A multidimensional superposition principle: classical solitons II

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November 19, 2002

Short title: A multidimensional superposition principle II

Abstract

A concept introduced previously as an approach for finding superposition formulas of nonlinear PDEs and explanation of various types wave interactions in such systems is developed further, both from the theoretical and technical point of view. In its framework, in the framework of the multidimensional superposition principle, a straightforward and self-consistent technique for constructing the related invariant manifolds in soliton cases is proposed.

The method is illustrated by simple examples, which, in particular, show principle generality between conventional linear PDEs and soliton nonlinear equations. The demonstration that so-called truncated singular expansions can be associated with some sort of the above soliton invariant manifolds is presented as well.

PACS codes: 02.30.Jr, 02.30.Ik, 05.45.Yv

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1 Introduction

Although the concept proposed in [1] for finding superposition laws of differential equations on principle imposes no restrictions on the related ingredients types (in particular, they may even be solutions to another equation), but advantages of that approach are the most clear, of course, in a case of superposition formulas with solutions namely of an original equation. Perhaps in the most obvious manner it manifests itself in explanation of soliton interactions mechanism only in its framework without involving of the Inverse Scattering Transform [2]. — In the same paper [1] a structure of *general solutions* describing interactions of a soliton with other, *arbitrary*, perturbations was practically shown for nonlinear PDEs which the truncated singular expansions technique [3, 4] is applicable for.

Although apparently among the last ones are the majority of known solitonic models integrable by the IST, nevertheless there exist also 'non-integrable' soliton systems, e.g. [5, 6], interesting for applications. So, from this point of view, both developing of a general technique for direct finding invariant manifolds for an equation under consideration in the framework of the multidimensional superposition approach and the elucidation of their correlation with properties of soliton solutions are very important. This is the goal of the present work.

2 The multidimensional superposition principle and invariant manifolds of the soliton type

2.1 The definition and properties of invariant manifolds of the soliton type

Suppose we have some differential equation, linear or nonlinear, for the definiteness of the evolution type in 1D and for the simplicity not depending explicitly on the independent variables

$$\frac{\partial}{\partial t}u = E\left(\frac{\partial}{\partial x}; u\right), \qquad u = u(x,t) \tag{1}$$

with the function $u(x,t)$ being *the projection*

$$u(x,t) = u(x_1, x_2, t)|_{x_1=x_2=x} \tag{2}$$

of another function, where the original spatial variable x is split. In so doing, the latter has to satisfy the following *adjoint* to (1) equation

$$\frac{\partial}{\partial t}u = E\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}; u\right), \qquad u = u(x_1, x_2, t) \tag{3}$$

Let this equation for $u(x_1, x_2, t)$ in its turn has *invariant manifolds* [7] *of some specific type* or, simply speaking, there are the differential relations

$$Q\left(u, u_{x_1}, \ldots, u_{kx_1}\right) = 0, \qquad k \in N \tag{4}$$

$$G_i\left(\frac{\partial}{\partial x_2}; u, u_{x_1}, \ldots, u_{(k-1)x_1}\right) = 0, \qquad i = \overline{1, m};\ m \in N \tag{5}$$

compatible with (3) and each other. (The last ones (5), however, may be absent at all.)

In view of (4), (5) the new variables x_1 and x_2 appear *to be separated*, and, moreover, the dependence of $u(x_1, x_2, t)$ on x_1 is fixed, while its dependence on x_2 is arbitrary. Really, (4) fully determines the form of u with respect to x_1, but simultaneously the new functions $\varphi_j(x_2, t)$, $j = \overline{0, k-1}$ arise as parameters. This equation will be called a *soliton envelope equation.* The rest equations, equations (5), in their turn impose differential linkages or restrictions on the dependence of φ_j on the variable x_2. As a result, the corresponding solutions (2) collapse to the ingredients not interacting spatially with one another, because they simply depend on the different independent variables. In so doing, a general structure of such solutions or *a Superposition Formula* is fixed and uniquely determined by (4), (5). Such a paradigm was called *a multi-dimensional superposition principle* [1]. The fact that here there exists some fixed component associated with (4) may be interpreted as the presence of a soliton in a solution. In such a manner the variables x_1 and x_2 are linked with the latter and its perturbation respectively, and the solution $u(x_1, x_2, t)$ itself can be considered as the soliton envelope with the parameters φ_j modulated by this perturbation. Because of this it is logically to call invariant manifolds of the above form by *Invariant Manifolds of the Soliton type.*

Note 1. All the foregoing are immediately generalized both to cases of any dimension and to systems of equations. In so doing, splitting can be performed for all or only for the part of the independent variables, as required for investigation of one or another phenomenon in a model of interest.

Note 2. Equations explicitly depending on their independent variables, e.g.

$$\frac{\partial}{\partial t}u = E\left(x; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}; u\right), \qquad u = u(x,t)$$

are considered in the same way. In the general case such a problem is, as usually, reduced to the previous one for an appropriate system by means of the formal introduction of the auxiliary dependent variable X, so that one has finally the adjoint system

$$\begin{aligned}
&\frac{\partial}{\partial t}u = E\left(X; \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial t}; u\right), \qquad u = u(x_1, x_2, t),\ X = X(x_1, x_2)\\
&X_{x_1} + X_{x_2} = 1,\\
&X_t = 0
\end{aligned}$$

instead of the only adjoint equation. While it is possible to investigate particular cases with some concrete dependence of the adjoint equation with respect to x_1, x_2

$$\frac{\partial}{\partial t}u = E\left(x_1, x_2; \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial t}; u\right), \qquad u = u(x_1, x_2, t)$$

compatible with the projection $x_1 = x_2 = x$.

Note 3. Of course the independent variables x_1 and t can be introduced into equation (4). However, such a case in its turn can be reduced to consideration of the initial type equation (4) but of the $(k+2)$-th order with the appropriate additional relations of the form (5) for two new parameters. From the computational point of view the first form is preferable, while the second, implicit, one is more suitable for theoretical analysis.

Although an evidence of real correlation between the above type invariant manifolds and the existence of solitons or, in other words, an evidence of the possibility of physical separation for the related ingredients in our observing x-space demands analysis of directly a superposition formula, nevertheless a number of *the characteristic and important features of possible soliton interactions can be determined already from the form of invariant manifolds* themselves.

Suppose that at $t \to \infty$ (before and after a collision) such separation takes place indeed, i.e. in the space there are some domain where the solution $u(x_1, x_2, t)$ depends only on x_1 (the variable associated with the soliton part) and another domain where it depends already only on x_2 (the variable associated with a perturbation). In these domains the subsystem (4), (5) obviously degenerate to the only equation of one of the following forms

$$Q_{\text{soliton}}\,(u, u_{x_1}, \ldots, u_{k_{\mathrm{d}} x_1}) = 0, \qquad 0 < k_{\mathrm{d}} < k \tag{6}$$

$$Q_{\text{soliton}}\,(u) = 0 \tag{7}$$

for $u = u(x_1, t)$ and

$$G_{\text{perturbation}}\,(u, u_{x_2}, \ldots, u_{\alpha_{\mathrm{d}} x_2}) = 0, \qquad \alpha_{\mathrm{d}} \in N \tag{8}$$

$$G_{\text{perturbation}}\,(u) = 0 \tag{9}$$

or the identity

$$0 \equiv 0 \tag{10}$$

for $u = u(x_2, t)$. Consider further these cases in details.

(*a*) Equation (6) describes the envelope of an undeformed soliton. The related parameters in the solution can be distinguished to two groups. The first of them is linked with really free constants such as, maybe, wave-numbers for classical solitons or something else for multi-parameter ones. The other corresponds to parameters being modulated during interactions by a perturbation, usually this is only a phase, but other soliton parameters may be among them as well [6]. Since the values of the last ones are determined by perturbation asymptotes, then in the general case the soliton before and after an interaction is in different states. A standard phase shift or changes of the kinks velocities and wave-numbers [6] are the examples of such *switching from one state to another*.

(*b*) Equation (7) means that the related soliton can be observed only during interactions (*'ghost-soliton'*) asymptotically gathering to some constants, the zeros of (7).

(*c*) Equations (8) and (9) mean that asymptotically already perturbations aspire to some class of solutions or gather to the constants, so that they can be observed only during interactions (*'ghost-perturbations'*).

(*d*) Finally, identity (10) in its turn indicates that there exists no limitation on the form of a perturbation (a classical case).

Note 4. In physical/computer experiments, when it is possible to specify a perturbation of any type and as a consequence of any amplitude but only at a finite distance from a soliton, the case (c) with equation (9) has to give rise to exponential grow of a 'ghost-perturbation' amplitude as a soliton is approaching.

Note 5. Although the case (a) imposes no restrictions to an asymptotic form of perturbations, nevertheless it does not also contradict to the existence of such limitations in reality, because the last ones may also be associated pearly with properties of soliton components (see *Note 6* further).

Below some simplest examples are cited, which, in spite of their triviality, well demonstrate the essence of all aforesaid.

