$

where $e' + e_1 \sim f' + f_1$ by axiom D, and $\mathrm{C}(e_2)\mathrm{C}(f_2) = 0$, whence (see the proof of 13.2) $u = \mathrm{C}(f_2)$ satisfies $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$. ♢

{This theorem is proved in [18, p. 87, Th. 57] with axiom E as an added hypothesis (redundant, as 13.9 shows).}

13.10. COROLLARY. [26, Th. 2.1] *If* A *is a Baer ∗-ring satisfying axiom* H *for* $\overset{*}{\sim}$, *then* A *has* GC *for* $\overset{*}{\sim}$.

Proof. Immediate from 11.2 and 13.9. ♢

13.11. COROLLARY. [17, p. 534, (10)] *Every regular Baer ∗-ring has* GC *for* $\overset{a}{\sim}$.

Proof. Immediate from 11.3 and 13.9. ♢

13.12. COROLLARY. *If* A *is a Baer ∗-ring satisfying the* SR-*axiom*, *then* A *has* GC *for* $\overset{*}{\sim}$ (cf. 6.10).

Proof. Since SR ⇒ axiom H for $\overset{*}{\sim}$ (12.13), the corollary is immediate from 13.10. {Incidentally, $\overset{*}{\sim}$ coincides with $\overset{a}{\sim}$ by 6.10.} ♢

13.13. COROLLARY. *If* A *is a Baer ∗-ring with no abelian summand*, *satisfying the* SR-*axiom*, *then partial isometries are addable in* A (cf. §10).

Sketch of proof. By 13.12, A has GC for $\overset{*}{\sim}$; since, moreover, A has no abelian summand, it follows that partial isometries are addable [2, p. 129, Th. 1]. ♢

{This corollary is proved in [18, p. 104, Th. 64] with the EP-axiom as an added hypothesis. We remark that in a Baer ∗-ring with no abelian summand and satisfying GC for $\overset{*}{\sim}$, EP ⇒ SR (14.32).}

13.14. PROPOSITION. [18, p. 85, Th. 55] *Let* A *be a Baer ∗-ring satisfying the hypotheses of* 13.9. *If* e, f *is any pair of projections in* A, *then there exists a central projection* u *such that*

$$ue \precsim uf \quad \textit{and} \quad (1-u)(1-e) \precsim (1-u)(1-f)\,.$$

Proof. Write $e_0 = e \cap (1-f)$, $f_0 = (1-e) \cap f$; by axiom H (12.4)

$$e - e_0 \sim f - f_0\,. \tag{i}$$

By 13.9, A has GC; applying it to the pair e_0, f_0 we find a central projection u such that

$$ue_0 \precsim uf_0\,, \ (1-u)f_0 \precsim (1-u)e_0\,. \tag{ii}$$

From (i) we have $ue - ue_0 \sim uf - uf_0$, which, when added to the first relation of (ii), yields $ue \precsim uf$. On the other hand, the substitutions $e \mapsto 1-e$, $f \mapsto 1-f$ transform e_0, f_0 into f_0, e_0, respectively, therefore (i) is transformed into

$$(1-e) - f_0 \sim (1-f) - e_0\,. \tag{i$'$}$$

From (i$'$) and the second relation of (ii), one argues as above that $(1-u)(1-e) \precsim (1-u)(1-f)$. $\diamondsuit$

The analogue of this propositon is valid for $\overset{a}{\sim}$ in a regular right self-injective ring [29, Prop. 1.3].

14. POLAR DECOMPOSITION

14.1. DEFINITION. A $*$-ring A is said to have **polar decomposition** (PD) if, for every $x \in \mathrm{A}$, one can write $x = wr$ with $r \in \{x^*x\}''$, $r^* = r$, $r^2 = x^*x$, and w a partial isometry such that $ww^* = \mathrm{LP}(x)$, $w^*w = \mathrm{RP}(x)$.

14.2. A $*$-ring with PD satisfies the SR-axiom of §10, as well as LP $\overset{*}{\sim}$ RP.

14.3. DEFINITION. Let A be a Baer $*$-ring. We say that **partial isometries are addable in** A if, whenever (w_i) is a family of partial isometries such that the projections $e_i = w_i^* w_i$ are pairwise orthogonal and the projections $f_i = w_i w_i^*$ are pairwise orthogonal, there exists a partial isometry $w \in \mathrm{A}$ such that $w^*w = \sup e_i$, $ww^* = \sup f_i$ and $we_i = w_i = f_i w$ for all i. {In particular, $*$-equivalence is (completely) additive in the sense of §10.}

14.4. Partial isometries are addable under each of the following hypotheses on a Baer $*$-ring A: (i) A has no abelian summand and has GC for $\overset{*}{\sim}$ [2, p. 129, Th. 1]; *(ii) A any AW*-algebra [same ref.]; (iii) $\mathrm{M}_2(\mathrm{A})$ is a Baer $*$-ring [2, p. 131, Exer. 1]; (iv) relative to $\overset{*}{\sim}$, A is properly infinite and satisfies axiom E of §10 [2, p. 131, Exer. 3].

14.5. The main result to be proved in this section (14.29) is the following theorem of L. Herman [14]: *If* A *is a Baer $*$-ring such that* (i) A *satisfies the* EP-*axiom of* §10, *and* (ii) *partial isometries are addable in* A, *then* A *has* PD.

The idea of the proof is to exhaust on the EP-axiom: given $x \in \mathrm{A}$ (whose polar decomposition we wish to effect) one considers a maximal orthogonal family of nonzero projections (e_i) such that for each i, there exists a self-adjoint $s_i \in \{x^*x\}''$ with $x^*x \cdot s_i^2 = e_i$. Replacing s_i by $e_i s_i$, one can suppose $e_i s_i = s_i$. Writing $w_i = xs_i$, one has $w_i^* w_i = s_i x^* x s_i = x^* x s_i^2 = e_i$; one shows easily that the projections $f_i = w_i w_i^*$ are pairwise orthogonal, and that $\sup f_i = \mathrm{LP}(x)$, $\sup e_i = \mathrm{RP}(x)$. By (ii) one forms a partial isometry w. The trick is to 'sum up' the s_i (more precisely, their 'relative inverses' r_i) to arrive at an element $r \in \{x^*x\}''$. A machinery for discussing such 'sums' was introduced by Loomis [20, p. 20ff] and elaborated by Herman [14]; for the reader's convenience, we reproduce the details here. {Herman's paper has not appeared in print; his elegant discussion deserves to be more widely known.}

14.6. DEFINITION. Elements a, b of a $*$-ring are said to be **orthogonal**, written $a \perp b$, if

$$ab^* = b^*a = 0\,.$$

{Taking $*$, one sees that $a \perp b \Leftrightarrow b \perp a \Leftrightarrow a^* \perp b^*$.} A family of elements (a_i) is said to be orthogonal (or 'pairwise orthogonal') if $a_i \perp a_j$ for $i \neq j$; this will be indicated by writing $(a_i)\perp$. {Not to be confused with the concept of 'independent family' in a lattice, for which the same notation is employed.}

14.7. For projections e, f in a $*$-ring, $e \perp f$ means $ef = 0$. For elements a, b of a Rickart $*$-ring, $a \perp b$ means that $\mathrm{RP}(a)\mathrm{LP}(b^*) = \mathrm{RP}(b^*)\mathrm{LP}(a) = 0$, that is, $\mathrm{RP}(a)\mathrm{RP}(b) = 0$ and $\mathrm{LP}(a)\mathrm{LP}(b) = 0$. {In particular, $(w_i)\perp$ for the family in 14.3.}

14.8. DEFINITION. An orthogonal family (a_i) in a $*$-ring A is said to be **summable** (to a) if there exists an element $a \in \mathrm{A}$ such that for each index j,

$$a_i x = 0 \text{ for all } i \neq j \ \Rightarrow\ ax = a_j x\,.$$

Expressed in terms of right annihilators, this means that for each j,

$$\bigcap_{i\neq j}\{a_i\}^r \subset \{a - a_j\}^r\,.$$

One then writes $a = \oplus a_i$, called the **sum** of the a_i. In a $*$-ring with proper involution, the sum is unique:

14.9. In a $*$-ring with proper involution, if $\oplus a_i = a$ and $\oplus a_i = b$, then $a = b$.

{Proof: For each j, by hypothesis

$$\bigcap_{i\neq j}\{a_i\}^r \subset \{a - a_j\}^r \quad\text{and}\quad \bigcap_{i\neq j}\{a_i\}^r \subset \{b - a_j\}^r\,,$$

so

$$\bigcap_{i\neq j}\{a_i\}^r \subset \{a - a_j\}^r \cap \{b - a_j\}^r\,;$$

but $a - b = (a - a_j) - (b - a_j)$ implies

$$\{a - a_j\}^r \cap \{b - a_j\}^r \subset \{a - b\}^r\,,$$

therefore

$$\bigcap_{i\neq j}\{a_i\}^r \subset \{a - b\}^r \text{ for each index } j\,. \tag{$*$}$$

By orthogonality and $(*)$,

$$a_j^* \in \bigcap_{i\neq j}\{a_i\}r \subset \{a - b\}^r\,,$$

so $(a - b)a_j^* = 0$, $a_j(a - b)^* = 0$. Thus $a_i(a - b)^* = 0$ for all i, that is,

$$(a - b)^* \in \bigcap_{i}\{a_i\}^r\,. \tag{$**$}$$

Fix any index j; from $(**)$ and $(*)$ we see that

$$(a-b)^* \in \bigcap_i \{a_i\}^r \subset \bigcap_{i \neq j} \{a_i\}^r \subset \{a-b\}^r ,$$

so $(a-b)(a-b)^* = 0$. Since the involution is proper, $a-b=0$.}

14.10. LEMMA. [20, p. 27, Lemma 47] *If* $a = \oplus a_i$ *in the sense of* 14.8, *then the following two conditions hold*:

(1) $aa_i^* = a_i a_i^* = a_i a^*$ *for all* i;

(2) *if* $a_i x = 0$ *for all* i, *then* $ax = 0$.

Proof. (1) For each j, by orthogonality one has

$$a_j^* \in \bigcap_{i \neq j} \{a_i\}^r \subset \{a - a_j\}^r ,$$

so $(a-a_j)a_j^* = 0$, $aa_j^* = a_j a_j^*$. In particular, aa_j^* is self-adjoint, so it is equal to its adjoint $a_j a^*$.

(2) Suppose $a_i x = 0$ for all i. Fix any index j. Then

$$x \in \bigcap_i \{a_i\}^r \subset \bigcap_{i \neq j} \{a_i\}^r \subset \{a - a_j\}^r ,$$

so $(a-a_j)x = 0$, $ax = a_j x = 0$. $\diamondsuit$

14.11. LEMMA. [20, p. 27, Lemma 48] *If*, *in a $*$-ring with proper involution*, $a = \oplus a_i$ *and* J *is a finite set of indices such that* $ba_i = 0$ *for all* $i \notin \mathrm{J}$, *then* $ba = \sum_{j \in \mathrm{J}} ba_j$.

Proof. {When $\mathrm{J} = \varnothing$, this says that the dual of 14.10, (2) for left annihilators holds.} Let

$$c = b\left(a - \sum_{j \in \mathrm{J}} a_j\right) ;$$

we are to show that $c = 0$, and it will suffice to show that $cc^* = 0$. One has

$$c^* = \left(a^* - \sum_{j \in \mathrm{J}} a_j^*\right) b^* .$$

We assert that $a_i c^* = 0$ for all i. Now,

$$\begin{aligned} a_i c^* &= \left(a_i a^* - \sum_{j \in \mathrm{J}} a_i a_j^*\right) b^* \\ &= \left(a_i a_i^* - \sum_{j \in \mathrm{J}} a_i a_j^*\right) b^* \qquad \text{by 14.10} \\ &= a_i (ba_i)^* - \left(\sum_{j \in \mathrm{J}} a_i a_j^*\right) b^* . \end{aligned}$$

If $i \in \mathrm{J}$ then by orthogonality $\sum_{j\in \mathrm{J}} a_i a_j^* = a_i a_i^*$, so

$$a_i c^* = (a_i a_i^* - a_i a_i^*)b^* = 0\,;$$

whereas if $i \notin \mathrm{J}$ then $ba_i = 0$ and $\sum_{j\in \mathrm{J}} a_i a_j^* = 0$ by orthogonality, so

$$a_i c^* = a_i(0)^* - (0)b^* = 0$$

and the assertion is proved. It then follows from 14.10 that $ac^* = 0$, so

$$\begin{aligned} cc^* &= b\left(a - \sum_{j\in \mathrm{J}} a_j\right)c^* = bac^* - \sum_{j\in \mathrm{J}} ba_j c^* \\ &= b(0) - \sum_{j\in \mathrm{J}} b\cdot 0 = 0 \end{aligned}$$

and the lemma is proved. ♢

14.12. LEMMA. [20, p. 27, Remark] *Let* A *be a* $*$*-ring with proper involution*, (a_i) *an orthogonal family in* A *and* $a \in \mathrm{A}$. *Then* $a = \oplus a_i$ *if and only if, for each index* j,

$$\bigcap_{i\neq j}\{a_i\}^l \subset \{a - a_j\}^l\,.$$

Proof. "Only if": Suppose $x \in \bigcap_{i\neq j}\{a_i\}^l$. Thus if $\mathrm{J} = \{j\}$ then $xa_i = 0$ for all $i \notin \mathrm{J}$. By 14.11, $xa = xa_j$, thus $x \in \{a - a_j\}^l$.

"If": Suppose the stated condition holds; in analogy with 14.8, let us express this by writing $a = \oplus' a_i$. By the dual of 14.11, if $b \in \mathrm{A}$ and J is a finite set of indices such that $a_i b = 0$ for all $i \notin \mathrm{J}$, then $ab = \sum_{i\in \mathrm{J}} a_i b$. Letting $\mathrm{J} = \{j\}$ we see that the conditions of 14.8 are fulfilled, in other words $a = \oplus a_i$. ♢

14.13. In view of 14.12, Definition 14.8 is 'left-right symmetric'; in other words, $a = \oplus a_i$ in A if and only if $a = \oplus a_i$ in the opposite ring A°.

14.14. PROPOSITION. [20, p. 27, Remark] *Let* A *be a* $*$*-ring with proper involution*, (a_i) *an orthogonal family in* A, *and* $a \in \mathrm{A}$. *In order that* $a = \oplus a_i$ (*in the sense of* 14.8) *it is necessary and sufficient that the following two conditions hold*:

(1) $aa_i^* = a_i a_i^*$ *for all* i;

(2) *if* $a_i x = 0$ *for all* i, *then* $ax = 0$.

One then has $aa_i^* = a_i a_i^* = a_i a^*$ *for all* i.

Proof. Necessity: This is 14.10.

Sufficiency: Suppose (1) and (2) hold. Fix an index j; by 14.12, it will suffice to show that

$$\bigcap_{i\neq j}\{a_i\}^l \subset \{a - a_j\}^l\,.$$

Suppose $xa_i = 0$ for all $i \neq j$; we are to show that $x(a - a_j) = 0$. We assert that

$$(xa - xa_j)a_i^* = 0 \text{ for all } i; \tag{$*$}$$

for, if $i \neq j$ then

$$\begin{aligned}(xa - xa_j)a_i^* &= xaa_i^* - xa_ja_i^* \\ &= xaa_i^* - 0 \qquad \text{by orthogonality} \\ &= xa_ia_i^* \qquad \text{by (1)} \\ &= 0a_i^* = 0\end{aligned}$$

whereas

$$\begin{aligned}(xa - xa_j)a_j^* &= x(aa_j^* - a_ja_j^*) \\ &= x(a_ja_j^* - a_ja_j^*) \quad \text{by (1)} \\ &= 0\,.\end{aligned}$$

Thus $(*)$ holds, in other words, $a_i(xa - xa_j)^* = 0$ for all i; then by (2), $a(xa - xa_j)^* = 0$, thus

$$\begin{aligned}&(xa - xa_j)a^* = 0\,, \\ &xaa^* = xa_ja^* = xaa_j^*\,, \\ &xa(a - a_j)^* = 0\,.\end{aligned}$$

Then

$$\begin{aligned}(xa - xa_j)(xa - xa_j)^* &= (xa - xa_j)(a - a_j)^*x^* \\ &= xa(a - a_j)^*x^* - xa_j(a - a_j)^*x^* \\ &= 0x^* - x(a_ja^* - a_ja_j^*)x^* \\ &= -x(a_ja_j^* - a_ja_j^*)x^* = 0\,,\end{aligned}$$

whence $xa - xa_j = 0$ (the involution is proper). ♢

14.15. PROPOSITION. [20, p. 28] *In a $*$-ring* A *with proper involution, suppose* $a = \oplus a_i$. *Then:*

(i) $a^* = \oplus a_i^*$.

(ii) *If* x *is an element such that the family* (a_ix) *is also orthogonal, then* $\oplus(a_ix) = ax$.

(iii) *If* x *is an element such that the family* (xa_i) *is also orthogonal, then* $\oplus(xa_i) = xa$.

(iv) $aa^* = \oplus(a_ia_i^*)$ *and* $a^*a = \oplus(a_i^*a_i)$.

(v) If x *and* y *are elements such that* $a_ix = ya_i$ *for all* i, *then* $ax = ya$.

Proof. (i) If $a = \oplus a_i$ in A, then $a^* = \oplus a_i^*$ in the opposite ring A° (because $x \mapsto x^*$ is an isomorphism of $*$-rings $\mathrm{A} \to \mathrm{A}^\circ$, A° being equipped with the same involution as A), therefore $a^* = \oplus a_i^*$ in A (14.13).

(ii) Fix j. If $y \in \bigcap_{i \neq j}\{a_ix\}^r$ then $a_i(xy) = (a_ix)y = 0$ for all $i \neq j$, whence $(a - a_j)xy = 0$ by the definition of a (14.8), thus $y \in \{ax - a_jx\}^r$.

(iii) If $(xa_i)\perp$ then also $((xa_i)^*)\perp$, so $(a_i^*x^*)\perp$. By (i) and (ii), $\oplus(a_i^*x^*) = a^*x^*$, therefore, citing (i), $xa = (a^*x^*)^* = \oplus(a_i^*x^*)^* = \oplus(xa_i)$.

(iv) By 14.14, $aa_i^* = a_ia_i^*$; but for $i \neq j$,

$$a_ia_i^* \cdot a_ja_j^* = a_i(a_i^*a_j)a_j^* = 0\,,$$

thus $(aa_i^*)\perp$. By (iii), $aa^* = \oplus(aa_i^*) = \oplus(a_ia_i^*)$. Then $a^*a = \oplus(a_i^*a_i)$ results on replacing a by a^*.

(v) Note that $(a_ix)\perp$: for $i \neq j$,

$$(a_ix)(a_jx)^* = (ya_i)(ya_j)^* = y(a_ia_j^*)y^* = 0\,,$$
$$(a_jx)^*(a_ix) = x^*(a_j^*a_i)x = 0\,.$$

So by (ii), $\oplus(a_ix) = ax$. Also $(ya_i) = (a_ix)\perp$, so $\oplus(ya_i) = ya$ by (iii). Thus $ax = \oplus(a_ix) = \oplus(ya_i) = ya$. ♢

14.16. COROLLARY. *Let* A *be a* $*$*-ring with proper involution*, $\mathrm{S} \subset \mathrm{A}$, *and* $a = \oplus a_i$ *with* $a_i \in \mathrm{S}'$ *for all* i. *Then* $a \in \mathrm{S}'$.

Proof. Immediate from (v) of 14.15. ♢

14.17. [14, Lemma 1.6] Let A be a $*$-ring with proper involution, (e_i) an orthogonal family of projections in A, and suppose that there exists $e = \oplus e_i$. Then e is a projection and $e = \sup e_i$.

{Proof: By 14.15, $e^*e = \oplus(e_i^*e_i) = \oplus e_i = e$, thus e is a projection. Citing 14.14, $ee_i = ee_i^* = e_ie_i^* = e_i$, thus $e_i \leq e$ for all i. On the other hand, if f is a projection such that $e_i \leq f$ for all i, then $e_i(1-f) = 0$ for all i, whence $e(1-f) = 0$ by 14.14, thus $e \leq f$. This proves that e serves as supremum for the family (e_i).}

14.18. Let A be a Rickart $*$-ring, (e_i) an orthogonal family of projections possessing a supremum e in the projection lattice of A. Then $e = \oplus e_i$.

{Proof: (1) One has $ee_i^* = ee_i = e_i = e_ie_i^*$ for all i. (2) If $e_ix = 0$ for all i, then $e_i\mathrm{LP}(x) = 0$, $e_i \leq 1 - \mathrm{LP}(x)$ for all i, $e \leq 1 - \mathrm{LP}(x)$, $e \cdot \mathrm{LP}(x) = 0$, $ex = 0$. By 14.14, $e = \oplus e_i$.}

14.19. Let A be a Rickart $*$-ring, (w_i) an orthogonal family of partial isometries in A (cf. 14.7). If (w_i) is summable in the sense of 14.8, then it is addable in the sense of 14.3. More precisely, if $w = \oplus w_i$ then w is a partial isometry; and if $e_i = \mathrm{RP}(w_i)$, $f_i = \mathrm{LP}(w_i)$, then $\sup e_i$ exists and is equal to w^*w, $\sup f_i$ exists and is equal to ww^*, and $we_i = w_i = f_iw$ for all i.

{Proof: Suppose $w = \oplus w_i$. Then (14.15) $w^*w = \oplus(w_i^*w_i) = \oplus e_i$; so writing $e = w^*w$, from 14.17 we know that e is a projection and serves as $\sup e_i$. Similarly, writing $f = ww^*$, one has $f = \oplus f_i$ and $f = \sup f_i$. Fix an index j. For all $i \neq j$, $f_jw_i = f_j(f_iw_i) = 0$; so by 14.11 one has $f_jw = f_jw_j = w_j$. Similarly $w^* = \oplus w_i^*$ yields $e_jw^* = w_j^*$, $we_j = w_j$. Thus the family (w_i) is addable in the sense of 14.3.}

Conversely:

14.20. Let A be a Baer $*$-ring, (w_i) an orthogonal family of partial isometries in A (cf. 14.7). If (w_i) is addable in the sense of 14.3, then it is summable in the sense of 14.8.

{Proof: Let $e_i = \mathrm{RP}(w_i)$, $f_i = \mathrm{LP}(w_i)$, $e = \sup e_i$, $f = \sup f_i$ (we are in a Baer $*$-ring). By hypothesis, there exists a partial isometry w with $w^*w = e$, $ww^* = f$ and $we_i = w_i = f_iw$ for all i. To show that $w = \oplus w_i$ we verify the conditions of 14.14. (1) For all i, $ww_i^* = w \cdot e_iw_i^* = we_i \cdot w_i^* = w_iw_i^*$. (2) If $w_ix = 0$ for all i, then $e_i = \mathrm{RP}(w_i) \leq 1 - \mathrm{LP}(x)$ for all i, so $e \leq 1 - \mathrm{LP}(x)$, $ex = 0$; therefore $wx = (we)x = w(ex) = 0$.}

14.21. [14, Lemma 1.5] In a Rickart $*$-ring A, suppose $a = \oplus a_i$ in the sense of 14.8. Let $e_i = \mathrm{RP}(a_i)$, $f_i = \mathrm{LP}(a_i)$. Then (e_i) is an orthogonal family of projections having $\mathrm{RP}(a)$ as supremum, and (f_i) is an orthogonal family of projections having $\mathrm{LP}(a)$ as supremum.

{Proof: That $(e_i)\perp$ and $(f_i)\perp$ results from 14.7. We assert that

$$\{a\}^r = \bigcap_i \{a_i\}^r . \tag{$*$}$$

The inclusion $\supset$ is (2) of 14.14. On the other hand, if $ax = 0$, then for all i one has $a_i^*a = a_i^*a_i$ (dual of 14.14) so

$$(a_ix)^*(a_ix) = x^*(a_i^*a_i)x = x^*(a_i^*a)x = (x^*a_i^*)(ax) = 0\,,$$

whence $a_ix = 0$. The formula ($*$) is verified. This means (1.7) that

$$[1 - \mathrm{RP}(a)]\mathrm{A} = \bigcap_i [1 - \mathrm{RP}(a_i)]\mathrm{A} = \bigcap_i (1 - e_i)\mathrm{A}\,;$$

it follows (1.18) that $\inf(1 - e_i)$ exists and is equal to $1 - \mathrm{RP}(a)$, whence $\sup e_i$ exists and is equal to $\mathrm{RP}(a)$. Similarly $\mathrm{LP}(a) = \sup f_i$.}

14.22. DEFINITION. (L. Herman [14]) A **Loomis $*$-ring** is a $*$-ring (with unity), satisfying the EP-axiom of §10, in which every orthogonal family of partial isometries is summable in the sense of 14.8. {Note: The EP-axiom trivially assures that the involution is proper.}

14.23. THEOREM. [14, Th. 1.7] *A $*$-ring* A *is a Loomis $*$-ring if and only if it satisfies the following three conditions*:

(i) A *is a Baer $*$-ring*;

(ii) A *satisfies the* EP-*axiom*;

(iii) *partial isometries are addable in* A.

Proof. "If": From (i), (iii) and 14.20, one sees that every orthogonal family of partial isometries in A is summable in the sense of 14.8, thus A is a Loomis $*$-ring (14.22).

"Only if": Suppose A is a Loomis $*$-ring. In particular, (ii) holds by the definition, hence the involution is proper.

claim 1: Every orthogonal family of projections (e_i) has a supremum.

For, by hypothesis (e_i) is summable, say $e = \oplus e_i$. By 14.17, e is a projection and $e = \sup e_i$.

claim 2: A is a Rickart $*$-ring.

