A multidimensional superposition principle: classical solitons III

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Abstract

The existence of new type solitons associated with truncated series analogous to traditional singular manifold expansions is shown in the framework of the multidimensional superposition principle. The characteristic features of the related solitonic interactions are demonstrated.

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1 Introduction

In the work [1] a concept called 'a multidimensional superposition principle' was proposed for finding superposition laws of nonlinear PDEs. In the same paper it was shown that so-called truncated singular expansions [2, 3] usually associating with the Painlevè-analysis are connected with the existence of solitons in the related models and directly lead to *general solution formulas* describing an interaction of the last ones with other, *arbitrary*, solutions. In so doing, all such interactions have unified features (like, e.g., a soliton phase shift as its result) and are fully determined by the structure of a superposition formula.

Although truncated series are historically linked with the Painlevè-analysis indeed, nevertheless it is an absolutely independent approach, and analogous presentations of solutions can be obtained with *various basis functions* satisfying to any suitable system of ODEs rather than only to the Riccaty ones [3]. Moreover, a similar technique, 'a generalized separation of variables' (see [4], e.g., for references) has been applying to nonlinear equations of the quite different types already long time. Some associated questions and the common (algebraic) view to a finite sums approach were discussed in [5]. In the same place it is shown how various problems arising in the framework of the traditional 'singular' interpretation can be successfully solved from this standpoint. The goal of the present paper is investigation of the existence and possible properties of solitons associated with the simplest generalizations of the technique which use expansions with respect to the powers of some basis functions satisfying to systems of the first-order polynomial right-hand side ODEs.

The plan of the paper is as follows. In the next Section 2 the main idea of the multidimensional superposition approach and the basic moments of the finite expansions technique are adduced. Solitons/kinks associated with the last ones are considered in Section 3. The final section, Section 4, contains the simplest examples of corresponding soliton interactions. At last in Conclusion the results obtained are discussed. Secondary technical details concerning analytical and numerical calculations carried out to Appendixes A–C.

2 General propositions

A multidimensional superposition principle. Suppose there is some differential equation, linear or nonlinear, for the definiteness in $(1+1)$ dimensions and for the simplicity not depending explicitly on the independent variables

$$E\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}; u\right) = 0, \qquad u = u(x,t) \tag{1}$$

Assume that, first, the function $u(x,t)$ is *a projection of another* $(2+2)$*-dimensional function*

$$u(x,t) = u(x_1, x_2, t_1, t_2)|_{x_1=x_2=x,\, t_1=t_2=t} \tag{2}$$

so that *the original variables* x, t *appear to be split*. In so doing, the latter has to satisfy the following *adjoint* to (1) equation

$$E\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}; u\right) = 0, \qquad u = u(x_1, x_2, t_1, t_2) \tag{3}$$

Than assume that this equation for $u(x_1, x_2, t_1, t_2)$ has the special property. — It admits the existence of *some invariant manifold* or, simply speaking, additional differential constraints (see, e.g., [6] and references therein) such that the equations only with the derivatives with respect to x_1, t_1

$$G_{1i_1}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial t_1}; u\right) = 0, \qquad i_1 = \overline{1, n_1},\ n_1 \in N \tag{4}$$

and/or x_2, t_2

$$G_{2i_2}\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial t_2}; u\right) = 0, \qquad i_2 = \overline{1, n_2},\ n_2 \in N \tag{5}$$

can be isolated from the last ones, possibly with other relations

$$G_{3i_3}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}; u\right) = 0, \qquad i_3 = \overline{1, n_3},\ n_3 \in N \tag{6}$$

Hence, in view of (3)–(6), the new variables x_1, t_1 and x_2, t_2 already appear *to be separated*, the full solution (process) $u(x_1, x_2, t_1, t_2)$ and respectively $u(x, t)$ can be presented as superposition of the two independent solutions (processes) proceeding in the different (x_1, t_1)- and (x_2, t_2)-spaces and could be obtained by consecutive solving with respect to x_1, t_1 and x_2, t_2. This paradigm was called *a multidimensional superposition principle*. Such a presentation in the terms of x_1, t_1 and x_2, t_2 with regard to the projection (2) for solutions of the original equation (1) can be used for constructing superposition formulas and explanation of the related solutions properties.

All the foregoing are immediately generalized both to cases of any dimension and to the systems of equations. In so doing, frequently it may be enough to introduce splitting of only the part from independent variables, e.g., splitting of spatial coordinates under investigation of waves collisions. Equations explicitly depending on the independent variables

$$E\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}; x, t; u\right) = 0, \qquad u = u(x, t)$$

are considered in the same way. In the general case such a problem as usually reduces to the previous one but already for a system by means of the formal introduction of the new auxiliary dependent variables $X(x_1, x_2)$ and $T(t_1, t_2)$

$$\begin{aligned}
&E\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}; X, T; u\right) = 0, \qquad u = u(x_1, x_2, t_1, t_2)\\
&X_{x_1} + X_{x_2} = 1, \qquad X = X(x_1, x_2)\\
&X_{t_1} = 0\\
&X_{t_2} = 0\\
&T_{t_1} + T_{t_2} = 1, \qquad T = T(t_1, t_2)\\
&T_{x_1} = 0\\
&T_{x_2} = 0
\end{aligned}$$

While it is possible to investigate particular cases with a concrete dependence of an adjoint equation with respect to x_1, t_1, x_2, t_2

$$E\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2},\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2};x_1,x_2,t_1,t_2;u\right)=0, \qquad u=u(x_1,x_2,t_1,t_2)$$

compatible with the projection $x_1 = x_2 = x$, $t_1 = t_2 = t$.

Presentation of solutions of NPDEs by truncated expansions. Consider the simplest generalization of the truncated singular expansions technique or, more precisely, its 'invariant' version [3].

Let there is a PDE with polynomial nonlinearity

$$F\left(U,U_x,U_t,\ldots;x,t\right)=0, \qquad U=U(x,t) \tag{7}$$

and some function $V(x,t)$ being determined by the following system (by the calibration $V \to (-a_n)^{\frac{1}{1-n}}V - \frac{a_{n-1}}{na_n}$ the coefficients at V^n and V^{n-1} in (8) are set to -1 and 0 without loss of generality)

$$V_x=-V^n+a_{n-2}V^{n-2}+\cdots+a_1V+a_0, \qquad a_i=a_i(x,t), \quad i=\overline{0,n-2} \tag{8}$$

$$V_t=b_nV^n+b_{n-1}V^{n-1}+\cdots+b_1V+b_0, \qquad b_j=b_j(x,t), \quad j=\overline{0,n},\ n\geq 2 \tag{9}$$

Where, according to its compatibility condition (see Appendix A), b_j with $j \neq n$ are expressed through a_i and b_n (the formulas (63), (65) for $n \geq 3$ or (67) for $n = 2$), the function $b_n(x,t)$ remains arbitrary, and $a_i(x,t)$ themselves satisfy to some equations ((66) or (68) respectively).

After substitution of a solution U in the form of the finite sum

$$U(x,t)=\sum_{i=m}^{0}w_i(x,t)V^i(x,t), \qquad m\in N \tag{10}$$

into (7) (as usually, all possible values of m in (10) are determined by the related dominant terms there) and equating all the coefficients at the different powers of V to zero or, strictly speaking, after the procedure of variables separation [5], we will have additional equations (both algebraic and differential) to the coefficients w_i, a_i, b_i in (8)–(10). In the case when the whole system is compatible, so that every solution U from (7) possibly except specific ones can be associated with some a_i and b_j, its investigation can be reduced to an investigation of the system (8), (9) for the function V with the found additional relations to a_i and b_j (analogs of 'Singular Manifold Equations' [3]).

In the next section such an investigation of (8), (9) is carried out to reveal the possible existence and properties of solitons in these systems and, as a consequence, in equations linked with them by series of the form (10).

3 The simplest solitons associated with the basis functions

Following the theory set forth in the previous section, consider the equations adjoint to (8), (9)

$$V_{x_1} + V_{x_2} = -V^n + a_{n-2}V^{n-2} + \cdots + a_1 V + a_0 = P \qquad (11)$$
$$V_{t_1} + V_{t_2} = b_n V^n + b_{n-1}V^{n-1} + \cdots + b_1 V + b_0 = Q \qquad (12)$$

Since here our interest is with invariant manifolds and respectively solitons common for the whole class of the equations possessing the solutions representation (10), then we have to set the coefficients-functions a_i, b_j not depending on x_1, t_1, i.e.

$$a_i = a_i(x_2, t_2), \qquad i = \overline{0, n-2}$$
$$b_j = b_j(x_2, t_2), \qquad j = \overline{0, n}$$

Really, for further investigation at a_i, b_j depending on x_1, t_1 the knowledge of 'SMEs' is already necessary, and this would do the results depending on a concrete form of the initial equation (7).