Example 1. Consider the conventional linear heat equation

$$u_t - u_{xx} = 0, \qquad u = u(x,t) \tag{11}$$

and its adjoint for the function $u(x_1, x_2, t)$

$$u_t - u_{x_1x_1} - 2u_{x_1x_2} - u_{x_2x_2} = 0 \tag{12}$$

It is simple to see that the latter, equation (12), possesses the existence of the following IMS

$$\begin{aligned} Q &= u_{x_1x_1} - ku_{x_1} = 0, \qquad k = \text{const} \\ G &= u_{x_1x_2} = 0 \end{aligned} \tag{13}$$

So, as a result, from (11) and (13) we have the SF

$$u = \varphi_1 \mathrm{e}^{kx_1 + k^2 t} + \varphi_0(x_2, t), \qquad \varphi_1 = \text{const}$$

where the 'free' function $\varphi_0(x_2,t)$ has to satisfy an equation of the original form (11). Since $\varphi_0(x_2,t)$ can, in particular, be equal to zero, the component $\varphi_1 \exp(kx_1 + k^2 t)$ is also the solution of the original equation.

Obviously, the above IMS or similar ones are suitable for any linear PDE in (1+1)D with constant coefficients and just means that the Fourier mode can be added to an arbitrary solution. Also, it is clear that for linear PDEs other IMSs corresponding to various superpositions can be constructed. Namely this IMS has been chosen to underline thereafter principal generality between linear equations and nonlinear soliton ones.

Example 2. The equation

$$u_t - \left(1 + \frac{u_x}{u}\right)^2 = 0, \qquad u = u(x,t)$$

presumably is the simplest nonlinearizable soliton equation. Its adjoint analogy

$$u_t - \left(1 + \frac{u_{x_1} + u_{x_2}}{u}\right)^2 = 0, \qquad u = u(x_1, x_2, t) \tag{14}$$

has the IMS

$$Q = u_{x_1x_1}u - 2u_{x_1}^2 - uu_{x_1} = 0$$
$$G = u_{x_1x_2}u - 2u_{x_1}u_{x_2} = 0 \tag{15}$$

which results in the following SF

$$u = \frac{1}{\varphi_1 e^{x_1} + \varphi_0^{-1}(x_2,t)}, \qquad \varphi_1 = \text{const} \tag{16}$$

with $\varphi_0(x_2,t)$ being a solution of the original form equation.

We see that (15) has the degenerations of the types (a) and (d), so that both a perturbation and the soliton part can exist independently. If $\varphi_{0x_2}(\infty,t) = 0$ or, according to (14)

$$\varphi_0(\pm\infty,t) = t + \theta_{\pm\infty}, \qquad \theta_{\pm\infty} = \text{const}$$

the solutions (16) are reduced to the expression

$$u_{\text{kink}} = \frac{1}{\varphi_1 e^{x_1} + (t+\theta_{\pm\infty})^{-1}}$$

corresponding to the chiral kink with the time-dependent velocity and amplitude. Simultaneously, we also have

$$u(+\infty,x_2,t) = 0$$
$$u(-\infty,x_2,t) = \varphi_0(x_2,t)$$

In other words, (16) may describe an interaction of the above kink with a localized perturbation. In so doing, absorption (or conversely exponential grow, see *Note 4*) takes place for the latter, while the kink is switched from one state to another.

In these examples both solutions in the SFs satisfy the initial equations. But as is demonstrated below, such a situation may be in not all cases.

Example 3. Let us consider the IMS

$$Q = -u_{x_1x_1}u + 2u_{x_1}^2 + kuu_{x_1} = 0, \qquad k = \text{const}$$
$$G = -u_{x_1x_2}u + 2u_{x_1x_2} - kuu_{x_1} = 0 \tag{17}$$

for the equation

$$u_t = u^2 + \left(\frac{u_{x_1}+u_{x_2}}{u}\right)^2, \qquad u = u(x_1,x_2,t) \tag{18}$$

adjoint to

$$u_t = u^2 + \left(\frac{u_x}{u}\right)^2, \qquad u = u(x,t)$$

When $k = 0$, we have from (17) and (18) the following SF

$$u = \frac{1}{\varphi_1 x_1 - (\varphi_1^2+1)t + \varphi_0(x_2,t)}, \qquad \varphi_1 = \text{const}$$

And this solution describes an interaction of the pole solution

$$u = \frac{1}{\varphi_1 x_1 - (\varphi_1^2 + 1)t} \tag{19}$$

of the initial equation with some other component that, however, is governed by

$$\varphi_{0_{t_2}} + (2\varphi_1)\varphi_{0_{x_2}} + \varphi_{0_{x_2}}^2 = 0 \tag{20}$$

that cannot be linked with (18). This is reflected in the fact that a perturbation is observed only during an interaction (in the domain when $u_{x_1} \neq 0$) and cannot exist separately (see *Note 5, 6*), so that

$$u(\infty, x_2, t) = 0$$

although the degenerations of (17) for $k = 0$ are of the type (a) and (d). In other words, we have a 'ghost-perturbation' here. It is not superfluous to underline that a perturbation and the pole solution are independent from each other. Moreover, if one looks to (19) and the linearized version of (20), it is seen that the pole and small enough perturbations move with the different velocities, $\varphi_1 + \varphi_1^{-1}$ and $2\varphi_1$ respectively.

The case with $k \neq 0$ leads to the SF

$$u = \frac{1}{\varphi_1 \mathrm{e}^{k(x_1 - x_2)} + \varphi_0^{-1}(x_2, t)}, \qquad \varphi_1 = \mathrm{const} \tag{21}$$

Where already only the function φ_0 satisfy the initial equation, and this SF has the quite other sense. Really, we will obtain after projection (2)

$$u(x, t) = \frac{1}{\varphi_1 + \varphi_0^{-1}(x, t)}$$

so that (21) corresponds just to the trivial transformation (one of the symmetries).

Note 6. As is seen from all the previous examples the existence of singular points for (4) and their types determine the type of the related SF, while their values determine character of an interaction, when it takes place. Together with an investigation of degenerations of (4) and (5) this makes it possible description of features of an interaction even when the equations for an IMS themselves cannot be integrated analytically.

2.2 Finding IMSs for equations with a polynomial nonlinearity

Now dwell on some questions associated with finding the invariant manifolds (4), (5) for equations (3) with a polynomial nonlinearity.

At the present moment the formal theory of overdetermined differential systems like (3)–(5) with the above type nonlinearity is imagined to be developed enough by itself. There exist a number of approaches and their computer implementations allowing one to work with such systems: prove their compatibility or, conversely, incompatibility, bring to some given form, in particular to the involutory form, and so on. Equations (3)–(5) are a typical example of such a system, compatible overdetermined system of NPDEs. The problem is that our aim here is namely determining the corresponding form of Q and G_i. Hence, for its solution we should in turn *at first derive a system of equations to Q and G_i* considering them as unknown functions and their arguments as independent variables after that work with the latter, e.g. by means of one of specialized computer algebra packages such as `CRACK` [8], `RifSimp` [9], or `DiffGrob2` [10]. As a result, the existing theory and methods have to be applied twice both when constructing the above-mentioned equations to Q, G_i and when their further solving. Unfortunately, the programs existing for these purposes allow us to work in an automatic mode only at the second stage.

Our purpose here is not and cannot be detail description of an algorithm for deriving equations to Q, G_i and determining their possible form, although such an algorithm indeed reduces to a finite number of cross-differentiations, excluding some derivatives from one equation by means of others, and so on. A description of such algorithms and the associated theory (see, e.g., [11, 12, 13]) is the separate, special field of the contemporary mathematics, on the one hand, being much beyond the scope of one or even several papers and, on the other hand, exhaustively enough elucidated in the related literature. However some specific moments associated with our problem specific and its main principles are necessary to be presented here. In so doing, for understanding them a knowledge of only basic notions (such as a weight or ranking of variables, etc) is needed, which cannot although be presented in this work but well enough and accessibly elucidated, in particular, in the manuals to the above-mentioned computer packages.

Below, the main moments and steps for finding IMSs, which at the present time in view of the form of the system (3)–(5) are imagined to be most optimal at the existing software, are presented.

In Section 2.1 the form of IMSs has been indicated in quite broad outline. It has just been said about the existence of two type equations compatible with an adjoint equation and with each other: a soliton envelope equation and linkage equations. In so doing, the last ones can be brought to various forms, and instead of some initial set of equations (5) we can use any equivalent system obtained by combination of these initial equations and their differential consequences. Because of this it is necessary to chose and fix some concrete form for them that would allow us to work effectively with such equations and clearly formulate their properties. For this it is needed first of all to introduce derivatives ranking fitting to our purposes. This is `LEX` ranking, such that differentiation with respect to t has the highest weight, and its order will be taken into account primarily, then analogously differentiation on x_2 and finally already differentiation on x_1. So, in our case the following ordering will take place

$$u_t \succ u_{\alpha x_2, k x_1} \succ u_{\alpha x_2, (k-1) x_1} \succ \ldots \succ u_{\alpha x_2} \succ u_{(\alpha-1) x_2, k x_1} \succ \ldots \succ u_{x_1} \succ u \tag{22}$$

where α is the maximal occurring order of differentiation on x_2 (for x_1 this is $(k-1)$ obviously). With regard to the introduced ranking the following form of IMSs for an adjoint equation

$$u_t = E\left[u, \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) u, \ldots, \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^n u\right], \qquad u = u(x_1, x_2, t),\ n \in N \tag{23}$$

will be called *canonical* and used on IMSs finding:
(1) A soliton envelope equation assumes to be solved with respect to its leading derivative

$$u_{kx_1} = q\Big(u_{(k-1)x_1}, u_{(k-2)x_1}, \ldots, u_{x_1}, u\Big), \qquad k \in N \tag{24}$$

(2) Linkage equations (if any)

$$G_i\Big(u_{\alpha_i x_2, ix_1}, \ldots, u\Big) = 0, \qquad \alpha_i \in N,\ i = \overline{l, k-1},\ 0 \leq l \leq k-1 \tag{25}$$

($u_{\alpha_i x_2, ix_1}$ corresponds to a leading, see (22), derivative from available ones in an equation) cannot already be simplified further by means of each other and equation (24) and contain all their differential consequences on x_1. — Without the loss of generality we will suppose further everywhere needed (in particular for *Proposition 1*) that they are presented in the form already solved with respect to these leading derivatives.