Let $x \in \mathrm{A}$; we seek a projection e such that $\{x\}^r = (1-e)\mathrm{A}$. If $x = 0$ then $e = 0$ fills the bill. Assume $x \neq 0$. Let (e_i) be a maximal orthogonal family of nonzero projections such that for each i, there exists $y_i \in \{x^*x\}''$ with $y_i^* = y_i$ and $x^*x \cdot y_i^2 = e_i$ (Zorn's lemma; get started by EP). Let $e = \oplus e_i$ ($= \sup e_i$, by claim 1); it will be shown that $\{x\}^r = (1-e)\mathrm{A}$. Suppose $xt = 0$. Then $x^*xt = 0$, $y_i^2 x^*xt = 0$, thus $e_i t = 0$ for all i. Since $e = \oplus e_i$, it follows that $et = 0$ (14.14), that is, $t \in (1-e)\mathrm{A}$. Thus $\{x\}^r \subset (1-e)\mathrm{A}$. If, conversely, $t \in (1-e)\mathrm{A}$, that is, $et = 0$, we are to show that $xt = 0$. It will suffice to show that $xe = x$ (for then $xt = xet = x \cdot 0 = 0$). Since $e = \oplus e_i$ and $e_i \in \{x^*x\}''$ for all i, one has $e \in \{x^*x\}''$ by 14.16, therefore $x^*x \cdot (1-e) \in \{x^*x\}''$. We wish to show that $x(1-e) = 0$. Assume to the contrary. Then by the EP-axiom, there exists an element

$$y \in \{[x(1-e)]^*[x(1-e)]\}'' = \{x^*x(1-e)\}'' \subset \{x^*x\}''$$

such that $y^* = y$ and $x^*x(1-e)y^2 = g$, g a nonzero projection. Replacing y by $(1-e)y$, we have $y \in \{x^*x\}''$, $y^* = y$, $ye = 0$ and $x^*x \cdot y^2 = g$. Obviously $ge = 0$, whence $ge_i = 0$ for all i, contradicting maximality.

From claims 1 and 2, one sees easily that A is a Baer $*$-ring [2, p. 20, Prop. 1], thus (i) holds. Finally, (iii) is immediate from 14.19. ♢

What is striking in this circle of ideas is that the conditions (i), (ii), (iii) imply SR (see 14.29 below).

***14.24.** COROLLARY. *A* C*-*algebra is an* AW*-*algebra if and only if it is a Loomis $*$-ring.*

Proof. "If": Immediate from (i) of 14.23 and the definition of AW*-algebra (1.38).

"Only if": Every AW*-algebra is a Baer $*$-ring (1.38) with EP [2, p. 43, Cor.], in which partial isometries are addable [2, p. 129, Th. 1]; thus the conditions (i), (ii), (iii) of 14.23 are fulfilled. ♢

14.25. COROLLARY. *If* A *is a Baer $*$-ring, with no abelian summand, satisfying the* EP-*axiom, in which* GC *holds for* $\overset{*}{\sim}$, *then* A *is a Loomis $*$-ring.*

Proof. Condition (iii) of 14.23 holds by [2, p. 129, Th. 1] (whose proof is long and tedious). ♢

For various reformulations of this result, see 14.32 below.

14.26. COROLLARY. *If* A *is a regular Baer $*$-ring with no abelian summand, satisfying the* SR-*axiom, then* A *is a Loomis $*$-ring.*

Proof. The SR-axiom implies that axiom H holds for $\overset{*}{\sim}$ (12.13), which in turn implies that GC holds for $\overset{*}{\sim}$ (13.10). So by 14.25, we need only verify the EP-axiom. Let $x \in \mathrm{A}$, $x \neq 0$. By the SR-axiom, write $x^*x = r^2$ with $r^* = r \in$

$\{x^*x\}''$, and let $e = \mathrm{RP}(x) = \mathrm{RP}(r)$. Let y be the relative inverse of r (2.7), in other words, y is the inverse of r in $e\mathrm{A}e$. Then $y^* = y$ (because $r^* = r$), $y \in \{x^*x\}''$ (4.7), and $x^*x \cdot y^2 = r^2y^2 = e \neq 0$. ♢

14.27. COROLLARY. *If* A *is a Baer* $*$*-ring satisfying the* EP-*axiom*, *properly infinite relative to* $\overset{*}{\sim}$, *then* A *is a Loomis* $*$*-ring*.

Proof. By 13.5, A satisfies axiom E for $\overset{*}{\sim}$; since, moreover, A is properly infinite for $\overset{*}{\sim}$, it follows that partial isometries are addable in A [2, p. 131, Exer. 3]. Thus the conditions (i), (ii), (iii) of 14.23 are satisfied. ♢

14.28. LEMMA. *In a* $*$*-ring satisfying the* EP-*axiom*, *if* x *is a nonzero element then there exists an element* s *such that*:

(i) $s^* = s$,
(ii) $s \in \{x^*x\}''$,
(iii) $x^*x \cdot s^2$ *is a nonzero projection* e,
(iv) xs *is a partial isometry*, *with* $(xs)^*(xs) = e$,
(v) $es = s$.

Proof. By the EP-axiom, there exists an element s satisfying (i), (ii), (iii). Moreover, $(xs)^*(xs) = sx^*xs = x^*xs^2 = e$. Since $e \in \{x^*x\}''$, replacing s by es $(= se)$ we can suppose $es = s$. ♢

14.29. THEOREM. (L. Herman [14, Th. 2.2]) *Every Loomis* $*$*-ring has polar decomposition*.

Proof. Let A be a Loomis $*$-ring, $x \in \mathrm{A}$. If $x = 0$ then the elements $w = r = 0$ meet the requirements of 14.1.

Assume $x \neq 0$. Let (e_i) be a maximal orthogonal family of nonzero projections such that for each i there exists $s_i \in \{x^*x\}''$ with $s_i^* = s_i$, $x^*x \cdot s_i^2 = e_i$, $e_is_i = s_i$ (Zorn's lemma, using 14.28 to get started). Set $w_i = xs_i$, a partial isometry with $w_i^*w_i = e_i$.

Write $e = \sup e_i$ (which exists by 14.23). We assert that $e = \mathrm{RP}(x)$. From $e_i = s_i^2x^*x$ it is clear that $e_i \leq \mathrm{RP}(x)$ for all i, therefore $e \leq \mathrm{RP}(x)$. To show that $e \geq \mathrm{RP}(x)$ it will suffice to show that $xe = x$, that is, $x(1-e) = 0$. At any rate, since $e_i \in \{x^*x\}''$ for all i, one has $e \in \{x^*x\}''$ (4.5), whence $x^*x(1-e) \in \{x^*x\}''$. If, on the contrary, $x(1-e) \neq 0$, by the lemma there exist a nonzero projection g and an element

$$y \in \{[x(1-e)]^*[x(1-e)]\}'' = \{x^*x(1-e)\}'' \subset \{x^*x\}''$$

such that $y^* = y$, $x^*x(1-e) \cdot y^2 = g$, $gy = y$. Evidently $ge = 0$, so $ge_i = 0$ for all i, contradicting maximality.

Let $f_i = w_iw_i^* = xs_i^2x^*$. The family (f_i) of projections is orthogonal: if $i \neq j$ then

$$f_if_j = xs_i^2x^* \cdot xs_j^2x^* = xs_i^2 \cdot e_jx^* = xs_i^2e_i \cdot e_jx^* = 0\,.$$

Let $f = \sup f_i$. From $f_i = xs_i^2x^*$ it is clear that $f_i \leq \mathrm{LP}(x)$ for all i, therefore $f \leq \mathrm{LP}(x)$. We assert that $f = \mathrm{LP}(x)$. Let $h = \mathrm{LP}(x) - f$. For all i one has

$(1-f)f_i = 0$, hence $hf_i = 0$, so

$$0 = 0x = hf_i x = hxs_i^2 x^*x = hxe_i\,;$$

thus $hxe_i = 0$ for all i, whence $0 = hxe = hx$. Therefore $h \cdot \mathrm{LP}(x) = 0$; but $h \le \mathrm{LP}(x)$, so $h = h \cdot \mathrm{LP}(x) = 0$, $\mathrm{LP}(x) = f$.

Since (e_i) and (f_i) are orthogonal families of projections, (w_i) is an orthogonal family of partial isometries (14.7). By hypothesis, (w_i) is summable, in other words (14.19) addable, and, writing $w = \oplus w_i$, we have $w^*w = \sup e_i = e$, $ww^* = \sup f_i = f$, and $we_i = w_i = f_i w$ for all i.

Set $r = w^*x$. Then $wr = ww^*x = fx = x$. Now, $w^* = \oplus w_i^*$ and $w_i^*x = s_i x^*x = x^*xs_i \in e_i\mathrm{A}e_i$ shows that (w_i^*x) is an orthogonal family, therefore by 14.15 one has $w^*x = \oplus(w_i^*x) = \oplus(x^*x \cdot s_i)$, that is, $r = \oplus(x^*x \cdot s_i)$. Since $x^*x \cdot s_i \in \{x^*x\}''$ for all i, it follows from 14.16 that $r \in \{x^*x\}''$. Also, since the x^*xs_i are self-adjoint, by 14.15 one has

$$r^* = \oplus(x^*x \cdot s_i)^* = \oplus(x^*x \cdot s_i) = r\,,$$

and

$$\begin{aligned} r^2 = r^*r &= \oplus[(x^*x \cdot s_i)^*(x^*x \cdot s_i)] \\ &= \oplus[x^*x \cdot x^*xs_i^2] = \oplus(x^*x \cdot e_i) \\ &= x^*x(\oplus e_i) = x^*xe = x^*x\,. \end{aligned}$$

Summarizing, we have $x = wr$ with $r^* = r \in \{x^*x\}''$, $r^2 = x^*x$, $w^*w = e = \mathrm{RP}(x)$, $ww^* = f = \mathrm{LP}(x)$; thus A has PD (14.1).

{Incidentally, writing $r_i = x^*xs_i \in \{x^*x\}''$, we have $r = \oplus r_i$. One has $r_i \in e_i\mathrm{A}e_i$, $r_is_i = s_ir_i = e_i$, thus r_i is the inverse of s_i in $e_i\mathrm{A}e_i$. Also, $r_i^2 = x^*x \cdot x^*xs_i^2 = x^*x \cdot e_i = e_i \cdot x^*x$, thus r_i is a 'square root' of x^*xe_i.} $\diamondsuit$

14.30. COROLLARY. *The following classes of Baer $*$-rings* (are Loomis $*$-rings, hence) *have polar decomposition*:

*(i) AW*-*algebras* (14.24);

(ii) *Baer $*$-rings without abelian summand, satisfying the* EP-*axiom and* GC *for* $\overset{*}{\sim}$ (14.25);

(iii) *regular Baer $*$-rings without abelian summand, satisfying the* SR-*axiom* (14.26);

(iv) *Baer $*$-rings satisfying the* EP-*axiom and properly infinite for* $\overset{*}{\sim}$ (14.27);

(v) *Baer $*$-rings without abelian summand, satisfying the* EP- *and* SR-*axioms.*

Proof. (v) [18, p. 104, Th. 64] From SR we know that axiom H holds for $\overset{*}{\sim}$ (12.13), therefore GC holds for $\overset{*}{\sim}$ (13.10). So we are in a Loomis $*$-ring (14.25), whence PD (14.29). $\diamondsuit$

14.31. COROLLARY. *The condition* LP $\overset{*}{\sim}$ RP *holds for the following classes of Baer $*$-rings*:

*(i) AW*-*algebras*;

(ii) *Baer $*$-rings satisfying the* EP-*axiom and* GC *for* $\overset{*}{\sim}$;

(iii) *regular Baer $*$-rings satisfying the* SR-*axiom*;

(iv) *Baer $*$-rings satisfying the* EP-*axiom and properly infinite for* $\overset{*}{\sim}$;

(v) *Baer $*$-rings satisfying the* EP- *and* SR-*axioms*.

Proof. Since $\mathrm{LP}(x) = \mathrm{RP}(x)$ for all x in an abelian ring (8.12), the corollary is immediate from 8.27 and 14.30. {We remark that the assertion for (iii) is also a trivial consequence of 5.8 and 6.10.} ◊

See 21.44 for a simpler proof of (ii).

14.32. COROLLARY. *For a Baer $*$-ring* A *without abelian summand, the following conditions are equivalent*:

(a) A *is a Loomis $*$-ring*;

(b) A *satisfies the* EP-*axiom and has* PD;

(c) A *satisfies the* EP- *and* SR-*axioms*;

(d) A *satisfies the* EP-*axiom and* $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$;

(e) A *satisfies the* EP-*axiom and axiom* H *for* $\overset{*}{\sim}$;

(f) A *satisfies the* EP-*axiom and* GC *for* $\overset{*}{\sim}$.

Proof. (a) ⇒ (b): 14.29.

(b) ⇒ (c), (d): trivial.

(c) ⇒ (e): 12.13 (theorem of S. Maeda).

(d) ⇒ (e): trivial (1.15).

(e) ⇒ (f): 13.10 (theorem of Maeda and Holland).

(f) ⇒ (a): 14.25. ◊ Cf. 21.44.

14.33. Abelian rings are genuinely pathological in this circle of ideas. For example, there exists a commutative Baer $*$-ring satisfying the EP- and SR-axioms, in which partial isometries are not in general addable ([18, p. 103], [2, p. 131, Exer. 4]). Refining Kaplansky's example, Herman [14] exhibited a Baer $*$-ring satisfying the EP-axiom and PD, in which partial isometries are not in general addable; thus, the implication (b) ⇒ (a) of 14.32 fails in general.

15. FINITE AND INFINITE RINGS

Throughout this section, A is a Baer $*$-ring and $\sim$ is an equivalence relation on its projection lattice *satisfying the axioms* A, B, C$'$, D, F *and generalized comparability* (GC) of §10. (The precise axioms actually needed for each proof will be noted parenthetically.) Finiteness and infiniteness are defined relative to $\sim$ (9.5, 9.9).

15.1. Axiom E holds.

{Proof (using GC and axioms A, D): Suppose $e\mathrm{A}f \neq 0$, in other words $\mathrm{C}(e)\mathrm{C}(f) \neq 0$ (3.21); we are to show that e, f are partially comparable (13.7). By GC, write $e = e_1 + e_2$, $f = f_1 + f_2$ with $e_1 \sim f_1$ and $\mathrm{C}(e_2)\mathrm{C}(f_2) = 0$ (13.2). Since $\mathrm{C}(e)\mathrm{C}(f) \neq 0$, either $e \neq e_2$ or $f \neq f_2$, thus $e_1 \neq 0$ or $f_1 \neq 0$, hence (axiom A) $e_1 \neq 0$ *and* $f_1 \neq 0$.}

Recall that $\overset{*}{\sim}$ always satisfies A–D and F (11.2); thus the present section (and following ones) is in a sense an exploitation of the consequences of GC. Since

$$\text{SR} \Rightarrow \text{axiom H for } \overset{*}{\sim} \Rightarrow \text{GC for } \overset{*}{\sim}$$

by the theorems of Maeda and Holland (12.13 and 13.10), GC is a natural axiom for many applications.

15.2. If f is a finite projection (relative to $\sim$; see 9.5) and $e \leq f$, then e is finite.

{Proof: (Axiom D) This is 9.8.}

15.3. If f is finite and $e \sim f$, then e is finite [18, p. 52, Th. 32].

{Proof: (Axioms A, C$'$) Suppose $g \leq e$, $g \sim e$. By axiom C$'$, there exists $h \leq f$ with $g \sim h$ and $e - g \sim f - h$. Then $f \geq h \sim g \sim e \sim f$, thus $f \sim h \leq f$; since f is finite, $h = f$, so $e - g \sim f - h = 0$, thus $e - g = 0$ by axiom A.}

15.4. If f is finite and $e \precsim f$, then e is finite.

{Proof: (Axioms A, C$'$, D) Immediate from 15.2, 15.3.}

15.5. PROPOSITION. [18, p. 88, Exer. 2] *The following conditions on* A *are equivalent*:

(a) A *is finite*;

(b) $e \sim f \Rightarrow 1 - e \sim 1 - f$.

Proof. (a) $\Rightarrow$ (b): (Axioms B, D and GC) Suppose $e \sim f$. Let u be a central projection with $u(1-e) \precsim u(1-f)$ and $(1-u)(1-f) \precsim (1-u)(1-e)$. Say

$u(1-e) \sim g \le u(1-f)$ and $(1-u)(1-f) \sim h \le (1-u)(1-e)$. Since $ue \sim uf$ (axiom B) one has

$$u = ue + u(1-e) \sim uf + g \le u$$

by axiom D; since u is finite (15.2) it follows that $uf + g = u$, thus $g = u(1-f)$, whence $u(1-e) \sim u(1-f)$. Similarly $(1-u)(1-e) \sim (1-u)(1-f)$, and another application of axiom D yields $1-e \sim 1-f$.

(b) $\Rightarrow$ (a): (Axiom A) Suppose $e \sim 1$. By (b), $1-e \sim 0$, so $1-e = 0$ by axiom A. $\diamondsuit$

Infiniteness is characterized in the next proposition.

15.6. LEMMA. [18, p. 58, Th. 38] *Let* $(e_i)_{i\in \mathrm{I}}$ *be an infinite family of pairwise orthogonal projections with* $e_i \sim e_j$ *for all* i *and* j, *let* J *be a subset of* I *with* $\operatorname{card}\mathrm{J} = \operatorname{card}\mathrm{I}$, *and let*

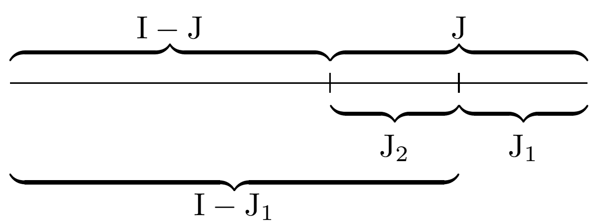
$$e = \sup\{e_i : \ i \in \mathrm{I}\}, \quad f = \sup\{e_j : \ j \in \mathrm{J}\}.$$

Then $e \sim f$.

Proof. (Axioms D, F) Write J as a disjoint union

$$\mathrm{J} = \mathrm{J}_1 \cup \mathrm{J}_2$$

with $\operatorname{card}\mathrm{J}_1 = \operatorname{card}\mathrm{J}_2 = \operatorname{card}\mathrm{J} = \operatorname{card}\mathrm{I}$.



One has $\mathrm{J}_2 \subset \mathrm{J}_2 \cup (\mathrm{I}-\mathrm{J}) = \mathrm{I} - \mathrm{J}_1 \subset \mathrm{I}$, so

$$\operatorname{card}\mathrm{I} = \operatorname{card}\mathrm{J}_2 \le \operatorname{card}(\mathrm{I}-\mathrm{J}_1) \le \operatorname{card}\mathrm{I},$$

therefore

$$\operatorname{card}(\mathrm{I}-\mathrm{J}_1) = \operatorname{card}\mathrm{I}$$

(Schröder-Bernstein theorem). Set

$$f_1 = \sup\{e_i : \ i \in \mathrm{J}_1\}, \quad f_2 = \sup\{e_i : \ i \in \mathrm{J}_2\};$$

one has $f_1f_2 = 0$, $f = f_1 + f_2$, and $f_2 \sim f_1$ by axiom F. Set

$$g = \sup\{e_i : \ i \in \mathrm{I} - \mathrm{J}_1\};$$

then $f_1g = 0$, $e = f_1 + g$, and $f_1 \sim g$ by axiom F. Then axiom D yields $f_1 + f_2 \sim g + f_1$, that is, $f \sim e$. ♢

15.7. PROPOSITION. [18, p. 62, Exer. 5] *The following conditions are equivalent*:

(a) A *is infinite* (*relative to* $\sim$);

(b) A *contains an infinite family* (e_i) *of pairwise orthogonal nonzero projections with* $e_i \sim e_j$ *for all* i *and* j.

Proof. (Axioms A, C′, D, F)

(b) ⇒ (a): (Axioms D, F) Dropping down to a subfamily, we can suppose the index set to be $\mathrm{I} = \{1, 2, 3, \ldots\}$. Let $\mathrm{J} = \{2, 3, 4, \ldots\}$. With notations as in 15.6, $e \sim f \le e$ but $f \ne e$, thus e is not finite, therefore neither is 1 (15.2).

(a) ⇒ (b): (Axioms A, C′) By hypothesis, there exists a projection $f_1 \ne 1$ with $f_1 \sim 1$. Also $f_1 \ne 0$ (axiom A). Write $e_1 = 1 - f_1$. From $e_1 + f_1 = 1 \sim f_1$ and axiom C′, one obtains an orthogonal decomposition $f_1 = e_2 + f_2$ with $e_1 \sim e_2$ and $f_1 \sim f_2$. Note that

$$1 = e_1 + f_1 = e_1 + (e_2 + f_2),$$

in particular $e_1e_2 = 0$. From $e_2 + f_2 = f_1 \sim f_2$ and axiom C′, one obtains an orthogonal decomposition $f_2 = e_3 + f_3$ with $e_2 \sim e_3$, $f_2 \sim f_3$. Note that

$$1 = e_1 + e_2 + f_2 = e_1 + e_2 + (e_3 + f_3),$$

in particular e_1, e_2, e_3 are pairwise orthogonal. Etc. ♢

15.8. LEMMA. [18, p. 63, Th. 42] (“Absorbing a scrap”) *Let* $(e_i)_{i\in\mathrm{I}}$ *be an infinite family of pairwise orthogonal projections with* $e_i \sim e_j$ *for all* i *and* j, *and let* $e = \sup\{e_i : \ i \in \mathrm{I}\}$. *Fix an index* $1 \in \mathrm{I}$ *and suppose that* f *is a projection such that* $f \precsim e_1$ *and* $fe = 0$. *Then* $e + f$ *is the supremum of an orthogonal family* $(h_i)_{i\in\mathrm{I}}$ *with* $h_i \sim e_1$ *for all* $i \in \mathrm{I}$.

Proof. (Axioms C′, D) Say $f \sim f_1 \le e_1$; write $g_1 = e_1 - f_1$. Then $f_1 + g_1 = e_1 \sim e_i$, so by axiom C′ there is a decomposition $e_i = f_i + g_i$ with $f_i \sim f_1 \sim f$ and $g_i \sim g_1$. Note that the f_i are pairwise orthogonal (because $f_i \le e_i$) and they are all orthogonal to f (because $ef = 0$); thus $\{f\} \cup \{f_i : \ i \in \mathrm{I}\}$ is an orthogonal family of pairwise equivalent projections, of cardinality $\operatorname{card}\mathrm{I}$. Also, (g_i) is an orthogonal family of pairwise equivalent projections, and each g_i is orthogonal to every f_j and to f.

Since I is infinite, there exists a bijection $\sigma : \mathrm{I} - \{1\} \to \mathrm{I}$. Define a bijection

$$\theta : \{g_i : \ i \in \mathrm{I}\} \to \{f\} \cup \{f_i : \ i \in \mathrm{I}\}$$

by

$$\theta(g_1) = f\,, \quad \theta(g_i) = f_{\sigma(i)} \ \text{ for } \ i \neq 1\,.$$

For each i, g_i and $\theta(g_i)$ are orthogonal; define

$$h_i = g_i + \theta(g_i) \quad (i \in \mathrm{I})\,.$$

Explicitly,

$$h_1 = g_1 + f\,, \quad h_i = g_i + f_{\sigma(i)} \ \text{ for } \ i \neq 1\,.$$

The family (h_i) is clearly orthogonal. By axiom D,

$$h_1 = g_1 + f \sim g_1 + f_1 = e_1\,,$$

and for $i \neq 1$,

$$h_i = g_i + f_{\sigma(i)} \sim g_i + f_i = e_i \sim e_1\,.$$

Thus $h_i \sim e_1$ for all $i \in \mathrm{I}$. And

$$\begin{aligned}
\sup\{h_i :\ i \in \mathrm{I}\} &= \sup\{h_i :\ i \neq 1\} + h_1 \\
&= \sup\{g_i + f_{\sigma(i)} :\ i \neq 1\} + g_1 + f \\
&= \sup\{g_i :\ i \neq 1\} + \sup\{f_{\sigma(i)} :\ i \neq 1\} + g_1 + f \\
&= \sup\{g_i :\ i \in \mathrm{I}\} + \sup\{f_j :\ j \in \mathrm{I}\} + f \\
&= \sup\{g_i + f_i :\ i \in \mathrm{I}\} + f \\
&= \sup\{e_i :\ i \in \mathrm{I}\} + f \\
&= e + f\,. \ \diamondsuit
\end{aligned}$$

15.9. LEMMA. [18, p. 64, Th. 43] *Suppose $(e_i)_{i\in\mathrm{I}}$ is a maximal family of pairwise orthogonal, nonzero projections in* A *such that $e_i \sim e_j$ for all i and j, and suppose that the index set* I *is infinite. Then there exist a nonzero central projection u and an orthogonal family $(h_i)_{i\in\mathrm{I}}$ with $u = \sup h_i$ and $h_i \sim ue_i$ for all $i \in \mathrm{I}$.*

Proof. Fix an index $1 \in \mathrm{I}$. Let $e = \sup e_i$ and apply GC to the pair $1-e, e_1$: let u be a central projection with

$$u(1-e) \precsim ue_1 \ \text{ and } \ (1-u)e_1 \precsim (1-u)(1-e)\,.$$

Note that $ue_1 \neq 0$. {For, $ue_1 = 0$ would imply that

$$e_1 = e_1 - ue_1 = (1-u)e_1 \precsim (1-u)(1-e) \leq 1 - e\,,$$

briefly $e_1 \precsim 1 - e$. Say $e_1 \sim e' \leq 1 - e$; then e' could be adjoined to the family (e_i), contradicting maximality.} The ue_i are pairwise equivalent by axiom B, and $u(1-e) \precsim ue_1$; so 15.8 provides a decomposition $(h_i)_{i\in\mathrm{I}}$ of the projection

$$\sup\{ue_i :\ i \in \mathrm{I}\} + u(1-e) = ue + u(1-e) = u$$

meeting the requirements. $\{$Since $1-e, e_1$ are orthogonal, all that is really needed is orthogonal GC; in view of 13.3, the proof works assuming B, C′, D, E, F.$\}$ $\diamondsuit$

15.10. LEMMA. [18, p. 64, Th. 44] *If* A *is properly infinite* (9.9), *then there exists an orthogonal sequence* (e_n) *of projections such that* $e_1 \sim e_2 \sim e_3 \sim \ldots$ *and* $\sup e_n = 1$.