Equation (11), (12) can be split in the obvious manner

$$V_{x_1} = P_1\,(x_1, x_2, t_1, t_2; V) \qquad (13)$$
$$V_{t_1} = Q_1\,(x_1, x_2, t_1, t_2; V) \qquad (14)$$

with, respectively,

$$V_{x_2} = P\,(x_2, t_2; V) - P_1\,(x_1, x_2, t_1, t_2; V) \qquad (15)$$
$$V_{t_2} = Q\,(x_2, t_2; V) - Q_1\,(x_1, x_2, t_1, t_2; V) \qquad (16)$$

without loss of generality, because (13) and (14) can simultaneously be considered as the intermediate integrals [7] of more common equations like $G(V_{kx_1}, \ldots, V; x_1, x_2, t_1, t_2) = 0$, $k \geq 2$. The compatibility conditions for them, namely

$$V_{x_i x_j} - V_{x_j x_i} = 0$$
$$V_{x_i t_j} - V_{t_j x_i} = 0$$
$$V_{t_i t_j} - V_{t_j t_i} = 0, \qquad i, j = 1, 2;\ i \neq j$$

are obviously equivalent to the following set of the relations

$$V_{xt} - V_{tx} = 0$$
$$V_{x_1 t_1} - V_{t_1 x_1} = 0$$
$$V_{x x_1} - V_{x_1 x} = 0$$
$$V_{x t_1} - V_{t_1 x} = 0$$
$$V_{t x_1} - V_{x_1 t} = 0$$
$$V_{t t_1} - V_{t_1 t} = 0$$

where now

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}$$

After the substitutions the last ones in turn directly give

$$P_{1t_1} - Q_{1x_1} + [P_1, Q_1] = 0 \tag{17}$$
$$P_t - Q_x + [P, Q] = 0 \tag{18}$$
$$P_{x_1} - P_{1x} + [P, P_1] = 0 \tag{19}$$
$$Q_{t_1} - Q_{1t} + [Q, Q_1] = 0 \tag{20}$$
$$P_{t_1} - Q_{1x} + [P, Q_1] = 0 \tag{21}$$
$$Q_{x_1} - P_{1t} + [Q, P_1] = 0 \tag{22}$$

where

$$[A, B] = B\frac{\partial}{\partial V}A - A\frac{\partial}{\partial V}B \qquad \forall A, B$$

It is easy to verify that when fulfilling (17), (18) the rest equations (19), (22) and (20), (21) are compatible, in other words the original equations always admit formal splitting. However, to solve the above system in the general form does not seem to be possible. Therefore one will further restrict ourselves to consideration of its particular solutions.

First of all, note that for P_1, Q_1 not depending on x_1, t_1 as well the trivial relation

$$Q_1(x_2, t_2) = -\lambda P_1(x_2, t_2), \qquad \lambda = \text{const}$$

takes place. Indeed, in this case the system (17)–(22) takes the form

$$[P_1, Q_1] = 0 \tag{23}$$
$$P_{t_2} - Q_{x_2} + [P, Q] = 0 \tag{24}$$
$$-P_{1x_2} + [P, P_1] = 0 \tag{25}$$
$$-Q_{1t_2} + [Q, Q_1] = 0 \tag{26}$$
$$-Q_{1x_2} + [P, Q_1] = 0 \tag{27}$$
$$-P_{1t_2} + [Q, P_1] = 0 \tag{28}$$

so that from (23) it follows that $Q_1 = -\lambda(x_2, t_2)P_1$, and (26) and (27) give that $\lambda_{t_2} = \lambda_{x_2} = 0$. Equation (24) is fulfilled by the condition. The rest equations (25), (28) are the homogeneous linear PDEs, from their form it is naturally to seek their special solutions primarily among polynomials of the nth order on V, so

$$P_1 = c_n V^n + c_{n-1}V^{n-1} + \cdots + c_1 V + c_0, \qquad c_i = c_i(x_2, t_2),\ i = \overline{0, n} \tag{29}$$
$$Q_1 = -\lambda\left(c_n V^n + c_{n-1}V^{n-1} + \cdots + c_1 V + c_0\right) \tag{30}$$

Equation (25) for P_1 (29) has the same structure as the compatibility condition (59) for (8), (9) (see Appendix A). At $n \geq 3$ the following relations between $a_i(x_2, t_2)$ and $c_j(x_2, t_2)$ take place

$$
\begin{aligned}
&c_i = -a_i c_n, \qquad i = 0; \overline{2, n-1} \\
&c_1 = -a_1 c_n - \frac{c_{nx_2}}{n-1}
\end{aligned}
$$

together with the equations

$$
\begin{aligned}
&c_n a_{lx_2} + \left(\frac{n-l}{n-1}\right) a_l c_{nx_2} = 0, \qquad l = 0; \overline{2, n-2} \\
&c_n a_{1x_2} + a_1 c_{nx_2} + \left(\frac{1}{n-1}\right) c_{nx_2x_2} = 0
\end{aligned}
$$

so that from the last ones we also have fixing some $a_k \not\equiv 0$ $(k \neq 1)$ (the case $a_l \equiv 0$ for all $l = 0; \overline{2, n-2}$ corresponds to the Bernoulli equation trivially linearizable by a point transformation and is not considered here)

$$
c_n = -\left(\frac{\lambda_k}{a_k}\right)^{\frac{n-1}{n-k}}, \qquad \lambda_k \neq 0,\ k \neq 1 \tag{31}
$$

$$
a_1 = \frac{1}{n-k}\left(\frac{a_{kx_2}}{a_k}\right) + \lambda_1 \left(\frac{a_k}{\lambda_k}\right)^{\frac{n-1}{n-k}} \tag{32}
$$

$$
a_l = \lambda_l \left(\frac{a_k}{\lambda_k}\right)^{\frac{n-l}{n-l}}, \qquad l = \overline{0, n-2};\ l \neq k, 1 \tag{33}
$$

$$
\lambda_i = \text{const}, \qquad i = \overline{0, n-2}
$$

Although in the general case $\lambda_i = \lambda_i(t_2)$, however it can be shown that $\lambda_{it_2} = 0$, because it would lead to the additional terms in (28) proportional to λ_{it_2} and violating its compatibility with (25).

So, as a result, taking into account also the expressions (63), (65) for b_i, we can finally rewrite (8), (9) as

$$
\begin{aligned}
&V_x = P(x, t; V) = -V^n + a_{n-2}V^{n-2} + \cdots + a_1 V + a_0, \quad n \geq 3 \\
&V_t = Q(x, t; V) = -b_n V_x - \frac{b_{nx} V}{n-1}
\end{aligned} \tag{34}
$$

with a_j (32), (33) and its splitting (13)–(16) as

$$
\begin{aligned}
&V_{x_1} = P_1(x_2, t_2; V) = -c_n V_{x_2} - \frac{c_{nx_2} V}{n-1}, \quad n \geq 3 \\
&V_{t_1} = Q_1(x_2, t_2; V) = -\lambda P_1(x_2, t_2; V) \\
&V_{x_2} = P(x_2, t_2; V) - P_1(x_2, t_2; V) \\
&Q_{t_2} = Q(x_2, t_2; V) - Q_1(x_2, t_2; V)
\end{aligned}
$$

with c_n (31).

It is easy to verify that the function $V(x_1,x_2,t_1,t_2)$ is linked with another function $v_n(x_1,x_2,t_1,t_2)$ determining by the system

$$v_{nx_1} = -v_n^n + \lambda_{n-2}v_n^{n-2} + \cdots + \lambda_1 v_n + \lambda_0, \quad n \geq 3 \tag{35}$$

$$v_{nt_1} = -\lambda v_{nx_1} \tag{36}$$

$$v_{nx_2} = \theta_{x_2} v_{nx_1} = \left[\left(\frac{a_k}{\lambda_k}\right)^{\frac{n-1}{n-k}} - 1\right] v_{nx_1} \tag{37}$$

$$v_{nt_2} = \theta_{t_2} v_{nx_1} = \left[\lambda - b_n \left(\frac{a_k}{\lambda_k}\right)^{\frac{n-1}{n-k}}\right] v_{nx_1} \tag{38}$$

$$v_n = v_n(x_1,x_2,t_1,t_2),\ \theta = \theta(x_2,t_2)$$

via the linear transformation

$$V = (\theta_{x_2}+1)^{n-1}\, v_n\big(x_1 - \lambda t_1 + \theta(x_2,t_2)\big) = \left(\frac{a_k}{\lambda_k}\right)^{\frac{1}{n-k}} v_n \tag{39}$$

Next, bounded real solution $v_n(\vartheta)$ are kinks, i.e.

$$\lim_{\vartheta\to\pm\infty} v_n(\vartheta) = r_\pm \tag{40}$$

where $r_\pm$ are the real roots of the polynomial

$$-r^n + \lambda_{n-2}r^{n-2} + \cdots + \lambda_1 r + \lambda_0 = 0$$

different from one another. Asymptotical behavior $v_n(\vartheta)$ depends on their multiplicity α_r. So, one has the classical exponential decay case

$$v_n(\vartheta) \sim r + \beta_1 \mathrm{e}^{\beta_2\vartheta}, \qquad r = r_+, r_-;\ \beta_{1,2} = \text{const}$$

at $\alpha_r = 1$ and

$$v_n(\vartheta) \sim r + \frac{\beta_1}{(\vartheta+\beta_2)^{\frac{1}{\alpha_r - 1}}}, \qquad r = r_+, r_-;\ \beta_{1,2} = \text{const}$$

when $\alpha_r \geq 2$. In other words, we deal with the soliton (with the envelope describing by the special function, the solution of (35)) with the amplitude and phase modulated by a perturbation. The obtained solution has the following properties. Firstly, if for $x_2 \to \pm\infty$ the phase θ has the asymptotic values

$$\theta_{\pm\infty} = \lim_{x_2\to\pm\infty} \theta(x_2,t_2) = \text{const}$$

then from (32)–(35), (37), (39) immediately follows

$$\lim_{\theta\to\theta_{\pm\infty}} a_j(x_2,t_2) = \lambda_j$$

and

$$V^{\text{kink}}_{\pm\infty}(x_1,t_1) = \lim_{x_2\to\pm\infty} V = v_n\big(x_1 - \lambda t_1 + \theta_{\pm\infty}\big) \tag{41}$$

(by the way, together with $\lim_{\theta\to\theta_{\pm\infty}} b_n = \lambda$ from (38), while λ (36), the speed of the unperturbed kink (41), is determined by the related 'ISMs', in other words, by an original equation under consideration. Secondly,

$$V^{\theta}_{\pm\infty}(x_2,t_2) = \lim_{x_1\to\pm\infty} V = r_{\pm}\left(\theta_{x_2}+1\right)^{n-1} = r_{\pm}\left(\frac{a_k(x_2,t_2)}{\lambda_k}\right)^{\frac{1}{n-k}} \tag{42}$$

see (39), (40).