Proposition 1. For equations (25) the relation

$$\alpha_l \geq \alpha_{l-1} \geq \ldots \geq \alpha_{k-1} \tag{26}$$

takes place.
Proof. This directly follows from the unsimplification demand for equations (25) under the above ranking. Indeed, assume that there exist some integers f and h ($l \leq f < h \leq k-1$) such that

$$\alpha_f < \alpha_h$$

Then differentiating the related equation $G_f = 0$ $(h - f)$ times with respect to x_1, one has an equation with the leading derivative $u_{\alpha_f x_2, hx_1}$. But, as a result, the equation $G_h = 0$ can be simplified because its leading derivative $u_{\alpha_h x_2, hx_1}$ can be eliminated using the latter.

Proposition 2. In the system (23)–(25) equation (24) and all of equations (25) with $\alpha_j \leq n_1$, where n_1 is some arbitrary integer, themselves constitute a compatible subsystem.
Proof. Since the compatibility conditions

$$\frac{\partial}{\partial x_1} G_j = 0, \qquad j = \overline{l_1, k-1},\ l \leq l_1$$

of such a subsystem are the part of the related conditions for the whole system (23)–(25) and do not contain the derivatives $u_{j_1 x_2, j_2 x_1}$ with $j_1 > n_1$, then they have to be satisfied without regard for the rest equations (25) and, of cause, without equation (23).

Proposition 3. The number l corresponds to a quantity of 'free' functions-parameters in a solution envelope.
Proof. Since the subsystem (25) contains all its differential consequences on x_1, then we can consider it as a system of $(k - l)$ differential already only on x_2 equations to k functions $v_i = u_{ix_1}$ ($i = \overline{0, k-1}$). By this means l functions v_i remain to

be arbitrary, that corresponds to the availability of l arbitrary, in this subsystem framework, functions of the variable x_2 (and t, of course) in the expression for u.

Note 7. Usually (soliton equations) $l = 1$, i.e. there exists the only 'free' function-parameter corresponding to arbitrariness in perturbation choosing. However, the cases $l > 1$ and $l = 0$ are possible as well. The latter means that the related SF describes superposition of a soliton only with some specific perturbations. It does not mean, nevertheless, that superposition with arbitrary perturbations is impossible at all, because for this it may be necessary to consider a higher order IMS (i.e. assume a higher number of the modulated parameters in a soliton envelope). For our purposes we will further suppose that $l \neq 1$.

Proposition 4. For the greatest value α_l (26) the relation

$$\alpha_{\max} = \alpha_l \leq n(k-l)$$

is just.
Proof. Indeed, the compatibility condition for (23) and (24)

$$u_{t,kx_1} - u_{kx_1,t} = 0$$

is the expression depending on the derivatives $u_{j_1x_2,j_2x_1}$ with $j_1 \leq n_1 \leq n$ (in a nondegenerate case — when it is not satisfied identically). Respectively, this demands the availability in (25) the equations $G_{i_1} = 0$ with $\alpha_{i_1} \leq n_1$. These equations are compatible with (24) (see *Proposition 2*) but optionally with (23). In the last case equations with $\alpha_{i_2} \leq n_2 \leq n_1 n \leq n$ must attend into (25) in order to satisfy the above-mentioned compatibility conditions $\frac{\partial}{\partial t} G_{i_1} = 0$. Analogously, if these new equations $G_{i_2} = 0$ are not compatible with (23) directly, the presence of the equations with $\alpha_{i_2} \leq n_2 n \leq 3n$ is necessary and so on. Since the quantity of equations (25) is limited (*Proposition 3*), then we arrive at the above estimation.

Next, on the construction strength of equation (23), when finding IMSs, it is more comfortable to work not directly with the differential operators $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ but with the operator $\frac{\partial}{\partial x_1}$ and operator $D_x = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$. (Further for the latter we will again use the familiar subscript notation

$$\begin{aligned} u_x &\equiv D_x = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \\ u_{ix} &\equiv D_x^i = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^i, \qquad i \in N \end{aligned} \tag{27}$$

for the convenience.) It is necessary to remember, however, that by this is meant now the whole differential operators rather than simply isolated derivatives. Transformation of equations (23)–(25) from one presentation to another is trivial and unique (with regard to (24)). It is essential here that under the chosen above ranking (22) and its D_x-operator version

$$u_t \succ u_{\alpha x,kx_1} \succ u_{\alpha x,(k-1)x_1} \succ \ldots \succ u_{\alpha x} \succ u_{(\alpha-1)x,kx_1} \succ \ldots \succ u_{x_1} \succ u$$

the transformation

$$u_{j_1x,j_2x_1} = u_{j_1x_2,j_2x_1} + R\left(u_{(j_1-1)x_2,(k-1)x_1},\dots,u_{(j_1-1)x_2};\dots;u_{(k-1)x_1},\dots,u\right) \quad (28)$$
$$j_1 \in N,\ 0 \le j_2 \le k-1$$

and the revise transformation

$$u_{j_1x_2,j_2x_1} = u_{j_1x,j_2x_1} - \tilde{R}\left(u_{(j_1-1)x,(k-1)x_1},\dots,u_{(j_1-1)x};\dots;u_{(k-1)x_1},\dots,u\right)$$

do not change leading terms in G_i. In other words, when finding IMSs, *instead of equations (23)–(25) we can work with the D_x-presentation*

$$u_t = E\Big(u,\dots,u_{nx}\Big), \qquad u = u(x_1,x_2,t),\ n \in N \quad (29)$$

$$u_{kx_1} = q\Big(u_{(k-1)x_1},u_{(k-2)x_1},\dots,u_{x_1},u\Big), \qquad k \in N \quad (30)$$

$$G_i\Big(u_{\alpha_i x, ix_1},\dots,u\Big) = 0, \qquad \alpha_i \in N,\ i = \overline{l,k-1},\ 0 \le l \le k-1 \quad (31)$$

considerably more compact and simple.

Now dwell on the concrete schemes for constructing IMSs (already in the presentation (29)–(31)) and first of all on a *direct approach*.

Step 1. Consider a system initially consisting of two equations only, namely, equation (29) with a known right-hand side, adjoint to an equation of interest, and a soliton envelope equation (30) with some predetermined number k of general, both modulated and unmodulated, parameters (here already the form of the right-hand side q is beforehand unknown). Calculate the compatibility condition

$$u_{t,kx_1} - u_{kx_1,t} = 0 \quad (32)$$

for them. This expression is a polynomial with respect to the kernels u_{j_1x,j_2x_1} ($j_1 = \overline{1,n}$; $j_2 = \overline{0,k-1}$) or, in view of (28), with respect to derivatives $u_{j_1x_2,j_2x_1}$ correspondingly. (The dependence on the derivatives $u_{j_2x_1}$ ($j_2 = \overline{0,k-1}$) is unknown because is associated with q.)

The next step depends on what supposition is done by us about the essence of (32). Two variants are possible:

(a) There exists a function q such that (32) becomes the identity $0 \equiv 0$, i.e. any equations $G_i = 0$ setting linkages between the parameters are absent, and we want to find its form.

(b) The equation obtained is satisfied for some function q with regard to *all or possibly only some of* the linkage equations (31) in the IMS sought.

Step 2(a). In this case we have to obtain the system of equations to the function q for further simplification and solving or to prove its incompatibility for any q. For obtaining such a system it is necessary and enough to equate to zero coefficients at the different monomials composed from the kernels u_{j_1x,j_2x_1} ($j_2 \ne 0$).

Really, assume that (32) was already transformed with regard to (28). Since the derivatives $u_{j_1x_2,j_2x_1}$ cannot be algebraically expressed in terms of each other, then the coefficients at their different powers have to be equal to zero identically, and in particular the coefficient at the highest power of a derivative with the maximal possible order of differentiation on x_1 under the maximal available order of differentiation on

x_2, say $u^l_{j_{1\max}x_2,j_{2\max}x_1}$. But on the strength of (28) the coefficient at such a derivative is identical to the coefficient at $u^l_{j_{1\max}x,j_{2\max}x_1}$, so that equating it to zero, we thus eliminate all the related terms from (32). Consider the remaining expression and again transform it based on (28). Now already a coefficient at $u^{l-1}_{j_{1\max}x_2,j_{2\max}x_1}$ in the untransformed version analogously have to be equal to zero, and so on. However the mentioned here coefficients themselves are expressions similar to the original ones (or already the sought equations to q), and all the above in its tern is also applicable to them. As a result, we leads to the first part of our statement. The reverse is obvious.