Proof. By 15.7, there exists an infinite family of pairwise orthogonal, equivalent, nonzero projections, and we can suppose the family to be maximal in these properties (Zorn). It then follows from 15.9 that there exist a nonzero central projection u and an orthogonal family $(f_i)_{i\in \mathrm{I}}$ of projections with $u = \sup f_i$, $f_i \sim f_j$ for all i and j, and I infinite. Since $\aleph_0 \cdot \operatorname{card} \mathrm{I} = \operatorname{card} \mathrm{I}$, one can write

$$\mathrm{I} = \bigcup_{n=1}^{\infty} \mathrm{I}_n$$

with the I_n pairwise disjoint and $\operatorname{card} \mathrm{I}_n = \operatorname{card} \mathrm{I}$ for all n. Defining $f_n = \sup\{f_i :\ i \in \mathrm{I}_n\}$ for $n = 1, 2, 3, \ldots$, one has $f_1 \sim f_2 \sim f_3 \sim \ldots$ by axiom F, and the f_n are pairwise orthogonal with $\sup f_n = u$.

Let $(u_\alpha)_{\alpha\in \mathrm{J}}$ be a maximal orthogonal family of nonzero central projections such that, for each α, there exists an infinite sequence of pairwise orthogonal projections $e_1^\alpha, e_2^\alpha, e_3^\alpha, \ldots$ with

$$u_\alpha = \sup\{e_n^\alpha :\ n = 1, 2, 3, \ldots\} \text{ and } e_1^\alpha \sim e_2^\alpha \sim e_3^\alpha \sim \ldots$$

(Zorn's lemma; get started by the preceding paragraph). Defining

$$e_n = \sup\{e_n^\alpha :\ \alpha \in \mathrm{J}\} \text{ for } n = 1, 2, 3, \ldots,$$

one has $e_1 \sim e_2 \sim e_3 \sim \ldots$ by axiom F, and $\sup e_n = \sup u_\alpha$; thus it will suffice to show that $\sup u_\alpha = 1$. Let $v = 1 - \sup u_\alpha$ and assume to the contrary that $v \neq 0$. Since $v\mathrm{A}$ is infinite (A is properly infinite), an application of the first paragraph of the proof contradicts maximality. $\{$Inspecting the proofs of 15.7 and 15.9, one sees that the present lemma also holds assuming axioms A, B, C′, D, E, F.$\}$ $\diamondsuit$

15.11. THEOREM. [18, p. 65, Th. 45] *If* A *is properly infinite* (9.9) *then*:

(i) *there exists an orthogonal sequence of projections* (f_n) *with* $\sup f_n = 1$ *and* $f_n \sim 1$ *for all* n;

(ii) *for each positive integer* m, *there exist orthogonal projections* $g_1, \ldots, g_m$ *with* $g_1 + \ldots + g_m = 1$ *and* $g_k \sim 1$ *for* $k = 1, \ldots, m$.

Proof. (i) Let $\mathrm{I} = \{1, 2, 3, \ldots\}$ and write

$$\mathrm{I} = \mathrm{I}_1 \cup \mathrm{I}_2 \cup \mathrm{I}_3 \cup \ldots \tag{$*$}$$

with the I_n pairwise disjoint and infinite. With notations as in 15.10, define

$$f_n = \sup\{e_i :\ i \in \mathrm{I}_n\};$$

then the f_n are pairwise orthogonal with $\sup f_n = 1$, and $f_n \sim 1$ by 15.6.

(ii) Same as (i), with just m terms on the right side of (*).

{The proof is valid assuming axioms A, B, C′, D, E, F.} $\diamondsuit$

If $\sim$ means $\overset{a}{\sim}$ in 15.11, (ii), then A is isomorphic to the matrix ring $\mathrm{M}_m(\mathrm{A})$, and when $\sim$ means $\overset{*}{\sim}$ this is a $*$-isomorphism (with $*$-transpose as the involution on the matrix ring) [cf. 2, p. 98, Prop. 1].

15.12. COROLLARY. *Under the hypotheses of* 15.11, *there exists a projection* g *with* $g \sim 1 \sim 1 - g$.

Proof. Take $m = 2$ in 15.11, (ii). $\diamondsuit$

15.13. THEOREM. [18, p. 86, Th. 56] *Suppose* A *is a Baer $*$-ring with an equivalence relation on its projection lattice satisfying the axioms* A, B, C′, D, F *and* H *of* §10.

If e *and* f *are finite projections in* A, *then the projection* $e \cup f$ *is also finite*.

Proof. A has GC by the theorem of Maeda and Holland (13.9), so we are under the hypotheses stated at the beginning of the section.

One has $e \cup f = (e \cup f - f) + f$, where, by axiom H, $e \cup f - f \sim e - e \cap f \le e$, thus $e \cup f - f \precsim e$; since $e \cup f - f$ is therefore finite (15.4), we are reduced to the case that $ef = 0$. Dropping down to $(e+f)\mathrm{A}(e+f)$, we can suppose that $e + f = 1$; we are to show that A is finite.

Assume to the contrary that A is infinite. Dropping down to a direct summand, we can suppose that A is properly infinite (9.11). Let g be a projection with $g \sim 1 \sim 1 - g$ (15.12). Apply 13.14 to the pair g, e: there is a central projection u such that

$$ug \precsim ue \ \text{ and } (1-u)(1-g) \precsim (1-u)(1-e) = (1-u)f\,.$$

From $u = u \cdot 1 \sim ug \precsim ue$ we see that u is finite (15.4); similarly, from

$$1 - u = (1-u) \cdot 1 \sim (1-u)(1-g) \precsim (1-u)f$$

we see that $1-u$ is finite. Since A is properly infinite, necessarily $u = 1-u = 0$, whence $1 = 0$, a contradiction. $\diamondsuit$

16. RINGS OF TYPE I

Throughout this section (as in §15) A is a Baer $*$-ring, $\sim$ is an equivalence relation on its projection lattice satisfying the axioms A, B, C′, D, F and GC of §10. As noted in 15.1, axiom E also holds. (The precise axioms needed for each proof will be noted parenthetically.)

16.1. LEMMA. [18, p. 53, Th. 34] *If* e, f *are projections in* A *with* f *abelian and* $e \precsim f$, *then* e *is abelian.*

Proof. (Axioms A, B, C′) In view of 8.5, it suffices to consider $e \sim f$. Since f is finite (9.17) so is e (15.3). Let g be a projection with $g \le e$; it will suffice (8.3) to show that $g = e\mathrm{C}(g)$. From $g \le e \sim f$ and axiom C′, one has $g \sim h \le f$ for suitable h. Since f is abelian, $h = f\mathrm{C}(h)$; but $\mathrm{C}(g) = \mathrm{C}(h)$ by 9.16, and $g \le e$, $g \le \mathrm{C}(g)$, therefore

$$g \le e\mathrm{C}(g) = e\mathrm{C}(h) \sim f\mathrm{C}(h) = h \sim g\,,$$

thus $e\mathrm{C}(g)$ is equivalent to its subprojection g. But $f\mathrm{C}(h)$ is abelian (it is $\le f$) hence finite (9.17); and $e\mathrm{C}(g) \sim f\mathrm{C}(h)$, so $e\mathrm{C}(g)$ is also finite (15.3), therefore $g = e\mathrm{C}(g)$. ♢

16.2. DEFINITION. A is said to be **homogeneous** if there exists an orthogonal family $(e_i)_{i\in \mathrm{I}}$ of abelian projections in A such that $\sup e_i = 1$ and $e_i \sim e_j$ for all i and j.

16.3. Homogeneous ⇒ type I.

{Proof: (Axioms A, B) With notations as in 16.2, fix an index j. One has $\mathrm{C}(e_i) = \mathrm{C}(e_j)$ for all i (9.16), therefore (3.22)

$$1 = \mathrm{C}(1) = \mathrm{C}(\sup e_i) = \sup \mathrm{C}(e_i) = \mathrm{C}(e_j)\,;$$

thus e_j is a faithful abelian projection, so A is of type I (8.14).}

16.4. If e, f are abelian projections with $\mathrm{C}(e) = \mathrm{C}(f)$, then $e \sim f$.

{Proof: (A, B, D and GC) Let u be a central projection with $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$. Say $ue \sim f' \le uf$. Then (3.17 and 9.16)

$$u\mathrm{C}(e) = \mathrm{C}(ue) = \mathrm{C}(f') \le \mathrm{C}(uf) = u\mathrm{C}(f) = u\mathrm{C}(e)\,,$$

so $C(f') = uC(f)$, whence

$$fC(f') = fuC(f) = uf\,.$$

But f is abelian, so $fC(f') = f'$ (8.3), therefore $uf = f'$. Thus $ue \sim f' = uf$. Similarly $(1-u)e \sim (1-u)f$, therefore $e \sim f$ by axiom D.}

16.5. If e, f are projections with e abelian and $C(e) \le C(f)$, then $e \precsim f$.

{Proof: (Axioms A, B, C′, D and GC): Let u be a central projection with $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$. Since e is abelian, so is $(1-u)e$, hence (16.1) so is $(1-u)f$; moreover (cf. 9.16)

$$C[(1-u)f] \le C[(1-u)e] = (1-u)C(e) \le (1-u)C(f) = C[(1-u)f]\,,$$

whence equality throughout, therefore $(1-u)e \sim (1-u)f$ by 16.4. But also $ue \precsim uf$, therefore $e \precsim f$.}

16.6. DEFINITION. A is said to be homogeneous of **order** $\aleph$ if, in 16.2, the cardinality of the index set I is $\aleph$. {We are not affirming that $\aleph$ is uniquely determined by A. Problem: Is it?[1]}

16.7. DEFINITION. A is said to be of **type** I_n (n a positive integer) if (i) A is finite (relative to $\sim$), and (ii) A is homogeneous of order n. {We see in 16.11 that n is unique. Problem: Is condition (i) redundant?} When $\sim$ means $\overset{a}{\sim}$ (resp. $\overset{*}{\sim}$), a ring of type I_n is isomorphic (resp. $*$-isomorphic) to an $n \times n$ matrix ring over an abelian ring [cf. 2, p. 98, Prop. 1].

16.8. DEFINITION. Let $\aleph$ be an infinite cardinal. If A is homogeneous of order $\aleph$, it is said to be of **type** $\mathrm{I}_\aleph$. {Note: (Axioms D, F) Such a ring is infinite by 15.7.}

16.9. Suppose A is of type $\mathrm{I}_\aleph$. If $\aleph$ is infinite then A is infinite (16.8); if $\aleph$ is finite then A is finite (by Definition 16.7) and $\aleph$ is unique (by 16.11 below).

16.10. Suppose A is homogeneous of order $\aleph$. If $\aleph$ is infinite, then A is infinite and is of type $\mathrm{I}_\aleph$ (16.8). If $\aleph$ is finite, we cannot conclude that A is of type $\mathrm{I}_\aleph$ (because we do not know that A is finite). In other words, it is conceivable that there exists an infinite ring that is homogeneous of order n (n an integer). {Problem: Does there? The crux of the matter is the following question: If A is properly infinite (cf. 15.12) can it be the ring of 2×2 matrices over an abelian ring?} This difficulty disappears if axiom H is added to the hypotheses (15.13).

16.11. PROPOSITION. [18, p. 68, Th. 47] *If* A *is of type* I_n (n *a positive integer*), *then* A *cannot contain* $n+1$ *pairwise orthogonal, equivalent, nonzero projections. In particular,* n *is unique.*

Proof. (Axioms A, B, D and GC) Write $1 = e_1 + \ldots + e_n$ with $e_1 \sim \ldots \sim e_n$ and the e_i abelian (16.7). Suppose $f_1, \ldots, f_m$ are pairwise orthogonal, equivalent,

[1] Not necessarily, even for AW*-algebras [M. Ozawa, "Nonuniqueness of the cardinality attached to homogeneous AW*-algebras", *Proc. Amer. Math. Soc.* 93 (1985), 681-684].

nonzero projections with $f_1 \sim \ldots \sim f_m$ and $m \geq n$; we are to show that $m = n$. Let $u = \mathrm{C}(f_1)$. Then $ue_1 \neq 0$ (e_1 is faithful). Dropping down to $u\mathrm{A}$, we can suppose that the f_j are also faithful. Then $e_i \precsim f_i$ for $1 \leq i \leq n$ (16.5), say $e_i \sim f_i' \leq f_i$, so

$$1 = e_1 + \ldots + e_n \sim f_1' + \ldots + f_n' \leq f_1 + \ldots + f_n \leq 1\,;$$

since A is finite, $f_1' + \ldots + f_n' = 1$, therefore $f_1 + \ldots + f_n = 1$, whence $n = m$. ♢

16.12. LEMMA. [18, p. 67, Th. 46] *If* A *contains a nonzero abelian projection* (i.e., A is not continuous in the sense of 8.15), *then* A *has a nonzero homogeneous direct summand.*

Proof. (Axioms A, B, C′, D and GC) Let $e \in \mathrm{A}$ be a nonzero abelian projection; dropping down to $\mathrm{C}(e)\mathrm{A}$ we can suppose that A is of type I and e is an abelian projection with $\mathrm{C}(e) = 1$. By Zorn, expand $\{e\}$ to a maximal family (e_i) of pairwise orthogonal, faithful abelian projections. By 16.4, $e_i \sim e_j$ for all i and j. Set $f = 1 - \sup e_i$. By maximality, f contains no faithful abelian projection. It follows that $\mathrm{C}(f) \neq 1$. {If $\mathrm{C}(f) = 1$ then $e \precsim f$ by 16.5, say $e \sim e' \leq f$; then e' is a faithful (9.16) abelian (16.1) subprojection of f, a contradiction.} Set $u = 1 - \mathrm{C}(f) \neq 0$. Then $uf = 0$, that is, $u(1 - \sup e_i) = 0$, so $u = \sup ue_i$ and $u\mathrm{A}$ is the desired homogeneous summand. ♢

16.13. THEOREM. [18, p. 67] *If* A *is of type* I, *then there exists an orthogonal family* (u_α) *of nonzero central projections such that* $\sup u_\alpha = 1$ *and every* $u_\alpha\mathrm{A}$ *is homogeneous.*

Proof. (Axioms A, B, C′, D and GC) Obvious exhaustion argument (Zorn) based on 16.12. ♢

Homogeneous summands of the same order can be combined:

16.14. LEMMA. [2, p. 114, Lemma] *Let* $(u_\alpha)_{\alpha\in\mathrm{K}}$ *be an orthogonal family of central projections such that every* $u_\alpha\mathrm{A}$ *is homogeneous of the same order* $\aleph$. *Let* $u = \sup u_\alpha$. *Then* $u\mathrm{A}$ *is homogeneous of order* $\aleph$.

Proof. (Axiom F) Let I be a set of cardinality $\aleph$ and for each $\alpha \in \mathrm{K}$ let $(e_{\alpha i})_{i\in\mathrm{I}}$ be a family of pairwise orthogonal, equivalent, abelian projections with supremum u_α. For each $i \in \mathrm{I}$, set

$$e_i = \sup\{e_{\alpha i} :\ \alpha \in \mathrm{K}\}\,.$$

The e_i are pairwise orthogonal; moreover, for each i, the $\mathrm{C}(e_{\alpha i})$ ($\alpha \in \mathrm{K}$) are orthogonal, therefore e_i is abelian (proof of 8.19). By axiom F, $e_i \sim e_j$ for all i and j (cf. the proof of 15.10). Finally, $\sup e_i = \sup u_\alpha = u$. ♢

16.15. THEOREM. [2, p. 115, Th. 3] *If* A *is of type* I, *then there exists an orthogonal family* $(u_\aleph)_{\aleph\leq\mathrm{card}\,\mathrm{A}}$ *of central projections, with supremum* 1, *such that for each* $\aleph \leq \mathrm{card}\,\mathrm{A}$ *either* $u_\aleph = 0$ $u_\aleph\mathrm{A}$ *is homogeneous of order* $\aleph$.

Proof. (Axioms A, B, C′, D, F and GC) Immediate from 16.13 and 16.14 (card A being trivially an upper bound on the cardinality of orthogonal families of nonzero projections). ♢

16.16. COROLLARY. [2, p. 115, Th. 2] *If* A *is finite and of type* I, *then there exists a unique orthogonal sequence* $(u_n)_{n\geq 1}$ *of central projections*, *with supremum* 1, *such that for each* n, *either* $u_n = 0$ *or* $u_n\mathrm{A}$ *is of type* I_n.

Proof. (Axioms A, B, C′, D, F and GC)

Existence: With notations as in 16.15, $u_{\aleph} = 0$ for all infinite $\aleph$ (16.8). And if, for a positive integer n, $u_n \neq 0$, then $u_n\mathrm{A}$ is of type I_n by definition (16.7).

Uniqueness: Suppose (v_n) is a sequence with the same properties. If $u_m v_n \neq 0$ then $u_m v_n \mathrm{A}$ is both of type I_m and type I_n, therefore $m = n$ (16.11). Thus $u_m v_n = 0$ when $m \neq n$, whence $u_m \leq v_m$ and $v_m \leq u_m$. ♢

16.17. PROPOSITION. [2, p. 117, Prop. 6] *If* A *is of type* I, *without abelian summand*, *and if* $e \in \mathrm{A}$ *is an abelian projection*, *then* $e \precsim 1 - e$.

Proof. (Axioms A, B, C′, D, F and GC) It will suffice to find an orthogonal family (u_α) of nonzero central projections, with $\sup u_\alpha = 1$, such that $u_\alpha e \precsim u_\alpha(1-e)$ for all α; for, by the orthogonality of e and $1-e$, the equivalences can be added (by Axiom F) to obtain $e \precsim 1 - e$.

We show first that there exists a nonzero central projection u such that $ue \precsim u(1-e)$. Since A is not abelian, $e \neq 1$; so $u = \mathrm{C}(1-e) \neq 0$. Then ue is abelian and $1-e$ is faithful in $u\mathrm{A}$, so $ue \precsim 1-e$ by 16.5, in other words $ue \precsim u(1-e)$.

Let (u_α) be a maximal orthogonal family of nonzero central projections such that $u_\alpha e \precsim u_\alpha(1-e)$ for all α. If $\sup u_\alpha = 1$ we are done. Let $u = 1 - \sup u_\alpha$ and assume to the contrary that $u \neq 0$. Then $u\mathrm{A}$ satisfies the hypotheses of A and ue is abelian; by the preceding paragraph, there exists a nonzero central projection $v \in u\mathrm{A}$ such that $v(ue) \precsim v(u - ue)$, that is, $ve \precsim v(1-e)$, contrary to the maximality of the family (u_α). ♢

16.18. LEMMA. [2, p. 81, Prop. 9] *Let* $(e_i)_{i\in\mathrm{I}}$ *be a family of projections in the Baer* ∗*-ring* A *such that*, *for every nonzero central projection* u, *the set*

$$\mathrm{I}(u) = \{ i \in \mathrm{I} : \ ue_i \neq 0 \}$$

is infinite. Then for each positive integer n, *there exist* n *distinct indices* $i_1, \ldots, i_n$ *and nonzero projections* $g_\nu \leq e_{i_\nu}$ $(1 \leq \nu \leq n)$ *such that* $g_1 \sim g_2 \sim \ldots \sim g_n$.

Proof. (Axioms C′, E) For $n = 1$, any index $i_1 \in \mathrm{I}(1)$ will do, with $g_1 = e_{i_1}$. Suppose $n \geq 2$ and assume inductively that $i_1, \ldots, i_{n-1}$ are distinct indices and $f_1, \ldots, f_{n-1}$ are nonzero projections with $f_\nu \leq e_{i_\nu}$ ($1 \leq \nu \leq n-1$) and $f_1 \sim \ldots \sim f_{n-1}$. Consider $u = \mathrm{C}(f_1) \neq 0$. By hypothesis $\mathrm{I}(u)$ is infinite, so we can choose $i_n \in \mathrm{I}(u) - \{i_1, \ldots, i_{n-1}\}$. Then $ue_{i_n} \neq 0$, so $u\mathrm{C}(e_{i_n}) \neq 0$, that is, $\mathrm{C}(f_1)\mathrm{C}(e_{i_n}) \neq 0$; therefore $f_1 \mathrm{A} e_{i_n} \neq 0$ (3.21). By axiom E, there exist nonzero projections $g_1 \leq f_1$ and $g_n \leq e_{i_n}$ with $g_1 \sim g_n$. For $1 \leq \nu \leq n-1$, the equivalence $f_1 \sim f_\nu$ and axiom C′ yield a projection $g_\nu \leq f_\nu$ with $g_1 \sim g_\nu$. Clearly $g_1, g_2, \ldots, g_{n-1}, g_n$ meet the requirements. ♢

16.19. THEOREM. [18, p. 69, Th. 48] *If* A *is finite and of type* I, *and if* $(e_i)_{i\in\mathrm{I}}$ *is any orthogonal family of projections in* A, *then there exists an orthogonal family* (u_α) *of nonzero central projections*, *with* $\sup u_\alpha = 1$, *such that for each* α *the set of indices* $\{ i \in \mathrm{I} : \ u_\alpha e_i \neq 0 \}$ *is finite.*

Proof. (Axioms A, B, C′, D, F and GC) We first show that there exists a nonzero central projection u such that the set

$$\mathrm{I}(u) = \{i \in \mathrm{I} : \ ue_i \neq 0\}$$

is finite. Suppose to the contrary that no such u exists. By 16.12, there exists a nonzero central projection v such that $v\mathrm{A}$ is homogeneous; since $v\mathrm{A}$ is finite it is homogeneous of some finite order n (16.8), hence is of type I_n. For every nonzero central projection $u \le v$, the set $\mathrm{I}(u)$ is, by supposition, infinite; this means that the family of projections $(ve_i)_{i\in\mathrm{I}}$ satisfies the hypothesis of 16.18 in the Baer $*$-ring $v\mathrm{A}$; since $v\mathrm{A}$ is of type I_n, the conclusion of 16.18 is clearly contradictory to 16.11.

Now let $(u_\alpha)_{\alpha\in\mathrm{K}}$ be a maximal orthogonal family of nonzero central projections such that for each α, the set

$$\mathrm{I}(u_\alpha) = \{i \in \mathrm{I} : \ u_\alpha e_i \neq 0\}$$

is finite (Zorn; get started by the preceding paragraph). It will suffice to show that $\sup u_\alpha = 1$. Let $w = 1 - \sup u_\alpha$ and assume to the contrary that $w \neq 0$. Then an application of the first paragraph of the proof to the Baer $*$-ring $w\mathrm{A}$ contradicts maximality. $\diamondsuit$

17. CONTINUOUS RINGS

In this section A is a Baer $*$-ring, $\sim$ is an equivalence relation on its projection lattice satisfying the axioms A, B, C$'$, E, F of §10. {Included are the rings satisfying the common hypotheses of §15 and §16 (cf. 15.1).} The precise axioms needed for each proof are noted parenthetically.

17.1. LEMMA. *The following conditions on* A *are equivalent*:

(a) A *is not abelian*;

(b) *there exist nonzero projections* e, f *with* $ef = 0$ *and* $e \sim f$.

Proof. (a) $\Rightarrow$ (b): (Axiom E) Let g be a projection in A that is not central (8.11), that is, $g\mathrm{A}(1-g) \neq 0$ (see the proof of 3.21). By axiom E there exist nonzero projections $e \le g$, $f \le 1-g$ with $e \sim f$.

(b) $\Rightarrow$ (a): (Axioms A, B) With notations as in (b), one has $\mathrm{C}(e) = \mathrm{C}(f)$ by 9.16. If e and f were both central, one would have $e = \mathrm{C}(e) = \mathrm{C}(f) = f$, whence $0 = ef = ee = e$, a contradiction. Thus one of e, f is noncentral, so A is not abelian (8.1). ♢

17.2. The concept of abelian ring is 'absolute' (i.e., independent of the axioms of §10); results like 17.1 (and 9.17) are of interest because they link properties of an 'absolute' (abelian) to a 'relative' (a postulated equivalence relation). If one *were* to define 'abelian' in terms of $\sim$, 17.1 suggests an appropriate definition: call A 'abelian relative to $\sim$' if the relations $e \sim f$ and $ef = 0$ imply $e = 0$ or $f = 0$.

17.3. LEMMA. *If* A *is continuous* (i.e., has no abelian projections other than 0) *then for every positive integer* n *there exist* n *pairwise orthogonal, nonzero projections* $e_1, \ldots, e_n$ *in* A *with* $e_1 \sim e_2 \sim \ldots \sim e_n$.

Proof. (Axioms A, C$'$, E) It clearly suffices to consider $n = 2^k$ $(k = 0, 1, 2, \ldots)$. For $k = 0$, $e_1 = 1$ fills the bill. Let $k \ge 1$, $m = 2^{k-1}$, and suppose inductively that $f_1, \ldots, f_m$ are pairwise orthogonal, equivalent, nonzero projections. Since A is continuous, f_1 is not abelian, therefore by 17.1 there exist nonzero projections $e_1 \le f_1$, $e_2 \le f_1$ with $e_1 e_2 = 0$ and $e_1 \sim e_2$. For $i = 2, 3, \ldots, m$, we see from axiom C$'$ and

$$e_1 + e_2 + [f_1 - (e_1 + e_2)] = f_1 \sim f_i$$

that f_i contains orthogonal 'copies' of e_1, e_2 (nonzero, by axiom A), and the induction is complete. ♢

17.4. THEOREM. [18, p. 70, Th. 49] *If* A *is continuous* (8.18) *then for every positive integer* n *there exist orthogonal projections* $e_1, \ldots, e_n$ *with* $e_1 + \ldots + e_n = 1$ *and* $e_1 \sim e_2 \sim \ldots \sim e_n$.

Proof. (Axioms A, C′, E, F) Consider n-ples of families of nonzero projections

$$(e_{1i})_{i \in \mathrm{I}}, \ldots, (e_{ni})_{i \in \mathrm{I}}$$

with a common index set I, where the $e_{\lambda i}$ are pairwise orthogonal ($e_{\lambda i} e_{\mu j} = 0$ if $\lambda \neq \mu$ or $i \neq j$) and $e_{1i} \sim e_{2i} \sim \ldots \sim e_{ni}$ for each $i \in \mathrm{I}$. We can suppose that the index set I cannot be enlarged (Zorn; get started by 17.3). Define

$$e_\lambda = \sup\{e_{\lambda i} : \ i \in \mathrm{I}\} \qquad (1 \leq \lambda \leq n).$$

The e_λ are pairwise orthogonal, and, by axiom F, one has $e_1 \sim e_2 \sim \ldots \sim e_n$, so it will suffice to show that $e_1 + \ldots + e_n = 1$. Let $e = 1 - (e_1 + \ldots + e_n)$ and assume to the contrary that $e \neq 0$. An application of 17.3 in the continuous ring $e\mathrm{A}e$ contradicts maximality. {For the case $n = 2$, axioms E, F are sufficient, by an evident simplification of the above proof.} ♢

17.5. In 17.4, if $\sim$ means $\overset{a}{\sim}$ then A is isomorphic to the matrix ring $\mathrm{M}_n(e_1 \mathrm{A} e_1)$, and when $\sim$ means $\overset{*}{\sim}$ this is a $*$-isomorphism (with $*$-transpose as the involution on the matrix ring) [cf. 2, p. 98, Prop. 1].