So, in the different regions the general solution degenerates to the solutions depending only on x_1, t_1 or x_2, t_2, namely, to the kink (41) or pure perturbation describing by the expression (42). In so doing, a perturbation on the left or right of the kink or before/after an interaction will have different amplitudes according to the values $r_{\pm}$, so that it will be increased or slackened after a collision depending on the coefficient

$$\eta = \frac{r_{\text{after}}}{r_{\text{befor}}}$$

including a possible change of the amplitude's sign.

This situation is depicted in Figure 1. It demonstrates the typical scenario for such kink-perturbation interactions. The fact that their features are common for all NPDEs with the same superposition formula because is fully determined by the last one. The case on the figure corresponds to the kink $v_3(x-t)$ with $(\lambda_0,\lambda_1)=(-8.16,7.69)$, i.e. $\eta=-32/15$, and the 'frozen' perturbation

$$\theta = -0.75\tanh(x-6.1)$$

The interaction's results are the phase shift for the former together with the overturning and slackening for the latter. Figure 2 presents the corresponding 2D function $V(x_1,x_2,t_1,t_2)$ at one moment, $t=2.8$, of the interaction, so that its projection $V(x,t)$ together with the reductions $V^{\text{kink}}_{\pm\infty}$, $V^{\theta}_{\pm\infty}$ are well seen there.

Return to the case $n=2$. Here another type relations take place, namely

$$c_0 = -a_0c_2 + \frac{c_{2x_2x_2}}{2}$$
$$c_1 = -c_{2x_2}$$
$$a_0 = \frac{c_{2x_2x_2}}{2c_2} - \frac{1}{4}\left(\frac{c_{2x_2}}{c_2}\right)^2 + \frac{\lambda_0}{c_2}$$

which lead to the following form of (8), (9)

$$V_x = P(x,t;V) = -V^2 + a_0$$
$$V_t = Q(x,t;V) = -b_2V_x - b_{2x}V + \frac{b_{2xx}}{2}$$

with the splitting

$$V_{x_1} = P_1(x_2,t_2;V) = -c_2V_{x_2} - c_{2x_2}V + \frac{c_{2x_2x_2}}{2}$$
$$V_{t_1} = Q_1(x_2,t_2;V) = -\lambda P_1(x_2,t_2;V)$$
$$V_{x_2} = P(x_2,t_2;V) - P_1(x_2,t_2;V)$$
$$Q_{t_2} = Q(x_2,t_2;V) - Q_1(x_2,t_2;V)$$

and the linear transformation

$$V = (\theta_{x_2} + 1)\, v_2\big(x_1 - \lambda t_1 + \theta(x_2, t_2)\big) - \frac{\theta_{x_2 x_2}}{2\,(\theta_{x_2} + 1)} = -{c_2}^{-1} v_2 + \frac{c_{2x_2}}{2c_2}$$

between V and the function $v_2(x_2, t_2)$ satisfying the equations

$$\begin{aligned}
v_{2x_1} &= -v_2^2 - \lambda_0 \\
v_{2t_1} &= -\lambda v_{2x_1} \\
v_{2x_2} &= \theta_{x_2} v_{2x_1} = -\left(\frac{1}{c_2} + 1\right) v_{2x_1} \\
v_{2t_2} &= \theta_{t_2} v_{2x_1} = \left(\frac{b_2}{c_2} + \lambda\right) v_{2x_1} \\
&v_2 = v_2(x_1, x_2, t_1, t_2),\ \theta = \theta(x_2, t_2)
\end{aligned}$$

This case was in details considered in [1].

In conclusion note that from the results obtained above it is seen now what the restrictions made in this section beginning corresponds to. So, the supposition that P_1 and Q_1 do not depend on x_1, t_1 means that we deal with the solitons with the constant velocities λ, while the restriction by the polynomials (29), (30) corresponds to its 'linear' deformation by a perturbation. (On the general condition for such a deformation, see Appendix B).

4 Computer simulation of some of soliton interactions for $\mathbf{n \geq 3}$

Nonlinear equations leading to the above systems for V with $n = 2$ are well known and wide-spread. Among them apparently are all known integrable models (at the assumption made before in 1D and associated with isospectral ISTs, of course). In [1] computer simulation and analytics are presented for several of such models comprehensively enough. While up to now cases with $n \geq 3$ have likely occurred neither in numerical nor especially analytical investigations. This does not seem to be surprising taking into account complexity and quantity of the additional constraints to the coefficients in (8) at $n \geq 3$. Therefore, both from the theoretical and applied viewpoint, it is interesting to carry out numeric modelling on concrete models possessing such soliton interactions. On the one hand, such structures are solitons in the strict sense. On the other hand, in contrast to known NPDEs they cannot be obviously described in the framework of techniques habitual for the last ones, such as, e.g., the Hirota's anzats, etc.

The case n = 3. The equations for V (34) together with the compatibility conditions (66) are as follows

$$\begin{aligned}
&V_x = -V^3 + a_1 V + a_0\\
&V_t = b_3 V^3 - \left(b_3 a_1 + \frac{b_{3x}}{2}\right) V - b_3 a_0\\
&a_{0t} + a_{0x} b_3 + \frac{3}{2} b_{3x} a_0 = 0\\
&a_{1t} + a_{1x} b_3 + b_{3x} a_1 + \frac{b_{3xx}}{2} = 0
\end{aligned} \tag{43}$$

with a_1 and a_0 have to be in addition linked by the relation $\lambda_0^2(3a_0a_1 - a_{0x})^3 = 27\lambda_1^3 a_0^5$ (32) for the existence of the above indicated type solitons.

For our purposes choose one of the simplest 'SME', namely

$$b_3 = -a_1 + \lambda', \qquad \lambda' = \mathrm{const}$$

Equation (43) together with the relation to a_0, a_1 can be simultaneously interpreted as the system of three PDEs to the functions $a_0(x,t)$, $a_1(x,t)$, $V(x,t)$

$$a_{0t} + (\lambda' - a_1)a_{0x} - \frac{3}{2} a_{1x} a_0 = 0 \tag{44}$$

$$a_{1t} + (\lambda' - 2a_1)a_{1x} - \frac{a_{1xx}}{2} = 0 \tag{45}$$

$$V_t + (\lambda' - a_1)V_x - \frac{a_{1x} V}{2} = 0 \tag{46}$$

with two extra constraints

$$V_x = -V^3 + a_1 V + a_0 \tag{47}$$

$$\lambda_0^2(3a_0a_1 - a_{0x})^3 = 27\lambda_1^3 a_0^5 \tag{48}$$

In so doing, only the function V may contents the soliton component. In principle, in view of (47), (48) the system (44)–(46) can be reduced to the only equation with respect to V only. However, in contrast to the wide-spread case $n = 2$ here a_0 and a_1 cannot be algebraically expressed through V. As a consequence, the resulting equation for V consists of several thousands addends including various mixed derivatives. Therefore we will just partially transform (44)–(46) using only (47) thereby decreased the quantity of the independent variables. One has as a result from (44), (46) the set of two NPDEs of the reaction-diffusion type

$$\begin{aligned}
&2a_{0t} - 2V^{-1}a_{0x}V_x + (2\lambda' - 2V^2 + 5a_0V^{-1})a_{0x} + 3a_0V^{-2}V_x^2\\
&\qquad - 3a_0V^{-2}(a_0 + 2V^3)V_x - 3a_0V^{-1}V_{xx} = 0\\
&2V_t + a_{0x} - V^{-1}V_x^2 + (a_0V^{-1} + 2\lambda' - 4V^2)V_x - V_{xx} = 0
\end{aligned} \tag{49}$$

with one constraint between a_0 and V

$$\lambda_0^2 \left(3V_x a_0 - 3{a_0}^2 - a_{0x}V + 3a_0V^3\right)^3 = 27\lambda_1^3 {a_0}^5 V^3 \tag{50}$$

Another system we will have if choose respectively equations (45) and (46)

$$
\begin{aligned}
a_{1t} + (\lambda' - 2a_1)a_{1x} - \frac{a_{1xx}}{2} &= 0 \\
V_t + (\lambda' - a_1)V_x - \frac{a_{1x}V}{2} &= 0
\end{aligned}
\tag{51}
$$

together with the constraint

$$\lambda_0^2 \left(4a_1V_x + 3a_1V^3 + a_{1x}V - 3V_xV^2 - V_{xx} - 3a_1^2V\right)^3 = 27\lambda_1^3 \left(V_x + V^3 - a_1V\right)^5 . \tag{52}$$

With regard to the related constraints, (47), (48) for (44)–(46), (50) for (49), (52) for (51), all these systems and the above-mentioned nonlinear equation only to V are equivalent each other. Any of these 'presentations' can be used for a computer simulation of the solitons.