Step 2(b). The expression for (32) written through the derivatives of q has the form (31) by itself, and our aim here is bringing it together with (30) to a compatible system. Differentiating it on x_1 needed quantity of times and mutually simplifying the resulting expressions, one has a set of the relations of the type (31) again. They can be separated to the linkage equations and the conditions of their compatibility with (30) (the determining equations to q). Since the first ones are optionally compatible with (29) yet, then further construction and consideration of the related equations is also necessary in turn. Here again two cases/steps are possible.

Step 3(a). The obtained set of the relations (31) is final, and therefore only fulfillment of the compatibility conditions with (29) is necessary in addition. The last ones are obtained by differentiating (31) on t and after simplification are added to the equation determining q.

Step 3(b). The part of the compatibility conditions (see *Step 3(a)*) is added to the relations of the type (31) obtained previously. After simplification all they together make the new candidate to (31) in the IMS sought. The rest relations are added to the determining equations to q.

Further the steps analogous *2(b)*, *3(b)* are repeating till the construction of the system (29)–(31) with needed number of equations (31) (*Note 7*). The process is finished at a step of the type *2(a)* or *3(a)*. We will also have the set of the equations to q necessary for its compatibility.

Proposition 5. Functions G_i (31) are polynomial with respect to the derivatives $u_{j_1x_2,j_2x_1}$ ($j_1 \neq 0$) and do not depend explicitly on x_1, x_2, t.
Proof. By the construction. This statement is a consequence of analogous polynomiality of equation (29) and the fact that neither (29) nor (30) depends explicitly on x_1, x_2 and t.

The direct scheme considered is complicated for realization already when $n, k > 2$. The matter is that although every step is reduced to operations standard for work with overdetermined systems, but simultaneously they demand a practically interactive mode with consideration of a large number of branches in effect repeating reasonings from *Proposition 4*. A *indirect approach* based on *Proposition 4* and *5* is more effective. In this case the various variants of all possible leading derivatives in G_i are assumed a priory and investigated separately. Without loss of generality we can assume solution with respect to the last ones, so that

$$u_t = E\Big(u, \dots, u_{nx}\Big), \qquad u = u(x_1, x_2, t),\ n \in N \tag{33}$$

$$u_{kx_1} = q\Big(u_{(k-1)x_1}, u_{(k-2)x_1}, \dots, u_{x_1}, u\Big), \qquad k \in N \tag{34}$$

$$u_{\alpha_i x, ix_1} = g_i\Big(u_{\alpha_i x,(i-1)x_1}, \dots, u\Big) = 0, \qquad \alpha_i \in N,\ i = \overline{l, k-1},\ 0 \le l \le k-1 \tag{35}$$

After that the compatibility conditions for (33)–(35)

$$u_{t,kx_1} - u_{kx_1,t} = 0$$
$$u_{t,\alpha_i x, ix_1} - u_{\alpha_i x, ix_1, t} = 0$$
$$u_{kx_1,\alpha_i x} - u_{\alpha_i x, kx_1} = 0$$

make the system for q and g_i suitable directly for automatic simlification/solving. Payment for this is the necessity to deal with all cases assumable by *Proposition 4*, while a number of them may be incompatible with a concrete form of E in (33) (see, e.g., Example 5 below). Such 'empty' branches, however, are comparatively quickly detected and discarded during calculations. For small n and k a mixed strategy is effective as well. It uses both steps from the direct scheme and the substitutions (35) reasonable in each step.

2.3 Multisoliton formulas of superposition. An universal form for IMSs. Collective and abstract variable differential operators

As a result of the procedures described above, in the case when the determining equations have a solution, we obtain the system (29)–(31) written in terms of the operator $D_x = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$. Taking into account the relation (28), this system is brought to the form (23)–(25) already with respect to the differential operators on x_1 and x_2 convenient for deriving a SF or an investigation of such. Inversely, the form (29)–(31) can be obtained from (23)–(25) by the formal replacement $u_{x_2} = D_x u - u_{x_1}$ and in turn *plays no less important role.*

First of all, since a perturbation in a SF is arbitrary ($l \neq 0$), it can again include in itself a soliton part or parts, so that the variable x_2 can be split further, say

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2'} + \dots + \frac{\partial}{\partial x_m'} + \frac{\partial}{\partial x_{m+1}'}, \qquad m \ge 2$$

where x_i' ($i = \overline{2, m}$) are associated with the solitons in this perturbation. On the other hand, x_1 can, in principle, be associated with any of the available solitons, and for each of them one can write respectively (the prime is discarded hereafter)

$$u_{kx_j} = q\Big(u_{(k-1)x_j}, u_{(k-2)x_j}, \dots, u_{x_j}, u\Big), \qquad j = \overline{1, m}$$
$$G_i\Big(u_{\alpha_i x, ix_j}, \dots, u\Big) = 0, \qquad i = \overline{l, k-1}$$
$$u = u(x_1, \dots; x_{m+1}; t)$$

Here the subscript 'x', e.g. u_x, already denotes the action of the new, 'extended', operator D_x

$$D_x = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_m} + \frac{\partial}{\partial x_{m+1}}$$

By this means, A construction of an IMS 'm solitons plus a perturbation' reduces to simple multiplication of relations already found for the case 'one soliton plus a perturbation' and further compatibility verification of this new set of the equations. Of course, in the same way one can couple IMSs corresponding to different type solitons (for instance bell-shape and kink-shape ones).

Note 8. It is necessary to remember that systems constructed in this manner may appear to be incompatible, that is multisoliton solutions may simply not exist, e.g. in the case of kinks with different asymptotes.

Next, the same form equations like (30), (31) with regard to the change $x_1 \longrightarrow t_1$

$$\begin{aligned} u_{kt_1} &= q\Big(u_{(k-1)t_1}, u_{(k-2)t_1}, \ldots, u_{t_1}, u\Big) \\ G_i\Big(u_{\alpha_i x, it_1}, \ldots, u\Big) &= 0, \qquad i = \overline{l, k-1} \\ u &= u(x, t_1, t_2) \end{aligned} \tag{36}$$

will take place, if the t variable splitting

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}$$

is considering instead of x (or together with x).

Finally, the equations

$$\begin{aligned} u_\tau &= q\Big(u_{(k-1)\tau}, u_{(k-2)\tau}, \ldots, u_\tau, u\Big) \\ G_i\Big(u_{\alpha_i x, i\tau}, \ldots, u\Big) &= 0, \qquad i = \overline{l, k-1} \\ u &= u(x, t, \tau) \end{aligned} \tag{37}$$

obtained from (30), (31) by the formal change $\frac{\partial}{\partial x_1} \longrightarrow \frac{\partial}{\partial \tau}$ ($D_x \equiv \frac{\partial}{\partial x}$ in this case) also has the sense and will be compatible with (29) and each other. Such a subsystem determines the dependence of u on some free parameter τ (see *Note 9* below).

In all three cases (the splitting of x or/and t with the related 'soliton' envelope equation, the case of a free parameter) the IMSs associated with the initial equation

$$u_t = E\Big(u, \ldots, u_{nx}\Big), \qquad n \in N \tag{38}$$

can be written in the unified form

$$u_{kz} = q\Big(u_{(k-1)z}, u_{(k-2)z}, \ldots, u_z, u\Big) \tag{39}$$

$$G_i\Big(u_{\alpha_i x, iz}, \ldots, u\Big) = 0, \qquad i = \overline{l, k-1}, \;\; k, l \in N \tag{40}$$

Here the notations u_t, u_x and so on correspond to differentiation on all the related split coordinates

$$u_{i_1 t} \equiv D_t^{i_1} u = \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_{j_1}} \right)^{i_1} u$$

$$u_{i_2 x} \equiv D_x^{i_2} u = \left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_{j_2}} \right)^{i_2} u$$

and the operator $\frac{\partial}{\partial z}$ (u_z and so on) correspond to any of the following real differential operators

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial t_i}, \qquad u = u(x_1, \ldots, x_{j_2}; t_1, \ldots, t_{j_1}),\ 1 \le i \le j_1 \tag{41}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x_j}, \qquad u = u(x_1, \ldots, x_{j_2}; t_1, \ldots, t_{j_1}),\ 1 \le j \le j_2 \tag{42}$$

$$j_1, j_1 \in N$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau}, \qquad u = u(x, t, \tau) \tag{43}$$

(t_i, x_j – one of the ‘soliton’ variables). Because of this, it is reasonable to call the unified form (38)–(40) as *the universal form for IMSs* and the operators D_x (D_t) and $\frac{\partial}{\partial z}$ respectively as *the collective and abstract variable differentiation operators.* (In so doing, however, it is important to remember that in terms of the concrete coordinates equations (39), (40) will be of the absolutely different sense and lead to different expressions for the function u.

Note 9. In the case (43), $\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau}$, (40) apparently corresponds to description of nonclassical symmetries of some special type. Really, equations (38) and (40) themselves, without (39), make the compatible subsystem. In so doing, they determine dependence of the function $u(x, t, \tau)$ on the parameter τ analogously to the equation

$$u_\tau = \sigma(u, u_x, u_{xx}, \ldots)$$

with σ being a classical symmetry [14]. (The latter is the particular case of (40).) While (39) means that this dependence is fixed.