18. ADDITIVITY OF EQUIVALENCE

This is the main result:

18.1. THEOREM. *Let* A *be a Baer* $*$*-ring*, $\sim$ *an equivalence relation on its projection lattice satisfying the axioms* A–D *and* F *of* §10. *Then the following conditions* (*relative to* $\sim$) *are equivalent*:

(a) A *has* GC;

(b) A *satisfies axiom* E, *and equivalence is additive* (*in the sense of* §10).

{Stripped to its essentials, this says (assuming A–F hold): GC $\Leftrightarrow$ equivalence is additive. Still another formulation (assuming A–E hold): equivalence is additive $\Leftrightarrow$ Axiom F and GC hold.}

18.2. The proof of (b) $\Rightarrow$ (a) is easy (same as 13.3, omitting the hypothesis $ef = 0$ and using complete additivity instead of the weaker axiom F). {One notes that axioms A, C, D are not needed for this part of the proof.}

Henceforth (through 18.11) A *denotes a Baer* $*$*-ring*, $\sim$ *an equivalence relation on its projection lattice satisfying the axioms* A–D, F *and* GC *of* §10. {Note that axiom E also holds (15.1).} *Our objective is to prove that equivalence is additive* (this will establish the validity of (a) $\Rightarrow$ (b)).

18.3. DEFINITION. Let each of $(e_i)_{i\in \mathrm{I}}$ and $(f_i)_{i\in \mathrm{I}}$ be an orthogonal family of projections such that $e_i \sim f_i$ for all $i \in \mathrm{I}$, and let $e = \sup e_i$, $f = \sup f_i$. We say that the equivalences

$$e_i \sim f_i \quad (i \in \mathrm{I})$$

are **addable** if $e \sim f$. {Caution: In the case of $\overset{*}{\sim}$, to say that a family of partial isometries is addable (14.3) means something more precise: the $*$-equivalence $e \overset{*}{\sim} f$ is required to 'induce' the given $*$-equivalences $e_i \overset{*}{\sim} f_i$ in an appropriate sense.}

Note: When I is finite, $e \sim f$ comes free of charge from axiom D; when $ef = 0$, it comes free of charge from axiom F.

18.4. LEMMA. *Let each of* $(e_i)_{i\in \mathrm{I}}$ *and* $(f_i)_{i\in \mathrm{I}}$ *be an orthogonal family of projections*, *such that*

$$e_i \sim f_i \ \textit{ for all } i \in \mathrm{I}. \tag{$*$}$$

For each $i \in \mathrm{I}$ *let* J_i *be an index set and let* $(e_{ij})_{j\in \mathrm{J}_i}$, $(f_{ij})_{j\in \mathrm{J}_i}$ *be orthogonal decompositions of* e_i, f_i, *respectively*, *such that*

$$e_{ij} \sim f_{ij} \ \textit{ for all } i \in \mathrm{I},\ j \in \mathrm{J}_i. \tag{$**$}$$

Then, the equivalences (*) *are addable if and only if the equivalences* (**) *are addable.*

Proof. $\sup\{e_{ij} : i \in \mathrm{I},\ j \in \mathrm{J}_i\} = \sup_{i\in\mathrm{I}}(\sup\{e_{ij} : j \in \mathrm{J}_i\}) = \sup_{i\in\mathrm{I}} e_i$, similarly $\sup f_{ij} = \sup f_i$. ♢

18.5. LEMMA. [18, p. 73, Proof of Th. 50] *With notations as in* 18.3, *if* $e \precsim 1 - f$ *then the equivalences* (*) *are addable* (*that is*, $e \sim f$).

Proof. (Axioms C, F) Say $e \sim f' \le 1 - f$. By axiom C, there is an orthogonal decomposition $f' = \sup f_i'$ with $e_i \sim f_i'$ for all i. Then $f_i' \sim e_i \sim f_i$; since $f'f = 0$, it follows from axiom F that $f' \sim f$. Thus $e \sim f' \sim f$. ♢

{Note: Until now, the weaker axiom C′ has sufficed; this is the first use of the full axiom C.}

18.6. LEMMA. [18, p. 73, Th. 50] *If* A *is properly infinite, then equivalence is additive in* A.

Proof. (Axioms A–F) Let g be a projection with $g \sim 1 \sim 1 - g$ (15.12). Adopt the notations of 18.3. From $e + (1 - e) = 1 \sim g$ we have $e \sim e' \le g$ for suitable e'. Similarly $f \sim f' \le 1 - g$ for suitable f'. It suffices to show that $e' \sim f'$; invoking axiom C, we can suppose $e \le g$ and $f \le 1 - g$. Then $ef = 0$ and an application of axiom F completes the proof. ♢

18.7. LEMMA. [18, p. 78, proof of Th. 52] *If* A *is continuous, then equivalence is additive in* A.

Proof. (Axioms A–D, F and GC; note that these imply axiom E by 15.1) With notations as in 18.3, write $e_i = e_i' + e_i''$ with $e_i' \sim e_i''$ (17.4). The equivalences $e_i \sim f_i$ induce decompositions $f_i = f_i' + f_i''$ with $e_i' \sim f_i'$ and $e_i'' \sim f_i''$, whence $f_i' \sim f_i''$. Set

$$e' = \sup e_i', \quad e'' = \sup e_i'';$$

since $e'e'' = 0$, one has $e' \sim e''$ (axiom F). Similarly, defining

$$f' = \sup f_i', \quad f'' = \sup f_i'',$$

one has $f'f'' = 0$, $f' \sim f''$. Evidently $e = e' + e''$, $f = f' + f''$, so by axiom D it will suffice to show that $e' \sim f'$ and $e'' \sim f''$.

Let u be a central projection with

$$ue' \precsim uf' \ \text{ and } \ (1-u)f' \precsim (1-u)e'.$$

Then

$$ue' \precsim uf' \sim uf'' \le u(1 - f') = u - uf' \le 1 - uf',$$

whence $ue' \precsim 1 - uf'$ (axiom C′); since $ue_i' \sim uf_i'$ for all i, it follows from 18.5 that $ue' \sim uf'$. Similarly,

$$(1-u)f' \precsim (1-u)e' \sim (1-u)e'' \le (1-u)(1-e') \le 1 - (1-u)e',$$

whence $(1-u)f' \sim (1-u)e'$; combined with $ue' \sim uf'$, this yields $e' \sim f'$. Similarly $e'' \sim f''$. ♢

18.8. LEMMA. *If* A *is abelian, then equivalence coincides with equality* (*hence is trivially additive*).

Proof. (Axioms A, B) If $e \sim f$ then, citing 9.16, one has $e = \mathrm{C}(e) = \mathrm{C}(f) = f$. ♢

Scanning 18.6–18.8, we see from structure theory (9.25, 8.27) that the remaining case to be considered is type $\mathrm{I}_{\mathrm{fin}}$ without abelian summand. This proves to be the most complicated case. First, a general lemma:

18.9. LEMMA. [2, p. 38, Prop. 5] *Let* $e_1, \ldots, e_n$ *be any projections in the Baer* $*$*-ring* A. *Then there exist orthogonal central projections* $u_1, \ldots, u_r$ *with sum* 1, *such that for each pair of indices* i *and* ν, *one has either* $u_\nu e_i = 0$ *or* $\mathrm{C}(u_\nu e_i) = u_\nu$.

Proof. (No axioms needed) Let $u_1 = 1 - \sup\{\mathrm{C}(e_i) : 1 \le i \le n\}$. Then $u_1 e_i = 0$ for all i. Dropping down to $(1 - u_1)\mathrm{A}$ we can therefore suppose that $\sup\{\mathrm{C}(e_i) : 1 \le i \le n\} = 1$.

Let $u_1, \ldots, u_r$ be orthogonal central projections with sum 1, such that each $\mathrm{C}(e_i)$ is the sum of certain of the u_ν. {E.g., let $r = 2^n$ and for each n-ple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, -1\}^n$ define

$$u_\varepsilon = \mathrm{C}(e_1)^{\varepsilon_1}\mathrm{C}(e_2)^{\varepsilon_2} \cdots \mathrm{C}(e_n)^{\varepsilon_n},$$

where $\mathrm{C}(e_i)^1 = \mathrm{C}(e_i)$ and $\mathrm{C}(e_i)^{-1} = 1 - \mathrm{C}(e_i)$; then let $u_1, \ldots, u_r$ be any enumeration of the u_ε.} If $u_\nu \mathrm{C}(e_i) \neq 0$ then u_ν must be one of the 'constituents' of $\mathrm{C}(e_i)$, so $u_\nu \mathrm{C}(e_i) = u_\nu$. Thus either $\mathrm{C}(u_\nu e_i) = 0$ or $\mathrm{C}(u_\nu e_i) = u_\nu$. ♢

18.10. LEMMA. [18, p. 84, proof of Th. 54] *If* A *is finite and of type* I, *then equivalence is additive in* A.

Proof. (Axioms A–D, F and GC)

(i) In view of 8.27 and 18.8, *we can suppose that* A *has no abelian summand*.

(ii) Adopt the notations of 18.3; we are to show that $e \sim f$. We can suppose the e_i (hence the f_i) to be nonzero.

(iii) We first observe that every nonzero projection g contains a nonzero abelian projection. {Proof: Let h be a faithful abelian projection (8.17). Then $g\mathrm{A}h \neq 0$ (3.21), therefore by axiom E (see 15.1) there exist nonzero projections $g_0 \le g$, $h_0 \le h$ with $g_0 \sim h_0$. Since h is abelian and $g_0 \precsim h$, g_0 is abelian (16.1).} By an obvious Zorn argument, it follows that g is in fact the supremum of an orthogonal family of abelian projections.

In particular, for each i one has

$$e_i = \sup\{e_{ij} : j \in \mathrm{J}_i\}$$

with the e_{ij} ($j \in \mathrm{J}_i$) an orthogonal family of abelian projections. Then $e_i \sim f_i$ induces an orthogonal decomposition

$$f_i = \sup\{f_{ij} : j \in \mathrm{J}_i\}$$

with $e_{ij} \sim f_{ij}$ and therefore f_{ij} abelian (16.1). It suffices to show that the equivalences $e_{ij} \sim f_{ij}$ are addable (18.4). So, changing notation, *we can suppose that the* e_i *and* f_i *are abelian.*

(iv) Let $(u_\alpha)_{\alpha \in \mathrm{K}}$ be an orthogonal family of nonzero central projections with $\sup u_\alpha = 1$, such that for each α the set

$$\mathrm{I}_\alpha = \{ i \in \mathrm{I} : \ u_\alpha e_i \neq 0 \}$$

is finite (16.19). Note that also $\mathrm{I}_\alpha = \{ i \in \mathrm{I} : \ u_\alpha f_i \neq 0 \}$.

(v) For each $\alpha \in \mathrm{K}$, there is a finite central partition $u_\alpha = v_1 + \ldots + v_r$ (of length depending on α) such that for $i \in \mathrm{I}_\alpha$ and $1 \leq \nu \leq r$, either $v_\nu e_i = 0$ or $\mathrm{C}(v_\nu e_i) = v_\nu$ (18.9). Partitioning every u_α in this way and replacing the u_α by the family of all v's obtained in this way, *we can suppose that for each* $\alpha \in \mathrm{K}$, *the set*

$$\mathrm{I}_\alpha = \{ i \in \mathrm{I} : \ u_\alpha e_i \neq 0 \}$$

is finite and $\mathrm{C}(u_\alpha e_i) = u_\alpha$ *for all* $i \in \mathrm{I}_\alpha$.

(vi) It follows that for each α, the finitely many projections

$$u_\alpha e_i \quad (i \in \mathrm{I}_\alpha)$$

are pairwise equivalent (16.4).

(vii) If (for some α) $\mathrm{I}_\alpha = \varnothing$, then $u_\alpha e_i = u_\alpha f_i = 0$ for all i, whence $u_\alpha e = u_\alpha f = 0$, thus $e \leq 1 - u_\alpha$, $f \leq 1 - u_\alpha$. Let K_0 be the set of all such α and let

$$v = \inf\{ 1 - u_\alpha : \ \alpha \in \mathrm{K}_0 \} = 1 - \sup\{ u_\alpha : \ \alpha \in \mathrm{K}_0 \}.$$

Then $e \leq v$, $f \leq v$, and dropping down to $v\mathrm{A}$ *we can suppose that every* I_α *is nonempty.*

(viii) Let $n(\alpha) = \operatorname{card} \mathrm{I}_\alpha$; thus $1 \leq n(\alpha) < \infty$. Write each I_α as a disjoint union

$$\mathrm{I}_\alpha = \mathrm{I}_\alpha^1 \cup \mathrm{I}_\alpha^2 \cup \mathrm{I}_\alpha^3$$

with the following properties:

(a) if $n(\alpha)$ is even, then $\mathrm{I}_\alpha^1, \mathrm{I}_\alpha^2$ each have $n(\alpha)/2$ elements and $\mathrm{I}_\alpha^3 = \varnothing$;

(b) if $n(\alpha)$ is odd, then $\mathrm{I}_\alpha^1, \mathrm{I}_\alpha^2$ each have $(n(\alpha)-1)/2$ elements and I_α^3 has one element.

(ix) For each α, define

$$\begin{aligned} e_\alpha^1 &= \sup\{ u_\alpha e_i : \ i \in \mathrm{I}_\alpha^1 \} \\ e_\alpha^2 &= \sup\{ u_\alpha e_i : \ i \in \mathrm{I}_\alpha^2 \} \\ e_\alpha^3 &= \sup\{ u_\alpha e_i : \ i \in \mathrm{I}_\alpha^3 \} \end{aligned}$$

(the sups are actually finite sums; and either $e_\alpha^3 = 0$, or $e_\alpha^3 = u_\alpha e_i$ where $\mathrm{I}_\alpha^3 = \{i\}$, thus e_α^3 is abelian). Since $\operatorname{card} \mathrm{I}_\alpha^1 = \operatorname{card} \mathrm{I}_\alpha^2$, it is clear from (vi) that

$e^1_\alpha \sim e^2_\alpha$. Moreover,

$$\begin{aligned} u_\alpha e &= u_\alpha \sup\{e_i :\ i \in \mathrm{I}\,\} \\ &= \sup\{u_\alpha e_i :\ i \in \mathrm{I}\,\} \qquad (3.23) \\ &= \sup\{u_\alpha e_i :\ i \in \mathrm{I}_\alpha\,\} \\ &= e^1_\alpha + e^2_\alpha + e^3_\alpha\,. \end{aligned}$$

Similarly, defining

$$f^t_\alpha = \sup\{u_\alpha f_i :\ i \in \mathrm{I}^t_\alpha\,\} \quad \text{for} \quad \alpha \in \mathrm{K}\,,\ 1 \le t \le 3\,,$$

we have

$$u_\alpha f = f^1_\alpha + f^2_\alpha + f^3_\alpha \quad \text{and} \quad f^1_\alpha \sim f^2_\alpha\,.$$

Moreover (axiom D)

$$e^t_\alpha \sim f^t_\alpha \quad \text{for} \quad \alpha \in \mathrm{K}\,,\ 1 \le t \le 3\,.$$

In particular,

$$e^1_\alpha \sim e^2_\alpha \sim f^2_\alpha \sim f^1_\alpha \quad \text{for all} \quad \alpha \in \mathrm{K}\,.$$

(x) For $t = 1, 2, 3$ (notations as in (ix)) define

$$e^t = \sup\{e^t_\alpha :\ \alpha \in \mathrm{K}\,\}\,, \quad f^t = \sup\{f^t_\alpha :\ \alpha \in \mathrm{K}\,\}\,.$$

Clearly e^1, e^2, e^3 are pairwise orthogonal, as are f^1, f^2, f^3, and

$$\begin{aligned} e = \sup\{u_\alpha e :\ \alpha \in \mathrm{K}\,\} &= \sup_\alpha \big(e^1_\alpha + e^2_\alpha + e^3_\alpha\big) \\ &= \sup_\alpha e^1_\alpha + \sup_\alpha e^2_\alpha + \sup_\alpha e^3_\alpha \\ &= e^1 + e^2 + e^3\,, \end{aligned}$$

similarly $f = f^1 + f^2 + f^3$.

(xi) Since $e^1 e^2 = 0$, $e^1 = \sup_\alpha e^1_\alpha$, $e^2 = \sup_\alpha e^2_\alpha$ and $e^1_\alpha \sim e^2_\alpha$ for all α, axiom F yields $e^1 \sim e^2$, and similarly $f^1 \sim f^2$.

(xii) Let $\mathrm{J} = \{(\alpha, i) :\ \alpha \in \mathrm{K}\,,\ i \in \mathrm{I}_\alpha\,\}$. For $t = 1, 2, 3$ define

$$\mathrm{J}^t = \{(\alpha, i) :\ \alpha \in \mathrm{K}\,,\ i \in \mathrm{I}^t_\alpha\,\}\,.$$

Since I_α is partitioned as $\mathrm{I}_\alpha = \mathrm{I}^1_\alpha \cup \mathrm{I}^2_\alpha \cup \mathrm{I}^3_\alpha$, it follows that

$$\mathrm{J} = \mathrm{J}^1 \cup \mathrm{J}^2 \cup \mathrm{J}^3$$

with $\mathrm{J}^1, \mathrm{J}^2, \mathrm{J}^3$ pairwise disjoint. For $t = 1, 2, 3$, one has

$$\begin{aligned} \sup\{u_\alpha e_i :\ (\alpha, i) \in \mathrm{J}^t\,\} &= \sup\{u_\alpha e_i :\ \alpha \in \mathrm{K}\,,\ i \in \mathrm{I}^t_\alpha\,\} \\ &= \sup_\alpha \big(\sup\{u_\alpha e_i :\ i \in \mathrm{I}^t_\alpha\,\}\big) \\ &= \sup_\alpha e^t_\alpha = e^t \end{aligned}$$

and similarly
$$f^t = \sup\{u_\alpha f_i :\ (\alpha, i) \in \mathrm{J}^t\} \qquad (1 \le t \le 3)\,,$$
where $u_\alpha e_i \sim u_\alpha f_i$ for all $(\alpha, i) \in \mathrm{J}$.

(xiii) Let u be a central projection such that
$$ue^1 \precsim uf^1 \ \text{ and } (1-u)f^1 \precsim (1-u)e^1\,.$$
Then since $f^1 \sim f^2$ one has
$$ue^1 \precsim uf^1 \sim uf^2 \le u(1-f^1) \le 1 - uf^1$$
so $ue^1 \sim uf^1$ by 18.5. Similarly
$$(1-u)f^1 \precsim (1-u)e^1 \sim (1-u)e^2 \le (1-u)(1-e^1) \le 1-(1-u)e^1\,,$$
so $(1-u)f^1 \sim (1-u)e^1$; combining this with $ue^1 \sim uf^1$ we get $e^1 \sim f^1$. Then $e^2 \sim e^1 \sim f^1 \sim f^2$, so also $e^2 \sim f^2$. Since, by (x), we have $e = e^1 + e^2 + e^3$ and $f = f^1 + f^2 + f^3$, it will suffice to show that $e^3 \sim f^3$.

(xiv) Note that e^3 is abelian. {For, $e^3 = \sup\{e^3_\alpha :\ \alpha \in \mathrm{K}\}$ where, as noted in (ix), the e^3_α are abelian, and $e^3_\alpha \le u_\alpha$; since the u_α are orthogonal, e^3 is abelian by the argument in the proof of 8.19.} Similarly, f^3 is abelian. For all α and i one has $u_\alpha e_i \sim u_\alpha f_i$, therefore $\mathrm{C}(u_\alpha e_i) = \mathrm{C}(u_\alpha f_i)$; it follows that
$$\begin{aligned}\mathrm{C}(e^3) &= \mathrm{C}\big(\sup\{u_\alpha e_i :\ (\alpha,i) \in \mathrm{J}^3\}\big)\\ &= \sup\{\mathrm{C}(u_\alpha e_i) :\ (\alpha,i) \in \mathrm{J}^3\} \qquad (3.22)\\ &= \sup\{\mathrm{C}(u_\alpha f_i) :\ (\alpha,i) \in \mathrm{J}^3\} = \mathrm{C}(f^3)\,,\end{aligned}$$
therefore $e^3 \sim f^3$ by 16.4. {We remark that $f^3 \precsim 1 - f^3$ (16.17)[1], therefore $e^3 \precsim 1 - f^3$; this is important for the theory of addability of partial isometries [cf. 2, p. 128, proof of Prop. 5].} ♢

18.11. *Completion of proof of* 18.1: Assuming A–D, F and GC, we have to show that equivalence is additive. As observed in 15.1, axiom E also holds (thus A–F and GC are in force).

The properly infinite case is covered by 18.6; so we can suppose A is finite (9.11). The continuous case is covered by 18.7, so we can suppose that A is type I (and finite), and the proof is finished off by 18.10. ♢

18.12. COROLLARY. [18, p. 82, Th. 54] *Let* A *be a Baer $*$-ring*, $\sim$ *an equivalence relation on its projection lattice satisfying the axioms* A–D, F *and* H *of* §10. *Then* A *has* GC *and equivalence is additive.*

Proof. GC holds by the theorem of Maeda and Holland (13.9), therefore equivalence is additive by 18.1 (and axiom E holds). {Since axiom G is a special case of additivity of equivalence, we see that axioms A–H are in force.} ♢

[1] Since A is finite, an alternative proof is available [2, p. 118, remark at the end of the proof of Prop. 6].

18.13. COROLLARY. [17, p. 534] *If* A *is any regular Baer* $*$*-ring then, for* $\overset{a}{\sim}$, A *has* GC *and equivalence is additive.*

Proof. Axioms A–F and H hold for $\overset{a}{\sim}$ (11.3); quote 18.12. ♢

18.14. THEOREM. [2, p. 129, Th. 1] *Let* A *be a Baer* $*$*-ring.*

(i) GC *holds for* $\overset{*}{\sim}$ *if and only if axiom* E *holds for* $\overset{*}{\sim}$ *and* $*$*-equivalence is additive.*

(ii) *If* GC *holds for* $\overset{*}{\sim}$ *and if* A *has no abelian summand, then partial isometries are addable in* A (*in the sense of* 14.3).

Proof. Axioms A–D and F hold for $\overset{*}{\sim}$ in any Baer $*$-ring (11.2), so (i) is immediate from 18.1; (ii) can be proved exactly as in [2, §20], essentially by the foregoing arguments (with minor supplementary remarks). ♢

18.15. If A is a Baer $*$-ring such that $\overset{*}{\sim}$ satisfies axiom H (in particular, if A is any Baer $*$-ring satisfying the SR-axiom) then $*$-equivalence is additive in A .

{Proof: SR ⇒ axiom H ⇒ GC by 12.13 and 13.10; quote 18.14, (i).}

18.16. THEOREM. [18, p. 78, Th. 52] *Let* A *be a Baer* $*$*-ring,* $\sim$ *an equivalence relation on its projection lattice satisfying the axioms* A–G *of* §10. *Then* A *has* GC *and equivalence is additive.*

Sketch of proof. We recall that axiom G is "central additivity" of equivalence. The properly infinite case is covered by 18.6, so we can suppose A is finite (9.11), hence a product of rings of type $\mathrm{I_{fin}}$ and type $\mathrm{II_{fin}}$ (9.25). Since axioms B, E and F hold, we have at least orthogonal GC (13.3).

Suppose A is of type $\mathrm{I_{fin}}$. Adopt the notations of 18.3; we are to show that $e \sim f$. There exists an orthogonal family (u_α) of central projections with $\sup u_\alpha = 1$, such that for each α the set $\mathrm{I}_\alpha = \{i \in \mathrm{I} :\ u_\alpha e_i \neq 0\}$ is finite. {For, axioms A, B, C′, E, F are sufficient for the proof of 16.19 [18, p. 69, Th. 48], in particular one can get by with orthogonal GC.} By axiom D, for every α ,

$$\sup\{u_\alpha e_i :\ i \in \mathrm{I}_\alpha\} \sim \sup\{u_\alpha f_i :\ i \in \mathrm{I}_\alpha\} ,$$

that is, $u_\alpha e \sim u_\alpha f$, therefore $e \sim f$ by axiom G.

There remains the case of type $\mathrm{II_{fin}}$ (messier than 18.7); we refer to Kaplansky's original proof for the details [18, pp. 79-80]. ♢

18.17. COROLLARY. *Let* A *be a Baer* $*$*-ring,* $\sim$ *an equivalence relation on its projection lattice satisfying the axioms* A–D *and* F *of* §10 (cf. 11.2). *Then the following conditions are equivalent*:

(a) A *has* GC;

(b) A *satisfies axiom* E *and equivalence is additive*;

(c) A *satisfies axioms* E *and* G (in other words, A–G).

Proof. (a) ⇔ (b) is 18.1.

(b) ⇒ (c) is trivial.

(c) ⇒ (b) is 18.16. ♢

In particular, the corollary characterizes the Baer $*$-rings satisfying GC for $\overset{*}{\sim}$ (cf. 11.2).

18.18. Let A be a regular, right self-injective ring, $\mathcal{L} = \mathcal{L}(\mathrm{A}_d)$ the lattice of principal right ideals[2] of A. For J, J' in $\mathcal{L}$ write $\mathrm{J} \sim \mathrm{J}'$ if $\mathrm{J} \cong \mathrm{J}'$ in $\operatorname{Mod} \mathrm{A}$ (that is, J and J' are isomorphic as right A-modules). Then $\sim$ is completely additive in $\mathcal{L}$ in the following sense: if each of $(\mathrm{J}_i)_{i \in \mathrm{I}}$, $(\mathrm{K}_i)_{i \in \mathrm{I}}$ is an independent family in $\mathcal{L}$ such that $\mathrm{J}_i \sim \mathrm{K}_i$ for all $i \in \mathrm{I}$, and if $\mathrm{J} = \bigvee \mathrm{J}_i$, $\mathrm{K} = \bigvee \mathrm{K}_i$ in $\mathcal{L}$, then $\mathrm{J} \sim \mathrm{K}$.