The series of computer experiments (on the technical details here and below see Appendix C) with the initial condition

$$V(x,0) = v_3(x,0) + v_{\text{perturbation}}(x)$$

was carried out to simulate collisions of localized perturbations and the various kinks v_3 of the family (35). At an initial moment the perturbations were set as

$$v_{\text{perturbation}}(x) = \frac{C_0}{(\mathrm{e}^{k_1(x-\varphi)} + \mathrm{e}^{k_2(x-\varphi)} + C_1)}, \qquad C_0, C_1, k_1, k_2, \varphi = \text{const}$$

Figures 3, 4 demonstrate the most characteristic of them. In so doing, λ' was choosing such that the kink itself was stationery, i.e. $\lambda' = \lambda_1$ and $\lambda = 0$ everywhere. As a result, any changes are well seen. Figures 3 corresponds to the kink-perturbation interaction when $(\lambda_0, \lambda_1) = (-3.75, 4.75)$ with the slackening coefficient $\eta = 2/3$ $(r_- = 1, r_+ = 1.5)$, while Figure 4 corresponds to the choice $(\lambda_0, \lambda_1) = (-0.25, 0.75)$ with $\eta = -1/2$ $(r_- = 0.5, r_+ = -1)$. In both cases the perturbations slackening take place, but the second case differs from the first one by the perturbation overturning according to the sign of η after passage through the kink's front and by the left asymptote $\sim 1/x$ because of the double root r_-. As seen from the experiments, the only change in the kinks is the phase shifts after the interactions, while the envelopes remain unchanged always, although the medium itself is dissipative, so that all localized disturbances are spreading and disappear with time.

While even to write the equation immediately for V appears to be not so trivial and any appropriate physical models seem to be unknown yet, to derive a similar equation to v (35)–(38) is much more easy. Indeed, from (35)–(38), summing v_{x_1}, v_{x_2} and v_{t_1}, v_{t_2}, one has

$$v_{nx} = \left(\frac{a_k}{\lambda_k}\right)^{\frac{n-1}{n-k}} p(v), \qquad p(v) = -v_n^n + \lambda_{n-2}v_n^{n-2} + \cdots + \lambda_1 v_n + \lambda_0 \tag{53}$$

$$v_{nt} = -b_n p(v) \tag{54}$$

If b_n is expressed in terms of the relation $\frac{a_k}{\lambda_k}$, it can be done in the pure algebraic manner. So setting

$$b_3 = -\left[\left(\frac{a_0}{\lambda_0}\right)^{\frac{2}{3}}\right]_x$$

for $n = 3$, we will arrive at the equation

$$v_t = 2\left(\frac{v_x}{p(v)}\right)^2 \left[v_{xx} - v_x p'_v(v)\left(\frac{v_x}{p(v)}\right)\right] \tag{55}$$

— $\left(\frac{a_0}{\lambda_0}\right)^{\frac{2}{3}}$ is finding from (53) and then directly substituting into (54).

In such 'marginal' solitonic models, when a perturbation modulates only a soliton phase, the interactions character slightly differs from the above. In these cases a perturbation always asymptotically (i.e. before and after a direct collision) disappears or, maybe, it will be better to say, gathers to zero. This is first. Second, a deformation of a soliton envelope consists in only formation of compression or stretching zones on a kink's front. For equation (55) such a situation is shown on Figure 5 when

$$\theta(x, 0) = \frac{7}{[\mathrm{e}^{-2(x-2.5)} + \mathrm{e}^{2(x-2.5)}]}$$

There exist one zone of compression and one of stretching, which form the left and right perturbation's fronts respectively. Since $\lim\limits_{x\to+\infty} \theta = \lim\limits_{x\to-\infty} \theta$ the phase shift is absent here in contrast to the similar interactions on Figure 8 below and in Figure 7 [1].

The case n = 4. Equations (34), the conditions of their compatibility (66) and the solitons existence (32), (33) can be also considered as the set of four PDEs

$$\begin{aligned}
&V_t + b_4 V_x + \frac{b_{4x} V}{3} = 0\\
&a_{0t} + a_{0x} b_4 + \frac{4}{3} b_{4x} a_0 = 0\\
&a_{1t} + a_{1x} b_4 + b_{4x} a_1 + \frac{b_{4xx}}{3} = 0\\
&a_{2t} + a_{2x} b_4 + \frac{2}{3} b_{4x} a_2 = 0
\end{aligned}$$

with three constraints to the independent variables a_0, a_1, a_2, V, if $\lambda_0 \neq 0$

$$\begin{aligned}
&V_x = -V^4 + a_2 V^2 + a_1 V + a_0\\
&\left(\frac{a_2}{\lambda_2}\right)^2 = \frac{a_0}{\lambda_0}\\
&\lambda_0^3 (4a_0 a_1 - a_{0x})^4 = 256 \lambda_1^4 a_0^7
\end{aligned} \tag{56}$$

and respectively

$$\begin{aligned}
&V_x = -V^4 + a_2 V^2 + a_1 V + a_0\\
&a_0 = 0\\
&\lambda_2^3 (2a_2 a_1 - a_{2x})^2 = 4 \lambda_1^2 a_2^5
\end{aligned} \tag{57}$$

if $\lambda_0 = 0$. Again, choosing the same trivial linkage of b_4 with the last ones

$$b_4 = -a_1 + \lambda', \qquad \lambda' = \mathrm{const}$$

one has the simple system of two equations

$$\begin{aligned} a_{1t} + (\lambda' - 2a_1)a_{1x} - \frac{a_{1xx}}{3} = 0 \\ V_t + (\lambda' - a_1)V_x - \frac{a_{1x}V}{3} = 0 \end{aligned} \tag{58}$$

to a_1 and V. Here, in principle, the constraints (56) or (57) can be changed by the only expression for a_1 and V, but because of the huge square roots the initial forms with a_0, a_2 as the auxiliary functions is preferable from many points of view.

Analogously to Figures 3, 4 Figures 6–8 illustrate the typical interactions of the v_4-kink (35) (stationary here, i.e. $\lambda' = \lambda_1$ was chosen) with the localized perturbations. Again for the kinks the only effect takes place — the phase shift. Figure 6 for $(\lambda_0, \lambda_1, \lambda_2) = (0.112, 0.592, 1.48)$ and the slackening coefficient $\eta = -5/7$ $(r_- = -1, r_+ = 1.4)$ demonstrates such the interaction for the kink with the nonmonotonic front. This kink is an analogy of a two-hump bell-shape soliton. Here the perturbation practically fully dissipates during the collision, nevertheless it effects to the phase shift arising. In the case shown on Figure 7, $(\lambda_0, \lambda_1, \lambda_2) = (0, -0.25, 0.75)$, we have the amplification of the perturbation after passage through the kink's front with the coefficient $\eta = 2.5$ $(r_- = 0.4, r_+ = 1)$; while Figure 8, $(\lambda_0, \lambda_1, \lambda_2) = (0, -0.25, 0.75)$ and $\eta = 0$ $(r_- = -1, r_+ = 0)$, corresponds to the full absorption of the perturbation.

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Appendix A

Equating to zero the coefficients at the different powers V^l in the compatibility condition

$$V_{xt} - V_{tx} = 0 \tag{59}$$

for (8), (9), we have the relations to the coefficients-functions a_i, b_j

$$a_{lt} - b_{lx} + \sum_{\substack{l+1-n \le i,j \le n \\ i+j-1=l}} (i-j)a_i b_j = 0, \qquad l = \overline{0, n-2} \tag{60}$$

$$-b_{lx} + \sum_{\substack{l+1-n \le i,j \le n \\ i+j-1=l}} (i-j)a_i b_j = 0, \qquad l = \overline{n-1, n} \tag{61}$$

for the cases $n \geq 3$ also with

$$\sum_{\substack{l+1-n\leq i,j\leq n\\ i+j-1=l}} (i-j)a_i b_j = 0, \qquad l = \overline{n+1, 2n-2} \tag{62}$$

This is the initial equations. Here and below, wherever it is needed for convenience, $a_n = -1$ and $a_{n-1} = 0$ have been again explicitly introduced into the formulas. Since the cases with $n = 2$ and $n \geq 3$ differ, further they should be consider separately.

The case $n \geq 3$. First, prove that from (62)

$$b_i = \frac{a_i}{a_n} b_n, \qquad i = \overline{2, n-1} \tag{63}$$

with $b_n(x,t)$ being arbitrary.

Indeed, (62) can be written down as

$$(l+1-2n)\left(a_{l+1-n}b_n - a_n b_{l+1-n}\right) + \sum_{l+1-n\leq j\leq n-1} (l+1-2j)\left(a_{l+1-j}b_j - a_j b_{l+1-j}\right) = 0 \tag{64}$$

Assume that for some k the equalities (63) with $j \geq k$ take place, then one has from (64) at $l = k+n+2$

$$\begin{aligned}&(k-1-n)\left(a_{k-1}b_n - a_n b_{k-1}\right)\\ &\quad = -\sum_{k\leq j\leq n-1} (k+n-1-2j)\left[a_{k+n-1-j}\left(\frac{a_j}{a_n}b_n\right) - a_j\left(\frac{a_{k+n-1-j}}{a_n}b_n\right)\right] = 0\end{aligned}$$

because every addend in the rhs sum is obviously identically equal to zero. As a result, (63) is true also for $j = k-1$. But at $l = 2n-2$ (62) directly have the form (63) with $j = n-1$, and on the mathematical induction we will have (63) for $j = \overline{n-2, 2}$ as well. While $b_n(x,t)$ itself remains arbitrary, because the coefficient at V^{2n-1} in the compatibility condition (59) is equal to zero identically.

Next, the sums in (61) have the same structure, so that equations (61) analogously give us

$$\begin{aligned} b_0 &= -a_0 b_n\\ b_1 &= -a_1 b_n - \frac{b_{nx}}{n-1}\end{aligned} \tag{65}$$

taking into account that $a_n = -1$, $a_{n-1} = 0$ and for them always $b_{n-1} = 0$.