Such symmetries are associated with the invariance of the IMSs when (41) or (42) with respect to the trivial transformations $x_j \longrightarrow \tilde{x}_j + \tau$ or $t_i \longrightarrow \tilde{t}_i + \tau$, and the related invariant solutions [14] correspond to solutions $u(x, t)$ without the solitonic components.

Note 10. Underline separately that on the strength of the universal form existence the above-mentioned IMSs and such nonclassical symmetries are directly linked to one another, but an IMS involves in itself the additional requirement — the splitting of some *concrete variable.*

3 Some examples of reaction-diffusion equations with the simplest IMSs

Below the techniques briefly outlined in the previous section is illustrated by the simple examples. In so doing, some others of its aspects, in particular the possibility for classification of equations, are considered as well.

Example 4. Consider the following problem. Assume that we have the general form second-order evolution equation or more precisely already its adjoint, see (27), version

$$u_t = f\left(u, u_x, u_{xx}\right), \qquad f_{u_{xx}} \neq 0 \tag{44}$$

and want to determine all possible f in (44) and the form of q in the simplest soliton envelope equation

$$u_{x_1 x_1} = q\left(u, u_{x_1}\right) \tag{45}$$

such that any linkage equations in the IMS are absent. In other words, their compatibility condition have to be satisfied identically. The last one is the polynomial with respect to u_{xxx_1}, i.e.

$$u_{tx_1x_1} - u_{x_1x_1t} = B_2 u_{xxx_1}^2 + B_1 u_{xxx_1} + B_0 = 0$$

where in the notations

$$\begin{aligned} u_{ix,jx_1} &= u_{ij}, \qquad i \geq 0,\ j > 1 \\ u_{ix} &= u_i \end{aligned} \tag{46}$$

the coefficients B_i $(i = \overline{0,2})$ are written as

$$\begin{aligned} B_2 &= f_{2u_2} \\ B_1 &= u_{01} f_{uu_2} + u_{11} f_{u_1 u_2} \\ B_0 &= 2u_{01}u_{11} f_{uu_1} + u_{01}^2 f_{2u} - u_{01} f_u q_{u_{01}} + f_u q + u_{11}^2 f_{2u_1} \\ &\quad + u_1 f_{u_1} q_u + 2u_1 u_{11} f_{u_2} q_{uu_{01}} + u_1^2 f_{u_2} q_{2u} + u_2 f_{u_2} q_u + u_{11}^2 f_{u_2} q_{2u_{01}} - f q_u \end{aligned}$$

Here, since obviously B_i do not depend on u_{2xx_1} $(u_{2x_2x_1})$, we can equate them to zero, i.e. $B_i = 0$, and one has from $B_2 = 0$

$$f(u, u_1, u_2) = f_0(u, u_1) + f_1(u, u_1) u_2, \qquad f_1 \neq 0 \tag{47}$$

In the same manner, after substituting (47) into the rest equations $B_1 = 0$ and $B_0 = 0$, we should equate to zero the coefficients at the different powers of u_2 and then u_{11}, so finally we arrive at the system

$$\begin{aligned} & f_{1u_1} = 0 \\ & f_{1u} = 0 \\ & u_{01} f_{12u} - u_{01} f_{1u} q_{u_{01}} + f_{1u} q + u_1 f_{1u_1} q_u = 0 \\ & f_{02u_1} + f_1 q_{2u_{01}} = 0 \\ & u_{01} f_{0uu_1} + u_1 q_{uu_{01}} f_1 = 0 \\ & u_{01}^2 f_{02u} - u_{01} f_{0u} q_{u_{01}} + f_{0u} q + u_1 f_{0u_1} q_u + u_1^2 f_1 q_{2u} - f_0 q_u = 0 \end{aligned} \tag{48}$$

As seen, f_1 is a constant

$$f_1(u, u_1) = c_1 = \text{const} \neq 0 \tag{49}$$

and the system (48) is simplified further

$$f_{02u_1} + c_1 q_{2u_{01}} = 0 \tag{50}$$
$$u_{01} f_{0uu_1} + c_1 u_1 q_{uu_{01}} = 0$$
$$u_{01}^2 f_{02u} - u_{01} f_{0u} q_{u_{01}} + f_{0u} q + u_1 f_{0u_1} q_u + c_1 u_1^2 q_{2u} - f_0 q_u = 0$$

Equation (50) leads to the additional separation of the variables, and we should set

$$f_0(u, u_1) = f_{00}(u) + f_{01}(u) u_1 + f_{02}(u) u_1^2 \tag{51}$$
$$q(u, u_{01}) = q_0(u) + q_1(u) u_{01} + q_2(u) u_{01}^2 \tag{52}$$

with

$$f_{02}(u) = -c_2 q_2(u), \qquad c_2 = \text{const} \tag{53}$$

After that it is already possible to equate to zero the coefficients in the rest equations at the u_1 powers and then u_{01}. So that one has the relations

$$f_{01}(u) = c_2 \tag{54}$$
$$q_1(u) = c_3, \qquad c_3 = \text{const} \tag{55}$$

together with the equations

$$q_{02u} - q_{0u} q_2 - q_{2u} q_0 = 0 \tag{56}$$
$$f_{002u} - f_{00u} q_2 - f_{00} q_{2u} = 0 \tag{57}$$
$$f_{00u} q_0 - f_{00} q_{0u} = 0 \tag{58}$$

and, as a result, taking into account (47), (49), (51)–(55), one obtains the final form of f (44) and q (45)

$$u_t = f_{00}(u) + c_2 u_x - c_1 q_2(u) u_x^2 + c_1 u_{xx}, \qquad c_1 \neq 0 \tag{59}$$
$$u_{x_1 x_1} = q_0(u) + c_3 u_{x_1} + q_2(u) u_{x_1}^2 \tag{60}$$

with the additional linkages (56)–(58) between $q_0(u)$, $q_2(u)$ and $f_{00}(u)$. It is easy to verify the last relations correspond to the fact that (59) and (60) can be obtained from the related second-order linear equations by the simple point transformation

$$u \longrightarrow g(u)$$

with the function $g(u)$ such that

$$q_2(u) = -\frac{g_{uu}}{g_u}$$

This result is not accidental. It can be shown that in the general case absence of linkage equations indicates linearization of an equation by means of a point transformation. This example also demonstrates that we can work with equations of the general enough form with the aim of their classification.

Example 5. As the next example try to find the IMSs associated again with the second-order soliton envelope equation

$$u_{x_1x_1} = q\,(u, u_{x_1}) \tag{61}$$

for the following equation

$$u_t = f(u)u_x + u_{xx}, \qquad f_u \neq 0 \tag{62}$$

and construct the related SFs.

In the first step, calculation of the compatibility condition for (61) and (62) brings us to the expression

$$\begin{aligned} &u_{11}^2 q_{2u_{01}} + 2u_{11}\,(u_{01}f_u + u_1 q_{uu_{01}}) \\ &\qquad + u_{01}^2 u_1 f_{2u} - u_{01}u_1 f_u q_{u_{01}} + u_1 f_u q + u_1^2 q_{uu} = 0 \end{aligned} \tag{63}$$

(Here the short notations (46) have been used again.) As a consequence, the linkage equation of the only type is possible

$$u_{xx_1} = g\,(u, u_{x_1}, u_x) \tag{64}$$

(According to *Proposition 4*, one more case $u_{xxx_1} = g(u, u_{x_1}, u_x)$ could be assumed.)

In the next step, we have to calculate already the compatibility conditions between (64) and (61) and between (64) and (62). The resulting equations are as follows

$$\begin{aligned} &u_2^2 g_{2u_1} + u_2\,(u_{01}f_u + 2u_1 g_{uu_1} + 2g_{u_{01}u_1}g) + u_{01}u_1^2 f_{2u} - u_{01}u_1 f_u g_{u_{01}} \\ &\qquad - u_1^2 f_u g_{u1} + 2u_1 f_u g + 2u_1 g_{uu_{01}} g + u_1^2 g_{2u} + g_{2u_{01}} g^2 = 0 \end{aligned} \tag{65}$$

$$u_1 q_u + q_{u_{01}} g - u_{01} g_u - g_{u_{01}} q - g_{u_1} g = 0 \tag{66}$$

for $u_{txx_1} - u_{xx_1t} = 0$ and $u_{x_1x_1x} - u_{xx_1x_1} = 0$ respectively.

In principle, the system (63), (65), (66) is already suitable for processing by such specialized packages like `CRACK` [8] or `RifSimp` [9]. But here it can be additionally preliminary simplified. We see that the coefficients at the powers of u_2 in (65) have to be equal to zero, so

$$g_{2u_1} = 0$$

i.e.

$$g(u, u_{01}, u_1) = g_0(u, u_{01}) + g_1(u, u_{01})u_1 \tag{67}$$

Taking into account the last expression (67) for g, `RifSimp`, e.g., gives the following simplified system to the unknown functions $g_0(u,u_{01})$, $g_1(u,u_{01})$, $q(u,u_{01})$ and $f(u)$

$$
\begin{aligned}
g_{1u} &= -\frac{g_1^2}{u_{01}}\\
g_{1u_{01}} &= \frac{g_1}{u_{01}}\\
g_{0u} &= \frac{g_0 g_1}{u_{01}}\\
g_{0u_{01}} &= \frac{g_0}{u_{01}}\\
q_u &= -2g_1^2\\
q_{u_{01}} &= \frac{2u_{01}g_1+q}{u_{01}}\\
f_u &= -2\frac{g_1 g_0}{u_{01}^2}
\end{aligned}
$$

which is easily integrated

$$q(u,u_{01}) = \frac{2u_{01}^2}{u+c_1} + \lambda u_{01} \qquad (68)$$

$$g_0(u,u_{01}) = -\frac{c_2(u+c_1)u_{01}}{2} \qquad (69)$$

$$g_1(u,u_{01}) = \frac{u_{01}}{u+c_1} \qquad (70)$$

$$f(u) = c_2 u + c_3, \qquad c_1, c_2, c_3, \lambda = \text{const} \qquad (71)$$

As seen from (71), there exists the only equation of the type (62) with the second-order soliton envelope equation, namely the well-known Burgers' equation.