{Proof: Recall that A is a regular Baer ring (1.30), so $\mathcal{L}$ is a complete lattice (1.22), consequently the indicated suprema exist in $\mathcal{L}$ [7, p. 162, Cor. 13.5]. If J is any right ideal of A, then J is an essential submodule of J^{lr}; for, one knows that J is an essential submodule of a principal right ideal K [7, p. 95, Prop. 9.1, (e)], and from $\mathrm{J} \subset \mathrm{J}^{lr} \subset \mathrm{K}^{lr} = \mathrm{K}$ we see that J^{lr} is essential in K; but J^{lr} is a principal right ideal (A is a Baer ring) hence is an injective right A-module, whence $\mathrm{J}^{lr} = \mathrm{K}$, which proves the assertion. Suppose now that (J_i), (K_i) are as given in the statement above. Write $\mathrm{J}_0 = \sum \mathrm{J}_i$, $\mathrm{K}_0 = \sum \mathrm{K}_i$; by independence, these can be viewed as module direct sums $\mathrm{J}_0 = \oplus \mathrm{J}_i$, $\mathrm{K}_0 = \oplus \mathrm{K}_i$, therefore the given isomorphisms $\mathrm{J}_i \cong \mathrm{K}_i$ induce an isomorphism $\mathrm{J}_0 \cong \mathrm{K}_0$ in $\operatorname{Mod} \mathrm{A}$. Now, $\mathrm{J} = \mathrm{J}_0^{lr}$ and $\mathrm{K} = \mathrm{K}_0^{lr}$ (1.21), so by the above remark we know that J_0, K_0 are essential submodules of J, K; hence the isormorphism $\mathrm{J}_0 \cong \mathrm{K}_0$ extends to an isomorphism $\mathrm{J} \cong \mathrm{K}$ in $\operatorname{Mod} \mathrm{A}$, in other words, $\mathrm{J} \sim \mathrm{K}$ as claimed.}

18.19. Let A be a directly finite, regular, left self-injective ring. J.-M. Goursaud and L. Jérémy have shown that A is right self-injective if and only if the analogue of axiom F holds in the lattice $\mathcal{L} = \mathcal{L}(\mathrm{A}_d)$ of principal right ideals of A (with independence playing the role of 'orthogonality') [8, Th. 3.2]; in view of 18.18, this says that for a ring A satisfying the initial assumptions, complete additivity in $\mathcal{L}$ is implied by its special case, axiom F.

[2] In the notation A_d, "d" abbreviates "*dexter*".

19. DIMENSION FUNCTIONS IN FINITE RINGS

Let A be a Baer $*$-ring, $\sim$ an equivalence relation on its projection lattice, satisfying the axioms A–D, F and GC of §10, relative to which A is finite (9.5). {Note: Axiom E also holds (15.1) and equivalence is additive (18.1).} Thus: *Axioms* A–F *and* GC *are in force*, *and* A *is finite* (relative to $\sim$).

In particular, the results of §§15–18 are available; with these in hand, the results to be noted in the present section may be proved verbatim as in [2, Ch. 6][1]. Thus, we shall simply state these results, with appropriate references to [2].

19.1. Let Z be the center of A. Let us write P(A) and P(Z) for the projection lattices of A and Z. In particular, P(Z) is a complete Boolean algebra (3.3, 3.8), with $u + v - uv$ as Boolean sum of $u, v \in \mathrm{P(Z)}$; by M. H. Stone's theory, it is isomorphic to the Boolean algebra of closed-open subsets of a Stonian space $\mathcal{X}$ (cf. 1.39). {One can identify $\mathcal{X}$ with the set of maximal ideals of the Boolean ring P(Z), and $u \in \mathrm{P(Z)}$ with the closed-open subset of $\mathcal{X}$ consisting of the maximal ideals that exclude u.} Let $\mathcal{C}(\mathcal{X})$ be the algebra of continuous complex-valued functions on $\mathcal{X}$; it is a commutative AW*-algebra (1.39). We shall view $\mathrm{P(Z)} \subset \mathcal{C}(\mathcal{X})$ by identifying $u \in \mathrm{P(Z)}$ with the characteristic function of the closed-open subset of $\mathcal{X}$ to which it corresponds. {A few other elements of Z can be identified with elements of $\mathcal{C}(\mathcal{X})$ [cf. 2, p. 157] but this is not needed for the statement of the following results.} The set of funtions

$$\mathcal{C}_1^+(\mathcal{X}) = \{ f \in \mathcal{C}(\mathcal{X}) :\ 0 \le f \le 1 \}$$

(the 'positive unit ball' of $\mathcal{C}(\mathcal{X})$) is a complete lattice. {Caution: The infinite lattice operations in general differ from the pointwise operations.}

19.2. THEOREM. [2, p. 181, Th. 1] *There exists a unique function* $\mathrm{D} : \mathrm{P(A)} \to \mathcal{C}(\mathcal{X})$ *with the following properties*:

(D1) $e \sim f \ \Rightarrow\ \mathrm{D}(e) = \mathrm{D}(f)$;

(D2) $\mathrm{D}(e) \ge 0$ *for all* e;

(D3) $\mathrm{D}(u) = u$ *for all central projections* u;

(D4) $ef = 0 \ \Rightarrow\ \mathrm{D}(e + f) = \mathrm{D}(e) + \mathrm{D}(f)$.

19.3. DEFINITION. The function D of 19.2 is called the **dimension function** of A.

[1] The case that $\sim$ is $\overset{*}{\sim}$, that is, A is a $*$-finite Baer $*$-ring satisfying GC relative to $\overset{*}{\sim}$. (Cf. 11.2).

19.4. THEOREM. [2, p. 160, Prop. 1; p. 181, Ths. 1,2] *The dimension function has, in addition, the following properties*:

(D5) $0 \le \mathrm{D}(e) \le 1$;

(D6) $\mathrm{D}(ue) = u\mathrm{D}(e)$ *when* u *is a central projection*;

(D7) $\mathrm{D}(e) = 0 \;\Leftrightarrow\; e = 0$;

(D8) $e \sim f \Leftrightarrow \mathrm{D}(e) = \mathrm{D}(f)$;

(D9) $e \precsim f \;\Leftrightarrow\; \mathrm{D}(e) \le \mathrm{D}(f)$.

(D10) *If* (e_i) *is an increasingly directed family of projections with supremum* e *(briefly,* $e_i \uparrow e$ *) then* $\mathrm{D}(e) = \sup \mathrm{D}(e_i)$ *in the lattice* $\mathcal{C}_1^+(\mathcal{X})$ [2, p. 184, Exer. 4].

(D11) *If* (e_i) *is an orthogonal family of projections with supremum* e, *then* $\mathrm{D}(e) = \sum \mathrm{D}(e_i)$ (the supremum of the family of finite subsums).

(D12) *If* A *is of type* II *then the range of* D *is all of* $\mathcal{C}_1^+(\mathcal{X})$ [2, p. 182, Th. 3].

19.5. Along the way one proves the following [2, p. 182, Th. 4; p. 183, Exer. 3]: Let $\aleph$ be an infinite cardinal. If every orthogonal family of nonzero central projections has cardinality $\le \aleph$, then the same is true for orthogonal families of not necessarily central projections. To put it more precisely: If there exists an orthogonal family $(e_i)_{i\in \mathrm{I}}$ of nonzero projections with $\operatorname{card} \mathrm{I} > \aleph$, then there exists an orthogonal family $(u_i)_{i\in \mathrm{I}}$ of nonzero central projections. In particular, if A is a 'factor' ($0, 1$ the only central projections) then every orthogonal family of nonzero projections is countable (i.e., has cardinality $\le \aleph_0$). {This is clear, for example, from (D5) and (D11).}

19.6. One can recover from the properties of dimension the axioms, listed at the beginning of the section, that permitted its construction. We illustrate with (i) GC, (ii) axiom C, and (iii) finiteness.

(i) Given any pair of projections e, f. Consider the (open) subset of $\mathcal{X}$ on which the function $\mathrm{D}(e)$ is $< \mathrm{D}(f)$, form its closure (which is closed-open) and let u be the corresponding central projection. Then $u\mathrm{D}(e) \le u\mathrm{D}(f)$ and $(1-u)\mathrm{D}(f) \le (1-u)\mathrm{D}(e)$; citing (D6), we have $\mathrm{D}(ue) \le \mathrm{D}(uf)$ and $\mathrm{D}((1-u)f) \le \mathrm{D}((1-u)e)$, so by (D9) we have $ue \precsim uf$ and $(1-u)f \precsim (1-u)e$.

(ii) Let $(e_i)_{i\in\mathrm{I}}$ be an orthogonal family with supremum e and let $e \sim f$; we seek an orthogonal family $(f_i)_{i\in\mathrm{I}}$ with $\sup f_i = f$ and $e_i \sim f_i$ for all i. We can suppose I to be infinite, and well-ordered, say $\mathrm{I} = \{\alpha :\ \alpha < \Omega\}$, Ω minimal (hence a limit ordinal). {The case of a finite index set is an obvious simplification of the following argument.} Suppose that $\beta < \Omega$ and that f_α has been defined for all $\alpha < \beta$. Let $e' = \sup\{e_\alpha :\ \alpha < \beta\}$, $f' = \sup\{f_\alpha :\ \alpha < \beta\}$. Then by (D11), $\mathrm{D}(e') = \mathrm{D}(f')$. Clearly $e'e_\beta = 0$, therefore $e_\beta \le e - e'$, so

$$\begin{aligned} \mathrm{D}(e_\beta) \le \mathrm{D}(e-e') &= \mathrm{D}(e) - \mathrm{D}(e') = \mathrm{D}(f) - \mathrm{D}(e') \\ &= \mathrm{D}(f) - \mathrm{D}(f') = \mathrm{D}(f - f') \end{aligned}$$

by (D9), (D4), (D1), whence $e_\beta \precsim f - f'$, say $e_\beta \sim f_\beta \le f - f'$. Thus, by transfinite induction, we can suppose defined an orthogonal family $(f_\beta)_{\beta<\Omega}$ with

$f_\beta \le f$ and $e_\beta \sim f_\beta$ for all β. Then

$$\begin{aligned}\mathrm{D}(f - \sup f_\beta) &= \mathrm{D}(f) - \mathrm{D}(\sup f_\beta) = \mathrm{D}(f) - \sum \mathrm{D}(f_\beta) \\ &= \mathrm{D}(e) - \sum \mathrm{D}(e_\beta) = 0\end{aligned}$$

whence $f - \sup f_\beta = 0$.

(iii) Suppose $e \sim 1$. Then $\mathrm{D}(e) = \mathrm{D}(1) = \mathrm{D}(e + (1-e)) = \mathrm{D}(e) + \mathrm{D}(1-e)$, therefore $\mathrm{D}(1-e) = 0$ and so $1 - e = 0$.

19.7. The results of this section are valid for the relation $\overset{a}{\sim}$ in any *regular* Baer $*$-ring.

{Sketch of proof: If A is a regular Baer $*$-ring, then $\overset{a}{\sim}$ satisfies A–F (11.3) and GC (13.11); by a theorem of Kaplansky [17, Th. 1], A is directly finite (this is the hard part!).}

19.8. The dimension theory on projection lattices sketched here has a parallel for the principal right ideal lattices of directly finite, regular right self-injective rings [7, Ch. 11].

20. CONTINUITY OF THE LATTICE OPERATIONS

20.1. DEFINITION. ([27], [21]) A **continuous geometry** is a lattice L that is complete, complemented, modular, and satisfies the following conditions (for increasingly directed and decreasingly directed families):

1° $e_i \uparrow e \;\Rightarrow\; e_i \cap f \uparrow e \cap f$ for all $f \in \mathrm{L}$;

2° $e_i \downarrow e \;\Rightarrow\; e_i \cup f \downarrow e \cup f$ for all $f \in \mathrm{L}$.

20.2. It is a theorem of I. Kaplansky that every orthocomplemented complete modular lattice is a continuous geometry [17]. The proof is long and difficult. In the application to projection lattices, some simplifications have been achieved, as we shall report here, but some heavy artillery from lattice theory still has to be called in to get the full results. The most difficult ring-theoretical argument in [17] is the proof that a regular Baer $*$-ring is directly finite [17, Th. 1]; the following result of I. Amemiya and I. Halperin offers an alternative path via lattice theory (intricate in its own way):

20.3. LEMMA. (I. Amemiya and I. Halperin [34, p. 516, Th.]) *Let* L *be an orthocomplemented*, *countably complete*, *modular lattice. If* (e_n) *is an independent sequence of pairwise perspective elements of* L, *then* $e_n = 0$ *for all* n.

20.4. PROPOSITION. [cf. 17, Th. 1] *If* A *is a Baer* $*$*-ring whose projection lattice is modular*, *then* A *is directly finite.*

Proof. Suppose to the contrary that A is not directly finite; then there exists a projection $e \in \mathrm{A}$ with $e \neq 1$ and $e \overset{a}{\sim} 1$ (7.17). By 11.1 and the proof of "(a) $\Rightarrow$ (b)" of 15.7, there exists an orthogonal sequence of nonzero projections (e_n) such that $e_1 \overset{a}{\sim} e_2 \overset{a}{\sim} \ldots$; the e_n are pairwise perspective by 5.20, which contradicts the lemma. {Remark: The proof remains valid for A a Rickart $*$-ring whose projection lattice is modular and countably complete (for example, a Rickart C*-algebra whose projection lattice is modular).} ♢

20.5. For use in the proof of the main theorem of the section, we review some concepts from lattice theory. Let L be a lattice with 0 and 1 (smallest and largest elements) and suppose there exist ordered sets L_1 and L_2 such that $\mathrm{L} \cong \mathrm{L}_1 \times \mathrm{L}_2$ as ordered sets ($\mathrm{L}_1 \times \mathrm{L}_2$ bears the product ordering). Then each of L_1, L_2 is a lattice with 0 and 1, and if $\varphi : \mathrm{L} \to \mathrm{L}_1 \times \mathrm{L}_2$ is an order-isomorphism then $\varphi(0) = (0, 0)$ and $\varphi(1) = (1, 1)$.

An element $z \in \mathrm{L}$ is said to be **central** if there exists such an isomorphism $\varphi : \mathrm{L} \to \mathrm{L}_1 \times \mathrm{L}_2$ with $\varphi(z) = (1,0)$ [21, p. 27, Def. 3.2]. It is easy to see that $(1,0)$ has $(0,1)$ as its only complement in $\mathrm{L}_1 \times \mathrm{L}_2$, therefore z has a unique complement z' in L, namely $z' = \varphi^{-1}(0,1)$. Write $\mathcal{Z}(\mathrm{L})$ for the set of all such z, and call it the **center** (or 'lattice-center') of L. If $z \in \mathcal{Z}(\mathrm{L})$ then $z' \in \mathcal{Z}(\mathrm{L})$ (let φ be as above and compose it with the canonical isomorphism $\mathrm{L}_1 \times \mathrm{L}_2 \to \mathrm{L}_2 \times \mathrm{L}_1$), consequently $(z')' = z$.

20.6. (S. Maeda [24, Th. 6.2]) Let A be a Rickart $*$-ring, $\mathrm{L} = \mathrm{P(A)}$ its projection lattice (1.15), Z the center of the ring A, and $\mathcal{Z}(\mathrm{L})$ the lattice-center of L (20.5). For $u \in \mathrm{L}$, the following conditions are equivalent: (a) $u \in \mathcal{Z}(\mathrm{L})$; (b) u has a unique complement in L; (c) $u \in \mathrm{Z}$. Thus $\mathcal{Z}(\mathrm{P(A)}) = \mathrm{P(Z)}$.

{Proof: (a) $\Rightarrow$ (b): See 20.5.

(b) $\Leftrightarrow$ (c): This is 3.9.

(c) $\Rightarrow$ (a): The map $e \mapsto (ue, (1-u)e)$ is an order-isomorphism $\mathrm{L} \to \mathrm{P}(u\mathrm{A}) \times \mathrm{P}((1-u)\mathrm{A})$, under which $u \mapsto (u,0)$, whence $u \in \mathcal{Z}(\mathrm{L})$ by definition (20.5).}

20.7. If A is a Baer $*$-ring and e is a projection in A, then its central cover $\mathrm{C}(e)$ is the smallest projection u in the center of Z such that $e \le u$ (3.8, 3.15); in view of 20.6, one sees that $\mathrm{C}(e)$ coincides with the 'central hull of e' defined in [21, p. 69, Def. 4.2].

20.8. THEOREM. [18, p. 117, Th. 69] *Let* A *be a Baer $*$-ring*, $\sim$ *an equivalence relation on its projection lattice*, *relative to which* A *is finite and satisfies axioms* A–D, F *and* H *of* §10. *Then*:

(1) *The projection lattice of* A *is a continuous geometry.*

(2) A *is directly finite.*

(3) $e \sim f \Leftrightarrow e, f$ *are perspective.*

Proof. The projections of A form a complete lattice L (1.24), with an orthocomplementation $e \mapsto 1-e$. By the theorem of Maeda and Holland, axiom E and GC also hold (13.9 and its proof); therefore equivalence is additive in A (18.1). {Since axiom G is a special case of additivity of equivalence, we thus see that *all of the axioms* A–H *are in force*.} Since, moreover, A is finite relative to $\sim$ (that is, $e \sim 1 \Rightarrow e = 1$), the results of §19 are available: A has a dimension function D.

(1) *Modularity*: Let e, f, g be projections with $g \ge e$ and write

$$h = g \cap (e \cup f), \quad k = e \cup (g \cap f);$$

we are to show that $h = k$. Obviously $k \le h$. From $e \le k \le h \le e \cup f$ one sees that

$$e \cup f = k \cup f = h \cup f,$$

and from $g \cap f \le k \le h = g \cap (e \cup f)$ one sees that

$$g \cap f = k \cap f = h \cap f.$$

Then, citing axiom H, one has

$$h - g \cap f = h - h \cap f \sim h \cup f - f = k \cup f - f \sim k - k \cap f = k - g \cap f,$$

thus $h - g \cap f \sim k - g \cap f$; adding this to $g \cap f = g \cap f$, axiom D yields $h \sim k$, which, combined with $k \le h$ and finiteness, yields $k = h$.

Continuity of the lattice operations: In view of the lattice anti-automorphism $e \mapsto 1 - e$, it suffices to verify 1° of 20.1. Suppose $e_i \uparrow e$ and f is any projection; setting $g = \sup(e_i \cap f)$, we are to show that $g = e \cap f$. Clearly $g \le e \cap f$. Citing axiom H, one has

$$e \cap f - e_i \cap f = e \cap f - e_i \cap (e \cap f) \sim (e \cap f) \cup e_i - e_i \le e - e_i\,;$$

invoking the properties of the dimension function D (19.2, 19.4), we therefore have

$$\mathrm{D}(e \cap f - e_i \cap f) \le \mathrm{D}(e - e_i)\,,$$

whence

$$\mathrm{D}(e_i) \le \mathrm{D}(e) + \mathrm{D}(e_i \cap f) - \mathrm{D}(e \cap f) \le \mathrm{D}(e) + \mathrm{D}(g) - \mathrm{D}(e \cap f)$$

for all i. Since $\sup \mathrm{D}(e_i) = \mathrm{D}(e)$, it follows that

$$\mathrm{D}(e) \le \mathrm{D}(e) + \mathrm{D}(g) - \mathrm{D}(e \cap f)\,,$$

whence $\mathrm{D}(e \cap f - g) \le 0$. Thus $\mathrm{D}(e \cap f - g) = 0$, $e \cap f - g = 0$.

(2) Immediate from (1) and 20.4.

(3) $\Leftarrow$: Let g be a common complement of e and f: $e \cup g = f \cup g = 1$, $e \cap g = f \cap g = 0$. Citing axiom H,

$$e = e - e \cap g \sim e \cup g - g = 1 - g\,,$$

similarly $f \sim 1 - g$, whence $e \sim f$.

$\Rightarrow$: By (1), the projection lattice L of A is a continuous geometry. Given any pair e, f in L, there exists, by the theory of continuous geometries, an element u in the lattice-center of L—that is (20.6), a projection u in the center of the ring A—such that $u \cap e$ is perspective to some $e' \le u \cap f$, and $(1-u) \cap f$ is perspective to some $f' \le (1-u) \cap e$ [21, p. 87, Satz. 1.1] (cf. 13.2). Thus, by the implication proved in the preceding paragraph, $u \cap e \sim e' \le u \cap f$ and $(1-u) \cap f \sim f' \le (1-u) \cap e$, in other words,

$$ue \sim e' \le uf \ \text{ and } (1-u)f \sim f' \le (1-u)e\,.$$

Suppose, in addition, that $e \sim f$. Then $ue \sim uf$ (axiom B) so $e' = uf$ by finiteness; then ue and uf ($= e'$) are perspective in L, say with common complement g:

$$(ue) \cup g = (uf) \cup g = 1\,, \quad (ue) \cap g = (uf) \cap g = 0\,.$$

Multiplying through by u, it follows (cf. 3.23) that ug is a common complement of ue and uf in the projection lattice of $u\mathrm{A}$. Similarly, $(1-u)e$ and $(1-u)f$

have a common complement h in L, hence they admit $(1-u)h$ as a common complement in the projection lattice of $(1-u)\mathrm{A}$. It is then elementary that $ug+(1-u)h$ is a common complement of e and f in L. Thus e,f are perspective in L. ♢

20.9. In a $*$-regular ring A , modularity of the projection lattice comes free of charge; for, $e \mapsto e\mathrm{A}$ is an order-isomorphism of the projection lattice onto the lattice of principal right ideals, and the principal right ideals of any regular ring form a modular lattice (1.16).

20.10. COROLLARY. (I. Kaplansky [17, Th. 3], [18, p. 120, Th. 71]) *Let* A *be a regular Baer* $*$*-ring.*

(i) *The relation* $\overset{a}{\sim}$ *satisfies axioms* A–H *and finiteness, and the projection lattice of* A *is a continuous geometry.*

(ii) $e \overset{a}{\sim} f \Leftrightarrow e,f$ *are perspective.*

(iii) *If* $\sim$ *is an equivalence relation on the projection lattice of* A *satisfying axioms* A–D, F, H *and finiteness, then* $\sim$ *coincides with* $\overset{a}{\sim}$ (*thus with perspectivity*).

Proof. (i) The relation $\overset{a}{\sim}$ satisfies A–F and H (11.3). Since the projection lattice of A is modular (20.9), A is directly finite by 20.4 (alternatively, cite [17, Th. 1]), that is, A is finite relative to $\overset{a}{\sim}$ (7.1, 7.17). It then follows from 20.8 that the projection lattice is a continuous geometry. Axiom G (indeed, additivity of equivalence) holds by the remarks at the beginning of the proof of 20.8.

(ii) Immediate from (i) and 20.8.

(iii) Immediate from 20.8 and (ii). ♢ Cf. 21.3.

20.11. COROLLARY. *Let* A *be a* $*$*-finite* ($xx^*=1 \Rightarrow x^*x=1$) *Baer* $*$*-ring such that* $e\cup f-f \overset{*}{\sim} e-e\cap f$ *for all projections* e,f .

(i) *The relation* $\overset{*}{\sim}$ *satisfies axioms* A–H *and finiteness, and the projection lattice of* A *is a continuous geometry. Moreover,* A *is directly finite.*

(ii) $e \overset{*}{\sim} f \Leftrightarrow e,f$ *are perspective.*

(iii) *If* $\sim$ *is an equivalence relation on the projection lattice of* A *satisfying axioms* A–D, F, H *and finiteness, then* $\sim$ *coincides with* $\overset{*}{\sim}$ (*thus with perspectivity*).

Proof. (i) For axioms A–D and F, see 11.2; by hypothesis, $\overset{*}{\sim}$ also satisfies axiom H and finiteness (9.5). By 20.8, the projection lattice of A is a continuous geometry and A is directly finite; moreover, as noted in the proof of 20.8, axiom E and axiom G (indeed, additivity of $*$-equivalence) also hold, thus A–H hold.

(ii) Immediate from (i) and 20.8.

(iii) Immediate from 20.8 and (ii). ♢

In particular, if A is a Baer $*$-ring for which $\overset{*}{\sim}$ satisfies the parallelogram law, then A is $*$-finite if and only if it is directly finite; in view of 12.13, this is a generalization of 6.13.

20.12. The proof of 20.8 shows that if $\sim$ satisfies axiom H and finiteness (and axiom D) then the projection lattice is modular. We now turn to a criterion for modularity that is independent of axiom H (indeed, no equivalence relation is postulated).

20.13. LEMMA. *If* e, f *are projections in a Rickart* $*$*-ring* A, *such that* $e \cap f = 0$ (*thus* e *and* f *are complementary in* $(e \cup f)A(e \cup f)$), *then the projection* $g = f + [1 - (e \cup f)]$ *is a complement of* e *in* A: $e \cup g = 1$, $e \cap g = 0$.

Proof. Since $g = f \cup [1 - (e \cup f)]$, it is clear that $e \cup g = 1$. Set

$$x = e(1-g) = e(e \cup f - f) = e - ef = e(1-f).$$

Citing 1.15 twice, we have

$$e \cap g = e - \mathrm{LP}(x) = e - (e - e \cap f) = e - e = 0. \diamondsuit$$

The following proposition was communicated to me by David Handelman, who attributes it to Israel Halperin:

20.14. PROPOSITION. *The following conditions on a Rickart* $*$*-ring* A *are equivalent*:

(a) *the projection lattice of* A *is modular*;

(b) *if* e, f *are projections in* A *such that* e, f *are perspective and* $e \le f$, *then* $e = f$.

Proof. (a) $\Rightarrow$ (b): Suppose $e \le f$ and g is a common complement of e and f. Citing modularity,

$$f \cap (e \cup g) = e \cup (f \cap g),$$

that is, $f \cap 1 = e \cup 0$, so $f = e$.