Finally, again in view of the structure of the sums in (60) we have from these relations

$$\begin{aligned} &a_{lt} + b_n a_{lx} + \left(\frac{n-l}{n-1}\right) a_l b_{nx} = 0, \qquad l = \overline{0, n-2};\ l \neq 1\\ &a_{1t} + b_n a_{1x} + a_1 b_{nx} + \left(\frac{1}{n-1}\right) b_{nxx} = 0\end{aligned} \tag{66}$$

i.e. the system of $n-1$ equations to $n-1$ functions $a_l(x,t)$.

The case $n = 2$. In this case, by the direct computation one obtains the relations for b_1, b_0 ($b_2(x,t)$ is arbitrary)

$$\begin{aligned} b_1 &= -b_{2x} \\ b_0 &= -a_0 b_2 + \frac{b_{2xx}}{2} \end{aligned} \tag{67}$$

together with the equation to the only function $a_0(x,t)$

$$a_{0t} + 2a_0 b_{2x} + b_2 a_{0x} - \frac{b_{2xxx}}{2} = 0 \tag{68}$$

as expected [3].

Appendix B

Suppose there is some compatible system of the form

$$\begin{aligned} V_{x_1} &= P_1(x_2, t_2; V), \qquad V = V(x_1, x_2, t_1, t_2) \\ V_{x_2} &= P_2(x_2, t_2; V) \end{aligned} \tag{69}$$

$$\begin{aligned} V_{t_1} &= Q_1(x_2, t_2; V) \\ V_{t_2} &= Q_2(x_2, t_2; V) \end{aligned} \tag{70}$$

Set the question: What do P_1, P_2, Q_1, Q_2 have to be so that the linear transformation

$$V = \alpha(x_2, t_2) v(x_1, x_2, t_1, t_2) + \beta(x_2, t_2) \tag{71}$$

reducing (69), (70) to another, simpler, system

$$\begin{aligned} v_{x_1} &= p(v) \\ v_{x_2} &= \gamma(x_2, t_2) p(v) \end{aligned} \tag{72}$$

$$\begin{aligned} v_{t_1} &= q(v) \\ v_{t_2} &= \delta(x_2, t_2) q(v) \end{aligned} \tag{73}$$

exists? The direct substitution (71) into (69) gives

$$\alpha p = P_1(x_2, t_2, \alpha v + \beta) \tag{74}$$

$$\alpha_{x_2} v + \gamma P_1 + \beta_{x_2} = P_2(x_2, t_2, \alpha v + \beta) \tag{75}$$

in view of (72). Next, taking into account that $v = (V - \beta)/\alpha$, one has from (75)

$$P_2 = \frac{\alpha_{x_2}}{\alpha}(V - \beta) + \beta_{x_2} + \gamma P_1 \tag{76}$$

In the same way, from (70), (73) one has

$$Q_2 = \frac{\alpha_{t_2}}{\alpha}(V - \beta) + \beta_{t_2} + \delta Q_1 \tag{77}$$

The conditions $p_{x_2} = p_{t_2} = 0$ for p (74) and analogous conditions for q are identical to the compatible conditions of the initial system (69), (70).

The relations (76), (77) are the relations sought between P_2, P_1 and Q_2, Q_1.

Appendix C

In order to simulate solitonic interactions we used the systems (51) or (58). In so doing, the following implicit finite-difference scheme ($A \equiv a_1$)

$$\begin{aligned}
&\frac{A^+ - A^-}{\tau} + \lambda' \left(\frac{A_x^+ + A_x^-}{2} \right) - (A^2)_x^\pm - \frac{1}{n-1} \left(\frac{A_{xx}^+ + A_{xx}^-}{2} \right) = 0 \\
&\frac{V^+ - V^-}{\tau} + \left(\lambda' - \frac{A^+ + A^-}{2} \right) \left(\frac{V_x^+ + V_x^-}{2} \right) \\
&\qquad - \frac{1}{n-1} \left(\frac{A_x^+ + A_x^-}{2} \right) \left(\frac{V^+ + V^-}{2} \right) = 0
\end{aligned} \tag{78}$$

was employed. In this symbolic notations the superscripts '+', '−' correspond to the mth and $(m+1)$th layers, and $V_x^\pm$, $V_{xx}^\pm$, $A_x^\pm$, $A_{xx}^\pm$ denote the standard approximations for the related derivatives of V, A on five points, so that for example

$$V_x^- = \frac{V_{i-2}^m - 8V_{i-1}^m + 8V_{i+1}^m - V_{i+2}^m}{12h}$$

The term $(A^2)_x^\pm$ analogously is

$$(A^2)_x^\pm = \frac{A_{i-2}^m A_{i-2}^{m+1} - 8A_{i-1}^m A_{i-1}^{m+1} + 8A_{i+1}^m A_{i+1}^{m+1} - A_{i+2}^m A_{i+2}^{m+1}}{12h}$$

The scheme for the mesh functions A_i^m, V_i^m ($i = \overline{0, N}$; $m \geq 0$ such that $t = \tau m$ and $x = hi$) of the accuracy $O(\tau^2) + O(h^4)$ is supplemented by the appropriate bounding condition

$$A(0,t), V(0,t), A(L,t), V(L,t) = \text{const}$$

and also

$$A_x(0,t), V_x(0,t), A_x(L,t), V_x(L,t) = \text{const}$$

approximated also with the order of accuracy $O(h^4)$ on five left/right points. Such a form of the last boundary conditions is necessary for the cases like depicted on Figure 2 because of the slowly failing down left asymptote and is very effective for stationary kinks and localized perturbations. The problem is reduced to solving the algebraic system, firstly, for A_i^m

$$a_i^m A_{i-2}^m + b_i^m A_{i-1}^m + c_i^m A_i^m + d_i^m A_{i+1}^m + e_i^m A_{i+2}^m = f_i^m$$

and then at the known already A_i^m for V_i^m

$$\tilde{a}_i^n V_{i-2}^n + \tilde{b}_i^n V_{i-1}^n + \tilde{c}_i^n V_i^n + \tilde{d}_i^n V_{i+1}^n + \tilde{d}_i^n V_{i+2}^n = \tilde{f}_i^n$$

both with the band matrixes.

For our purposes here the above scheme appears to be economical, simple in the implementation and has the high accuracy when $|V_{kx}| \lesssim 1$. For bigger values of V_{kx} in (78) the residual begins to play an essential role, that visibly decreases the accuracy. In particular such high value derivatives arise at the moment of the kink-perturbation touching in the cases depicted in Figures 4, 7, 8. The calculations were performed with $h = 0.1$ (Figures 3, 6) and $h = 0.05$ (Figures 4, 7, 8) at $\tau = 0.01$ with long double precision (18 figures). This ensured the error from $\varepsilon = 1 \cdot 10^{-5}$ (Figure 3) up to $\varepsilon = 2 \cdot 10^{-3}$ (Figure 4) in the most critical case from mentioned before, i.e. 0.0005–0.2%, and was in the agreement with the theoretical estimations. An interesting reader can find more about various schemes for similar equations in [8].

It is necessary separately to dwell on setting the initial conditions. The task here is to determine A (a_1) satisfying the constraints at a specified V. For the initial conditions of the form

$$V(x,0) = v_n(x,0) + v_{\text{perturbation}}(x) \tag{79}$$

when overlapping between a kink's front and localized disturbance is absent, it can be done very simply. Indeed, since $V^{\theta}_{\pm\infty} \equiv v_{\text{perturbation}}$, from (42), (32) one directly obtains

$$A(x,0) \equiv a_1(x,0) = \frac{v_{\text{perturbation}_x}}{v_{\text{perturbation}}} + \lambda_1 \left(\frac{v_{\text{perturbation}}}{r}\right)^{n-1} \tag{80}$$

where the root r corresponds to the appropriate kink's asymptote. By this means the initial conditions for $V(x,0)$ (79) are set directly, and $A(x,0)$ is calculated according to the formulas (80). The derivative in the last one was approximated on 5 steps for the adequate accuracy $O(h^4)$.

For the simulation of equation (55), because of its complex nonlinearity, the following simplest explicit scheme of the accuracy $O(\tau) + O(h^2)$ appeared to be efficient

$$\frac{v^+ - v^-}{\tau} = 2D^2 \left[v^-_{xx} - v^-_x p'_v(v^-) D\right], \qquad D = \frac{v^-_x}{p(v)}$$

(to avoid loss of the accuracy near the zeros of $p(v)$ it is reasonable to calculate the relation $v_x/p(v)$ separately — $\lim\limits_{v \to r} \frac{v_x}{p(v)} = 1$) with the boundary conditions

$$\begin{aligned} v(0,t) &= v(0,0) \\ v(L,t) &= v(L,0) \end{aligned}$$

In spite of the rigid limitation to τ for the stability, namely $\tau \lesssim 0.02$–$0.1h$, computations are fast enough for not very small h because of its realization simplicity. In the experiments $\tau = 0.00005$ and $h = 0.05$ were used, that ensured the error $\varepsilon = 1 \cdot 10^{-3}$ (0.1%) or even less.

Finally note that when calculating v_n we used the explicit fifth-order Runge-Kutta method with the same step h as in the difference schemes.