Taking into account the expressions (68)–(71) obtained and without loss of generality setting $c_3 = 0$, $c_2 = -1$ as usually for the Burgers' equation, in the $\frac{\partial}{\partial x_1}$-$\frac{\partial}{\partial x_2}$-presentation one finally has the IMS

$$u_t + (u_{x_1} + u_{x_2})u - u_{x_1x_1} - 2u_{x_1x_2} - u_{x_2x_2} \qquad (72)$$

$$u_{x_1x_1} = \frac{2u_{x_1}^2}{u+c_1} + \lambda u_{x_1} \qquad (73)$$

$$u_{x_2x_1} = u_{x_1}\left(\frac{u_{x_2}-u_{x_1}}{u+c_1} + \frac{u+c_1}{2} - \lambda\right) \qquad (74)$$

It has the following degenerations: the first

$$u_t + uu_{x_2} - u_{x_2x_2} = 0, \qquad u = u(x_2,t)$$

and the second

$$
\begin{aligned}
&u_t + uu_{x_1} - u_{x_1x_1} = 0, \qquad u = u(x_1,t)\\
&u_{x_1} = \frac{(c_1 - 2\lambda + u)(c_1+u)}{2}
\end{aligned}
$$

It is obvious that the first of them corresponds to arbitrariness of a perturbation in the related SF, while the latter indicates the form of the unperturbed soliton (kink).

The IMS (73), (74) when $\lambda \neq 0$ leads to the following SF (see Appendix for the calculation details)

$$u(x_1, x_2, t) = -2\left[\frac{\lambda e^{\lambda x_1 + \lambda(c_1-\lambda)t} + \theta_{x_2}}{\lambda e^{\lambda x_1 + \lambda(c_1-\lambda)t} + \theta}\right] - c_1 + 2\lambda, \qquad \theta = \theta(x_2, t),\ \lambda \neq 0 \tag{75}$$

In so doing, the function $\theta(x_2, t)$ satisfies the equation

$$\theta_t - \theta_{x_2x_2} + (2\lambda - c_1)\theta_{x_2} = 0$$

and (75) has the limits

$$\lim_{x_1\to+\infty} u(x_1, x_2, t) = -c_1$$

$$\lim_{x_1\to-\infty} u(x_1, x_2, t) = -2\frac{\theta_{x_2}}{\theta} - c_1 + 2\lambda \tag{76}$$

In other words, the combination (76) itself satisfies the Burgers' equation.

Analogously, for $\lambda = 0$, one obtains the SF

$$u(x_1, x_2, t) = -\frac{2(1 + \theta_{x_2})}{(x_1 + c_1 t + \theta)} - c_1, \qquad \theta = \theta(x_2, t) \tag{77}$$

when $\theta(x_2, t)$ satisfies the equation

$$\theta_t - \theta_{x_2x_2} - c_1\theta_{x_2} = 0$$

Here, however, one has

$$\lim_{x_1\to\infty} u(x_1, x_2, t) = -c_1$$

and we obtain no combination of $\theta(x_2, t)$ satisfying the initial equation.

So, from the IMS (73), (74) we have the two SFs: one (75) with the degeneration of the types (a), (d) (Section 2.1) and another (77) with the degenerate (a), (c), because, setting $\theta = 0$, in the both cases we have the unperturbed solitonic solutions.

Example 6. Although the theory allows us to set a problem of finding IMSs generally enough, e.g., 'find all the third-order evolution equations with the forth-order envelope equations', however at the present moment the similar problem already for the second-order evolution PDEs and the second-order soliton envelope equations is likely to be insolvable for reasonable CPU time at the most personal computers usually available. Moreover, in so doing, on the one hand, in the framework of a computer algorithms it is necessary to process and analyse the results of plenty branches, and, on the other hand, far not all of finding structures are associated with interesting dynamics in solutions.

In this example we will consider the following family of reaction-diffusion equations

$$u_t = f_0(u) + f_1(u)u_x + f_2(u)u_x^2 + u_{xx} \tag{78}$$

f_i and u are real-value functions. Such type modified diffusion appears in a number of interesting models (see, e.g., [15, 16]). Our aim here will be to determine all such equations having the simplest kink solution analogous to the Burgers' one associated with the soliton envelope equation

$$u_{x_1x_1} = \frac{u_{x_1}}{u}\left(2u_{x_1} + \lambda u\right), \qquad \lambda \neq 0 \tag{79}$$

i.e. the same type equation like (73) but with $\lambda \neq 0$, because $\lambda = 0$ (the pole case) here leads just to a 'ghost' perturbation ($c_1 = 0$ without loss of generality).

The compatibility condition $u_{tx_1x_1} - u_{x_1x_1t} = 0$ for (78) and (79) are as follows

$$\begin{aligned} &u_{11}^2\left(q_2 u_{01} + 2f_2\right) + 2u_{11}\left(u_{01}f_{1u} + 2u_{01}u_1 f_{2u} + u_1 q_{uu_{01}}\right) \\ &+u_{01}^2 f_{02u} - u_{01}f_{0u}q_{u_{01}} + f_{0u}q + u_{01}^2 u_1 f_{12u} - u_{01}u_1 f_{1u}q_{u_{01}} + u_1 f_{1u} q \\ &+u_{01}^2 u_1^2 f_{22u} - u_{01}u_1^2 f_{2u}q_{u_{01}} + u_1^2 f_{2u} q + u_1^2 q_{2u} - q_u f_0 + u_1^2 q_u f_2 = 0 \end{aligned} \tag{80}$$

so that only the simplest linkage equation is possible

$$u_{xx_1} = g\left(u, u_{x_1}, u_x\right) \tag{81}$$

(Another of the linkage equations according to *Proposition 4*, $u_{xxx_1} = g(u, u_{x_1}, u_x, u_{xx_1})$, appears to be incompatible with the concrete form (78).) And we have

$$\begin{aligned} &u_2^2 g_{2u_1} + u_2\left(u_{01}f_{1u} + 2u_{01}u_1 f_{2u} + 2u_1 g_{uu_1} + 2g_{u_{01}u_1}g + 2f_2 g\right) \\ &+u_{01}u_1 f_{02u} - u_{01}f_{0u}g_{u_{01}} - u_1 f_{0u}g_{u_1} + f_{0u}g + u_{01}u_1^2 f_{12u} - u_{01}u_1 f_{1u}g_{u_{01}} \\ &-u_1^2 f_{1u}g_{u_1} + 2u_1 f_{1u}g + u_{01}u_1^3 f_{22u} - u_{01}u_1^2 f_{2u}g_{u_{01}} - u_1^3 f_{2u}g_{u_1} + 3u_1^2 f_{2u} g \\ &+2u_1 g_{uu_{01}}g + u_1^2 g_{2u} - g_u f_0 + u_1^2 g_u f_2 + g_{2u_{01}}g^2 = 0 \end{aligned} \tag{82}$$

$$u_1 q_u + q_{u_{01}}g - u_{01}g_u - g_{u_{01}}q - g_{u_1}g = 0 \tag{83}$$

as its compatibility conditions with (78) and (79) respectively.

The equations (80), (82), (83) with g instead of u_{11} in (80) make the system for determining the functions f_0, f_1, f_2 and q, g and can be simplified by the above-mentioned computer packages. To avoid consideration of the trivial cases of (78) linearizable by point transformations (see, e.g., (57) and (59)), we will immediately introduce the related inequality to the system under consideration

$$\left(f_{02u} + f_{0u}f_2 + f_0 f_{2u}\right)^2 + f_{1u}^2 \neq 0 \tag{84}$$

before performing the calculations.

Simplification of (80), (82)–(84) gives us several different variants. The further analysis shows that only two of them may have dynamics interesting from the physical viewpoint, namely

$$\begin{aligned} g_{0u_{01}} &= \frac{g_0}{u_{01}} \\ g_{0u} &= \frac{g_0(uf_2 + 2)}{2u} \\ g_1 &= \frac{u_{01}(2 - uf_2)}{2u} \\ f_0 &= 0 \\ f_{1u} &= -\frac{g_0(uf_2 + 2)}{uu_{01}} \\ f_{2u} &= \frac{f_2^2}{2} \end{aligned} \tag{85}$$

and

$$
\begin{aligned}
g_{0u} &= 0\\
g_{0u_{01}} &= \frac{g_0}{u_{01}}\\
g_1 &= 2\frac{u_{01}}{u}\\
f_{02u} &= \frac{2}{u^2u_{01}^2}\left(uu_{01}^2f_{0u} + u^2f_2g_0^2 + 2ug_0^2 - u_{01}^2f_0\right)\\
f_{1u} &= -2\frac{g_0(uf_2+2)}{uu_{01}}\\
f_{2u} &= -2\frac{(1+uf_2)}{u^2}
\end{aligned}
\tag{86}
$$

with

$$
g(u,u_{01},u_1) = g_0(u,u_{01}) + g_1(u,u_{01})u_1 \tag{87}
$$

Proceeding in the same way as in the previous example, the form of $f_i(u)$ $(i=\overline{0,2})$ in (78), the related IMSs and SFs are easily found. Below the final results are presented.