(b) $\Rightarrow$ (a): Let e, f, g be projections with $e \le g$ and set

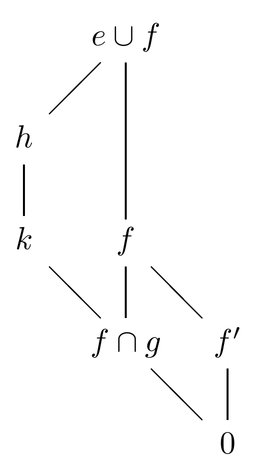
$$h = (e \cup f) \cap g, \quad k = e \cup (f \cap g);$$

the problem is to show that $h = k$. Evidently $k \le h$. As in the proof of 20.8, we see that

$$h \cup f = k \cup f = e \cup f, \tag{i}$$

$$h \cap f = k \cap f = f \cap g. \tag{ii}$$

Set $f' = f - f \cap g$;



then

$$h \cap f' = 0\,; \tag{iii}$$

for, citing (ii),

$$\begin{aligned} h \cap f' &= h \cap (f \cap f') \\ &= (h \cap f) \cap f' \\ &= (f \cap g) \cap f' \\ &= (f \cap g) f' = 0\,. \end{aligned}$$

Also,

$$k \cup f' = e \cup f\,; \tag{iv}$$

for,

$$\begin{aligned} k \cup f' &= [e \cup (f \cap g)] \cup f' = e \cup [(f \cap g) \cup f'] \\ &= e \cup [(f \cap g) + f'] = e \cup f\,. \end{aligned}$$

Now, $k \le h \le e \cup f$. Set

$$\overline{h} = h + [1 - (e \cup f)]\,, \quad \overline{k} = k + [1 - (e \cup f)]\,.$$

Then $\overline{k} \le \overline{h}$ and it will suffice to show that $\overline{k} = \overline{h}$. In view of (b), it will suffice to show that f' is a common complement of $\overline{h}$ and $\overline{k}$; since $\overline{k} \le \overline{h}$, it is sufficient to show

$$\overline{k} \cup f' = 1\,, \tag{v}$$

$$\overline{h} \cap f' = 0\,. \tag{vi}$$

Re (v): Citing (iv) at the appropriate step,

$$\begin{aligned} \overline{k} \cup f' &= (k \cup [1 - (e \cup f)]) \cup f' \\ &= (k \cup f') \cup [1 - (e \cup f)] \\ &= (e \cup f) \cup [1 - (e \cup f)] = 1\,. \end{aligned}$$

Re (vi): Let $a = f'$, $b = h$; then $a \cap b = 0$ by (iii), so, setting $c = b + [1 - (a \cup b)]$, it follows from the lemma (20.13) that a and c are complementary, in particular $a \cap c = 0$. Note that $a \cup b = e \cup f$. {For, $a \cup b = f' \cup h \ge f' \cup k = e \cup f \ge a \cup b$.} Therefore

$$c = h + [1 - (e \cup f)] = \overline{h}\,,$$

thus $0 = a \cap c = f' \cap \overline{h}$. $\diamondsuit$

21. EXTENDING THE INVOLUTION

This section is an exposition of results contained in a paper of D. Handelman [11]. {Handelman's paper contains in addition a wealth of material on matrix rings over Baer $*$-rings.}

21.1. DEFINITION. A $*$-ring A will be said to be $*$-**extendible** if its involution is extendible to an involution of its maximal ring of right quotients. {For the general theory of rings of quotients, which is due to Y. Utumi, I draw on the exposition of J. Lambek [19, §4.3–§4.5].}

21.2. PROPOSITION. ([28], [11]) *Let* A *be a $*$-extendible Baer $*$-ring,* Q *its maximal ring of right quotients. Then*:

(i) Q *is also the maximal ring of left quotients of* A.

(ii) Q *is regular and self-injective* (both right and left).

(iii) Q *is unit-regular, hence directly finite* (hence A is directly finite).

(iv) *The extension of the involution of* A *to* Q *is unique.*

(v) *The involution of* Q *is proper.*

(vi) Q *is a regular Baer $*$-ring, all of whose projections are in* A.

(vii) *A projection of* A (i.e., of Q) *is in the center of* A *if and only if it is in the center of* Q.

(viii) *The projection lattice of* A (i.e., of Q) *is a continuous geometry* (in particular, it is modular).

(ix) *For projections* e, f *in* A, *the following conditions are equivalent*: (a) e, f *are perspective in* A; (b) e, f *are perspective in* Q; (c) $e \overset{a}{\sim} f$ *in* Q.

Proof. (i) The ring isomorphism $a \mapsto a^*$ of A onto the opposite ring A° extends, by hypothesis, to a ring isomorphism $Q \to Q^\circ$.

(ii) By 1.29, A is nonsingular (right and left), so Q is regular and right self-injective [19, p. 106, Prop. 2 and its corollary]; since Q possesses an involution, it is also left self-injective (or cite (i)).

(iii) Immediate from (ii) and [7, p. 105, Th. 9.29].

(iv) Given any two involutions of Q extending that of A, their composition is a ring automorphism of Q leaving fixed every element of A; since Q is a ring of right quotients of A, this automorphism must be the identity mapping [cf. 19, p. 99, Prop. 8].

(v) Write $x \mapsto x^*$ for the involution of Q extending that of A (iv). For each $x \in Q$, the set $I = (A : x) = \{a \in A : xa \in A\}$ is a dense right ideal of A [cf. 19, p. 96, Lemma 2]. If $x^*x = 0$ then for all $a \in \mathrm{I}$ one has $xa \in \mathrm{A}$ and $(xa)^*(xa) = a^*(x^*x)a = 0$, therefore $xa = 0$ (the involution of A is proper by 1.10); thus $x\mathrm{I} = 0$, so $x = 0$ by the density of I [19, p. 96, Prop. 4]. (See also the proof of 21.8 below.)

(vi) By (ii) and 1.30, Q is a regular Baer ring; since its involution is proper (v) it is $*$-regular (1.14) hence is a regular Baer $*$-ring (1.25). To complete the proof of (vi), given any $x \in \mathrm{Q}$ it will suffice to show that $\mathrm{LP}(x)$ (as computed in the Baer $*$-ring Q) is in A; the following argument is taken from [28, Th. 3.3]. Let $\mathrm{I} = (\mathrm{A} : x) = \{a \in \mathrm{A} :\ xa \in \mathrm{A}\}$, which is a dense right ideal of A, and let

$$e = \sup\{\mathrm{LP}(xa) :\ a \in \mathrm{I}\},$$

where $\mathrm{LP}(xa)$ is computed in A and the supremum is taken in the projection lattice of A; it will suffice to show that $\{x\}^l = \mathrm{Q}(1-e)$, where $\{x\}^l$ denotes the left annihilator of x in Q (cf. 1.7). For all $a \in \mathrm{I}$ one has $e \geq \mathrm{LP}(xa)$, so $e(xa) = xa$, $(1-e)xa = 0$; thus $(1-e)x\mathrm{I} = 0$, therefore $(1-e)x = 0$ by the density of I. This shows that $\mathrm{Q}(1-e) \subset \{x\}^l$. On the other hand, suppose $y \in \{x\}^l$ and let $\mathrm{J} = \{b \in \mathrm{A} :\ by \in \mathrm{A}\}$; in view of (i), J is a dense left ideal of A. Let $b \in \mathrm{J}$; for all $a \in \mathrm{I}$, $(by)(xa) = b(yx)a = 0$, thus $by\mathrm{LP}(xa) = 0$ for all $a \in \mathrm{I}$, consequently $bye = 0$; varying $b \in \mathrm{J}$, we have $\mathrm{J}ye = 0$, so $ye = 0$ by the density of J, whence $y = y(1-e) \in \mathrm{Q}(1-e)$. This shows that $\{x\}^l \subset \mathrm{Q}(1-e)$ and completes the proof of (vi). {It follows that if $x \in \mathrm{A}$ then $\mathrm{LP}(x)$ is unambiguous—it is the same whether calculated in A or in Q. And of course the lattice operations are unambiguous.}

(vii) Since A and Q have the same projection lattice (vi), this is immediate from 3.9 (it is also easy to see from the theory of rings of quotients (cf. 21.30)).

(viii) The projection lattice of Q is isomorphic to the lattice of principal right ideals of Q (via $e \mapsto e\mathrm{Q}$) hence is modular (cf. 1.16); the continuity of the lattice operations follows from self-injectivity [7, p. 162, Cor. 13.5]. So the requirements of Definition 20.1 are fulfilled. {One could also cite 20.10; but note that in the proof of 20.10 one had to cite Kaplansky's theorem (or its generalization by Amemiya and Halperin) to get direct finiteness, whereas in the present context we have an easier route to direct finiteness via (iii). Caution: We do not have here an easier proof of Kaplansky's theorem; indeed, we do not have here an alternative proof of Kaplansky's theorem at all.}

(ix) The equivalence (a) $\Leftrightarrow$ (b) is trivial since the projection lattices are the same, whereas (b) $\Leftrightarrow$ (c) by unit-regularity [7, p. 39, Cor. 4.4]. {One could also cite 20.10 for the equivalence of (b) and (c), but again this would entail Kaplansky's theorem.} ♢

21.3. PROPOSITION. [13, Prop. 3] *Let* A *be a $*$-extendible Baer $*$-ring,* Q *its maximal ring of right quotients, and view* A *as a $*$-subring of* Q (21.2). *Then the following conditions on* A *relative to* $\overset{a}{\sim}$ (written briefly $\sim$) *are equivalent*:

(a) A *satisfies* $\mathrm{LP} \sim \mathrm{RP}$;

(b) A *satisfies axiom* H;
(c) A *has* GC;
(d) $e \sim f$ *in* Q $\Rightarrow$ $e \sim f$ *in* A;
(d′) $e \sim f$ *in* Q $\Leftrightarrow$ $e \sim f$ *in* A;
(e) e, f *perspective in* A $\Rightarrow$ $e \sim f$ *in* A;
(e′) e, f *perspective in* A $\Leftrightarrow$ $e \sim f$ *in* A.

Proof. (a) $\Rightarrow$ (b): Trivial (1.15).

(b) $\Rightarrow$ (d): Suppose $e \sim f$ in Q. By (ix) of 21.2, e and f are perspective in A, therefore $e \sim f$ in A by (b) (see the proof of 20.8).

(d) $\Leftrightarrow$ (d′): The reverse implication in (d) is trivial.

(d′) $\Leftrightarrow$ (e′): The left sides of (d′) and (e′) are equivalent by (ix) of 21.2.

(e′) $\Rightarrow$ (e): Trivial.

(e) $\Rightarrow$ (a): Let $a \in$ A and write $e = \mathrm{LP}(a)$, $f = \mathrm{RP}(a)$. Since Q is $*$-regular, $e \sim f$ in Q (5.8), therefore e, f are perspective in A by (ix) of 21.2, whence $e \sim f$ in A by (e).

So far, we know that all conditions other than (c) are equivalent.

(d) $\Rightarrow$ (c): One knows that GC holds for $\sim$ in the regular Baer $*$-ring Q (13.11); it is then immediate from (d) that GC holds for $\sim$ in A (recall that Q and A have the same projection lattices and the same central projections).

(c) $\Rightarrow$ (d): Suppose $e \sim f$ in Q. By (c) there is a central projection u such that, *in* A, one has

$$ue \sim e' \leq uf \quad \text{and} \quad (1-u)f \sim f' \leq (1-u)e$$

for suitable projections e', f'. In Q we have

$$uf \sim ue \sim e' \leq uf\,,$$

so $e' = uf$ by the direct finiteness of Q; thus $ue \sim uf$ in A. Similarly $(1-u)e \sim (1-u)f$ in A, so $e \sim f$ in A. ♢

21.4. PROPOSITION. ([11, Prop. 3.3], [13]) *Let* A, Q *be as in* 21.2. *The following conditions are equivalent*:†

(1) A *satisfies* LP $\overset{*}{\sim}$ RP;
(2) *for all* $x \in$ Q, $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x)$ *in* A;
(3) A *satisfies axiom* H *for* $\overset{*}{\sim}$;
(4) e, f *perspective in* A $\Rightarrow e \overset{*}{\sim} f$ *in* A;
(4′) e, f *perspective in* A $\Leftrightarrow$ $e \overset{*}{\sim} f$ *in* A;
(5) $e \overset{a}{\sim} f$ *in* Q $\Rightarrow$ $e \overset{*}{\sim} f$ *in* A;

†To put it more succinctly: In a $*$-extendible Baer $*$-ring, the conditions LP $\overset{*}{\sim}$ RP, axiom H for $\overset{*}{\sim}$, and GC for $\overset{*}{\sim}$ are equivalent. When these conditions are verified, the relations (in A or in Q, it does not matter) $e \overset{a}{\sim} f$, $e \overset{*}{\sim} f$, e and f perspective, are all equivalent. On the other hand, let A be the ring of all 2×2 matrices over the field of 3 elements. With transpose as involution, A is a regular Baer $*$-ring, trivially $*$-extendible (A = Q), in which $e \overset{a}{\sim} f$ does not imply $e \overset{*}{\sim} f$ [18, p. 39]. Of course, axiom H fails for $\overset{*}{\sim}$ in this example [2, p. 75, Exer. 1].

(6) A *has* GC *for* $\overset{*}{\sim}$.

Moreover, these conditions imply each of the following two:

(7) $e \overset{*}{\sim} f$ *in* Q $\Rightarrow$ $e \overset{*}{\sim} f$ *in* A ;

(8) $e \overset{a}{\sim} f$ *in* A $\Rightarrow$ $e \overset{*}{\sim} f$ *in* A .

Proof. (1) $\Rightarrow$ (2): Let $x \in \mathrm{Q}$, $e = \mathrm{LP}(x)$, $f = \mathrm{RP}(x)$. Since Q is $*$-regular, $e \overset{a}{\sim} f$ in Q (5.8). But from (1) one has *a fortiori* LP $\overset{a}{\sim}$ RP in A, hence the equivalent conditions of 21.3 are verified; in particular, condition (d) of 21.3 yields $e \overset{a}{\sim} f$ in A. Say $r \in e\mathrm{A}f$, $s \in f\mathrm{A}e$ with $rs = e$, $sr = f$; it is then elementary that $e = \mathrm{LP}(r)$, $f = \mathrm{RP}(r)$, therefore $e \overset{*}{\sim} f$ in A by (1). That is, $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x)$ in A.

(2) $\Rightarrow$ (3): Immediate from 1.15.

(3) $\Rightarrow$ (4): Same as in the proof of 20.8.

(4) $\Leftrightarrow$ (4′): If $e \overset{*}{\sim} f$ in A, then *a fortiori* $e \overset{a}{\sim} f$ in Q, hence e, f are perspective in A by (ix) of 21.2. Thus the reverse implication in (4) always holds, so (4) and (4′) say the same thing.

(4) $\Rightarrow$ (5): Suppose $e \overset{a}{\sim} f$ in Q. Then e, f are perspective in A by (ix) of 21.2, so $e \overset{*}{\sim} f$ in A by (4).

(5) $\Rightarrow$ (1): Let $r \in \mathrm{A}$, $e = \mathrm{LP}(r)$, $f = \mathrm{RP}(r)$. Then $e \overset{a}{\sim} f$ in Q by $*$-regularity (5.8), so $e \overset{*}{\sim} f$ in A by (5).

Thus all conditions from (1) through (5) are equivalent.

(3) $\Rightarrow$ (6): This holds, for example, by the theorem of Maeda and Holland (13.10). {Alternatively, one could show (5) $\Rightarrow$ (6) using 13.11. Or show (5) $\Rightarrow$ (6) using general comparability for regular, right self-injective rings [7, p. 102, Cor. 9.15].}

(6) $\Rightarrow$ (5): Let $e \overset{a}{\sim} f$ in Q. By (6), there exists a central projection u with $ue \overset{*}{\sim} f' \le uf$ in A and $(1-u)f \overset{*}{\sim} e' \le (1-u)e$ in A. *A fortiori*,

$$ue \overset{a}{\sim} f' \le uf \quad \text{in } \mathrm{Q};$$

but $ue \overset{a}{\sim} uf$ in Q, so $f' = uf$ by direct finiteness of Q (21.2). Thus

$$ue \overset{*}{\sim} uf \quad \text{in } \mathrm{A};$$

similarly $(1-u)e \overset{*}{\sim} (1-u)f$ in A, and addition yields $e \overset{*}{\sim} f$ in A.

Summarizing, all conditions from (1) to (6) are equivalent. The implications (5) $\Rightarrow$ (7) and (5) $\Rightarrow$ (8) are trivial, whence the last statement of the proposition. ♢

21.5. DEFINITION. (D. Handelman [11, p. 7]) A ring A is said to be **strongly modular** if, for $x \in \mathrm{A}$,

$$\{x\}^r = 0 \quad \Rightarrow \quad x\mathrm{A} \text{ is an essential right ideal of } \mathrm{A}.$$

Handelman proved [11, p. 12 and Th. 2.3]: *A Baer $*$-ring is $*$-extendible if and only if it is strongly modular.* We give here an exposition of his proof, culminating in 21.22 below. {Observe that in the applications of strong modularity in the results leading up to 21.22, the implication "$\{x\}^r = 0 \Rightarrow xA$ essential" is cited only for self-adjoint x; this remark is exploited in 21.23 and 21.24.}

21.6. A strongly modular Rickart ring is directly finite.

{Proof: Suppose $yx = 1$ and let $e = xy$, which is idempotent; we are to show that $e = 1$. One has

$$xt = 0 \Rightarrow yxt = 0 \Rightarrow t = 0,$$

so $\{x\}^r = 0$; by the hypothesis, xA is an essential right ideal. But $x = x1 = xyx = ex$, so $xA \subset eA$; therefore eA is also an essential right ideal, whence $eA = A$, $e = 1$.}

21.7. A Rickart $*$-ring A is strongly modular if and only if $\mathrm{RP}(x) = 1 \Rightarrow xA$ essential; in such a ring, $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$.

{Proof: $\{x\}^r = (1-\mathrm{RP}(x))A$, thus $\mathrm{RP}(x) = 1 \Leftrightarrow \{x\}^r = 0$, whence the first assertion. On the other hand, $xA \subset \mathrm{LP}(x)A$, so xA essential $\Rightarrow \mathrm{LP}(x) = 1$.} [1] Cf. 21.26.

21.8. LEMMA. *Let* A *be a right nonsingular ring,* Q *its maximal ring of right quotients,* $x \in Q$. *Then, as right* A*-modules,* xA *is essential in* xQ.

Proof. The assertion is that $(xA)_A \subset_e (xQ)_A$. At any rate, one knows that $A_A \subset_e Q_A$ [19, p. 99, proof of Prop. 8]. Moreover, if $y \in Q$ and J is an essential right ideal of A with $yJ = 0$, then $y = 0$. {For, since A is right nonsingular, Q is regular and right self-injective [19, p. 106, Prop. 2 and its corollary]; moreover, if I_A is the injective hull of A_A, then the singular submodule of I_A is 0. But $I_A \cong Q_A$ [19, p. 95, Prop. 3], so the singular submodule of Q_A is also 0. This means that if $y \in Q$ and $y \neq 0$, then the right ideal $(0 : y) = \{a \in A :\ ya = 0\}$ of A cannot be essential.}

Now let N be an A-submodule of $(xQ)_A$ with $N \cap xA = 0$; we must show $N = 0$. Let $y \in N$. Since $N \subset xQ$, $y = xz$ for some $z \in Q$. Since $A_A \subset_e Q_A$, the right ideal $I = (A : z) = \{a \in A :\ za \in A\}$ is essential; and $zI \subset A$, so $yI = xzI \subset xA$. But also $yI \subset yA \subset N$, thus $yI \subset N \cap xA = 0$, $yI = 0$. By the preceding paragraph, $y = 0$. Thus $N = 0$. ♢

21.9. LEMMA. [11, p. 12] *If* A *is a* $*$*-extendible Baer* $*$*-ring, then* A *is strongly modular.*

Proof. Let $x \in A$ with $\mathrm{RP}(x) = 1$; we are to show that xA is an essential right ideal of A (21.7). Let I be a right ideal of A with $I \cap xA = 0$; we must show that $I = 0$. It suffices to consider the case that $I = yA$ for some $y \in A$. Then $xA \cap yA = 0$ with $\mathrm{RP}(x) = 1$, and we are to show that $y = 0$.

Let Q be the maximal ring of right quotients of A (cf. 21.2). Then $\mathrm{LP}(x) \overset{a}{\sim} \mathrm{RP}(x) = 1$ in the regular ring Q (5.8), so $\mathrm{LP}(x) = 1$ by direct finiteness.

Now, A is right nonsingular (1.29), so 21.8 applies: $xA \subset_e xQ$ and $yA \subset_e yQ$ as right A-modules. By elementary module theory, $xA \cap yA \subset_e xQ \cap yQ$, thus

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[1] For a $*$-regular ring A, strong modularity means $Ax = A \Rightarrow xA = A$, that is, x left-invertible $\Rightarrow$ x right-invertible. Thus, a $*$-regular ring (known to be $*$-finite—see the footnote to 9.5) is strongly modular if and only if it is directly finite.

$0 \subset_e x\mathrm{Q} \cap y\mathrm{Q}$, whence $x\mathrm{Q} \cap y\mathrm{Q} = 0$. Then, by $*$-regularity (1.13, 1.16),

$$0 = x\mathrm{Q} \cap y\mathrm{Q} = (\mathrm{LP}(x)\mathrm{Q}) \cap (\mathrm{LP}(y)\mathrm{Q}) = \mathrm{Q} \cap (\mathrm{LP}(y)\mathrm{Q}) = \mathrm{LP}(y)\mathrm{Q}\,,$$

so $\mathrm{LP}(y) = 0$, $y = 0$. $\diamondsuit$

To prove the converse, we now develop the consequences of strong modularity.

21.10. LEMMA. [11, pp. 7-8] *In a Rickart $*$-ring* A, *if* e, f *are projections with* $e \cap f = 0$, *then* $\mathrm{RP}(e+f) = e \cup f$.

Proof. Let $k = \mathrm{RP}(e+f)$. Obviously $(e+f)(e \cup f) = e + f$, so $k \le e \cup f$. But $\{e+f\}^r = (1-k)\mathrm{A}$, so $0 = (e+f)(1-k)$, and citing 1.18 we have

$$e(1-k) = -f(1-k) \in e\mathrm{A} \cap f\mathrm{A} = (e \cap f)\mathrm{A} = 0\,,$$

thus $e(1-k) = f(1-k) = 0$, $e \le k$ and $f \le k$, whence $e \cup f \le k$. $\diamondsuit$

21.11. LEMMA. [11, Prop. 2.2, (b)] *If* A *is a strongly modular Rickart $*$-ring* (21.5) *then the projection lattice of* A *is modular.*

Proof. Assuming e, f perspective with $e \le f$, it will suffice by 20.14 to show that $e = f$. Let g be a common complement of e and f: $e \cup g = f \cup g = 1$, $e \cap g = f \cap g = 0$. Since $e \cap g = 0$, by 21.10 we have $\mathrm{RP}(e+g) = e \cup g = 1$, thus $\{e+g\}^r = 0$; by strong modularity, $(e+g)\mathrm{A}$ is an essential right ideal. So to show that $f - e = 0$, it will suffice to show that

$$(f-e)\mathrm{A} \cap (e+g)\mathrm{A} = 0\,. \qquad (*)$$

Suppose $t = (f-e)a = (e+g)b$. In particular $t \in (f-e)\mathrm{A} \subset f\mathrm{A}$, also $e\mathrm{A} \subset f\mathrm{A}$, so

$$t - eb = gb \in f\mathrm{A} \cap g\mathrm{A} = (f \cap g)\mathrm{A} = 0\,,$$

whence $t - eb = gb = 0$. Then $eb = t \in e\mathrm{A} \cap (f-e)\mathrm{A} = 0$, whence $t = 0$, thus (*) is verified. $\diamondsuit$

21.12. [11, p. 9] If A is a strongly modular Baer $*$-ring, then its projection lattice is a continuous geometry.

{Proof: In view of 21.11, the projections of A form an orthocomplemented, complete modular lattice; such a lattice must be a continuous geometry by Kaplansky's theorem [17, Th. 3]. We remark that if A satisfies axiom H for $\overset{*}{\sim}$, then since A is $*$-finite (even directly finite, by 21.6) a proof of continuity avoiding Kaplansky's theorem is available via the proof of 20.11, (i).}

21.13. The next target (21.21) is Handelman's theorem that if A is a strongly modular Baer $*$-ring and I is any right ideal of A, then I is essential in I^{lr} (as right A-modules). The strategy is to prove it first for a principal right ideal, then for a finitely generated right ideal, and then infer the general case from the finitely generated case by an application of the continuity of the lattice operations (21.12).

21.14. LEMMA. [11, Prop. 2.2, (a)] *If* A *is a strongly modular Rickart $*$-ring, then for every* $x \in \mathrm{A}$, $x\mathrm{A}$ *is essential in* $(x\mathrm{A})^{lr}$ (*as right* A-*modules*).

Proof. Let $x \in \mathrm{A}$. Set $e = \mathrm{LP}(x) = \mathrm{LP}(xx^*)$. Then $(x\mathrm{A})^l = \{x\}^l = \mathrm{A}(1-e)$, so $(x\mathrm{A})^{lr} = e\mathrm{A}$; thus the problem is to show that $x\mathrm{A}$ is essential in $e\mathrm{A}$. Set $z = xx^* + (1-e)$, which is self-adjoint.

claim 1: $\{z\}^r = 0$.

Write $\{z\}^r = g\mathrm{A}$, g a projection. Then $0 = zg = xx^*g + (1-e)g$, so

$$xx^*g = -(1-e)g \in x\mathrm{A} \cap (1-e)\mathrm{A} \subset e\mathrm{A} \cap (1-e)\mathrm{A} = 0\,,$$

thus $xx^*g = (1-e)g = 0$. Thus $g = eg = ge$ and $gxx^* = 0$; since $e = \mathrm{LP}(xx^*)$, one has $ge = 0$, thus $g = ge = 0$.

Since A is strongly modular, we infer from claim 1 that $z\mathrm{A}$ is an essential right ideal of A. To show that $x\mathrm{A}$ is essential in $e\mathrm{A}$, let J be a right ideal of A with $\mathrm{J} \subset e\mathrm{A}$ and $\mathrm{J} \cap x\mathrm{A} = 0$; we are to show that $\mathrm{J} = 0$. Let $y \in \mathrm{J}$; then $y\mathrm{A} \subset \mathrm{J}$, so $y\mathrm{A} \cap x\mathrm{A} = 0$. We must infer that $y = 0$; since $z\mathrm{A}$ is essential, it will suffice to establish the following:

claim 2: $z\mathrm{A} \cap y\mathrm{A} = 0$.