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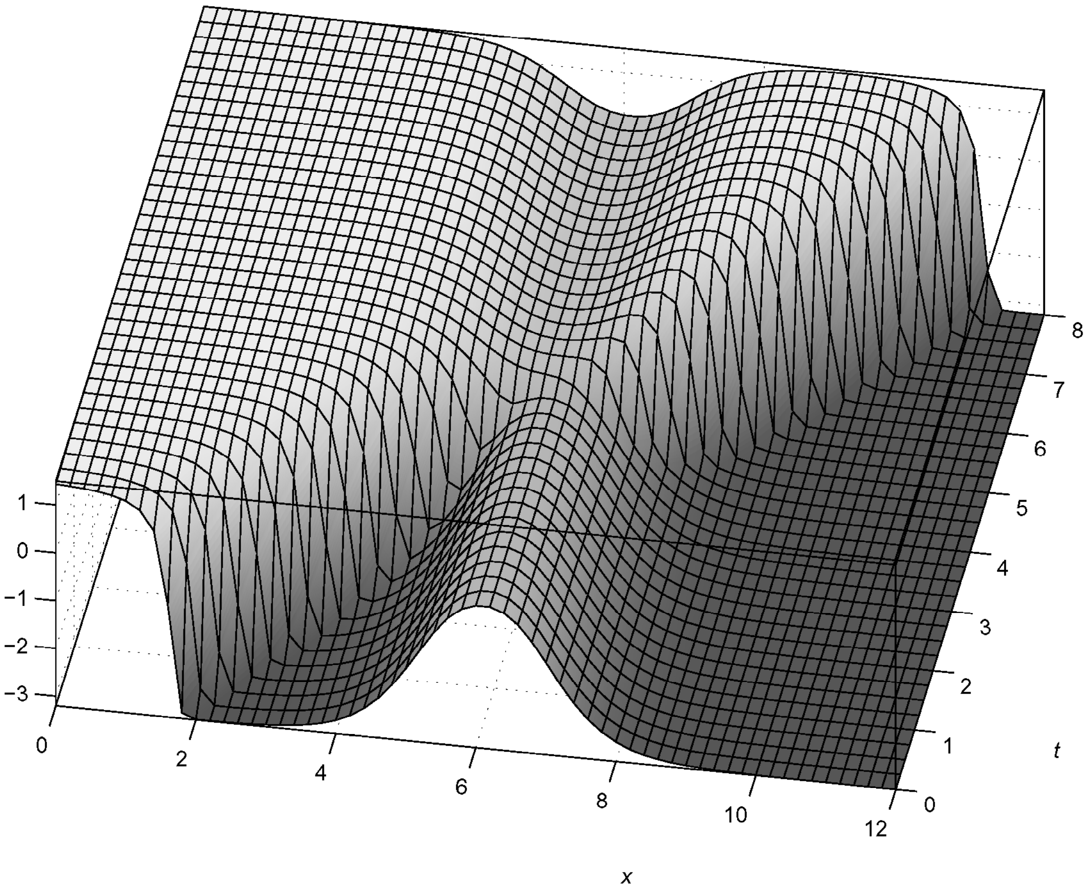

Figure 1. Typical scenario of a solitonic interaction.

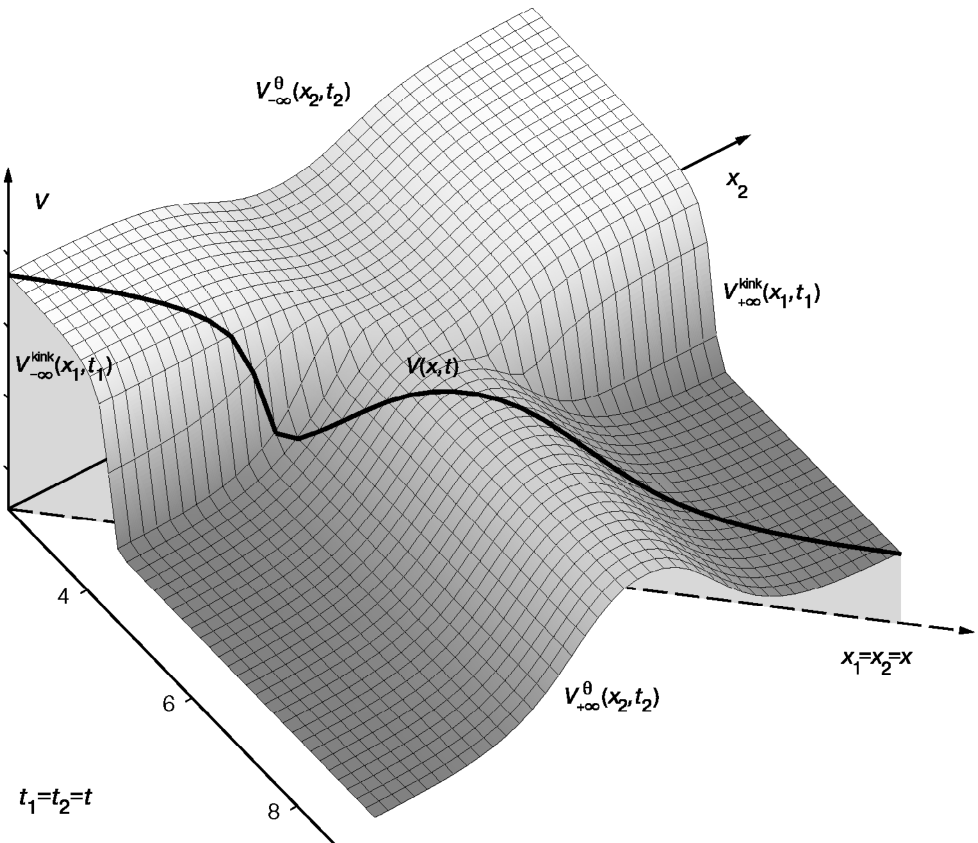

Figure 2. Form of the 2D function V at one of the moments (t=2.8) of the kink-perturbation interaction depicted in Figure 1.

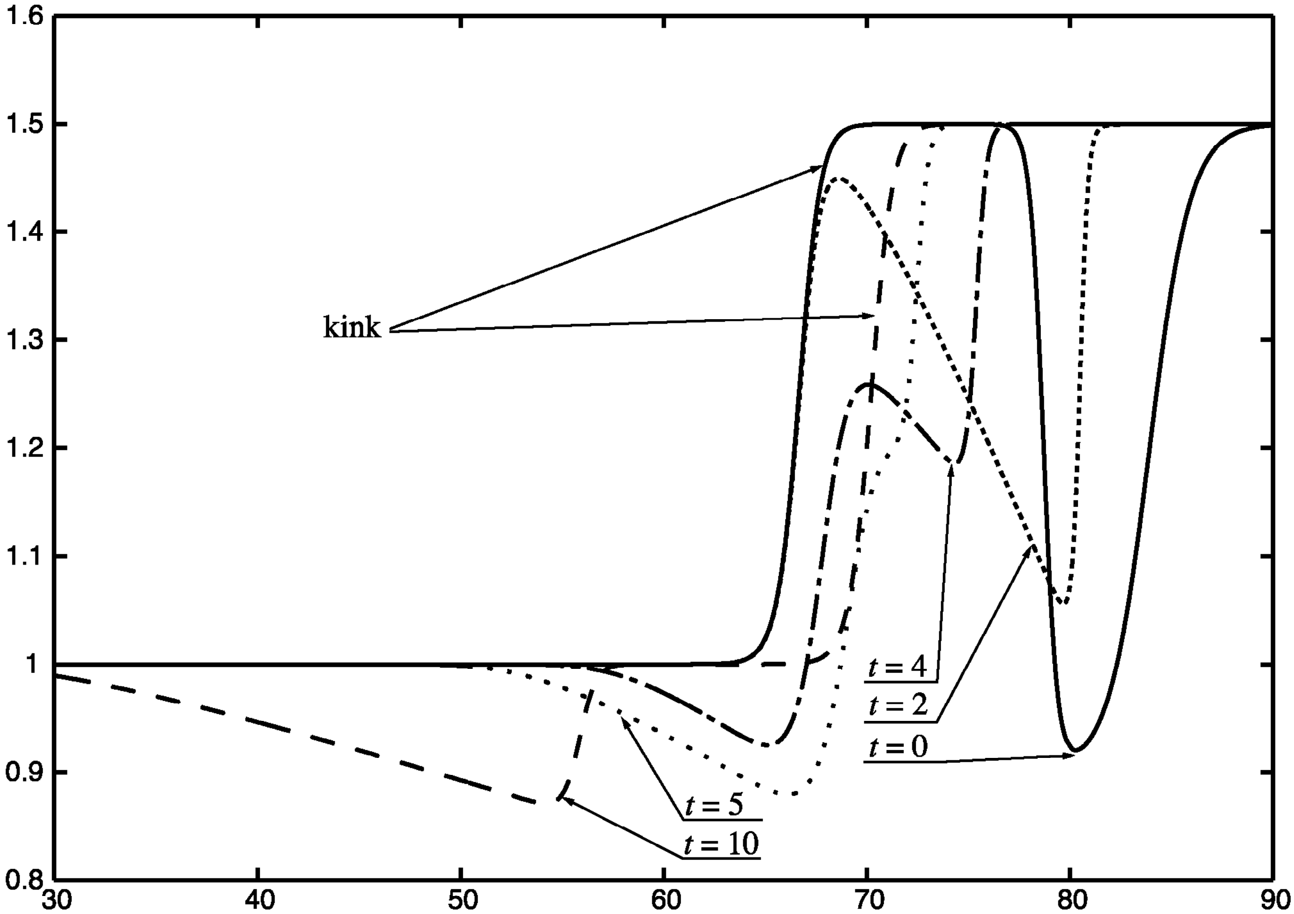

Figure 3. Interaction of the v_3-kink (λ_0,λ_1)=(-3.75, 4.75) and the localized perturbation.