The system (85) with (87) in the general case (it also contains the Burgers case as a degeneration) leads to the following expression

$$
u_t = \left(\frac{2c_1c_2}{c_1-u} + c_3\right)u_x + \left(\frac{2}{c_1-u}\right)u_x^2 + u_{xx}, \quad c_1, c_2, c_3 = \text{const},\ c_1c_2 \neq 0 \tag{88}
$$

(at $c_1c_2 = 0$, according to (84), it is linearizable by a point transformation) with the IMS (79), (81) in the D_x-presentation

$$
\begin{aligned}
u_{x_1x_1} &= \frac{u_{x_1}}{u}\left(2u_{x_1} + \lambda u\right), \qquad \lambda \neq 0\\
u_{xx_1} &= \frac{u_{x_1}}{u-c_1}\left[c_2u + (2u-c_1)\frac{u_x}{u}\right]
\end{aligned}
$$

the latter corresponds to the SF

$$
u(x_1,x_2,t) = \frac{A(x_2,t)}{e^{\lambda[x_1+(\lambda-2c_2-c_3)t]+\varphi(x_2,t)}+1} \tag{89}
$$

where the phase $\varphi(x_2,t)$ are linked by the integral relation with the amplitude $A(x_2,t)$

$$
\varphi(x_2,t) = \int\left(\frac{A_{x_2}+c_2A}{A-c_1} - \lambda\right)dx_2
$$

The solutions (89) have the properties (here $\lambda > 0$ for the definiteness)

$$
\begin{aligned}
&\lim_{x_1\to+\infty} u(x_1,x_2,t) = 0\\
&\lim_{x_1\to-\infty} u(x_1,x_2,t) = A(x_2,t)
\end{aligned}
$$

Simply speaking, the function A is an arbitrary solution of the original equation (88).

The solving of the other system (86) brings in the simpler result. The NPDEs (78) is as follows

$$u_t = c_1^2 c_2 + c_4 u + c_3 u^2 + \left(2\frac{c_1 c_2}{u} + c_5\right) u_x + \left(\frac{c_2 - 2u}{u^2}\right) u_x^2 + u_{xx} \tag{90}$$
$$c_i = \text{const}, \; i = \overline{1,5}$$

and is not linearizable directly if ($c_i \in \mathbb{R}$ on the condition)

$$(c_1 c_2)^2 + [c_2(c_4 - 2c_1^2)]^2 \neq 0$$

While the IMS has the form

$$u_{x_1 x_1} = \frac{u_{x_1}}{u}\left(2u_{x_1} + \lambda u\right), \qquad \lambda \neq 0$$
$$u_{x x_1} = c_1 u_{x_1} + 2\frac{u_{x_1} u_x}{u}$$

and at $c_1 = \lambda$ corresponds to the simplest nontrivial SF

$$u(x_1, x_2, t) = \frac{1}{e^{\lambda x_1 + (c_4 - c_5\lambda - \lambda^2)t} + \varphi(x_2, t)} \tag{91}$$

with the properties ($\lambda > 0$)

$$\lim_{x_1 \to +\infty} u(x_1, x_2, t) = 0$$
$$\lim_{x_1 \to -\infty} u(x_1, x_2, t) = \frac{1}{\varphi(x_2, t)}$$

So, $\varphi(x_2, t)^{-1}$ satisfies the initial equation (90). As seen from (91), the specific of the present SF is that in contrast to (89) in certain situations appearance of a singularity on the real axis is possible.

4 IMSs and truncated singular expansions

In the work [1] not only do the multidimensional superposition principle itself was introduced, but with its help the linkage between presentation of solutions by truncated expansions in the framework of the singular manifold method [3, 4] and existence of solitons in the related equations under consideration was shown as well. In so doing, from the standpoint of the multidimensional superposition principle not NPDEs themselves were considering but the equations of the following system

$$V_x = -V^2 - \frac{S}{2}$$
$$V_t = CV^2 - C_x V + \frac{CS + C_{xx}}{2}, \qquad V = V(x,t), \; S = S(x,t), \; C = C(x,t)$$

(S and C are subject of the formal compatibility condition $S_t + C_{xxx} + 2SC_x + CS_x = 0$) to the basic function V of the above-mentioned expansions

$$u(x,t) = \sum_{i=m}^{0} w_i\left(S, C, S_x, C_x, S_t, C_t, \ldots\right) V^i, \qquad m \in N \tag{92}$$

The SF for V obtained in such a way together with the auxiliary expressions for C and S thereafter lead, in view of (92), to SFs for concrete NPDEs.

To be more precise, it was demonstrated that the functions V and S, C can be presented in the following manner (t was also split there)

$$\begin{aligned} V &= \left(\frac{k+\theta_{x_2}}{2}\right)\tanh\left(\frac{kx_1+\omega t_1+\theta}{2}\right) - \frac{\theta_{x_2x_2}}{2\,(k+\theta_{x_2})}, \qquad \theta=\theta(x_2,t_2) \\ S &= -\frac{(k+\theta_{x_2})^2}{2} - \frac{3}{2}\left(\frac{\theta_{x_2x_2}}{k+\theta_{x_2}}\right)^2 + \frac{\theta_{x_2x_2x_2}}{k+\theta_{x_2}} \\ C &= -\left(\frac{\omega+\theta_{t_2}}{k+\theta_{x_2}}\right), \qquad k,\omega=\text{const};\ k\neq 0 \end{aligned} \tag{93}$$

or

$$\begin{aligned} V &= \frac{1+\theta_{x_2}}{x_1+\omega t_1+\theta} - \frac{\theta_{x_2x_2}}{2\,(1+\theta_{x_2})}, \qquad \theta=\theta(x_2,t_2) \\ S &= -\frac{3}{2}\left(\frac{\theta_{x_2x_2}}{1+\theta_{x_2}}\right)^2 + \frac{\theta_{x_2x_2x_2}}{1+\theta_{x_2}} \\ C &= -\left(\frac{\omega+\theta_{t_2}}{1+\theta_{x_2}}\right), \qquad \omega=\text{const} \end{aligned} \tag{94}$$

if the related expression (92) with $\theta = 0$ is a solution of an equation of interest, and simultaneously $\theta = 0$ satisfies its singular manifold equation. In so doing, the expression (92) is directly the SF sought with a real (not 'ghost') soliton.

Hence, in view of the form of (92), all such SFs will have the following structure

$$u = \sum_{i=m}^{0} \varphi_{i+1}(x_2,t)\left(\frac{\mathrm{e}^{kx_1+\varphi_0(x_2,t)}}{\mathrm{e}^{kx_1+\varphi_0(x_2,t)}+1}\right)^i, \qquad k\neq 0$$

for the case (93) and

$$u = \sum_{i=m}^{0} \frac{\varphi_{i+1}(x_2,t)}{(x_1+\varphi_0(x_2,t))^i}$$

for (94) respectively, that determines a soliton envelope equation. The last one can easily be derived from the appropriate 'generating' equation

$$u_{(m+1)\xi} = 0$$

where

$$\xi = \frac{\mathrm{e}^{kx_1+\varphi_0}}{\mathrm{e}^{kx_1+\varphi_0}+1}, \qquad k\neq 0,\ \varphi_0=\varphi_0(x_2,t)$$

or

$$\xi = \frac{1}{x_1+\varphi_0}, \qquad \varphi_0=\varphi_0(x_2,t)$$

In particular, when $m = 1$ one has for the first case of ξ

$$2u_{x_1x_1x_1}u_{x_1} - 3u_{x_1x_1}^2 + k^2u_{x_1}^2 = 0$$

With setting here $k = 0$, we also arrive at the equation corresponding to the second case. As a result, we can avoid the use of truncated series and consider a suitable soliton envelope equation

$$Q\left(u, u_{x_1}, \ldots, u_{(m+1)x_1}\right) = 0$$

instead. The latter together with an original NPDEs (more precisely its adjoint equation) make an initial set of equations which can be processed by any of the existing specialized computer algebra programs.

As examples, one will adduce the results for the two well-known equations, namely, the KdV and MKdV equations.