Since $z\mathrm{A} = [xx^* + (1-e)]\mathrm{A} \subset x\mathrm{A} + (1-e)\mathrm{A}$, it is enough to show that

$$[x\mathrm{A} + (1-e)\mathrm{A}] \cap y\mathrm{A} = 0\,.$$

Suppose $t = xa + (1-e)b = yc$ with $a, b, c \in \mathrm{A}$. Then

$$xa - yc = -(1-e)b \in e\mathrm{A} \cap (1-e)\mathrm{A} = 0\,,$$

so $xa - yc = (1-e)b = 0$. Then

$$t = xa + (1-e)b = xa = yc \in x\mathrm{A} \cap y\mathrm{A} = 0\,,$$

so $t = 0$. ♢

21.15. LEMMA. *If* A *is a strongly modular Rickart $*$-ring, then for every pair of projections* e_1, e_2 *in* A, $e_1\mathrm{A} + e_2\mathrm{A}$ *is essential in* $(e_1 \cup e_2)\mathrm{A}$.

Proof. Write $e = e_1 \cup e_2$, $f = e_1 \cap e_2$. Then $f \le e_i$, so

$$e_i\mathrm{A} = (e_i - f)\mathrm{A} \oplus f\mathrm{A} \quad (i = 1, 2)\,. \tag{1}$$

Note that

$$e_1\mathrm{A} + e_2\mathrm{A} = (e_1 - f)\mathrm{A} \oplus e_2\mathrm{A}\,. \tag{2}$$

For, the inclusion $\supset$ is obvious, and the inclusion $\subset$ follows from $e_1 = (e_1 - f) + f$ and $f \in e_2\mathrm{A}$; finally,

$$(e_1 - f)\mathrm{A} \cap e_2\mathrm{A} \subset e_1\mathrm{A} \cap e_2\mathrm{A} = (e_1 \cap e_2)\mathrm{A} = f\mathrm{A}$$

whereas $(e_1 - f)\mathrm{A} \subset (1-f)\mathrm{A}$, whence $(e_1 - f)\mathrm{A} \cap e_2\mathrm{A} = 0$. From (1) and (2) we have

$$e_1\mathrm{A} + e_2\mathrm{A} = (e_1 - f)\mathrm{A} \oplus (e_2 - f)\mathrm{A} \oplus f\mathrm{A}\,. \tag{3}$$

Next we observe that the (self-adjoint) element

$$z = (1 - e) + (e_1 - f) + (e_2 - f) + f$$

has right annihilator 0 (hence $z\mathrm{A}$ is an essential right ideal by strong modularity). For, write $\{z\}^r = g\mathrm{A}$, g a projection. From $zg = 0$ it follows that

$$-(1 - e)g = (e_1 - f)g + (e_2 - f)g + fg\,; \tag{$*$}$$

since the right side of (*) is in $e\mathrm{A}$, both sides must be 0, and citing directness of the decomposition (3) we infer

$$(1 - e)g = (e_1 - f)g = (e_2 - f)g = fg = 0\,.$$

It follows that $e_1g = fg = 0$, $e_2g = fg = 0$. Therefore $ge_1 = ge_2 = 0$, whence

$$g \in (e_1\mathrm{A} + e_2\mathrm{A})^l = (e_1\mathrm{A} + e_2\mathrm{A})^{lrl} = ((e_1 \cup e_2)\mathrm{A})^l = (e\mathrm{A})^l = \mathrm{A}(1 - e)\,;$$

thus $g \le 1 - e$. But $(1 - e)g = 0$, so $g \le e$; consequently $g = 0$, thus $\{z\}^r = 0$ as claimed.

To prove $e_1\mathrm{A} + e_2\mathrm{A}$ essential in $e\mathrm{A}$, it will suffice to show that if $x \in e\mathrm{A}$ with $x\mathrm{A} \cap (e_1\mathrm{A} + e_2\mathrm{A}) = 0$, then $x = 0$. We first note that

$$x\mathrm{A} \cap [(e_1\mathrm{A} + e_2\mathrm{A}) + (1 - e)\mathrm{A}] = 0\,. \tag{4}$$

For, suppose $xa = e_1b + e_2c + (1 - e)d$. Then

$$xa - (e_1b + e_2c) = (1 - e)d \in e\mathrm{A} \cap (1 - e)\mathrm{A} = 0\,,$$

whence

$$xa = e_1b + e_2c \in x\mathrm{A} \cap (e_1\mathrm{A} + e_2\mathrm{A}) = 0\,;$$

thus $xa = 0$, proving (4).

Now, the element z defined earlier obviously belongs to

$$(1 - e)\mathrm{A} + (e_1 - f)\mathrm{A} + (e_2 - f)\mathrm{A} + f\mathrm{A} = (1 - e)\mathrm{A} + (e_1\mathrm{A} + e_2\mathrm{A})\,,$$

therefore $x\mathrm{A} \cap z\mathrm{A} = 0$ by (4); since $z\mathrm{A}$ is essential, $x\mathrm{A} = 0$. $\diamondsuit$

21.16. In a nonsingular module (right, say) over a ring A, if L, M are submodules with L essential in M (briefly, $\mathrm{L} \subset_e \mathrm{M}$), then $\mathrm{L} + \mathrm{N} \subset_e \mathrm{M} + \mathrm{N}$ for every submodule N.

{Proof: Let us first show that if $y \in \mathrm{M}$, $z \in \mathrm{N}$ with $y + z \ne 0$, then $(y + z)\mathrm{A} \cap (\mathrm{L} + \mathrm{N}) \ne 0$. For, let $\mathrm{I} = (\mathrm{L} : y) = \{a \in \mathrm{A} :\ ya \in \mathrm{L}\}$; since $\mathrm{L} \subset_e \mathrm{M}$, I is an essential right ideal of A. Since $y + z \ne 0$, $(y + z)\mathrm{I} \ne 0$ by nonsingularity. Choose $a \in \mathrm{I}$ with $(y + z)a \ne 0$. Then $ya \in \mathrm{L}$, and $0 \ne (y + z)a = ya + za \in \mathrm{L} + \mathrm{N}$, whence $(y + z)\mathrm{A} \cap (\mathrm{L} + \mathrm{N}) \ne 0$.

Now let K be a nonzero submodule of $\mathrm{M}+\mathrm{N}$; we are to show that $\mathrm{K}\cap(\mathrm{L}+\mathrm{N})\neq 0$. Choose $x\in\mathrm{K}$, $x\neq 0$; say $x=y+z$, $y\in\mathrm{M}$, $z\in\mathrm{N}$. Then $(y+z)\mathrm{A}\cap(\mathrm{L}+\mathrm{N})\neq 0$ by the preceding paragraph, whence the assertion.}

It follows that if $\mathrm{L}_i\subset_e\mathrm{M}_i$ for $i=1,\ldots,n$, then $\mathrm{L}_1+\ldots+\mathrm{L}_n\subset_e\mathrm{M}_1+\ldots+\mathrm{M}_n$; for example, $\mathrm{L}_1+\mathrm{L}_2\subset_e\mathrm{M}_1+\mathrm{L}_2\subset_e\mathrm{M}_1+\mathrm{M}_2$ by two applications of the preceding assertion.

21.17. LEMMA. *If* A *is a strongly modular Rickart ∗-ring, then* $\mathrm{I}\subset_e\mathrm{I}^{lr}$ *for every right ideal* I *that is generated by two elements.*

Proof. Say $\mathrm{I}=x_1\mathrm{A}+x_2\mathrm{A}$. Let $e_i=\mathrm{LP}(x_i)$; thus $(x_i\mathrm{A})^{lr}=e_i\mathrm{A}$ for $i=1,2$, and

$$\begin{aligned}\mathrm{I}^l&=(x_1\mathrm{A})^l\cap(x_2\mathrm{A})^l=\mathrm{A}(1-e_1)\cap\mathrm{A}(1-e_2)\\&=\mathrm{A}\cdot(1-e_1)\cap(1-e_2)=\mathrm{A}(1-e_1\cup e_2),\end{aligned}$$

so $\mathrm{I}^{lr}=(e_1\cup e_2)\mathrm{A}$. By 21.14, $x_i\mathrm{A}$ is essential in $e_i\mathrm{A}$, therefore I is essential in $e_1\mathrm{A}+e_2\mathrm{A}$ by the remark preceding the lemma (the requisite nonsingularity being provided by 1.29). We thus have

$$\mathrm{I}\subset_e e_1\mathrm{A}+e_2\mathrm{A},$$

whereas

$$e_1\mathrm{A}+e_2\mathrm{A}\subset_e(e_1\cup e_2)\mathrm{A}$$

by 21.15, therefore $\mathrm{I}\subset_e(e_1\cup e_2)\mathrm{A}=\mathrm{I}^{lr}$. ♢

21.18. PROPOSITION. [11, Lemma 2.5] *If* A *is a strongly modular Rickart ∗-ring, then for every finitely generated right ideal* I *of* A, I *is essential in* I^{lr}.

Proof. Say $\mathrm{I}=x_1\mathrm{A}+\ldots+x_n\mathrm{A}$; the proof is by induction on n. For $n=1$, quote 21.14; for $n=2$, 21.17.

Suppose $n\geq 3$ and assume inductively that all's well with $n-1$. Writing $\mathrm{J}=x_1\mathrm{A}+\ldots+x_{n-1}\mathrm{A}$, we then have $\mathrm{I}=\mathrm{J}+x_n\mathrm{A}$ with $\mathrm{J}\subset_e\mathrm{J}^{lr}$ and $x_n\mathrm{A}\subset_e(x_n\mathrm{A})^{lr}$, whence (21.16)

$$\mathrm{I}\subset_e\mathrm{J}^{lr}+(x_n\mathrm{A})^{lr}. \tag{i}$$

Let $e_i=\mathrm{LP}(x_i)$; by the argument in 21.17,

$$\mathrm{I}^{lr}=(e_1\cup\ldots\cup e_n)\mathrm{A},\quad \mathrm{J}^{lr}=(e_1\cup\ldots\cup e_{n-1})\mathrm{A},\quad (x_n\mathrm{A})^{lr}=e_n\mathrm{A},$$

in particular

$$\mathrm{J}^{lr}+(x_n\mathrm{A})^{lr}=(e_1\cup\ldots\cup e_{n-1})\mathrm{A}+e_n\mathrm{A}. \tag{ii}$$

By 21.15,

$$(e_1\cup\ldots\cup e_{n-1})\mathrm{A}+e_n\mathrm{A}\subset_e(e_1\cup\ldots\cup e_{n-1}\cup e_n)\mathrm{A}=\mathrm{I}^{lr}. \tag{iii}$$

Combining (i), (ii), (iii) we get $\mathrm{I}\subset_e\mathrm{I}^{lr}$. ♢

21.19. For submodules of a module, suppose $\mathrm{L}\subset_e\mathrm{L}'$ and $\mathrm{M}\subset_e\mathrm{M}'$; then $\mathrm{L}\cap\mathrm{M}\subset_e\mathrm{L}'\cap\mathrm{M}'$, and in particular $\mathrm{L}\cap\mathrm{M}=0\ \Rightarrow\ \mathrm{L}'\cap\mathrm{M}'=0$.

{Proof: Let K be a submodule of $\mathrm{L}' \cap \mathrm{M}'$ with $\mathrm{K} \cap (\mathrm{L} \cap \mathrm{M}) = 0$. Then $(\mathrm{K} \cap \mathrm{L}) \cap \mathrm{M} = 0$, where $\mathrm{K} \cap \mathrm{L} \subset \mathrm{K} \subset \mathrm{M}'$, therefore $\mathrm{K} \cap \mathrm{L} = 0$ (because $\mathrm{M} \subset_e \mathrm{M}'$). Thus $\mathrm{K} \cap \mathrm{L} = 0$, where $\mathrm{K} \subset \mathrm{L}'$, therefore $\mathrm{K} = 0$ (because $\mathrm{L} \subset_e \mathrm{L}'$).}

21.20. LEMMA. *In a strongly modular Rickart $*$-ring, if* I *and* J *are finitely generated right ideals such that* $\mathrm{I} \cap \mathrm{J} = 0$, *then* $\mathrm{I}^{lr} \cap \mathrm{J}^{lr} = 0$.

Proof. $\mathrm{I} \subset_e \mathrm{I}^{lr}$ and $\mathrm{J} \subset_e \mathrm{J}^{lr}$ by 21.18; quote 21.19. ♢

21.21. THEOREM. (D. Handelman [11, Lemma 2.6]) *If* A *is a strongly modular Baer $*$-ring, then* $\mathrm{I} \subset_e \mathrm{I}^{lr}$ *for every right ideal* I (*and* $\mathrm{J} \subset_e \mathrm{J}^{rl}$ *for every left ideal* J).

Proof. Write $\mathrm{I} = \bigcup \mathrm{I}_i$ with (I_i) an increasingly directed family of finitely generated right ideals (e.g., consider the set of all finitely generated right ideals $\subset \mathrm{I}$, ordered by inclusion). Then $\mathrm{I}_i^l \downarrow$, so $\mathrm{I}_i^{lr} \uparrow$. In the lattice of right annihilators (1.21),

$$\bigvee(\mathrm{I}_i^{lr}) = (\bigcup \mathrm{I}_i^{lr})^{lr} = (\bigcap \mathrm{I}_i^{lrl})^r = (\bigcap \mathrm{I}_i^l)^r = (\bigcup \mathrm{I}_i)^{lr} = \mathrm{I}^{lr},$$

briefly $\mathrm{I}^{lr} = \bigvee \mathrm{I}_i^{lr}$.

Let K be a right ideal with $\mathrm{K} \subset \mathrm{I}^{lr}$ and $\mathrm{K} \cap \mathrm{I} = 0$; we are to show that $\mathrm{K} = 0$. We can suppose K finitely generated (even principal). From $\mathrm{K} \cap \mathrm{I} = 0$ we have $\mathrm{K} \cap \mathrm{I}_i = 0$ for all i, therefore $\mathrm{K}^{lr} \cap \mathrm{I}_i^{lr} = 0$ by 21.20. Citing continuity of the lattice operations (21.12),

$$\begin{aligned} \mathrm{K}^{lr} \cap \mathrm{I}^{lr} &= \mathrm{K}^{lr} \wedge \mathrm{I}^{lr} = \mathrm{K}^{lr} \wedge (\bigvee \mathrm{I}_i^{lr}) \\ &= \bigvee(\mathrm{K}^{lr} \wedge \mathrm{I}_i^{lr}) = \bigvee(0) = 0, \end{aligned}$$

briefly $\mathrm{K}^{lr} \cap \mathrm{I}^{lr} = 0$. But $\mathrm{K} \subset \mathrm{I}^{lr}$ yields $\mathrm{K}^{lr} \subset \mathrm{I}^{lrlr} = \mathrm{I}^{lr}$, so $\mathrm{K}^{lr} = \mathrm{K}^{lr} \cap \mathrm{I}^{lr} = 0$, whence $\mathrm{K} = 0$. Thus $\mathrm{I} \subset_e \mathrm{I}^{lr}$.

If J is a left ideal of A, then J^* is a right ideal, so by the preceding,

$$\mathrm{J}^* \subset_e (\mathrm{J}^*)^{lr} = (\mathrm{J}^{r*})^r = (\mathrm{J}^{rl})^*,$$

thus $\mathrm{J}^* \subset_e (\mathrm{J}^{rl})^*$ as right A-modules, whence $\mathrm{J} \subset_e \mathrm{J}^{rl}$ as left A-modules. {Alternatively, one can show by similar arguments that the ring A° opposite A is also strongly modular.} ♢

Note that the appeal to Kaplansky's theorem via 21.12 can be avoided if one has an alternate proof of the continuity axioms (cf. 20.8).

21.22. COROLLARY. [11, p. 12] *A Baer $*$-ring is $*$-extendible* (21.1) *if and only if it is strongly modular* (21.5).

Proof. "Only if": This is 21.9.

"If": Let A be a strongly modular Baer $*$-ring, and suppose I is a right ideal with $\mathrm{I}^l = 0$. Then $\mathrm{A} = \mathrm{I}^{lr}$, so I is an essential right ideal of A by 21.21. Briefly, for right ideals I, $\mathrm{I}^l = 0 \Rightarrow \mathrm{I}$ essential; for a nonsingular $*$-ring, this condition is equivalent to $*$-extendibility by a theorem of Y. Utumi [cf. 28, Th. 3.2]. ♢

21.23. PROPOSITION. *For a Rickart $*$-ring* A, *the following conditions are equivalent*:

(a) $z^* = z$, $\{z\}^r = 0 \Rightarrow z\mathrm{A}$ *essential*;

(b) $\mathrm{I} \subset_e \mathrm{I}^{lr}$ *for all finitely generated right ideals* I;

(c) $z\mathrm{A} \subset_e (z\mathrm{A})^{lr}$ *for all* $z \in \mathrm{A}$ *with* $z^* = z$;

(d) $x\mathrm{A} \subset_e (x\mathrm{A})^{lr}$ *for all* $x \in \mathrm{A}$;

(e) $\{x\}^l = 0 \Rightarrow x\mathrm{A}$ *essential* (that is, $\mathrm{LP}(x) = 1 \Rightarrow x\mathrm{A}$ essential).

Proof. (a) $\Rightarrow$ (b): Inspecting the proof of 21.18 and its preliminaries (especially 21.11, 21.14, 21.15) we see that the condition (a) is all that is needed (the full force of 21.5 is not used).

(b) $\Rightarrow$ (c): Trivial.

(c) $\Rightarrow$ (d): Let $x \in \mathrm{A}$. Then $\{x\}^l = \{xx^*\}^l$ because the involution is proper, thus $(x\mathrm{A})^{lr} = (xx^*\mathrm{A})^{lr}$. Since xx^* is self-adjoint, $xx^*\mathrm{A} \subset_e (xx^*\mathrm{A})^{lr}$ by the hypothesis (c), thus $xx^*\mathrm{A} \subset_e (x\mathrm{A})^{lr}$; but $xx^*\mathrm{A} \subset x\mathrm{A} \subset (x\mathrm{A})^{lr}$, therefore also $x\mathrm{A} \subset_e (x\mathrm{A})^{lr}$.

(d) $\Rightarrow$ (e): If $\{x\}^l = 0$ then $(x\mathrm{A})^{lr} = \mathrm{A}$, so $x\mathrm{A}$ is an essential right ideal by (d).

(e) $\Rightarrow$ (a): For $z^* = z$, $\{z\}^r = 0$ if and only if $\{z\}^l = 0$. ♢

21.24. COROLLARY. *For a Baer $*$-ring* A, *the following conditions are equivalent*:

(1) A *is strongly modular* (i.e., $\{x\}^r = 0 \Rightarrow x\mathrm{A}$ essential);

(2) $\mathrm{I} \subset_e \mathrm{I}^{lr}$ *for all right ideals* I;

(3) $x\mathrm{A} \subset_e (x\mathrm{A})^{lr}$ *for all* $x \in \mathrm{A}$ (i.e., $x\mathrm{A} \subset_e \mathrm{LP}(x)\mathrm{A}$ for all $x \in \mathrm{A}$);

(4) $\{x\}^l = 0 \Rightarrow x\mathrm{A}$ *essential* (i.e., $\mathrm{LP}(x) = 1 \Rightarrow x\mathrm{A}$ essential).

Proof. (1) $\Rightarrow$ (2): This is 21.21.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (2): Condition (4) is just (e) of 21.23, so we know that (a) also holds. Inspecting the proof of 21.21 and its preliminaries (especially 21.11, 21.14, 21.15, 21.18), we see that condition (a) is sufficient to establish (2).

(2) $\Rightarrow$ (1): By the proof of 21.22, A is $*$-extendible, therefore it is strongly modular (21.9). ♢

It is tempting to replace the definition of "strongly modular" by condition (4); but then the easy proof of direct finiteness (21.6) would not work. {Another definition one might contemplate: $\{x\}^r = \{x\}^l = 0 \Rightarrow x\mathrm{A}$ and $\mathrm{A}x$ essential.[2] }

21.25. A $*$-ring A is said to have *sufficiently many projections* if for every nonzero element $x \in \mathrm{A}$ there exists $y \in \mathrm{A}$ with xy a nonzero projection. It is the same to say that every nonzero right (or left) ideal contains a nonzero projection. {For example, a $*$-ring satisfying the EP-axiom (§10) has sufficiently many projections.}

[2] Suppose A has the stated property. Then $\{x\}^l = 0 \Rightarrow \{xx^*\}^l = \{x\}^l = 0 \Rightarrow \{xx^*\}^r = 0$, so $xx^*\mathrm{A}$ is essential, therefore so is $x\mathrm{A}$; thus $\{x\}^l = 0 \Rightarrow x\mathrm{A}$ essential, so A is strongly modular by 21.24.

21.26. PROPOSITION. [11, Prop. 2.10] *If* A *is a Rickart* $*$*-ring with sufficiently many projections*, *then the following conditions are equivalent*:

(a) A *is strongly modular* (i.e., $\mathrm{RP}(x) = 1 \Rightarrow x\mathrm{A}$ is essential);

(b) *for* $x \in \mathrm{A}$, $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$.

Proof. (a) $\Rightarrow$ (b): See 21.7.

(b) $\Rightarrow$ (a): Let $x \in \mathrm{A}$ with $\mathrm{RP}(x) = 1$; we are to show that $x\mathrm{A}$ is essential. (By hypothesis, $\mathrm{LP}(x) = 1$.) Suppose to the contrary that there exists a nonzero right ideal J with $\mathrm{J} \cap x\mathrm{A} = 0$. Since A has sufficiently many projections, we can suppose $\mathrm{J} = e\mathrm{A}$, e a projection. Thus $e\mathrm{A} \cap x\mathrm{A} = 0$.

Note that $\{x - ex\}^r = 0$. For, if $(x-ex)y = 0$ then $xy = exy \in x\mathrm{A} \cap e\mathrm{A} = 0$, so $xy = 0$; since $\mathrm{RP}(x) = 1$, it follows that $y = 0$. Thus $\mathrm{RP}(x - ex) = 1$; by the hypothesis (b), we have $1 = \mathrm{LP}(x - ex) = \mathrm{LP}[(1 - e)x] \leq 1 - e$, whence $e = 0$, $\mathrm{J} = e\mathrm{A} = 0$, a contradiction. ♢

Note that the version of 21.5 limited to self-adjoint x is not sufficient for the proof of 21.26.

21.27. COROLLARY. (cf. [28, Th. 5.2], [10, Th. 2], [11, Prop. 2.10]) *If* A *is a* $*$*-finite Rickart* $*$*-ring*, *with sufficiently many projections*, *satisfying* $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ (cf. 14.31, 14.32), *then* A *is strongly modular* (hence directly finite by 21.6).

Proof. If $\mathrm{RP}(x) = 1$ then $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x) = 1$, so $\mathrm{LP}(x) = 1$ by $*$-finiteness; quote 21.26. {Incidentally, it is elementary that a $*$-finite Rickart $*$-ring[3] satisfying $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ is directly finite [2, p. 210, Prop. 1]; the proof is an easy variation on 21.6.} ♢

21.28. COROLLARY. *If* A *is a directly finite Rickart* $*$*-ring with sufficiently many projections*, *satisfying* $\mathrm{LP} \overset{a}{\sim} \mathrm{RP}$, *then* A *is strongly modular.*

Proof. Formally the same as 21.27. ♢

Indeed, since 21.26 makes no reference to a specific equivalence relation, we have the following: If A is a Rickart $*$-ring with sufficiently many projections, and $\sim$ is a relation on its projection lattice relative to which A is finite ($e \sim 1 \Rightarrow e = 1$) and $\mathrm{LP} \sim \mathrm{RP}$, then A is strongly modular (hence directly finite!, by 21.6).

21.29. COROLLARY. *Let* A *be a Baer* $*$*-ring satisfying the* EP-*axiom* (§10) *such that*, *relative to* $\overset{*}{\sim}$, A *is finite and satisfies* GC. *Then* A *is strongly modular* (hence directly finite).

Proof. From EP we know that A has sufficiently many projections; by 14.31, $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$; and A is $*$-finite by hypothesis. Quote 21.27. ♢

21.30. Let A be a right nonsingular ring, Q its maximal ring of right quotients, u a central idempotent of A. Then u is central in Q, and $u\mathrm{Q}$ is the maximal ring of right quotients of $u\mathrm{A}$.

{Proof: Let $x \in \mathrm{Q}$; let us show that $ux - xu = 0$. Let $\mathrm{I} = (\mathrm{A} : x) = \{a \in \mathrm{A} : xa \in \mathrm{A}\}$; as noted in the proof of 21.8, I is an essential right ideal of A. For all $a \in \mathrm{I}$, since $xa \in \mathrm{A}$ and u is in the center of A, we have

$$(ux - xu)a = u(xa) - x(ua) = (xa)u - x(au) = 0,$$

[3] For example, a $*$-regular ring (see the footnote to 9.5).

briefly $(ux - xu)\mathrm{I} = 0$; since A is right nonsingular, it follows that $ux - xu = 0$ (proof of 21.8). Thus u is indeed central in Q. Since Q is a ring of right quotients of A [19, p. 99, Prop. 8], it follows easily that $u\mathrm{Q}$ is a ring of right quotients of $u\mathrm{A}$; and $\mathrm{Q} = u\mathrm{Q} \times (1-u)\mathrm{Q}$ shows that $u\mathrm{Q}$ is also a right self-injective ring, consequently it is the maximal ring of right quotients of the (right nonsingular) ring $u\mathrm{A}$.}

21.31. PROPOSITION. *Let* A *be a Baer $*$-ring,* (u_α) *an orthogonal family of central projections in* A *with* $\sup u_\alpha = 1$. *In order that* A *be $*$-extendible, it is necessary and sufficient that* $u_\alpha\mathrm{A}$ *be $*$-extendible for every* α.

Proof. Let Q be the maximal ring of right quotients of A. From 21.30, we know that (u_α) is an (orthogonal) family of central idempotents in Q, and that $u_\alpha\mathrm{Q}$ is the maximal ring of right quotients of $u_\alpha\mathrm{A}$. The following assertion shows that $\sup u_\alpha = 1$ in the complete Boolean algebra B of central idempotents of Q (cf. 3.3, 1.31):

claim 1: If $x \in \mathrm{Q}$ and $u_\alpha x = 0$ for all α, then $x = 0$.