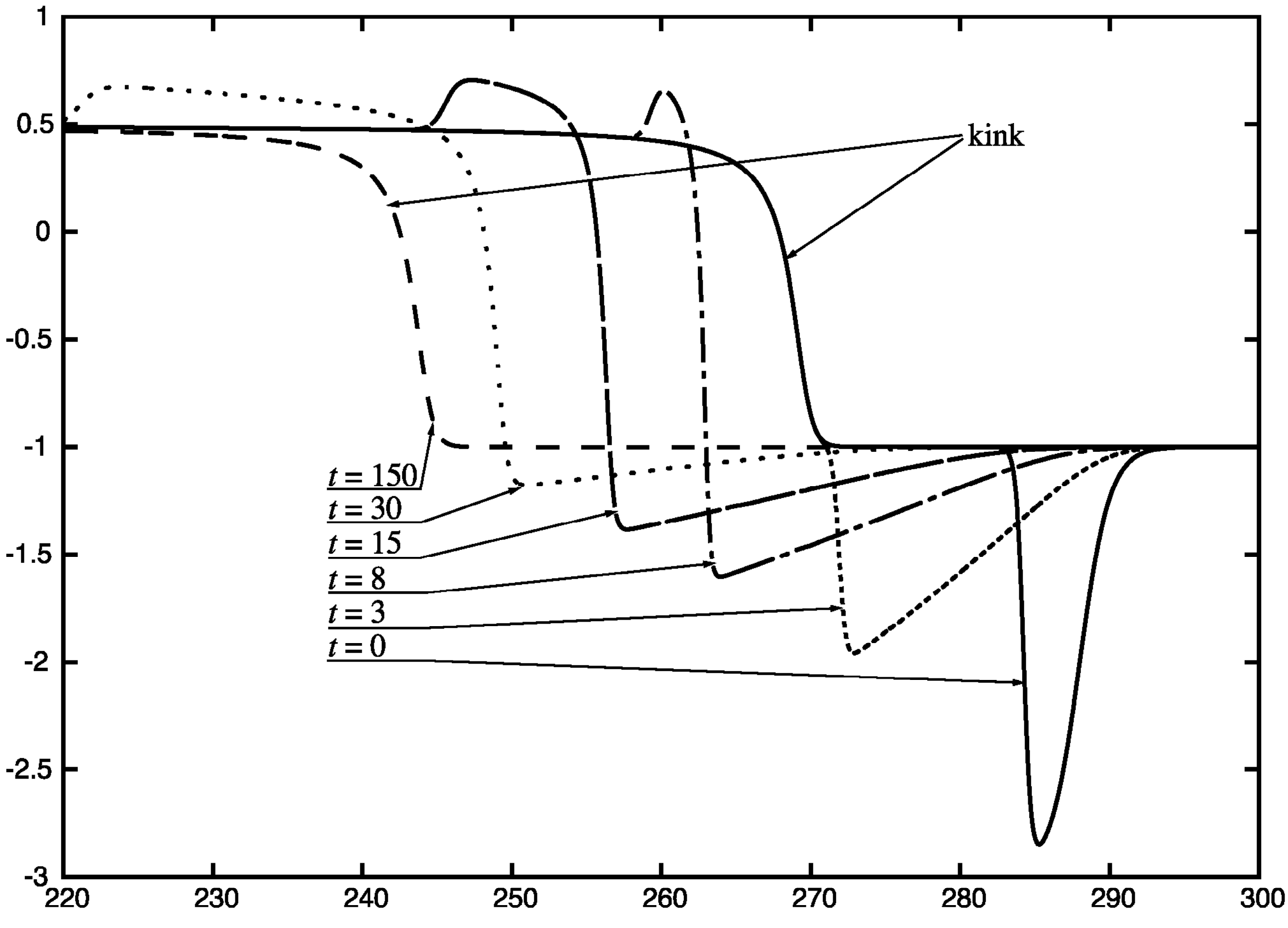

Figure 4. Interaction of the v_3-kink (λ_0,λ_1)=(-0.25, 0.75) and the localized perturbation.

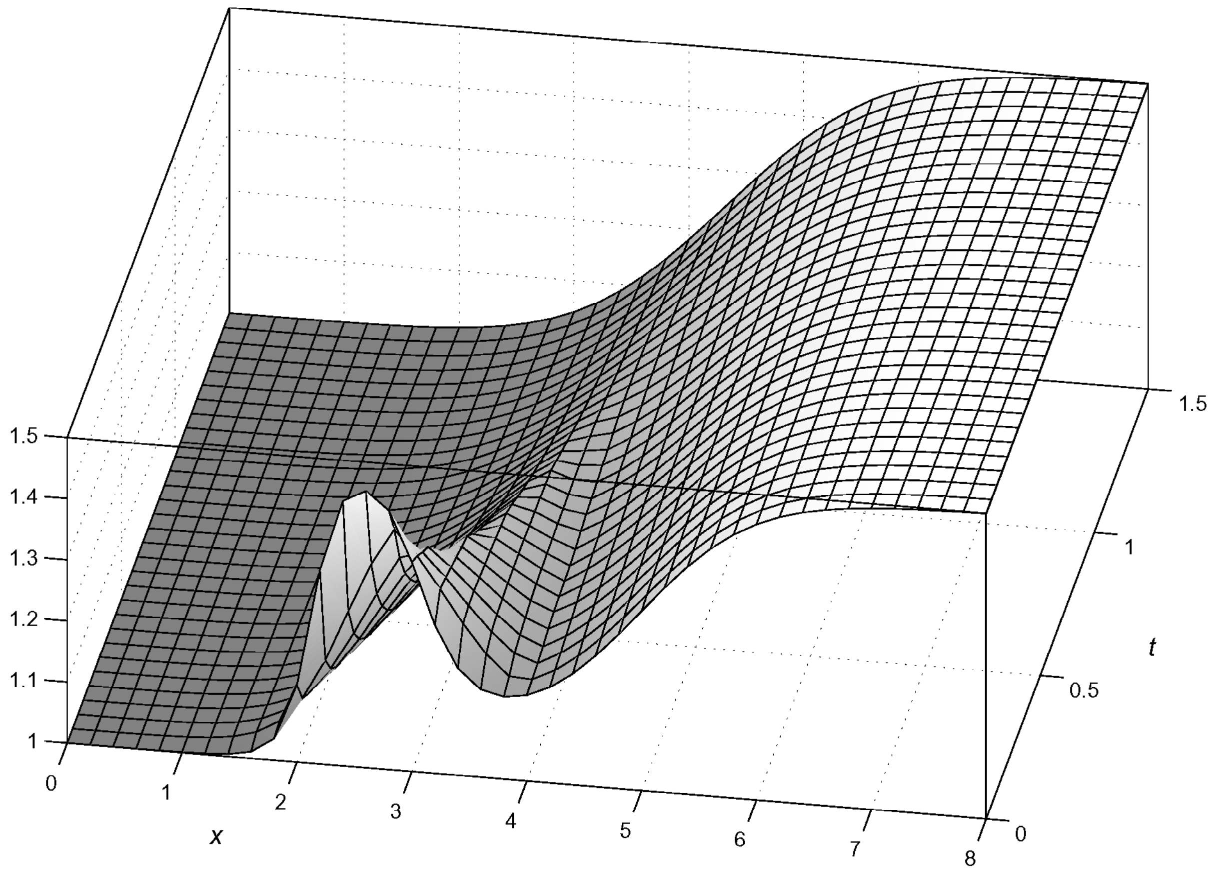

Figure 5. Picture of the kink-perturbation interaction in the case of the phase modulation only (λ_0,λ_1)=(-3.75, 4.75).