Example 7. (The MKdV equation.) Since the MKdV has the truncated singular expansion with $m = 1$ [3], the initial system for $u(x_1, x_2, t)$ has the form

$$\begin{aligned} &u_t - 6u^2 u_x + u_{xxx} = 0, \qquad u = u(x_1, x_2, t) \\ &2u_{x_1x_1x_1}u_{x_1} - 3u_{x_1x_1}^2 + k^2 u_{x_1}^2 = 0 \end{aligned} \tag{95}$$

(we will consider only the kink case and set $k > 0$ for the definiteness without loss of generality) and is closed by the following equation

$$\begin{aligned} &u_{x_1x} \mp 2uu_{x_1} = 0 \\ &u_{x_1x_1x} \mp 2\left(u_{x_1}^2 + uu_{x_1x_1}\right) = 0 \end{aligned}$$

describing the linkages to the three 'parameters' associated with the soliton envelope equation (95). The related SF

$$u(x_1, x_2, t) = \pm\left[\left(\frac{k + \theta_{x_2}}{2}\right)\tanh\left(\frac{kx_1 + \frac{k^3}{2}t + \theta}{2}\right) - \frac{\theta_{x_2x_2}}{2\left(k + \theta_{x_2}\right)}\right],$$
$$\theta = \theta(x_2, t), \;\; k > 0$$

with θ satisfying the equation

$$2\theta_t + 2\theta_{x_2x_2x_2} - \theta_{x_2}^3 - 3k\theta_{x_2}^2 - 3k^2\theta_{x_2} - \frac{3\theta_{x_2x_2}^2}{\theta_{x_2} + k} = 0, \quad \theta = \theta(x_2, t)$$

(the calibration

$$\varphi_0(x_2, t) = \frac{k^3}{2}t + \theta(x_2, t)$$

has been used for the clearness like before in Example 5, Appendix) corresponds to the superposition of the kink

$$\lim_{\theta \to 0} u(x_1, x_2, t) = \pm\frac{k}{2}\tanh\left(\frac{kx_1 + \frac{k^3}{2}t}{2}\right)$$

and an arbitrary perturbation

$$\begin{aligned} &\lim_{x_1 \to +\infty} u(x_1, x_2, t) = \pm\left[\left(\frac{k + \theta_{x_2}}{2}\right) - \frac{\theta_{x_2x_2}}{2\left(k + \theta_{x_2}\right)}\right] \\ &\lim_{x_1 \to -\infty} u(x_1, x_2, t) = \pm\left[-\left(\frac{k + \theta_{x_2}}{2}\right) - \frac{\theta_{x_2x_2}}{2\left(k + \theta_{x_2}\right)}\right] \end{aligned}$$

It shows that in the process of their interaction the latter modulates the kink's amplitude and phase and also leads to appearance of the additional additive component in the solution.

Example 8. (The KdV equation.) Since for the KdV

$$u_t + 2uu_x + u_{xxx} = 0, \qquad u = u(x_1, x_2, t)$$

$m = 2$ [3], and it requires much more computational time then in the case with $m = 1$, consider its potential version ($u = v_x$) instead. The initial system has the form

$$\begin{aligned} &v_t + v_x^2 + v_{xxx} = 0, \qquad v = v(x_1, x_2, t) \\ &2v_{x_1x_1x_1}v_{x_1} - 3v_{x_1x_1}^2 + k^2v_{x_1}^2 = 0 \end{aligned}$$

(as before let $k > 0$) and is closed by the equations

$$\begin{aligned} &3v_{x_1xxx}v_{x_1}^2 + 3v_{x_1x}^3 + 2v_{x_1}^3v_{xx} - 6v_{x_1}v_{x_1x}v_{x_1xx} = 0 \\ &3v_{x_1x_1x}v_{x_1} + v_{x_1}^3 - 3v_{x_1x}v_{x_1x_1} = 0 \end{aligned}$$

As a result, already for the KdV one has the following expression ($u = v_{x_1} + v_{x_2}$)

$$\begin{aligned} u(x_1, x_2, t) = &-\frac{3}{2}\left(k + \theta_{x_2}\right)^2 \tanh^2\left(\frac{kx_1 - k^3t + \theta}{2}\right) + 3\theta_{x_2x_2}\tanh\left(\frac{kx_1 - k^3t + \theta}{2}\right) \\ &+\frac{3}{4}\left(k + \theta_{x_2}\right)^2 + \frac{3}{4}\left(\frac{\theta_{x_2x_2}}{k + \theta_{x_2}}\right)^2 - \frac{3}{2}\left(\frac{\theta_{x_2x_2x_2}}{k + \theta_{x_2}}\right) + \frac{3}{4}k^2, \\ &\theta = \theta(x_2, t), \ k > 0 \end{aligned}$$

after the calibration

$$\varphi_0(x_2, t) = -k^3t + \theta(x_2, t)$$

with $\theta(x_2, t)$ being a solution of the equation

$$2\theta_t + 2\theta_{x_2x_2x_2} - \theta_{x_2}^3 - 3k\theta_{x_2}^2 - \frac{3\theta_{x_2x_2}^2}{\theta_{x_2} + k} = 0, \quad \theta = \theta(x_2, t)$$

Again, seen that a perturbation modulates the phase, but the general deformation of the envelope is much more complicated here. In so doing, the separated soliton and a localized ($\theta(\pm\infty, t) = \theta_{\pm\infty} = \text{const}$) perturbation will have the form (before and after an interaction)

$$\lim_{x_2 \to \pm\infty} u(x_1, x_2, t) = \frac{3}{2}k^2\left[1 - \tanh^2\left(\frac{kx_1 - k^3t + \theta_{\pm\infty}}{2}\right)\right]$$

and

$$\lim_{x_1 \to \pm\infty} u(x_1, x_2, t) = \pm 3\theta_{x_2x_2} - \frac{3}{4}\left(k + \theta_{x_2}\right)^2 + \frac{3}{4}\left(\frac{\theta_{x_2x_2}}{k + \theta_{x_2}}\right)^2 - \frac{3}{2}\left(\frac{\theta_{x_2x_2x_2}}{k + \theta_{x_2}}\right) + \frac{3}{4}k^2$$

respectively.

Note 11. Point out that from the viewpoint of the singular manifold method constructing of an IMS with the above types soliton envelopes can be considered as an alternative to the standard technique with truncated expansions substitution. Moreover, for equations possessing the Painlevé property but not admitting the solution presentation by the above expansions IMSs can be applied as an approach for summing the related infinite Laurent series and studying their properties.

5 Conclusion

The approach been demonstrated above is a formal, direct method for working with soliton and soliton-like NPDEs, simultaneously flexible and general enough because of its ground ideology. One of its advantages is that it is absolutely self-consistent and is associated with no known methods for so-called integrable equations. The next one is naturally embedding of solitonic solutions and possibility for direct and detail description of their interactions with other waves. The main difficulty of its wide application at the present moment is the necessity to deal with systems of overdetermined PDEs. The fact that the existing now computer algorithms for this are still implemented only on general purposes computer algebra systems, that does not allow one to use resources of modern powerful computers appropriately and, as a consequence, to consider complicated physical models and applied problems.

Acknowledgements

The author is indebted to many people and organizations for their help, support and attention, which made this research possible.

He is especially grateful to Professor W J Zakrzewski for the kind hospitality during the visit and work in University of Durham, his permanent participation and stimulative discussions, Drs T Wolf and A Wittkopf for the provided copies of `CRACK`, `RifSimp/CDiffElim` packages and numerous consultations, as well as to Professor A Jeffrey for invariable encouragement and help. It would also like to thank Professor A P Veselov, Dr C Athorne, and again Professor A Jeffrey for organization of the seminars in Universities of Loughborough, Glasgow and Newcastle, to Dr T Wolf and Professor P M Santini for the invitations to take part in the programme 'Integrable Systems' of the Isaac Newton Institute, Cambridge and one of the workshops in its framework. At last, the author must express his deepest acknowledgment to the sponsors, the Royal Society, the Foreign and Commonwealth Office, and NATO, for their joint financial support of the visit and research as well as the UMS MEDICIS service at the CNRS/Polytechniqe for the providing computing facilities.

Appendix

Integrating equation (73), the soliton envelope equation, one has

$$u(x_1,x_2,t) = -2\left[\frac{\lambda e^{\lambda x_1}+\varphi_1(x_2,t)}{e^{\lambda x_1}+\varphi_0(x_2,t)}\right]-c_1+2\lambda \tag{96}$$

so that the functions φ_0 and φ_1 are still undefined here. After substitution of (96) into (74), we have the linkage between them

$$\varphi_1 = \varphi_{0x_2}$$

while after the separation of the variables x_2 and x_1 (72) leads to the two equations

$$-\varphi_{0tx_2}\varphi_0 + \varphi_{0t}\varphi_{0x_2} + \varphi_{03x_2}\varphi_0 - \varphi_{02x_2}\varphi_{0x_2} + (c_1-2\lambda)\left(\varphi_{02x_2}\varphi_0 - \varphi_{0x_2}^{2}\right) = 0$$
$$-\varphi_{0tx_2} + \lambda\varphi_{0t} + \varphi_{03x_2} + (c_1-3\lambda)\varphi_{02x_2} + \lambda(3\lambda-2c_1)\varphi_{0x_2} + \lambda^2(c_1-\lambda)\varphi_0 = 0$$

which reduce to the only equation to φ_0

$$\varphi_{0t} = \varphi_{02x_2} + (c_1 - 2\lambda)\varphi_{0x_2} + \lambda(\lambda - c_1)\varphi_0, \qquad \varphi_0 = \varphi_0(x_2, t)$$

Next, introduce another function according to the relation

$$\varphi_0(x_2, t) = e^{\lambda(\lambda - c_1)t}\theta(x_2, t)$$

so that the new function $\theta(x_2, t)$ will already satisfy the equation

$$\theta_t = \theta_{x_2x_2} + (c_1 - 2\lambda)\theta_{x_2}, \qquad \theta = \theta(x_2, t)$$

possessing the trivial solution $\theta = 0$.

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