For, let $\mathrm{I} = (\mathrm{A} : x) = \{a \in \mathrm{A} :\ xa \in \mathrm{A}\}$, which is an essential right ideal of A (proof of 21.8). Let $a \in \mathrm{I}$; for all α we have

$$u_\alpha(xa) = (u_\alpha x)a = 0 \cdot a = 0\,,$$

therefore $xa = 0$ (because $xa \in \mathrm{A}$ and $\sup u_\alpha = 1$ in A). Thus $x\mathrm{I} = 0$, whence $x = 0$ because I is essential (proof of 21.8). {A similar argument shows that if (e_i) is any family of projections in A and $e = \sup e_i$ in A, then for $x \in \mathrm{Q}$ one has $ex = 0$ if and only if $e_i x = 0$ for all i. In particular, if (v_i) is any family of central projections in A, and $v = \sup v_i$ in A, then v is also the supremum of the v_i in the complete Boolean algebra B.}

claim 2: $x \mapsto (u_\alpha x)$ is a ring isomorphism $\mathrm{Q} \to \prod u_\alpha\mathrm{Q}$.

For, in view of claim 1, it is a monomorphism of rings. Since Q is a regular, right self-injective ring (1.31) and since, as noted above, $\sup u_\alpha = 1$ in the complete Boolean algebra B, claim 2 follows from [7, p. 99, Prop. 9.10].

Write $\varphi : \mathrm{Q} \to \prod u_\alpha\mathrm{Q}$ for the isomorphism $\varphi(x) = (u_\alpha x)$.

"Sufficiency": If the $u_\alpha\mathrm{A}$ are all $*$-extendible, and if $\sigma_\alpha : u_\alpha\mathrm{Q} \to u_\alpha\mathrm{Q}$ is the involution of $u_\alpha\mathrm{Q}$ extending that of $u_\alpha\mathrm{A}$ (cf. 21.2) then $x \mapsto \varphi^{-1}((\sigma_\alpha(u_\alpha x)))$ defines an involution of Q extending that of A.

"Necessity": The proof is equally straightforward, utilizing the following observation: the $u_\alpha \in \mathrm{A}$ are self-adjoint, so an involution of Q extending that of A must leave each of the $u_\alpha\mathrm{Q}$ invariant (as sets), hence induces an involution of $u_\alpha\mathrm{Q}$ extending that of $u_\alpha\mathrm{A}$. ♢

The significance of this proposition is that $*$-extendibility for a Baer $*$-ring A can be reduced, via structure theory, to proving it for rings of various special types (this is illustrated in 21.36 below). The following proposition can be omitted, but in view of 21.22 it provides an interesting alternative proof for 21.31:

21.32. Let A be a Baer ring, B its complete Boolean algebra of central idempotents (3.3), (u_α) an orthogonal family in B with $\sup u_\alpha = 1$. In order that A be strongly modular, it is necessary and sufficient that every $u_\alpha \mathrm{A}$ be strongly modular.

{Proof: "Sufficiency": Let $x \in \mathrm{A}$ with $\{x\}^r = 0$. For each α, it is clear that the right annihilator of $u_\alpha x$ in $u_\alpha \mathrm{A}$ is also 0, so by hypothesis $u_\alpha x \cdot u_\alpha \mathrm{A} = u_\alpha x \mathrm{A}$ is an essential right ideal of $u_\alpha \mathrm{A}$. Let J be a right ideal of A with $\mathrm{J} \cap x\mathrm{A} = 0$; we are to show that $\mathrm{J} = 0$. For all α, $u_\alpha \mathrm{J}$ is a right ideal of $u_\alpha \mathrm{A}$ with

$$u_\alpha \mathrm{J} \cap u_\alpha x \mathrm{A} \subset \mathrm{J} \cap x\mathrm{A} = 0\,,$$

whence $u_\alpha \mathrm{J} = 0$. Thus for $y \in \mathrm{J}$ one has $u_\alpha y = 0$ for all α, therefore $y = 0$ (3.4); that is, $\mathrm{J} = 0$.

"Necessity": Assuming A strongly modular and u any central idempotent, let us show that $u\mathrm{A}$ is strongly modular. Let $x \in u\mathrm{A}$ with 0 right annihilator in $u\mathrm{A}$; we are to show that $x \cdot u\mathrm{A} = x\mathrm{A}$ is an essential right ideal of $u\mathrm{A}$. Let $y = x + (1-u)$. If $z \in \mathrm{A}$ and $yz = 0$, then $0 = yz = xz + (1-u)z$, so

$$xz = -(1-u)z \in u\mathrm{A} \cap (1-u)\mathrm{A} = 0\,,$$

whence $xz = (1-u)z = 0$. But $0 = xz = (xu)z = x(uz)$ yields $uz = 0$ by the hypothesis on x, whence $0 = (1-u)z = z - uz = z$. Thus $\{y\}^r = 0$ in A. Since A is strongly modular, $y\mathrm{A}$ is an essential right ideal of A. Suppose now J is a right ideal of $u\mathrm{A}$ with $\mathrm{J} \cap x\mathrm{A} = 0$; J is also a right ideal of A, and $\mathrm{J} \cap y\mathrm{A} = 0$, for if

$$t \in \mathrm{J} \cap y\mathrm{A} = u\mathrm{J} \cap [x + (1-u)]\mathrm{A}\,,$$

say $t = [x + (1-u)]a$, then left multiplication by u yields $t = ut = uxa = xa \in \mathrm{J} \cap x\mathrm{A} = 0$. Since $y\mathrm{A}$ is essential, $\mathrm{J} = 0$.}

21.33. LEMMA. [11, p. 12] *Let* A *be a* $*$*-extendible Baer* $*$*-ring* (cf. 21.22), Q *its maximal ring of right quotients* (21.2), *and* $n \geq 1$ *an integer. Let* $\mathrm{M}_n(\mathrm{A})$, $\mathrm{M}_n(\mathrm{Q})$ *be the rings of* $n \times n$ *matrices.*

(i) $\mathrm{M}_n(\mathrm{Q})$ *is the maximal ring of right* (*and left*) *quotients of* $\mathrm{M}_n(\mathrm{A})$;

(ii) $\mathrm{M}_n(\mathrm{Q})$ *is unit-regular, hence directly finite*; *therefore* $\mathrm{M}_n(\mathrm{A})$ *is directly finite*;

(iii) $\mathrm{M}_n(\mathrm{Q})$ *is self-injective* (*right and left*);

(iv) $\mathrm{M}_n(\mathrm{A})$ *is nonsingular* (*right and left*);

(v) *each involution of* $\mathrm{M}_n(\mathrm{A})$ *is uniquely extendible to an involution of* $\mathrm{M}_n(\mathrm{Q})$.

(vi) *If* $\mathrm{M}_n(\mathrm{A})$ *is a Baer* $*$*-ring relative to some involution* $\#$ (not necessarily the natural involution of $*$-transposition that it inherits from A), *then* $(\mathrm{M}_n(\mathrm{A}), \#)$ *is a* $*$*-extendible Baer* $*$*-ring* (hence $\mathrm{M}_n(\mathrm{A})$ is a strongly modular ring by 21.22).

Proof. Write $\mathrm{B} = \mathrm{M}_n(\mathrm{A})$ for the ring of $n \times n$ matrices over A.

(i) By a general theorem of Y. Utumi, $\mathrm{M}_n(\mathrm{Q})$ is the maximal ring of right quotients of B ([32, p. 5], [19, p. 101, Exer. 8]). But Q is also the maximal ring of left quotients of A (21.2), therefore $\mathrm{M}_n(\mathrm{Q})$ is the maximal ring of left quotients of B.

(ii) Since Q is unit-regular (21.2), so is $M_n(Q)$ [7, p. 38, Cor. 4.7]; therefore $M_n(Q)$ is directly finite [7, p. 50, Prop. 5.2], hence so is its subring B.

(iii), (iv) $M_n(Q)$ is the maximal ring of right quotients of B and is regular, therefore B is right nonsingular [19, p. 106, Prop. 2] and $M_n(Q)$ is right self-injective [19, p. 107, Cor. of Prop. 2]. Similarly for "left".

(v) Let # be any involution of B. Since $M_n(Q)$ is both the left and right maximal ring of quotients of B by (i), it follows that # is uniquely extendible to an involution of $M_n(Q)$ (the proof is written out in [28, Th. 3.2]).

(vi) Suppose # is an involution on B that makes it a Baer $*$-ring; then # extends to its maximal right quotient ring $M_n(Q)$ by (v), thus $(B, \#)$ is a $*$-extendible Baer $*$-ring (21.1), hence is strongly modular by 21.22. {Note, incidentally, that if $e \in M_n(Q)$ is a projection relative to the extension of # to $M_n(Q)$, then $e \in M_n(A)$ by 21.2.} $\diamondsuit$

In the reverse direction, we remark that if A is a $*$-extendible Baer $*$-ring, then so is every corner of A. {Proof: Let Q be the maximal ring of right quotients of A, and view A as a $*$-subring of Q (21.2). Let $e \in A$ be a projection. Since A is semiprime (3.20) and nonsingular (1.29), eQe is the maximal ring of right quotients of eAe [33, p. 135, Prop. 0.2]; the proof is completed by the observation that eAe is a $*$-subring of eQe.}

21.34. THEOREM. (D. Handelman [11, Prop. 2.9]) *Let* A *be a Baer $*$-ring in which* $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$ (in other words, $\{x\}^r = 0 \Rightarrow \{x\}^l = 0$) *and suppose that, for some integer* $n \geq 2$, A *is isomorphic as a ring to the ring of* $n \times n$ *matrices over some ring* (i.e., as a ring "A has $n \times n$ matrix units" for some $n \geq 2$). *Then* A *is $*$-extendible* (cf. 21.22).

Proof. Let $e_1, \ldots, e_n$ be orthogonal idempotents in A with $1 = e_1 + \ldots + e_n$ and $e_1 \overset{a}{\sim} e_2 \overset{a}{\sim} \ldots \overset{a}{\sim} e_n$. Writing $B = e_1 A e_1$, we have $A \cong M_n(B)$ as rings. {On occasion we identify $A = M_n(B)$, but we note that while A is a Baer $*$-ring for its given involution, B is merely a Baer ring (2.2), thus the involution of A need not arise from an involution of B via $*$-transposition.}

Let $f = \mathrm{LP}(e_1)$; then $e_1 A = fA$ (5.6), so $e_1 \overset{a}{\sim} f$ (indeed, e_1, f are similar by 5.5), hence there exists a ring isomorphism $\varphi : e_1 A e_1 \to fAf$ (5.4). We know that fAf is a Baer $*$-ring for the involution of A (2.6). Let # be the unique involution on $B = e_1 A e_1$ that makes φ a $*$-isomorphism; then $(B, \#)$ is a Baer $*$-ring (but the involution on $A = M_n(B)$ that it induces by "#-transposition" need not coincide with the given involution on A).

Let us show that B is a strongly modular ring. Suppose $b \in B$ with $\{b\}^r = 0$ (where the "r" means right annihilator in B); we are to show that bB is an essential right ideal of B. Thus if $c \in B$ with $cB \cap bB = 0$, it is to be shown that $c = 0$. Since B is a Baer ring, there exists an idempotent $e \in B$ (thus $e \leq e_1$ as idempotents) such that $eB = \{c\}^r$ (the right annihilator of c in B).

Let $x \in \mathrm{A} = \mathrm{M}_n(\mathrm{B})$ be the element given by the matrix

$$\left(\begin{array}{cc|cccc} b & c & & & & \\ 0 & e & & \mathbf{0} & & \\ \hline & & e_1 & & & \\ & & & e_1 & \mathbf{0} & \\ & \mathbf{0} & & \mathbf{0} & \ddots & \\ & & & & & e_1 \end{array}\right) = \begin{pmatrix} \mathrm{P} & 0 \\ 0 & \mathrm{I} \end{pmatrix},$$

where $\mathrm{P} = \begin{pmatrix} b & c \\ 0 & e \end{pmatrix} \in \mathrm{M}_2(\mathrm{B})$ and $\mathrm{I} \in \mathrm{M}_{n-2}(\mathrm{B})$ is the identity matrix. {If $n = 2$ the following argument applies with appropriate simplifications.} We assert that $\{x\}^r = 0$ (here "r" means right annihilator in A). For, suppose $y \in \mathrm{A}$ with $xy = 0$, say

$$y = \begin{pmatrix} \mathrm{Q} & \mathrm{S} \\ \mathrm{R} & \mathrm{T} \end{pmatrix}$$

where $\mathrm{Q} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{M}_2(\mathrm{B})$ and $\mathrm{R}, \mathrm{S}, \mathrm{T}$ are matrices over B of the appropriate sizes. Then

$$0 = xy = \begin{pmatrix} \mathrm{PQ} & \mathrm{PS} \\ \mathrm{R} & \mathrm{T} \end{pmatrix} = \left(\begin{array}{cc|c} br + ct & bs + cu & \\ & & \mathrm{PS} \\ et & eu & \\ \hline \mathrm{R} & & \mathrm{T} \end{array}\right)$$

where $\mathrm{R} = 0$, $\mathrm{T} = 0$, $\mathrm{PS} = 0$ and

$$br + ct = bs + cu = et = eu = 0\,. \tag{$*$}$$

Then $br = -ct \in b\mathrm{B} \cap c\mathrm{B} = 0$ and $bs = -cu \in b\mathrm{B} \cap c\mathrm{B} = 0$, so $br = ct = bs = cu = 0$. Thus $r, s \in \{b\}^r = 0$, so $r = s = 0$; and $t, u \in \{c\}^r = e\mathrm{B}$, so $t = et$ and $u = eu$. But $et = eu = 0$ by (*), so $t = u = 0$. Thus $\mathrm{Q} = 0$ and so

$$y = \begin{pmatrix} 0 & \mathrm{S} \\ 0 & 0 \end{pmatrix},$$

where $\mathrm{PS} = 0$. Say $\mathrm{S} = (s_{ij})$, where $i = 1, 2$ and $1 \le j \le n-2$. Thus

$$\begin{aligned} 0 = \mathrm{PS} &= \begin{pmatrix} b & c \\ 0 & e \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1,n-2} \\ s_{21} & s_{22} & \dots & s_{2,n-2} \end{pmatrix} \\ &= \begin{pmatrix} bs_{11} + cs_{21} & bs_{12} + cs_{22} & \dots \\ es_{21} & es_{22} & \dots \end{pmatrix}. \end{aligned} \tag{$**$}$$

Then for $j = 1, \dots, n-2$, one has $bs_{1j} + cs_{2j} = 0$,

$$bs_{1j} = -cs_{2j} \in b\mathrm{B} \cap c\mathrm{B} = 0\,,$$

so $bs_{1j} = cs_{2j} = 0$; thus $s_{1j} \in \{b\}^r = 0$, so that the top row of S has all entries 0. Also $s_{2j} \in \{c\}^r = e\mathrm{B}$ shows that $s_{2j} = es_{2j}$; but (**) shows that $es_{2j} = 0$, thus $s_{2j} = 0$ and we conclude that $\mathrm{S} = 0$, consequently $y = 0$. Thus $\{x\}^r = 0$, therefore $\{x\}^l = 0$ by the hypothesis on A. Let $z \in \mathrm{A} = \mathrm{M}_n(\mathrm{B})$ be the element

$$z = \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & e_1 - e & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Then

$$zx = \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & e_1 - e & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)\left(\begin{array}{cc|c} b & c & \\ 0 & e & 0 \\ \hline \multicolumn{2}{c|}{0} & \mathrm{I} \end{array}\right) = \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right),$$

thus $zx = 0$, $z \in \{x\}^l = 0$, $z = 0$, $e_1 = e$, therefore $\{c\}^r = e\mathrm{B} = e_1\mathrm{B} = \mathrm{B}$, whence $c = 0$. This completes the proof that B is a strongly modular ring.

Therefore the ring $f\mathrm{A}f$ (isomorphic to B) is also strongly modular; it follows that the Baer $*$-ring $f\mathrm{A}f$ (with the involution of A) is $*$-extendible (21.22).

Now, we have ring isomorphisms

$$\mathrm{M}_n(f\mathrm{A}f) \cong \mathrm{M}_n(e_1\mathrm{A}e_1) = \mathrm{M}_n(\mathrm{B}) \cong \mathrm{A},$$

whence an isomorphism of rings

$$\psi : \mathrm{M}_n(f\mathrm{A}f) \to \mathrm{A};$$

viewing A as equipped with its original involution, let $\#$ be the involution on $\mathrm{M}_n(f\mathrm{A}f)$ that makes ψ a $*$-isomorphism. Then $(\mathrm{M}_n(f\mathrm{A}f), \#)$ is also a Baer $*$-ring ($*$-isomorphic to A via ψ); since $f\mathrm{A}f$, with its original involution, is a $*$-extendible Baer $*$-ring, it follows from (vi) of 21.33 that $(\mathrm{M}_n(f\mathrm{A}f), \#)$ is a $*$-extendible Baer $*$-ring, therefore (via ψ) so is A. $\diamondsuit$

21.35. In the wake of 21.34, whose proof requires the use of both halves of the equivalence 21.22, there is no point in using *both* of the terms "$*$-extendible" and "strongly modular" in the context of Baer $*$-rings. The term "strongly modular" is attractive because the definition applies also to non-involutive rings and is intrinsic to the ring (i.e., does not require reference to rings of quotients); but "$*$-extendible" tells us what we want to know in the Baer $*$-ring setting (cf. 21.2-21.4). I vote for "$*$-extendible"; the following results are stated accordingly.

21.36. COROLLARY. *Let* A *be a Baer $*$-ring without abelian summand, such that* $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$. *Assume that ordinary equivalence* $\overset{a}{\sim}$ *satisfies axiom* F *and* GC (cf. §10). *Then* A *is $*$-extendible* (cf. 21.22).

Proof. Axioms A–D hold by 11.1; moreover, axiom E holds by 15.1, thus A–F and GC are in force. Note that A is directly finite (i.e., finite for $\overset{a}{\sim}$). {For, suppose $yx = 1$. Then $xt = 0 \Rightarrow yxt = 0 \Rightarrow t = 0$, so $\{x\}^r = 0$, therefore $\{x\}^l = 0$ by the hypothesis; but $(1-xy)x = x - xyx = x - x1 = 0$, so $1 - xy = 0$.}

We can write $\mathrm{A} = \mathrm{A}_1 \times \mathrm{A}_2$ with A_1 of type $\mathrm{I}_{\mathrm{fin}}$ and A_2 of type $\mathrm{II}_{\mathrm{fin}}$ (cf. 8.26 or 9.25), so the two cases can be considered separately.

case 1: A is of type $\mathrm{I}_{\mathrm{fin}}$.

By 16.13, there exists an orthogonal family (u_α) of central projections such that every $u_\alpha\mathrm{A}$ is homogeneous (relative to $\overset{a}{\sim}$). In view of 21.31 we are reduced to the case that A is homogeneous, say of order $\aleph$. By 15.7, $\aleph$ is finite (this is where axiom F is needed!); and $\aleph > 1$ because A has no abelian summand. Then A is isomorphic as a ring to an $\aleph \times \aleph$ matrix ring [cf. 2, p. 98, Prop. 1]. Thus A is $*$-extendible by 21.34.

case 2: A is of type $\mathrm{II}_{\mathrm{fin}}$.

Since axioms A–F hold, 17.4 applies: A is isomorphic as a ring to a 2×2 matrix ring. Therefore A is $*$-extendible by 21.34. ♢

21.37. COROLLARY. *Let* A *be a directly finite Baer $*$-ring without abelian summand, and suppose* $\overset{a}{\sim}$ *satisfies axiom* F *and* LP $\overset{a}{\sim}$ RP. *Then* A *is $*$-extendible.*

Proof. Axioms A–D hold by 11.1; from LP $\overset{a}{\sim}$ RP we infer axiom H (cf. 1.15), therefore E and GC hold (13.9 and its proof). So axioms A–F and GC are in force. Moreover, if $\mathrm{RP}(x) = 1$ then $\mathrm{LP}(x) \overset{a}{\sim} \mathrm{RP}(x) = 1$, so $\mathrm{LP}(x) = 1$ by direct finiteness. Thus the hypotheses of 21.36 are fulfilled. ♢

21.38. COROLLARY. *Let* A *be a Baer $*$-ring without abelian summand, satisfying* GC *for $*$-equivalence* $\overset{*}{\sim}$, *in which* $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$. *Then* A *is $*$-extendible.*

Proof. As noted in the proof of 21.36, A is directly finite, hence $*$-finite (i.e., finite for $\overset{*}{\sim}$). By 11.2, $\overset{*}{\sim}$ satisfies axioms A–D and F; by 15.1, it satisfies E. The proof of 21.36 may now be taken over word for word (with $\overset{a}{\sim}$ replaced by $\overset{*}{\sim}$). ♢

21.39. Abelian rings are truly exceptional in this circle of ideas (21.36–21.38). For example, let A be a $*$-ring with no divisors of 0 (thus A is a Baer $*$-ring whose only projections are 0 and 1, hence it is abelian), and suppose A is $*$-extendible. By 21.2, the maximal ring of right quotients Q of A is a regular Baer $*$-ring whose only projections are 0 and 1, thus Q is a division ring (1.45). It follows that if a, b are nonzero elements of A, then $a\mathrm{A} \cap b\mathrm{A} \neq 0$; indeed, as noted in the proof of 21.9, $a\mathrm{A} \cap b\mathrm{A}$ is an essential right A-submodule of $a\mathrm{Q} \bigcap b\mathrm{Q} = \mathrm{Q} \cap \mathrm{Q} = \mathrm{Q}$. Thus, A satisfies the "right Ore condition" (and, because of the involution, also the "left Ore condition").

However, there exist $*$-rings A without divisors of 0, satisfying neither of the Ore conditions (which explains why abelian summands were banned from the preceding corollaries). {For example [35, p. 436, Exer. 8], let F be the free group with two generators x, y (written multiplicatively), and let $\mathrm{A} = \mathbb{Z}\mathrm{F}$ be the group algebra of F over the ring of integers, equipped with the involution $a \mapsto a^*$ for which $x^* = x$ and $y^* = y$ (therefore $(xy)^* = yx$, etc.).} For such a ring A, Q is a factor of type III by a theorem of J.-E. Roos ([31, Prop. 1], [33, Cor. 2.3]). Incidentally, the projection lattice of such a ring A — namely $\{0,1\}$ — is trivially modular, and axiom H is trivially verified for every equivalence relation $\sim$ (there are only two!), thus these conditions are not sufficient to assure $*$-extendibility.

21.40. The key to proving "strongly modular $\Rightarrow$ $*$-extendible" (21.22) is Theorem 21.21, whose proof depends on Kaplansky's proof of continuity of the lattice operations (via 21.12). If one is willing to augment strong modularity with additional hypotheses (available in many applications, such as AW*-algebras) then Kaplansky's proof can be circumvented by materially simpler arguments; this is illustrated in the concluding results of this section.

21.41. PROPOSITION. *If* A *is a strongly modular Baer $*$-ring satisfying axiom* H *for* $\overset{*}{\sim}$, *then* A *is $*$-extendible.*

Proof. {We emphasize that for a higher price one can omit axiom H (21.22).}

Strong modularity implies direct finiteness (21.6) hence $*$-finiteness, that is, finiteness for $\overset{*}{\sim}$. Therefore the projection lattice of A is a continuous geometry by (i) of 20.11 (whose proof is based on (1) of 20.8 hence avoids Kaplansky's theorem). One can now repeat the proof of 21.21 (omitting the reference to 21.12), and then the proof of $*$-extendibility in the "If" part of 21.22. ◊

21.42. COROLLARY. *If* A *is a Baer $*$-ring with sufficiently many projections, satisfying axiom* H *for* $\overset{*}{\sim}$, *such that* $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$, *then* A *is $*$-extendible.*

Proof. By 21.26, A is strongly modular, thus the hypotheses of 21.41 are fulfilled. ◊

21.43. COROLLARY. (I. Hafner [10, Th. 2]) *If* A *is a $*$-finite Baer $*$-ring with sufficiently many projections, satisfying* $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$, *then* A *is $*$-extendible.*

Proof. From $*$-finiteness and $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ it is clear that $\mathrm{RP}(x) = 1 \Rightarrow \mathrm{LP}(x) = 1$, and $\overset{*}{\sim}$ satisfies axiom H by 1.15; quote 21.42. ◊

21.44. LEMMA. *If* A *is a Baer $*$-ring satisfying* GC *for* $\overset{*}{\sim}$ *and the* EP-*axiom, then* A *satisfies* $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$.

Proof. Let $x \in A$, $x \neq 0$, $e = \mathrm{RP}(x)$, $f = \mathrm{LP}(x)$. Using the EP-axiom, the first part of the proof of 14.29 yields orthogonal decompositions $e = \sup e_i$, $f = \sup f_i$ with $e_i \overset{*}{\sim} f_i$ for all i; therefore $e \overset{*}{\sim} f$ by (i) of 18.14. {Note: This is simpler than the proof of the same result in 14.31.} ◊

21.45. COROLLARY. *If* A *is a $*$-finite Baer $*$-ring satisfying* GC *for* $\overset{*}{\sim}$ *and the* EP-*axiom, then* A *is $*$-extendible.*

Proof. By EP, A has sufficiently many projections; in view of the lemma, the hypotheses of 21.43 are fulfilled. ◊

21.46. COROLLARY. *If* A *is a $*$-finite Baer $*$-ring satisfying axiom* H *for* $\overset{*}{\sim}$ *and the* EP-*axiom, then* A *is $*$-extendible.*

Proof. GC holds for $\overset{*}{\sim}$ by the theorem of Maeda and Holland (13.10); quote 21.45. ◊

21.47. COROLLARY. (J.-E. Roos [31]) *If* A *is a* $*$*-finite Baer* $*$*-ring satisfying the* EP- *and* SR-*axioms*, *then* A *is* $*$*-extendible.*

Proof. By Maeda's theorem (12.13), SR $\Rightarrow$ Axiom H for $\overset{*}{\sim}$; quote 21.46. $\diamondsuit$

21.48. COROLLARY. (cf. [31], [2, p. 237, Cor.], [3]) *If* A *is a finite* AW*-*algebra*, *then* A *is* $*$*-extendible.*

Proof. {Note that A is directly finite by 6.11, 6.13.} By spectral theory, A satisfies the EP-axiom [2, p. 43, Cor.] and the SR-axiom (which holds in every C*-algebra [2, p. 70]); quote 21.47. $\diamondsuit$

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