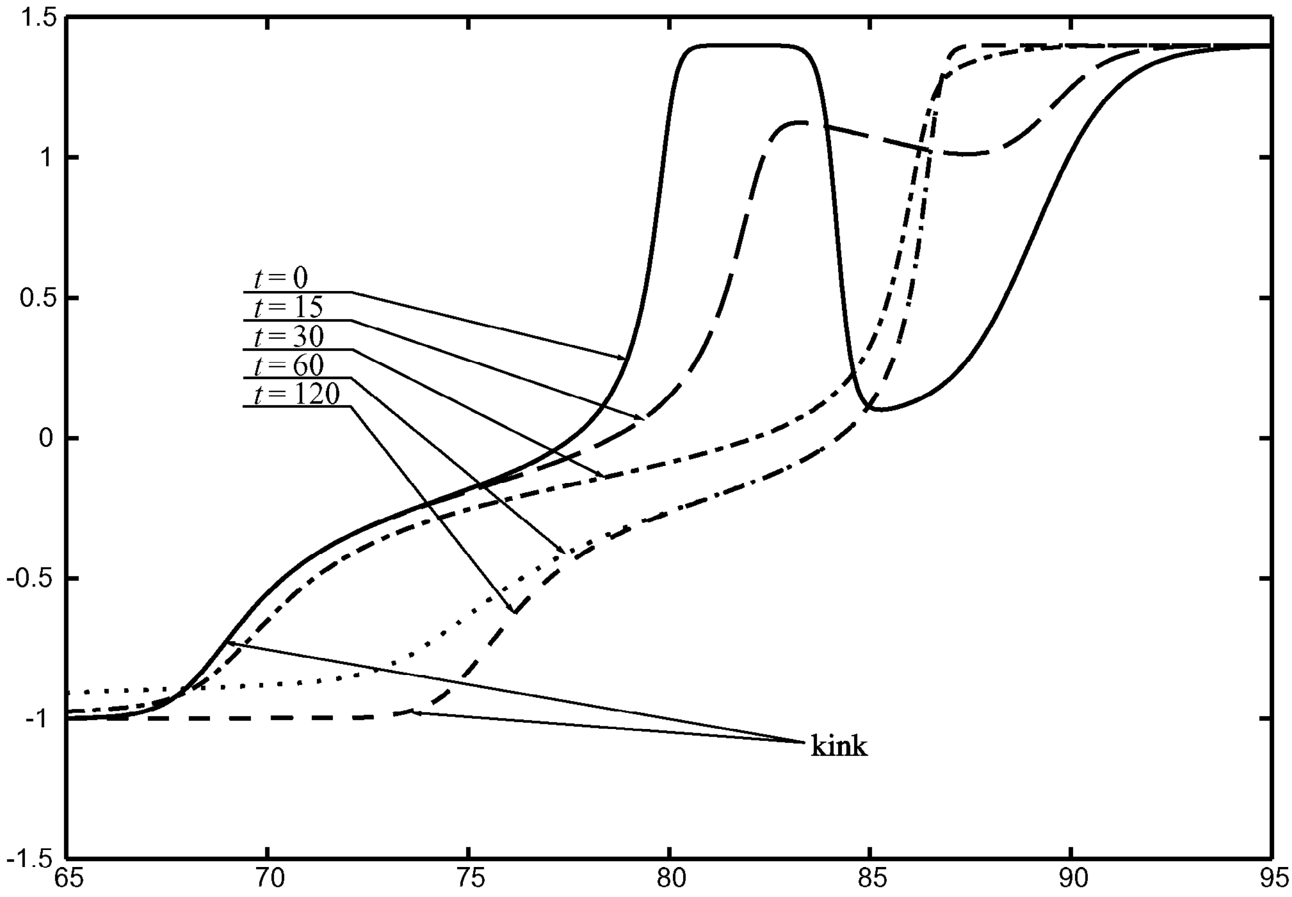

Figure 6. Interaction of the v_4-kink $(\lambda_0,\lambda_1,\lambda_2)$=(0.112, 0.592, 1.48) and the localized perturbation.

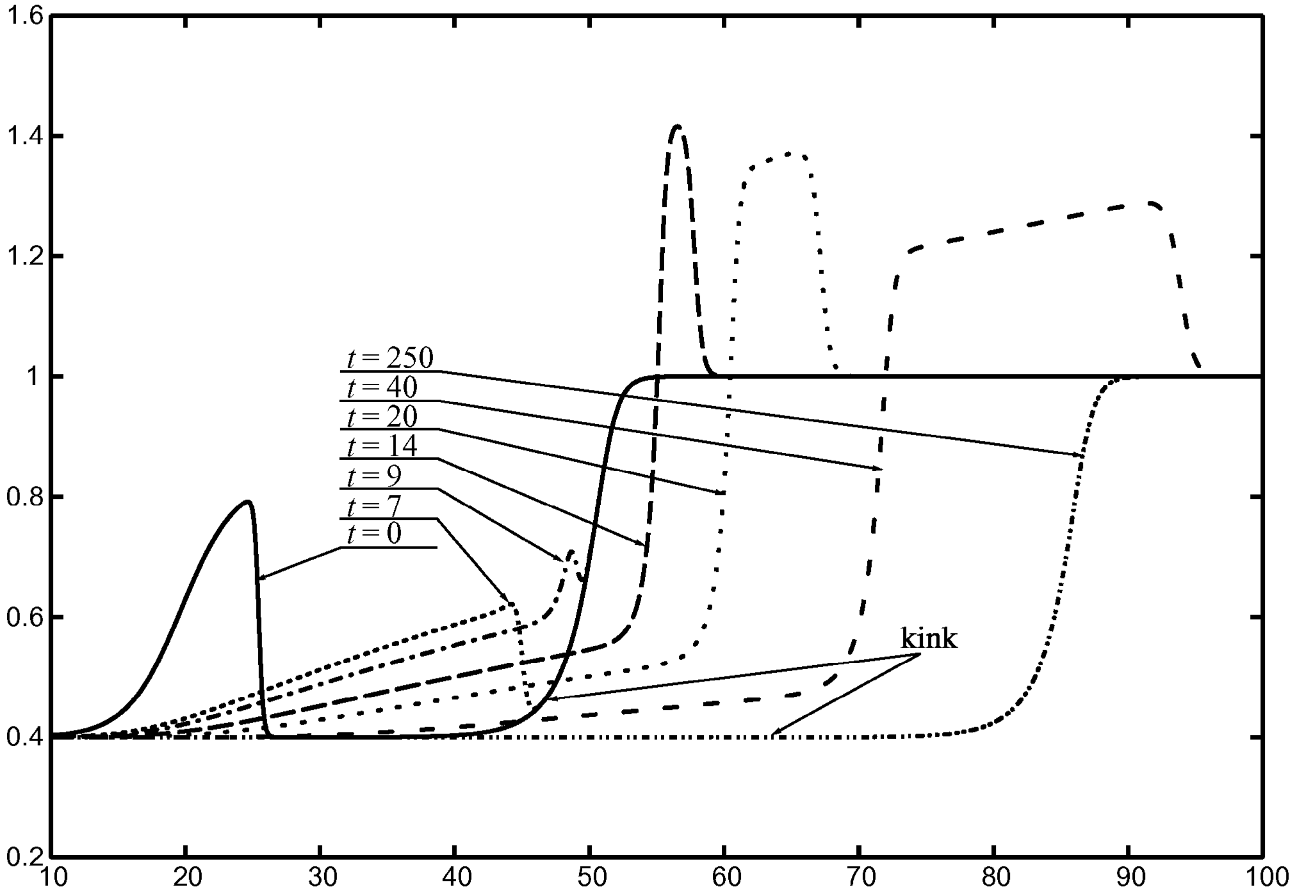

Figure 7. Interaction of the v_4-kink $(\lambda_0,\lambda_1,\lambda_2)$=(0, -0.56, 1.56) and the localized perturbation with the amplification of the latter.

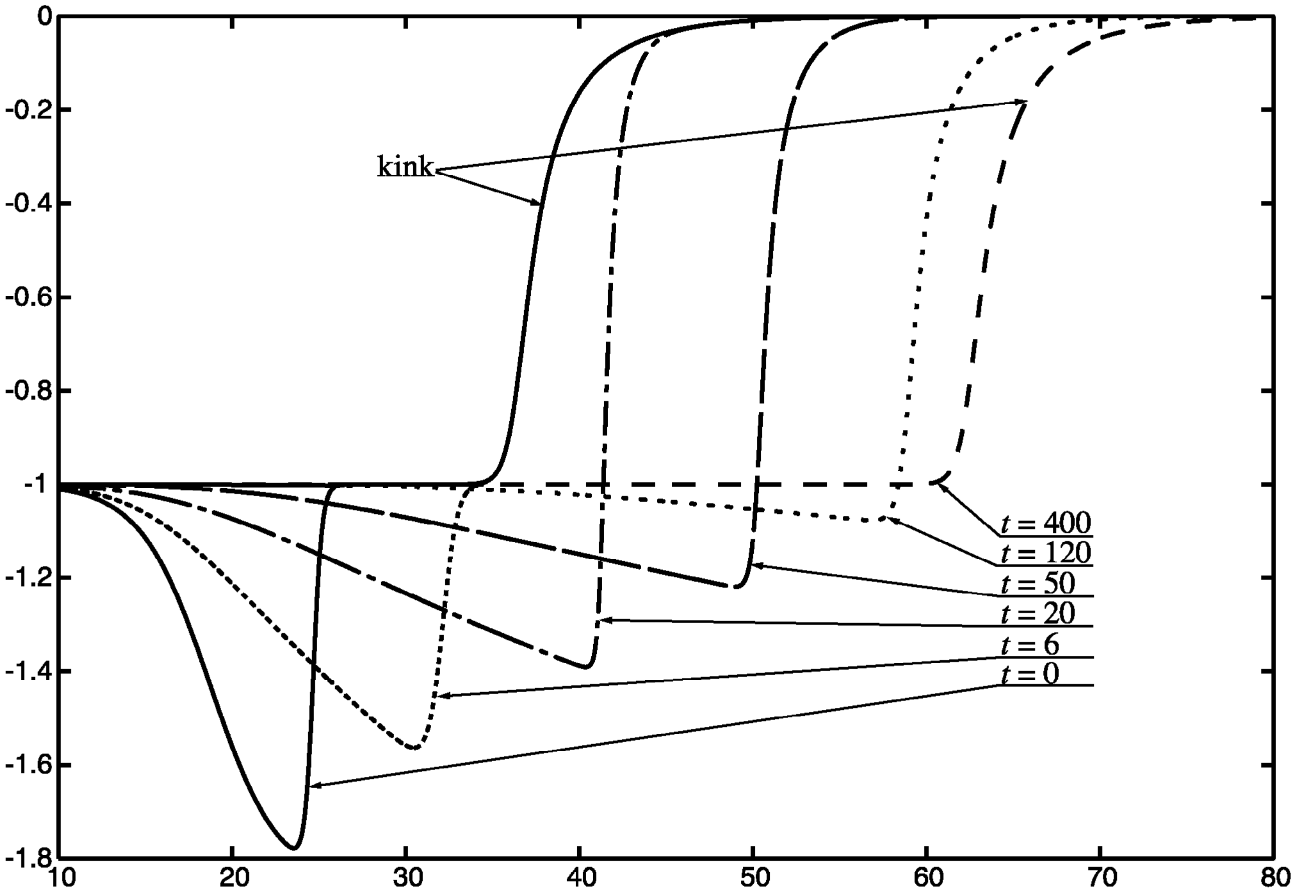

Figure 8. Interaction of the v_4-kink $(\lambda_0,\lambda_1,\lambda_2)$=(0, -0.25, 0.75) and the localized perturbation with the full absorption of the latter.